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**BEAM HARDENING CORRECTION FOR STANDARD
COMPUTED TOMOGRAPHY USING DUAL-MATERIAL DECOMPOSITION**

**by
JULIA K. OLDER, B.Sc.**

**A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of**

Master of Science

Department of Physics

Carleton University

Ottawa, Ontario

November 25, 1991

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**BEAM HARDENING CORRECTION FOR STANDARD
COMPUTED TOMOGRAPHY USING DUAL-MATERIAL DECOMPOSITION**

submitted by Julia K. Older, B.Sc.

in partial fulfilment of the requirements for
the degree of Masters of Science

Thesis Supervisor

Chair, Department of Physics

Carleton University

January 17, 1992

ABSTRACT

A non-linear algorithm for beam-hardening correction in Computed Tomography is described. A least square matrix formulation which avoids the necessity for matrix inversion is developed to provide a framework for analysis of point spread functions and convergence of iterative algorithms. The reconstruction algorithm incorporates the scanner energy spectrum and the material energy dependence using a mapping from the linear attenuation coefficients to the region of the dual-material plane defined by the human body. In the process a water-aluminum decomposition is also generated. Convergence to a visually acceptable monoenergetic image usually occurs within two to three iterations. Simulations of a 20 cm diameter water phantom with two symmetrically placed 5 cm diameter bone insets showed a reduction in the root mean square difference (in CT number) by a factor of 13.7 overall and 14.4 in the bone.

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CHAPTER 1: INTRODUCTION

1.1 OVERVIEW

The contents of the Introduction are intended to provide enough background material to orient readers unfamiliar with Computed Tomography (CT). The first two sections briefly cover how the data is taken and the theory of image reconstruction. The next section covers beam hardening artefacts and methods used to correct the image for these artefacts. The last section covers dual-energy radiography as it is an important component of this project.

1.2 GEOMETRY

Since its introduction by EMI Ltd. in 1973, the medical CT scanner has undergone many changes¹. The first generation CT scanners used only one x-ray pencil beam source and one detector and therefore had a rather simple and easily understood geometry. Today 3rd and 4th generation scanners are the norm (with a 5th generation now being introduced into clinical use) with multiple detectors and using fan beam geometry. Fortunately there exists a geometric transformation of the more advanced systems' data sets to that of the first generation². This transformation thereby allows us to investigate any new form of manipulation of the data set by considering only the simpler geometry of the first generation.

The basic geometry of the first generation scanner can be seen

in Fig. 1.1a. For a given transverse slice, an object has a linear attenuation coefficient distribution $\mu(x,y)$ defined for all points P in the slice. It is this map that CT attempts to determine as accurately as possible. If we consider the x - y axes to be rotated by an angle ϕ we define the rotated x and y axes to be x_r and y_r respectively. A line L drawn between the source and detector will be parallel to the y_r axis.

The data consist of a set of translational readings in x_r of the intensities transmitted through the object for a fixed angle. The source and detector are rotated by an angle $\Delta\phi$ and a new set of translational readings is recorded (see Fig. 1.1b). This is repeated until the source and detector have been rotated through 180° (Fig. 1.2). In theory projections of greater angles need not be considered as these are duplicates of those in our original data set; in reality data is taken from 360° and the opposite values averaged so as to minimize magnification. A more detailed discussion of the data will be covered in the section describing the simulation program.

1.3 THEORY OF IMAGE RECONSTRUCTION

1.3.1 Theory

The theory of CT image reconstruction is based on the assumptions that we have a continuous data set and have a monoenergetic source. For a monoenergetic pencil beam the photon number fluence Φ , along the line L , observed at the detector is

b

a

Fig. 1.1 first generation CT geometry. a) The set of translation ray paths for angles 0 and ϕ . b) All rays in a translational set defined by the angle ϕ are parallel to the rotated y axis.

Fig. 1.2 Sinogram - image of simulated data set for a cylindrical 20 cm diameter water phantom with two 5 cm diameter bone insets. One row of pixels in image corresponds to one translational data set. The rotational angle ϕ increases from one row to the next in steps of one $\Delta\phi$. (Grey level numbers are actual output values times 10^4 .)

$$\Phi = \Phi_0 e^{-\int \mu(r) dl} \quad (1.1)$$

where Φ_0 is the fluence observed with no object present. With respect to our rotated frame of reference this then becomes

$$\Phi_\phi(x_r) = \Phi_0 e^{-\int_L \mu(r) dy_r} \quad (1.2)$$

The Radon transform (or the projection) $\lambda_\phi(x_r)$ of $\mu(r)$ is defined as $\int_L \mu(r) dy_r$, therefore

$$\lambda_\phi(x_r) = \ln\left(\frac{\Phi_0}{\Phi_\phi(x_r)}\right) \quad (1.3)$$

or as we are dealing with a monoenergetic fluence

$$\lambda_\phi(x_r) = \ln\left(\frac{I_0}{I_\phi(x_r)}\right) \quad (1.4)$$

where $\frac{I_0}{I_\phi(x_r)}$ is the ratio of the respective intensities.

As we can take $\mu(r)$ to be zero outside of the scanning area then

$$\lambda_\phi(x_r) = \int_{-\infty}^{\infty} \mu(x_r, y_r) dy_r \quad (1.5)$$

and by application of the Central Slice Theorem³ of Fourier analysis

$$\mathcal{F}_1\{\lambda_\phi(x_r)\} = \Lambda_\phi(\xi_r) = \mathcal{F}_2\{\mu(x_r, y_r)\}|_{\eta_r=0}, \quad (1.6)$$

where ξ_r and η_r are respectively the Fourier transform variables of x_r and y_r . Hence a superposition of all $\Lambda_\phi(\xi_r)$ gives the complete two dimensional Fourier transform of the object. In other words the object can be uniquely reconstructed (for a continuous data set) either

by a linear superposition of the $\Lambda_\phi(\xi_r)$ set in Fourier space or equivalently by a linear superposition of the $\lambda_\phi(x_r)$ set in real space.

1.3.2 Image Reconstruction

There exist several different techniques for CT image reconstruction⁴. These can be divided into three categories: (i) those employing an iterative algebraic formulation, namely Algebraic Reconstruction Technique⁵ (ART), Simultaneous Iterative Reconstruction Technique⁶ (SIRT), and Iterative Least Squares Technique⁷ (ILST), (ii) those using discrete Fourier transforms, and (iii) convolution backprojection.

The iterative methods treat the data as $N_\phi * N_d$ linear equations in N_p unknowns to be solved where N_ϕ , N_d and N_p are respectively the number of angular increments, the number of translational detector positions and the number of image pixels. The distinction between them is that ART corrects ray-by-ray (all pixels within a given ray path are updated together), whereas SIRT and ILST correct on a point-by-point basis. (SIRT updates a given pixel immediately from the discrepancy between all the calculated and measured rays passing through a pixel. ILST updates the pixel at the end of the iteration with the least square minimized discrepancy). All of the algebraic methods are computationally expensive, have convergence problems (due to noise, etc.), and suffer from other problems that will be discussed in Chapter 2.

In Section 1.3.1 it was shown that a linear superposition of either the $\Lambda_\phi(\xi_r)$ sets in Fourier space (discrete Fourier transform method) or the $\lambda_\phi(x_r)$ sets in real space (convolution backprojection) could be used to reconstruct the image. These two techniques are essentially the same except that one is done in Fourier space and the other in real space. Due to the method in which the data is collected it is highly correlated; many rays from different angles pass through the same pixel, with the density of rays decreasing as the radius increases. Theoretic analysis in the continuous case of the point spread function gives the result that the correlation is equivalent to the data being convolved with $\frac{1}{\pi r}$. Therefore in either the Fourier or the convolution backprojection method one requires a filter to remove this $\frac{1}{\pi r}$ dependence (for a further discussion on this see Appendix 2). The choice of the type of filter will be discussed in Chapter 3.

The Fourier method consists of applying a one dimensional Fourier transform with respect to x_r on the incoming data and summing over the angles so that we obtain the function

$$B(\rho) = \frac{1}{\pi} \int_0^\pi \Lambda_\phi(\xi_r) \delta(\eta_r) d\phi. \quad (1.7)$$

One then multiplies by the filter function $\rho A(\rho)$, where $A(\rho)$ is some apodizing function (in frequency space a function consisting of an abrupt cutoff multiplied by some shaping function, e.g. $\sin(\rho/\rho') \text{circ}(\rho/\rho')$) to remove the higher frequency components, and applies a two dimensional inverse transform to the result to obtain

$$\hat{\mu}(\mathbf{r}) = \frac{1}{\pi} \mathcal{F}_2^{-1} \left\{ \rho A(\rho) \int_0^\pi \Lambda_\phi(\xi_r) \delta(\eta_r) d\phi \right\} \quad (1.8)$$

where $\hat{\mu}(\mathbf{r})$ is the estimated image⁸.

Backprojection essentially consists of smearing back the projections over their respective y_r (Fig. 1.3) giving us the function

$$b(\mathbf{r}) = \frac{1}{\pi} \int_0^\pi \lambda_\phi(\mathbf{x}_r) d\phi. \quad (1.9)$$

In analogy to the Fourier transform method we must first apply an appropriate filter function to the data to remove the $\frac{1}{\pi r}$ dependence. Where in Fourier space this was done by multiplication by $\rho A(\rho)$ in real space this is accomplished by a convolution with $a(\mathbf{x}_r)$, the one-dimensional equivalent Fourier transform of $\rho A(\rho)$, therefore

$$\hat{\mu}(\mathbf{r}) = \frac{1}{\pi} \int_0^\pi (\lambda_\phi(\mathbf{x}_r) * a(\mathbf{x}_r)) d\phi \quad (1.10)$$

where again $\hat{\mu}(\mathbf{r})$ is the estimated image.

At first glance it would appear that the Fourier transform method would be preferable to convolution backprojection. The direct implementation of the transform via FFT (Fast Fourier transform) is extremely efficient and the convolution is replaced by a simple multiplication. But in both cases a major portion of the computing time is that of interpolation (the Fourier transformed data or its image is in polar coordinates and therefore must be interpolated into Cartesian). If high resolution is required the Fourier method is only about twice as efficient⁹. Due to the simplicity of implementa-

Fig. 1.3 Convolution Backprojection of data set of Fig. 1.2. Upper left is the image after backprojection summation of angles up to 45° , upper right angles up to 90° , lower left up to 135° , and lower right the complete image. Numbers at right are in CT numbers (attenuation relative to water at some effective energy (here 60 keV) times one thousand).

tion and the fact that it is used in most commercial scanners, convolution backprojection was chosen for this project.

1.4 BEAM HARDENING

As was stated in the Section 1.3.1 two major assumptions about the data are usually made in reconstruction:

- 1) the data is acquired continuously, and
- 2) the source is monoenergetic.

These assumptions are rarely met in practice, and these failures can lead to degradation of the image by the presence of artefacts. This project is concerned with correction of one type of artefact - beam hardening - resulting from the use of a polychromatic source.

For a monoenergetic beam the relationship between $\ln(I)$ and object thickness is that of a straight line. In reality the source fluence often extends over many energies, with a large percentage at the low end. The low energy photons are preferentially attenuated i.e. the softer (lower) energies are disproportionately removed from the beam within a relatively short distance. This is known as beam hardening. Such behavior results in the relationship between $\ln(I)$ and thickness being nonlinear, with the change of $\ln(I)$ with respect to thickness decreasing as thickness increases (see Fig. 1.4). This leads to a depression in CT numbers (attenuation relative to water at some effective energy times one thousand) for rays through a thick portion of the object relative to those having short path lengths. This effect is often termed a cupping artefact (see Fig. 1.5).

Secondly, the energy dependence of $\mu(E)$ varies from material to material. Therefore bone, having a large calcium content, attenuates more strongly in the low energy range than does soft tissue such as muscle. This behavior leads to a further reduction of CT values in the region of bone (Fig. 1.6).

Fig. 1.4 $\ln\left[\frac{I_0}{I}\right]$ versus object thickness: (1) Monoenergetic, (2) Polyenergetic source.

Finally, CT scanners suffer from scattered radiation¹⁰ with a scatter/primary ratio of up to 10%. When considering scatter the intensities may be described in terms of the scatter and primary components. Letting S_0 and P_0 equal respectively the scatter and primary observed with no object present then

$$\begin{aligned} \ln\left[\frac{I_0}{I}\right] &= \ln\left[\frac{S_0 + P_0}{S + P}\right] \\ &= \ln\left[\left(\frac{P_0}{P}\right) \frac{1 + (S_0/P_0)}{1 + (S/P)}\right] \end{aligned} \quad (1.11)$$

$$\text{as } \frac{S_0}{P_0} \rightarrow 0 \quad \approx \ln\left[\frac{P_0}{P}\right] - \ln\left[1 + \frac{S}{P}\right]$$

Fig. 1.5 Left image is of a homogeneous 20 cm diameter water phantom with a 120 kVp polyenergetic source, profile of middle line shows cupping artefact. Image on right is as before but now with a 5% scatter to primary ratio, note increase in cupping artefact in profile. Image generated using a simulated data set and convolution backprojection reconstruction.

Fig. 1.6 Image of water phantom with two bone insets. Images are reconstructed using simulated data of a 120 kVp source reconstructed using convolution backprojection. Left image has no scatter, right with 5% scatter to primary ratio. Note marked depression of values between bone insets.

$$\ln\left[\frac{I_0}{I}\right] \approx \ln\left[\frac{P_0}{P}\right] - \frac{S}{p} \quad (1.12)$$

as $\frac{S}{p}$ is much less than one.

This means that as scatter is essentially constant across the object cupping of the image is enhanced (see Figs. 1.5 and 1.6).

1.5 METHODS OF BEAM HARDENING CORRECTION

1.5.1 Single Material Component

One of the most commonly used forms of beam-hardening correction approximates the object by assuming only one material is present¹¹, such as water (as the energy dependence of $\mu(E)$ for soft tissues is very close to that of water), but of varying densities. Define the mass attenuation

$$\mu^m(E) = \frac{\mu(E)}{\rho} \quad (1.13)$$

where ρ is the density. In this approach $\mu^m(E)$ is assumed constant across the object and $\rho(r)$, the density function defined at all points $P(x,y)$, is to be determined. Therefore for an object composed of only a single component the polyenergetic projection is

$$\lambda_{\text{poly}} = \ln \left(\frac{\int_0^{E_{\text{max}}} \Phi_o(E) E dE}{\int_0^{E_{\text{max}}} \Phi_o(E) E \exp[-\mu^m(E) \int \rho(r) dl] dE} \right). \quad (1.14)$$

In general λ_{poly} is almost linear with respect to $\int \rho(r)dl$, thus we may define the corrected polyenergetic transform (the equivalent monoenergetic transform) as

$$\lambda'_{\text{poly}} = k \int \rho(r)dl = f(\lambda_{\text{poly}}). \quad (1.15)$$

This relationship can be used for correction by computing a look-up table or equivalent interpolating polynomial to apply to the initial data (λ_{poly}). For the polynomial correction method we may choose the constant k such that $\frac{\lambda_{\text{poly}}}{\lambda'_{\text{poly}}}$ goes to 1 as the integral of ρ goes to zero. Using such a k and noting that λ_{poly} and λ'_{poly} are analytic functions of each other that pass through zero together, the mapping may be approximated by

$$\lambda'_{\text{poly}} = \lambda_{\text{poly}} + a_2 \lambda_{\text{poly}}^2 + \dots \quad (1.16)$$

Therefore $\frac{d\lambda_{\text{poly}}}{d\lambda'_{\text{poly}}}$ is equal to 1 in the limit as ρ goes to zero. Substituting this into the derivative of Eq. (1.14) gives

$$k = \frac{\int_0^{E_{\text{max}}} \mu^w(E) \Phi(E) E dE}{\int_0^{E_{\text{max}}} \Phi(E) E dE} = \overline{\mu^w} \quad (1.17)$$

i.e., the first moment of μ^w with respect to the spectrum energies.

In most instances when bone is present beam-hardening artefacts are greatly reduced using this approximation¹² with λ'_{poly} set equal to

$$\lambda'_{\text{poly}} = a_0 + a_1 \lambda_{\text{poly}} + a_2 \lambda_{\text{poly}}^2 \quad (1.18)$$

Fig. 1.7 Image of pelvic region using one material correction. Note depression of values between bones.

a second order polynomial of λ_{poly} . But in the presence of large amounts of bone, e.g. the pelvic region (see Fig. 1.7), this approximation breaks down. The attenuation spectrum for bone is too different from that of water-like tissue to be approximated in this manner. A more general correction would have to incorporate both materials.

1.5.2 Two Material Component

An extension of the previous method¹³ assumes that the materials comprising the object are bone mineral (taken to be $3\text{Ca}_3(\text{PO}_4)_2 \cdot \text{Ca}(\text{OH})_2$), water or mixtures of both. Setting $S(E)$ equal to the normalized source energy fluence then we define the correction factor $R(S(E))$ such that

$$\lambda_{\text{mono}} = R(S(E))\lambda_{\text{poly}} \quad (1.19)$$

where λ_{mono} is the monoenergetic λ at an effective energy E_{eff} .

The variable R is then successively approximated by the variables

$$R_w = \frac{-\mu_w^m(E_{\text{eff}})d}{\ln \int_0^{E_{\text{max}}} S(E)e^{-\mu_w^m(E)d}dE} \quad (1.20)$$

and

$$R_b = \frac{-(\mu_w^m(E_{\text{eff}})d + \mu_b^m(E_{\text{eff}})t)}{\ln \int_0^{E_{\text{max}}} S(E)\exp[-(\mu_w^m(E)d + \mu_b^m(E)t)]dE} \quad (1.21)$$

where d is the water equivalent material the ray passes through, t the bone mineral equivalent material, $\mu_w^m(E_{eff})$ is the mass attenuation coefficient of water at E_{eff} , and $\mu_b^m(E_{eff})$ the mass attenuation coefficient of bone mineral at E_{eff} .

Corrections to the image are done iteratively as follows: 1) Assume that the object is composed entirely of water and apply the correction factor R_w taken from lookup tables to the data and construct an image (this is equivalent to the one material correction of section 1.5.1). 2) Scan the image for pixels having a $\mu(E_{eff})$ value greater than that of water at E_{eff} . Compute the fractional bone mineral and water components of the pixel composition and their effective densities. These computed values are then used to find the appropriate R_b values to apply to the data. 3) reconstruct the image and assess change in $\mu(E_{eff})$, repeat steps 2) and 3) until the change in $\mu(E_{eff})$ is acceptably small.

1.6 DUAL ENERGY RADIOLOGY

Dual energy is based^{14,15} on the concept that in the diagnostic region and away from the K-edge the mass attenuation coefficient may be represented as a linear combination of two functions of energy

$$\mu_x^m(E) = a_1^m f_1(E) + a_2^m f_2(E) \quad (1.22)$$

where a_1^m and a_2^m are characteristic energy independent constants of

the material, f_1 is the energy dependence of the photoelectric effect, and f_2 is the energy dependent Klein-Nishina function. This equation may be written in several alternative ways, the most convenient of which is in terms of a material basis

$$\mu_{\xi}^m(E) = a_{\alpha}^m(\xi)\mu_{\alpha}^m(E) + a_{\beta}^m(\xi)\mu_{\beta}^m(E) \quad (1.23)$$

where $\mu_{\xi}^m(E)$ is the mass attenuation coefficient of material ξ , and α and β refer to the two basis materials, most frequently taken to be lucite and aluminum (see Fig. 1.8). In this form the energy dependence is now contained in the basis materials alone.

Fig. 1.8 Dual-material plane showing the position of material ξ . The intensity of beam after passing through a unit volume of material ξ is equal to the intensity after passing through a unit volume with density a_{lu}^m of lucite and a_{al}^m of aluminum.

Eq.(1.23) is not convenient for use with the Radon transform which is in terms of the linear attenuation coefficient. Rewriting the equation in terms of the linear attenuation¹⁵ we have

$$\mu_{\xi}(E) = a_{\alpha}(\xi)\mu_{\alpha}(E) + a_{\beta}(\xi)\mu_{\beta}(E) \quad (1.24)$$

where $a_i = a_i^m(\xi) \frac{\rho(\xi)}{\rho(i)}$. The coefficients a_{α} and a_{β} may now be thought of as the equivalent thicknesses of material α and β that give the same attenuation over all energies in the diagnostic range as a unit thickness of material ξ .

For a given ray path with polyenergetic sources the observed intensities can now be written as

$$I_t(\Lambda_{\alpha}, \Lambda_{\beta}) = \int_0^{E_{\max}} \Phi_t(E) e^{-(\Lambda_{\alpha}\mu_{\alpha}(E) + \Lambda_{\beta}\mu_{\beta}(E))} dE \quad (1.25)$$

$$\text{and} \quad I_h(\Lambda_{\alpha}, \Lambda_{\beta}) = \int_0^{E_{\max}} \Phi_h(E) e^{-(\Lambda_{\alpha}\mu_{\alpha}(E) + \Lambda_{\beta}\mu_{\beta}(E))} dE \quad (1.26)$$

$$\text{where} \quad \Lambda_{\alpha} = \int_L a_{\alpha}(r) dL \quad \text{and} \quad \Lambda_{\beta} = \int_L a_{\beta}(r) dL. \quad (1.27)$$

Letting $T_j = \ln\left(\frac{I_j(\Lambda_{\alpha}, \Lambda_{\beta})}{I_0}\right)$, then Λ_i may be approximated^{15,16} by the equation

$$\Lambda_i = k_{i,0} + k_{i,1}T_t + k_{i,2}T_h + k_{i,3}T_t^2 + k_{i,4}T_h^2 + k_{i,5}T_t T_h. \quad (1.28)$$

Having determined the Λ_i for each ray path, these computed data sets are used separately to reconstruct the two image maps (in the case of CT a_{α} and a_{β}). These maps may then be used to construct a final monoenergetic image via Eq. (1.24) at any chosen effective energy. But of more importance is the ability of this method to make fine soft-tissue discrimination. Soft-tissue discrimination in CT is of great diagnostic use, e.g. the ability to distinguish between normal

and cancerous tissue. This is not possible for standard CT implementations, as there is not enough information available with one data set alone to make fine distinctions. In the absence of *a priori* knowledge, two pieces of information are required, e.g. data at two sufficiently different energies. In Chapter 3 another method that takes into account *a priori* knowledge and requires only one data set will be discussed.

1.7 OVERVIEW OF THE THESIS

In Chap. 2 a theoretical description of CT reconstruction using matrix theory is presented. In the matrix formulation the relationship between the different methods of reconstruction is clarified. The filter component of convolution backprojection and the convergence of iterative algorithms is analyzed within this model.

Chap. 3 presents a nonlinear iterative algorithm, based on a direct physical model using the dual-material plane. The problems encountered in implementing this algorithm and solutions are discussed. The results of the algorithm for various simulated test cases are presented in terms of comparison to their respective monoenergetic images and convergence rates are discussed.

Chap. 4 contains a summary of the thesis and several proposed directions for future work.

Since this project is concerned with systematic errors it was

decided that the correction algorithm should be developed under ideal conditions. Therefore a program was written to provide ideal test data, i.e. no photon statistics and perfect detectors. In Appendix 1 a discussion of the theory behind and the algorithm that produces the test data is given.

In Appendix 2 the theory developed in Chapter 2 is used to evaluate the point spread function for various interpolation schemes. The variation of these point spread functions from the Fourier ideal is investigated and their effects on the convolution kernel is described.

Appendix 3 gives a listing of the beam-hardening correction program described in Chapter 3.

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CHAPTER II : MATRIX FORMULATION OF CT

2.1 OVERVIEW

A useful tool for investigating methods of Computed Tomography (CT) image reconstruction for overdetermined systems is that of least squares matrix operator formulation. This formulation provides first of all a framework in which the divergent methods of reconstruction can be compared and evaluated. In CT image reconstruction theory there are several different forms of reconstruction: iterative algebraic formulations (consisting of Algebraic Reconstruction Technique (ART), Simultaneous Iterative Reconstruction Technique (SIRT), and Iterative Least Squares Technique (ILST)), those using discrete Fourier transforms, and convolution backprojection. Generally, discussions of reconstruction theory divide the different methods into two camps - the algebraic and the analytic formulations - but give no theoretical connection between them. This missing link is disconcerting: since they both attempt to solve the same problem, one expects that there should be some theoretical relationship. The matrix formulation gives a simple and elegant connection between them. A correction term, usually omitted, is required to correct for the fact that we are dealing with a finite and discrete data set rather than the idealizations of Fourier analysis (see Appendix 2).

If one considers the functional relationship of the projection data ($\ln(\frac{I_0}{I})$) to the final image in matrix form, then least squares gives the minimum mean square error of reconstruction, given the

original discrete data. The usual filtered backprojection reconstruction gives almost the same result, and can be considered as an approximate solution to the least squares formulation of the reconstruction problem.

Within the convolution backprojection scheme, the matrix formulation provides a better understanding of how finiteness and quantization affect the model. The matrix formulation also has the advantage of corresponding more closely to actual implementations, thus helping to bridge the gap between theory and practice. Using an algorithm to simulate the behavior about the image origin we have approximated the effect of the various methods of interpolation and the interactions between filters and interpolation algorithms (see Appendix 2).

2.2 LEAST SQUARES MINIMIZATION

In this discussion we are assuming a linear model, i.e. one which physically requires monoenergetic photons. We take our input projection data vector Y to be $N_d N_\phi$ long, where N_d is the number of detector positions for a given angle and N_ϕ is the number of angular increments. The image vector X is $N_x N_y$ long, where N_x and N_y are, respectively, the number of pixels in a row and column. Thus M , the operator that takes X (a vector of the image space $\mathcal{X}$) into the vector Y of the observation space $\mathcal{Y}$, is an $N_d N_\phi$ by $N_x N_y$ matrix (note the system is overdetermined, i.e. $N_d N_\phi$ is greater than $N_x N_y$). We will further restrict ourselves to the case of a square image, $N_x = N_y$. We

now have the relationship (see Fig. 2.1)

$$MX = Y. \quad (2.1)$$

Fig. 2.1 An example of a matrix operation of CT reconstruction $MX = Y$ (M is equal to the nearest neighbor interpolation).

This model reflects the computational form imposed by reconstruction algorithms. An entry in the image vector X is the averaged value of the linear attenuation coefficient (relative to background) for that pixel volume in the object, i.e. it is the ideal image of the actual object in which the coefficient varies continuously. The matrix M is the operational inverse of the reconstruction algorithm. The elements of M are dependent on the form of the reconstruction used.

In reality the data set Y is equal to $MX + \epsilon$, where ϵ is some error vector due to variations such as nonlinearities, photon statistics, detector response, etc.; the least squares method minimizes the norm of ϵ . As we are dealing with vectors we must also take into consideration the norms and associated inner products of the two spaces, because in general a linear mapping from one space to another does

not preserve inner products. This is accomplished by defining the inner product $\langle X|Y \rangle$ in the most general case by the matrix expression $X^T H Y$ with H being a symmetric positive definite matrix. The common assumption that $H = I$ or diagonal (with the entries equal to the error variances) in the least square minimization is tantamount to assuming that observations are uncorrelated (orthogonal) and have comparable standard deviations, so that the usual Euclidean notion of distance applies in the observation space ($\mathbf{y}$). If this were true then, for two vectors X and X' in the image space the "true" inner product should be calculated by mapping them first, i.e. $\langle MX | MX' \rangle = X^T M^T M X'$, which shows that the matrix $M^T M$ represents the induced inner product in the image space in this case. But the image space is the one where the Euclidean metric ought to be applied -- the pixels are logically independent whereas the observations are highly correlated.

Least squares minimizes the squared error ϵ^2 , given M and Y and defined in terms of the inner product on Y by

$$\begin{aligned}\epsilon^2 &= \|Y - MX\|^2 \\ &= \langle Y - MX | Y - MX \rangle \\ &= \langle Y | Y \rangle - \langle Y | MX \rangle - \langle MX | Y \rangle + \langle MX | MX \rangle.\end{aligned}\tag{2.2}$$

By symmetry of inner products

$$\begin{aligned}\epsilon^2 &= \|Y\|^2 - 2\langle Y | MX \rangle + \|MX\|^2 \\ &= Y^T H Y - 2Y^T H M X + X^T M^T H M X.\end{aligned}$$

Minimizing gives

$$d\epsilon^2 = -2Y^T H M dX + (dX)^T M^T H M X + X^T M^T H M dX = 0.$$

Again by symmetry of inner products

$$\begin{aligned} d\epsilon^2 &= -2Y^T H M dX + X^T M^T H M dX + X^T M^T H M dX \\ &= 2(X^T M^T H M - Y^T H M) dX = 0. \end{aligned}$$

Since dX is arbitrary, we have

$$X^T M^T H M - Y^T H M = 0$$

and so

$$X^T M^T H M = Y^T H M$$

assuming $(M^T H M)^{-1}$ exists,

$$X^T = Y^T H M (M^T H M)^{-1}$$

on transposing, the image which best estimates the object is

$$X = (M^T H M)^{-1} M^T H Y. \quad (2.3)$$

If the Euclidean inner product is in the image space, then we must have agreement between the inner product as calculated in the observation space $(X^T M^T H M X)$ and the Euclidean inner product $(X^T X)$. It follows that (unless $M^T M = I$) one needs to use an inner product in $\mathcal{U}$ given by H such that $M^T H M = I$. If such an H can be found then the necessity for a matrix inversion is eliminated and

$$X = M^T H Y. \quad (2.4)$$

This is essentially filtered backprojection, with H being the filter.

From this perspective it can be seen the conceptional dichotomy between filtered backprojection and algebraic reconstruction is misleading, but the principal problem of ART et al. is that they^{1,2} placed the Euclidean metric in the wrong space, thus requiring the inversion of $M^T M$. Not only is $M^T M$ uncomfortably large (for an image of 512 by 512 pixels $M^T M$ would have on the order of $2^{36} \approx 69$ billion entries) but it tends to be ill conditioned.

If the Euclidean norm is the norm of the image space, then H

gives us a natural measure of convergence in the image. As $MX = Y$ and if we have an H such that $M^T H M = I$, then

$$\begin{aligned} Y^T H Y &= X^T M^T H M X \\ &= X^T X \\ &= \langle X | X \rangle. \end{aligned}$$

Thus the magnitude of the image error is equivalent to $\sqrt{e^T H e}$ (the L_2 norm $\langle X_e | X_e \rangle$ of the image space).

2.3 THE FILTER H

The H of Eq. (2.4) is of course not unique: the unknown coefficients ($\frac{1}{2} N_d N_\phi (N_d N_\phi + 1)$ the number of independent entries in H) outnumber the independent conditions ($\frac{1}{2} N_x N_y (N_x N_y + 1)$ the number of independent equations in the formula $M^T H M = I$). In order to analyze H we can decompose it relative to the range of M . For this we can use the projection operator P defined by $M(M^T M)^{-1} M^T$, which has the same range as M , and the same null space as M^T . P is the projection operator onto the range of M since

$$1) \text{ } P \text{ is self adjoint - } (M(M^T M)^{-1} M^T)^T = M(M^T M)^{-1} M^T$$

and 2) P is idempotent - $(M(M^T M)^{-1} M^T M(M^T M)^{-1} M^T = M(M^T M)^{-1} M^T)$. Note also that $PM = M$ and $M^T P = M^T$. We can therefore decompose H as follows

$$\begin{aligned} H &= H H \\ &= (I - P + P) H (I - P + P) \\ &= (I - P) H (I - P) + (I - P) H P + P H (I - P) + P H P \\ &= H_c + H_c + H_c^T + H_m. \end{aligned} \quad (2.5)$$

H_c is the operator induced by the inner product of the errors of $\mathbf{y}$ and the H_c operators represent correlations between the data and the errors. The matrix $H_m = PHP$ is that portion of the H matrix operating on the range of M . If $M^T H M = I$ then the matrix H_m is equal to $PH P = M(M^T M)^{-1}(M^T H M)(M^T M)^{-1} M^T = M(M^T M)^{-2} M^T$ and is thus uniquely determined by M . In the standard implementation of convolution backprojection the filter can be described by H_m as no other components of H are considered. Note that as H_c and H_c^T take vectors into the null space of M^T they do not affect the reconstruction, whereas H_c^T takes vectors into the range of M^T and therefore does have an effect.

If we substitute the decomposition of H of Eq. (2.5) into Eq. (2.4) and apply $M^T(I - P) = 0$ we get

$$\begin{aligned}
 X &= M^T H Y \\
 &= M^T (PH(I - P) + PHP) Y \\
 &= M^T (P^2 H(I - P)^2 + P^2 H P^2) Y \\
 &= M^T P (PH(I - P)(I - P) + (PHP)P) Y \\
 &= M^T H_c^T (I - P) Y + M^T H_m M (M^T M)^{-1} M^T Y \\
 &= M^T H_c^T \epsilon + (M^T M)^{-1} M^T Y. \quad (2.6)
 \end{aligned}$$

The second term is just the standard set of normal equations for least squares (this is equivalent to assuming that $H=I$), therefore if $H_c^T = 0$ then Eq. (2.4) is equivalent to the usual least squares method and ART, et al in principle give results equivalent to standard convolution backprojection. This leaves the question of interpreting this first term.

The first term of Eq. (2.6) is peculiar in the conventional sense as it states that the image is partially reconstructed from the "errors" if error is defined as that which is projected out in the normal least squares sense. The point is that the concept of error with respect to H is different than that corresponding to the Euclidean norm of U . What this states is that if we assume H_c^T is 0 then some of the "errors" that would have been removed in the presence of H have been retained and some that should have been retained will have been removed. This results in a systematic error pattern in the reconstruction. If we consider the problem of beam hardening we see that there is a systematic pattern of errors that is image dependent. To the extent that this can be approximated by a linear model then we can introduce a matrix L which maps error vectors into correction terms for the data. This can be accomplished by applying the following iterative algorithm:

$$\begin{aligned}
 X_0 &= (M^T M)^{-1} M^T Y && \text{(conventional reconstruction)} \\
 \epsilon &= Y - M X_0 && \text{(errors in projections)} \\
 Y_1 &= L \epsilon && \text{(corrections to data)} \\
 X_1 &= M^T Y_1 && \text{(backproject correction)} \\
 X_2 &= X_1 + X_0 && \text{(corrected image).}
 \end{aligned}$$

This system is equivalent to Eq. (2.6) when L is equal to H_c^T . This suggests that if we have the right form of H that we can correct mild types of beam hardening directly with the filter.

In reality we must deal with an imperfect filter. At best we can only approximate H (see Appendix 2). The question then is how

to correct for this imperfection. We will now show that this can be done iteratively for the ideal linear case³.

Suppose $M^T H M = I - R$ where R is a matrix such that the norm of R is less than one, then $(M^T H M)^{-1} = (I - R)^{-1} = I + R + R^2 + R^3 + \dots$. Let $X_{i+1} = X_i + M^T H(Y - M X_i)$ for $i \geq 0$, then

$$X_0 = M^T H Y \quad (\text{standard convolution backprojection})$$

$$X_1 = X_0 + M^T H(Y - M X_0) \quad (\text{add in the convolved backprojected difference between the initial data and the reprojected image})$$

$$\begin{aligned} &= X_0 + M^T H(Y - M M^T H Y) \\ &= X_0 + M^T H Y - (M^T H M) M^T H Y \\ &= X_0 + (I - (I - R)) M^T H Y \\ &= (I + R) M^T H Y \end{aligned}$$

and

$$\begin{aligned} X_2 &= X_1 + M^T H(Y - M X_1) \quad (\text{repeat above step}) \\ &= X_1 + M^T H(Y - M(I + R) M^T H Y) \\ &= X_1 + M^T H Y - (M^T H M)(I + R) M^T H Y \\ &= X_1 + (I - (I - R)(I + R)) M^T H Y \\ &= X_1 + (I - I + R^2) M^T H Y \\ &= (I + R + R^2) M^T H Y. \end{aligned}$$

Suppose that this is true for n iterations, then

$$\begin{aligned} X_n &= (I + R + R^2 + \dots + R^n) M^T H Y \\ X_{n+1} &= X_n + M^T H(Y - M X_n) \\ &= X_n + M^T H(Y - M(I + R + R^2 + \dots + R^n) M^T H Y) \\ &= X_n + M^T H Y - (M^T H M)(I + R + R^2 + \dots + R^n) M^T H Y \\ &= X_n + (I - (I - R)(I + R + R^2 + \dots + R^n)) M^T H Y \end{aligned}$$

$$\begin{aligned}
&= X_n + (I - (I + R + R^2 + \dots + R^n) + (R + R^2 + \dots + R^{n+1}))M^T H Y \\
&= X_n + R^{n+1} M^T H Y \\
&= (I + R + R^2 + \dots + R^n + R^{n+1}) M^T H Y.
\end{aligned}$$

Therefore it is true for any n , and in the limit X converges to Eq. (2.3). We can therefore approach the ideal H results by iteratively adding the convolved backprojected error data to the image as long as our initial filter is sufficiently close.

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CHAPTER III: THE ITERATIVE ALGORITHM

3.1 OVERVIEW

As described in Chapter 1 there have been various algorithms developed to correct beam hardening artefacts. Generically these methods have been oriented to a ray-by-ray correction of the input data. This is accomplished by mapping each input value for the polyenergetic source to a corresponding monoenergetic value, given the assumption that the object is composed of only water or of water and bone¹. These mappings are complex functions dependent on the source spectrum, the effective energy, and in the water-bone case the estimated amount of bone present along the path. Since the final corrected data corresponds to the linear monoenergetic model, convolution backprojection is used to construct the final image.

As was stated in Chapter 2 in the ideal monoenergetic case an iterative scheme would converge, even in the presence of a less than ideal convolution kernel. This suggests that an iterative algorithm for the polyenergetic case would also converge, if the nonlinearity due to such a source is taken into account. The iterative method described here estimates a decomposition of the image on a pixel by pixel basis by a mapping to the dual material plane. We can then improve this estimate iteratively by comparing the original input projection data to a reprojection over the scanner's spectrum of this decomposition. This method not only provides a correction for beam hardening artefacts, but the decomposition of the image provides the

capability to make use of techniques developed for dual energy technology such as providing equivalent monoenergetic images, removal of contrast between materials, and electron density data for radiotherapy treatment planning. A similar iterative method based on the two function decomposition discussed in Section 1.6 was given by Stonestrom, Alvarez, and Macovski² in 1981, but no further development has been found in the literature.

The overall algorithm will be presented first, followed by discussions of the various parts. The program itself is given in Appendix 3.

3.2 THE ALGORITHM

Following the flow chart given in Fig. 3.1 the initial data set is first used to reconstruct an image using convolution backprojection. Using a mapping from the attenuation coefficients to the dual material plane, equivalent virtual density maps of water and aluminum (virtual is advised in this description as values may be negative) are constructed from the image. Equivalent path lengths of aluminum and water are computed and then used to reproject over the energy spectrum of the scanner. The resulting data set is then compared to a 'smoothed' version (to compensate for the point spread function) of the initial data set to provide an error data set.

The error data set is smoothed, then reconstructed and added to the initial image. It has been found using simulated data that two

Fig. 3.1 Flow chart for beam-hardening correction algorithm.

to three iterations are usually sufficient to produce highly accurate monoenergetic images, even in the presence of large amounts of bone. The resulting water and aluminum density maps can be used to generate material-based or variable monoenergetic images.

3.3 CONVOLUTION, BACKPROJECTION AND INTERPOLATION

One important point of an imaging algorithm that iterates is that an imperfect PSF (point spread function) feeds back on itself, i.e. it would spread, producing a loss of resolution on each iteration. In particular we must minimize as much as possible the effects of the Gibbs phenomenon. The standard S-L filter (Shepp and Logan)³ and the R-L filter (Ramachandran and Lakshminarayanan)⁴ have a discontinuity in frequency space, producing an oscillation at edges in the image. It was found that even on a standard reconstruction such oscillations were present. These filters also suffer from other problems (see Appendix 2). In order to remove these problems the convolution kernel devised was one that smoothly approached zero at the cutoff frequency (see Fig. 3.2). In Fourier space the filter equation is

$$Q(\xi_r) = \pi |\xi_r| \text{rect}(\xi_r/2\xi_m) \text{sinc}(\xi_r/\xi_m) \quad (3.1)$$

where ξ_m , the inverse of the detector width W (for a discussion on choice of sampling steps see Appendix 1), is the cutoff frequency, and ξ_r is the Fourier transform variable corresponding to x_r , the translational detector position. The real space equation for this filter is

$$q(x_r) = \frac{-4\xi_m^2}{\pi} \frac{\cos^2(2\pi\xi_m x_r)}{(2\xi_m x_r)^2 - 1} \quad (3.2)$$

This gives the convolution summation coefficients as

$$q(x_i) = \begin{cases} 0 & x_i = \frac{(2n+1)W}{2} \\ \frac{4}{W^2\pi} \frac{1}{1-4(i)^2} & x_i = nW \end{cases} \quad (3.3)$$

where x_i is with respect to sampling distances. Data values were taken at intervals of one detector width, therefore only the second half of Eq. (3.3) was applicable.

Fig. 3.2 Comparison in Fourier space of the filters developed by Ramachandran & Lakshminarayanan, Shepp & Logan and the one devised for this project to remove the Gibbs phenomenon. The figure axes are in units of the maximum allowed frequency.

The constant that appears within the program (Appendix 3, page 105) is not the coefficient that appears in Eq. (3.3). This stems from the following reasons: 1) Eq. (3.3) was developed for sampling at half detector widths, but since alternating coefficients are zero the summation can be taken over multiples of the detector width but the dx of the Riemann sum of the convolution is still $\frac{W}{2}$, so that the j^{th} convolved term $\lambda_j = \frac{2}{W^2\pi} \sum_{i=-N_x/2}^{N_x/2} a_i \mu_{j-i} W$, where the a_i 's are the ap-

appropriate weights. 2) The backprojection is a Riemann sum replacing the integral over the angles, that is normalized with the constant $\frac{1}{\pi}$ resulting for a given pixel the estimated μ value $\hat{\mu} = \frac{d\phi}{\pi} = \frac{1}{N_\phi}$ times the summation value, where N_ϕ is the number of angles. 3) The results are displayed in terms of mm and the incoming data is in cm therefore requiring a conversion factor of $\frac{1}{10}$ numerically equal to W . These considerations when taken together give the final leading coefficient as $(\frac{4}{W^2\pi})(\frac{W}{2})(\frac{1}{N_\phi})W = \frac{2}{\pi N_\phi}$.

For simplicity the same interpolation scheme was used in both backprojection and reprojection. Three methods of interpolation were tested: nearest neighbor, linear and area weighting. Nearest neighbor assumes that if a beam crosses more than one pixel in a row then the pixel whose center is nearest the beam center is the only contributor. As expected this form of interpolation did not perform well, producing a highly patterned artifact on iteration (see Fig. 3.3a).

Initial work done with linear interpolation showed that it introduced a 'herring bone' artefact when used iteratively (see Fig. 3.3b). Linear interpolation assumes that the pixel presents the same cross section to the detector for all angles. The cross section for an angle ψ is in fact $(\sin(\psi) + |\cos(\psi)|)$ times the pixel width. As the pixel size is the same as the detector width, this implies that for pixels where $m(\sin(\psi) + |\cos(\psi)|) \approx n$ where n and m are integers, their contribution will be overestimated and that for neighboring pixels will be underestimated, with the discrepancy maximized at 45° and 135° .

Fig. 3.3 a) Highly patterned image produced by use of nearest neighbor interpolation in iterative scheme. Image is that produced by two iterations on a 20 cm. diameter homogeneous water phantom. b) Herringbone artefact produced by use of linear interpolation on the same phantom as in a).

For a simple convolution backprojection reconstruction the discrepancies of nearest neighbor and linear interpolation do not affect the image visually. Unfortunately when used in an iterative back-projection-reprojection algorithm the discrepancies feed back on themselves, producing artefacts. Interpolation weighting schemes dependent on the area of overlap between the beam and pixel do not suffer from this problem, making them much more suitable for this type of algorithm. For this reason such a weighting scheme was chosen.

An interpolation algorithm based on area weighting cannot be as efficient as linear interpolation, although the one developed for this project is not substantially more expensive. The algorithm can be understood by referring to Figs. 3.4a and 3.4b. As the pixel width and the detector width are the same we see in Fig. 3.4a that a given pixel can be intersected by at most three beams; therefore for a given pixel we need to calculate at most three areas. What allows these computations to be accomplished efficiently is to observe (see Fig. 3.4b), that for a given angle ϕ these areas are completely determined by the five variables S , L , D , Q , and R which are functions only of ϕ and X , where X is the distance between the center of the pixel and the nearest ray (N).

As the ratio of the average distance a beam traverses across a pixel to the pixel width is equal to the ratio of the pixel area to the area of the pixel covered by the beam, we do not need to take into account the actual detector or pixel width (which are set equal).

Since we are only interested in the fractional area that a given beam and pixel overlap, therefore we take the pixel and beam width to be one unit of length. As an example, consider $\phi \leq \frac{\pi}{4}$. Let D be a line parallel to the rays for ϕ that passes through the pixel center. Since the pixel is 1 unit wide then the intersection distance D of the line D and the pixel is $\frac{1}{\cos(\phi)}$. S is defined to be the shortest perpendicular distance between a corner of the pixel and the line D and L is defined to be the longest perpendicular distance. From Fig. 3.4b we see that in this case $L + S = \cos(\phi)$ units and $L - S = \sin(\phi)$ units. Therefore we have $L = \frac{\cos(\phi) + \sin(\phi)}{2}$ and $S = \frac{\cos(\phi) - \sin(\phi)}{2}$. As the angle ϕ increases the functional forms in respect to ϕ of D , L and S change, e.g. for $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$ D is now $\frac{1}{\cos((\pi/2) - \phi)} = \frac{1}{\sin(\phi)}$, but their interpretation remain the same.

The line D bisects the pixel so that the area above the line is equal to that below equal to 0.5 units². Q is the perpendicular distance from the pixel center to the left edge of ray N and R is the perpendicular distance to the right edge, and we now define the variable $C = \frac{D}{2(L - S)}$. If $Q \geq L$ then the fractional area of overlap, AL , with ray $N - 1$ is zero, else if $S \geq Q$ ($L \geq S$ is always true) then $AL = 0.5 - QD$, else $AL = C(L - Q)^2$. If $R \geq L$ then the fractional area of overlap, AR , with ray $N + 1$ is zero, else if $S \geq R$ then $AR = 0.5 - RD$, otherwise $AR = C(L - R)^2$. The overlap fraction of ray N is just $1 - (AR + AL)$. As may be seen we need only to calculate D , S , and L once for each angle; Q and R must be calculated for each pixel with at most two comparisons for both followed by the calculation of AL and AR . The ac

tual number of comparisons and evaluations is much less than this would imply as in the vast majority of cases either AL or AR is zero. The distances Q and R correspond to the distances X (the fractional distance between the pixel center and ray N 's center) and $1 - X$ that would normally be calculated in linear interpolation, so that the extra cost of this interpolation as opposed to linear is essentially that of computing AL , AR and $1 - (AL + AR)$.

Fig. 3.4 a) Up to three rays can overlap one pixel, the line D is parallel to the ray path and passes through the pixel center thereby bisecting the pixel. b) Showing the distances S , L , Q , and R for a given pixel and nearest ray N .

The filter and backprojection interpolation combination used gives a PSF (for a normalized delta function located at the origin as input) as shown in Fig. 3.5.

Y\X	-3	-2	-1	0	1	2	3
3	0	0	0	0	0	0	0
2	0	0	1	0	1	0	0
1	0	1	19	154	19	1	0
0	0	0	154	996	154	0	0
-1	0	1	19	154	19	1	0
-2	0	0	1	0	1	0	0
-3	0	0	0	0	0	0	0

Fig. 3.5 The PSF of the convolution backprojection used. Values are response to an input of 1000 at the origin (0,0). (Note values are not CT numbers.)

3.4 WATER AND ALUMINUM DENSITY MAPS

In Section 1.6 it was shown that over the diagnostic range of energies the linear attenuation coefficient of a material could be written as a linear combination of two basis materials. An alternative basis to that of Eq. (1.23) is to rewrite it in terms of the mass attenuation coefficients

$$\mu_{\xi}(E) = b_{\alpha}(\xi)\mu_{\alpha}^m(E) + b_{\beta}(\xi)\mu_{\beta}^m(E), \quad (3.4)$$

where $b_{\alpha}(\xi) = \rho_{\xi}a_{\alpha}^m(\xi)$ and $b_{\beta}(\xi) = \rho_{\xi}a_{\beta}^m(\xi)$ respectively. The b_i coefficients maybe thought of as virtual densities such that the transmission through a unit thickness of materials α and β , with respective densities b_{α} and b_{β} , is the same as the transmission through a unit thickness of material ξ over all diagnostic energies.

In this study, elemental compositions of fat, muscle, and gen-

eral soft tissue were taken from Ref. 5. The corresponding densities were 0.92 g cm^{-3} for adipose⁵, 1.04 for muscle⁵, and 1.03 for soft tissue⁶. The elemental composition of bone was taken from Woodard⁷. Elemental composition of blood was taken from Johns and Cunningham⁸. Linear attenuation coefficients were generated using the coefficients for the elements from Plechaty et al⁹. Within this scheme the coefficients for materials of the human body used are given in Table 3.1. With respect to these values the maximum relative errors for the materials and their corresponding energy are listed.

Table 3.1 Accuracy of decomposition in terms of the dual-material plane over the energy range of 25 to 150 keV. Coefficients are derived using relative error minimization.

Material of the human body	b_{al} g cm^{-3}	b_w g cm^{-3}	RMS error of μ cm^{-1}	Max Rel. error of μ and the energy
Air	2.429×10^{-5}	1.143×10^{-3}	1×10^{-7}	7×10^{-4} , 57 keV
Adipose Tissue	-8.201×10^{-2}	0.9953	1×10^{-4}	1×10^{-3} , 33 keV
Muscle	1.022×10^{-2}	1.020	4×10^{-5}	3×10^{-4} , 125 keV
Soft Tissue	-1.478×10^{-2}	1.036	6×10^{-5}	6×10^{-4} , 33 keV
Blood	1.229×10^{-2}	1.038	6×10^{-5}	4×10^{-4} , 149 keV
Bone	2.244	-0.2320	1×10^{-2}	5×10^{-2} , 50 keV

Conventional dual energy imaging requires two pieces of information, to determine the position of a given material in the plane. But with few exceptions in the human body, air, bone and soft tissue do not appear as a homogeneous mixture¹⁰ in a voxel. Referring to Fig. 3.6, taking our basis materials to be aluminum and water, in the b_w, b_{al} plane we see that the coefficients described by the materials

Fig. 3.6 a) Region in the dual-material plane defined by homogeneous mixtures of materials of the human body. Region may be approximated to first order by a piecewise linear map. Graph shows lines of constant $\mu = 0.25 \text{ cm}^{-1}$ for energies of 40, 60, and 100 keV. b) Expanded region from a) showing relationship between adipose, soft tissue, muscle, blood and water; broken line indicates mapping.

TABLE 3.II End points of the piecewise linear mapping between the linear attenuation coefficients and the region defined by the materials of the human body in the dual-material plane for an effective energy of 60 keV.

Material of the human body	$\mu(60\text{keV})$ cm^{-1}	b_w g cm^{-3}	b_{al} g cm^{-3}
Air	2.4060×10^{-4}	1.1426×10^{-3}	2.42900×10^{-5}
*	0.166429	0.82879	-0.014251
Adipose	0.180964	0.99529	-0.082008
†	0.211714	1.03130	0.002574
Bone	0.573685	-0.23202	2.244200

* arbitrary point chosen so that line segment between air-soft tissue and bone-adipose regions is not parallel to the constant μ lines for $E > 35$ keV.

† average value of muscle, soft tissue, and blood; their points in the plane are too close to distinguish clearly.

of the human body are confined to a very small region. This region has the property that, to first order, it may be approximated by the broken line through the plane. This implies that for a given energy E in the diagnostic range there exists a piecewise linear mapping between the $\mu_{\xi}(E)$ values and the plane. If we consider the equal μ lines we find that the mapping is unique as long as the slope of these lines do not approach that of the lines in the piecewise mapping (which does not occur in the energy range of interest). A 120 kVp spectrum¹¹ and an effective energy of 60 keV (the approximate average of the reconstructed values of a simulated 20 cm diameter homogeneous water phantom derived for a 120 kVp spectrum) was used. This gives the piecewise mapping shown in Table 3.II. This mapping to the dual-material plane, together with the tabulated energy spectrum of the X-ray source, can then be used successively to improve our estimate of the object's spatial μ distribution. Use of the b_w, b_{al} plane mapping does not give the fine tissue discrimination of

dual energy. The b_w, b_{al} function maps all points along an equal μ line to the same point, therefore if two points happen to fall on the same line they are indistinguishable.

3.5 REPROJECTION

In a first generation type of scanner all the x-ray beam paths have the same length L . If we compute separately the interpolation weighted sum of the b_w and b_{al} coefficients times the thickness of one pixel along a given beam path k we now have a good approximation for the effective density-lengths of aluminum and water that the beam passed through. Multiplication by the appropriate mass attenuation coefficients for a fixed energy E is then a good approximation for the Radon transform at energy E along path k , i.e.

$$\int_k \mu(r, E) dy_r \approx \mu_w^m(E) dy \sum_{j=1}^n w_{jk} b_{wj} + \mu_{al}^m(E) dy \sum_{j=1}^n w_{jk} b_{alj} \quad (3.5)$$

where w_{jk} is the weighting of the j^{th} pixel for the k^{th} ray and n is the number of pixels in the image. The quantity $\frac{\Phi_o(E)E}{\sum_{E_{min}}^{E_{max}} \Phi_o(E)E}$ times the exponential of the negative of Eq. (3.5) gives that portion of the intensity ratio due to photons of energy E . Summing over all contributing energies gives the reprojected datum for the k^{th} path as

$$\ln \left[\frac{I_o}{I(k)} \right] \approx \ln \left\{ \frac{\sum_{i=E_{min}}^{E_{max}} \Phi_o(E_i) E_i}{\sum_{i=E_{min}}^{E_{max}} \Phi_o(E_i) E_i \exp \left[-\mu_w^m(E_i) dy \sum_{j=1}^n w_{jk} b_{wj} - \mu_{al}^m(E_i) dy \sum_{j=1}^n w_{jk} b_{alj} \right]} \right\} \quad (3.6)$$

The spectrum over which the algorithm reprojects does not have to exactly match the true spectrum of the scanner in order to correct for beam hardening, but there are limits. When the simulated data for a 20 cm diameter phantom with two bone insets of 5 cm diameter placed 5 cm to either side of the origin is produced using the spectrum¹¹ for 120 kVp, W target with 4 mm Al filtration, then using this same data in the reprojection but grouped in 5 keV steps no appreciable change in image quality or output CT number is observed, but for the spectrum taken in 10 keV steps a drop of about 10 CT numbers is observed in the bone, for steps of 20 keV the drop becomes approximately 40 CT numbers. For an input data set computed with a spectrum of 100 kVp, W target with 4 mm Al filtration, all image values are shifted upwards by about 10 CT numbers, but the overall visual quality is essentially unchanged.

3.6 THE ERROR DATA SET

The reprojected data are then compared to the smoothed version of the input data to produce the error data set. The initial data were smoothed in order to compensate for the point spread function differing from a delta function which gives rise to oscillation at boundaries in the reprojected reconstruction. The variation from a delta function for this convolution backprojection-reprojection algorithm is not large but is sufficient to contaminate the data, even on one iteration. The reprojected data of the PSF may be approximated by convolving the initial data with the set of coefficients

Relative Pixel Position	PSF Smoothing Coefficient
0	0.6410256
1	0.1730769
2	0.0064103

which were found empirically by observing the reprojected data of an input image of 1000 at the origin and zero otherwise.

The error data set is then smoothed with a Gaussian with $\sigma = 0.609$ ($3.29\sigma = 2$) modified so that the sum of the terms used was 1. This convolution was introduced to remove high frequency components that may have been introduced during the data manipulation. This data set was then convolved and backprojected, with the resultant image added to the initial reconstruction. The reprojection, error comparison, reconstruction and image addition loop is iterated until appropriate convergence criteria are met. The choice of convergence criteria will be discussed in the next section.

3.7 RESULTS

We illustrate the behavior of the algorithm for various material compositions and geometries. Fig. 3.7 displays the image of a phantom composed of water and adipose tissue; the phantom is a 20 cm diameter cylinder containing two 5 cm diameter cylinders of adipose tissue placed symmetrically about the origin, at a radius of 5 cm. Fig. 3.7a is the reconstruction of the phantom with a 60 keV monoenergetic source, Fig. 3.7b shows the reconstruction with a 120 kVp

polyenergetic source before correction and Fig. 3.7c after the algorithm is applied. Fig. 3.7d displays the differences in CT numbers of the reconstructions in Fig. 3.7c and Fig. 3.7a. The image of Fig. 3.7b displays the typical cupping artefact due to beam hardening, here 69 CT numbers. Fig. 3.7c shows the corrected reconstruction after two iterations. The maximum and minimum difference values between the corrected version and the monoenergetic were 25 and -4. These extremes occurred at boundaries between differing materials due to imperfections of the point spread function. Referring to Table 3.III we see that the root mean square (RMS) average difference taken over pixels located within the phantom diameter has been reduced by a factor of 8.53, a reduction factor of 8.78 in the region composed of water and a factor of 4.83 within the insets.

Table 3.III RMS averages of the differences between polyenergetic reconstructed images and their corresponding monoenergetic image. Comparisons are in CT numbers.

Image	Entire Phantom	Water Region	Left Inset	Right Inset
Fig. 3.7b	20.1	21.3	8.95	8.95
Fig. 3.7c	2.36	2.42	1.85	1.85
Fig. 3.8b	179	32.6	494	494
Fig. 3.8c	13.1	4.90	34.4	34.4
Fig. 3.9b	114.	21.9	4.55	446
Fig. 3.9c	8.19	4.25	1.59	28.3

Figs. 3.8 and 3.9 have similar geometries to Fig. 3.7, but the phantom materials are water and bone for Fig. 3.8 and water, bone, and soft tissue for Fig. 3.9.

Fig. 3.7 a) (upper left) The reconstructed image of a 20 cm diameter water phantom with two 5 cm diameter adipose insets, for a 60 keV monoenergetic spectrum. b) (upper right) The reconstructed image of the phantom in a) but with a 120 kVp polyenergetic source. c) (lower left) Corrected image produced after two iterations of the algorithm. d) The CT number difference between images c) and a).

Fig. 3.8 Phantom is the same as in Fig. 3.7, but insets are now composed of solid bone.

Fig. 3.9 Phantom is the same as in Fig. 3.7, but now one inset is composed of soft tissue and the other of solid bone.

$$\begin{aligned}
x_c &= x \cos(\phi) + y \sin(\phi) \\
&= A \cos(\theta) \cos(\phi) + A \sin(\phi) \sin(\theta) \\
x_c &= A \cos(\theta - \phi) \quad (\text{A1.18})
\end{aligned}$$

and

$$\begin{aligned}
y_c &= -x \sin(\phi) + y \cos(\phi) \\
&= -A \cos(\theta) \sin(\phi) + A \cos(\phi) \sin(\theta) \\
y_c &= A \sin(\theta - \phi). \quad (\text{A1.19})
\end{aligned}$$

By circular symmetry the projection of the inset onto the rotated x axis extends from $A \cos(\theta - \phi) - R_2$ to $A \cos(\theta - \phi) + R_2$. Therefore any ray path whose translational x is between $(A \cos(\theta - \phi) - R_2, W)$ and $(A \cos(\theta - \phi) + R_2 + W)$ must cross the inset. As with the homogeneous phantom we will leave discussion of partial volumes until later and now only consider those x positions such that $|x - x_c| < R_2$. The equation for the edge of the inset is $R_2^2 = (x_0 - x_c)^2 + (y - y_c)^2$ hence the chord length across the inset is

$$D_2 = 2|y - y_c| = 2\sqrt{R_2^2 - (x_0 - x_c)^2}. \quad (\text{A1.20})$$

We now have $g(x_0) = \mu_a h(x_0) + \mu_w j(x_0) + \mu_\xi k(x_0)$ where $h(x_0)$ is as for the homogeneous phantom, $j(x_0) = D - D_2$, with $k(x_0) = D_2$. Again L is equal to $h + j + k$ and the derivatives may be simplified by removing the most troublesome function -- $j(x_0)$. This gives in the monoenergetic case

$$\begin{aligned}
F(x_0) &= e^{[-\mu_1 D - \mu_2 k]} \left\{ 1 + \frac{W^2}{3!} [(-\mu_1 h'(x_0) + \mu_2 k'(x_0))^2 \right. \\
&\quad \left. - (-\mu_1 h''(x_0) + \mu_2 k''(x_0))] + C'' \right\} \quad (\text{A1.21})
\end{aligned}$$

where $\mu_2 = \mu_\xi - \mu_w$, and C'' is the correction term with respect to h and k .

curves are nearly linear in the logarithmic scale indicating a steady convergence rate down to the knee of the curve, after which the convergence slows significantly. Iterations after the knee thus yield little improvement, but the graph indicates that the iteration remains stable well past the natural stopping point. Note that the knee in the image space occurs first, indicating that the image becomes visually stable even though fine adjustments continue in the data space.

Fig. 3.10 Logarithmic graph of both the RMS of the error set (giving a measure of the convergence of the data) and the error metric $\epsilon^T H \epsilon$ (giving a measure of the image convergence), where H is the filter component of the convolution backprojection and ϵ is the error data-set. Error measures are for the water and bone phantom in Fig. 3.8.

Fig. 3.11 shows the values in the dual-material plane of selected pixels within the right bone inset of Fig. 3.8 over several iterations. The graphs of the pixels (27,0), (55,0), and (73,0) (positions are with respect to the image center and are in pixel units) show the convergence across the right inset. The monochromatic value for these pixels is that of cortical bone $(-0.2320, 2.244)$. The first two pixels reach the value for bone in six iterations but the third pixel

Fig. 3.11 Image values in the dual-material plane of points a) (27,0), b) (55,0), and c) (73,0) for Fig. 3.8, taken over six iterations. Pixel positions are relative to the image center and are located within the right bone inset. Note that the values rapidly approach their correct values in an apparent geometric progression.

lies within the region affected by the boundary effect. This results in the pixel steadily approaching the values for bone up to the fifth iteration, after which the value gradually descends back along the line. At three iterations the water component for all three pixels is within 90.2% of that of bone and the aluminum component is within 98.2%. Visually there is little difference between the images of the 3rd and 4th iterations.

Figs. 3.12 and 3.13 show the water and aluminum decomposition corresponding to Fig. 3.8 and 3.9, respectively. These would be the principal output from the algorithm in sophisticated applications involving synthesis of attenuation maps for other source spectra, for selective elimination of material, or for background contrast compression^{12,13}. Since the patient is imaged with only one beam, this method does not have the same amount of information as a rigorous dual-energy technique. Consequently in comparison the fine tissue discrimination is limited but may still be of diagnostic value.

Fig. 3.12 Upper images are the same as in Figs. 3.8b and c. Lower images are the water and aluminum density decomposition of the corrected image; density values are relative to water. Aluminum values displayed are actual values plus 1000.

Fig. 3.13 Upper images are the same as in Figs. 3.9b and c. Lower images are the water and aluminum density decomposition of the corrected image; density values are relative to water. Aluminum values displayed are actual values plus 1000.

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CHAPTER IV: CONCLUSIONS

4.1 SUMMARY

Chapter 1 covered the basic methodology of CT imaging, including the Radon transform and convolution-backprojection reconstruction. Beam hardening is a principal artefact of this imaging technology, due to the nonlinear response to a polyenergetic source. Several techniques for removing the artefact were discussed. In particular, when two data sets of sufficiently different spectral distributions are available, then the dual-material decomposition may be applied. In addition to beam-hardening correction, the dual-material decomposition provides a basis for various image manipulation techniques that are useful in diagnostic procedures.

In Chapter 2 CT reconstruction was formulated as a least squares problem, as was done in the Algebraic Reconstruction Techniques (ART) et al. Within this model it was shown that if the Euclidean norm is imposed in the image space instead of the data space (as is done in ART) then the massive matrix inversion problem is removed and we are left with a discretized version of convolution backprojection in a form directly suitable for computer implementation. In this form it was shown that, in the linear case, an iterative application would converge rapidly, suggesting the viability of a nonlinear iterative algorithm.

Such an algorithm was presented in Chapter 3. This algorithm

required the development of a new filter to minimize oscillation at boundaries and the use of an efficient interpolation technique that incorporates specific geometrical relationships between pixels and rays. Since it is necessary to reproject the image in an iterative algorithm, the energy dependence must be known. We estimate this using a piecewise linear dual-material decomposition of the image based on *a priori* knowledge of the location of biological materials in the water-aluminum plane, thus eliminating the requirement for two data sets.

Preliminary tests with simulated data have shown the convergence and stability of the algorithm. The absolute accuracy of the corrected reconstruction in the ideal case suggests possible applications to quantitative CT.

4.2 FUTURE WORK

The study reported here has used noise-free simulated projection data. In order to observe how well the algorithm behaves under normal conditions, tests of the algorithm using simulated noisy data and real scan data of phantoms and patients are now required.

At present it requires approximately 50 minutes per iteration on a Vax/VMS Version V5.4 System, an excessive amount of time when compared to that of the standard reconstruction time. For most applications it is necessary to reduce the run time. The first method would be to initially apply the standard one material correc

tion. This should eliminate one or two iterations. Secondly, the present implementation uses double precision arithmetic. If suitable precautions were taken, single precision should be sufficient. Thirdly, at present the program does an explicit integration over the spectrum in each reprojection phase; this could be replaced by two linear reprojections (for water and aluminum separately) and a table lookup for a correction term. This should reduce the time for this phase by a factor of at least 20 by removing the need for exponentials and logarithms (see Appendix 1). Finally, it was observed that the convergence trajectory for most pixels appears to be a geometric progression in a straight line in the water-aluminum plane. This would allow a direct computation of the final value after two iterations in most instances (as can be seen in Fig. 3.11).

APPENDIX 1: PHANTOM SIMULATION

A1.1 OVERVIEW

In an investigation of CT reconstruction algorithms it is necessary to provide a program that generates simulated projection data sets. First, the simulation must provide as correctly as necessary the behavior that the investigator is interested in, but no more. The addition of a too "realistic" model may in fact obscure those characteristics that one wishes to investigate. The use of an approximate anatomical phantom may so clutter the image that it is difficult to see if any improvement is accomplished in the area of interest (such as the base of the skull, see Fig. A1.1). Alternatively, the anatomical geometry may be such that the problem is minimized already by the geometry (the presence of the skull in many instances minimizes beam hardening as most of the attenuation takes place in the bone) leading to erroneous conclusions.

Second, the interactions between the data simulator and the reconstruction programs are complex and with a simple geometry if a problem arises it is much easier to spot and correct. The introduction of photon statistical fluctuation, a detector response function, and magnification due to beam spread at the early stages adds nothing to the development of a correction algorithm; it is much better to first develop it under ideal conditions.

In an investigation of beam hardening the simulation program must meet certain criteria: 1) The data set must be constructed at a

Fig. A1.1 CT image of the base of the skull. Note the amount of detail, making it difficult to ascertain the degree of improvement.

level of complexity higher than that assumed by the reconstruction algorithm and if possible using different basic assumptions. 2) It must simulate a physical phantom with a sufficiently small error. 3) It should be able to produce data sets that are constructed with different levels of complexity, i.e. monoenergetic or polyenergetic x ray beams at varying energies and within limits, varying geometries, with and without scattered radiation, etc. 4) At the same time the program should not take an excessive period of time to run.

Having decided what error is acceptable, then from those methods that guarantee this level or better we must make a trade off of geometric flexibility versus efficiency. This choice is more difficult than one might suppose at first glance. If we consider that the number of measurements for one monoenergetic, nonscatter data set consists of on the order of 10^5 evaluations of the intensity observed at a detector, we see that efficiency is a very important consideration. On the other hand if we wish to keep efficiency levels high then we must sacrifice geometric flexibility -- we cannot introduce an arbitrary geometry and arbitrary number of components. Fortunately, in the case of beam hardening, models built on simple cylinders usually suffice. This simplification gives us flexibility to meet efficiency demands.

A1.2 GENERAL THEORY

For this project the geometry of a first generation scanner was assumed. At this point to maintain simplicity we will restrict the

discussion to the monoenergetic case. Detectors have a finite width, hence the response is an average across the detector face. The detector width is small relative to the geometric features of interest; as the detector width is equal to $2W$, W may be considered a small variation. The average intensity recorded by the detector, relative to the intensity with no object present ($\frac{1}{I_0}$) is approximated by considering the integral of the intensity across the detector face

$$F(x_0) = \frac{1}{2W} \int_{x_0-W}^{x_0+W} e^{-g(x)} dx \quad (A1.1)$$

for small variations W . Here $g(x)$ is the line integral of the phantom $\int_L \mu(x,y)dy$ defined with respect to the rotated detector array axes. Certainly, there exists more than one algorithm that one may use to numerically evaluate Eq. (A1.1). The question that remains is that given the geometry of the phantom, which algorithm is most appropriate?

The most obvious choice to evaluate this integral is some form of numerical quadrature¹, i.e. $F(x) \approx \sum_{i=0}^M \Lambda_i f(x_i)$, for Λ_i 's = weights, and where M is some sufficiently large positive integer. These methods have the advantage of being simple to program. In particular the Gauss Legendre (and variations) quadrature and the Romberg method offer stability at higher order approximations and greatest accuracy. Unfortunately, considering the large number of integrals that must be computed for one data set, except for the simplest phantom geometry, these methods would require an excessive amount of time.

When a general function $n(x)$, given analytically, is to be integrated for small variations Δx then we can expand the function

$$N(x_0) = \int_{x_0 - \Delta x}^{x_0 + \Delta x} n(x) dx \quad (A1.2)$$

about the central value x_0 . Setting $N(x)$ equal to the indefinite integral of $n(x)$ we see that $N(x_0) = N(x_0 + \Delta x) - N(x_0 - \Delta x)$. If we now do a Taylor expansion on both parts then we may evaluate the integral as

$$[N(x_0 + \Delta x) - N(x_0 - \Delta x)] + [n(x_0)\Delta x - n(x_0)(-\Delta x)] + \frac{n'(x_0)(\Delta x)^2}{2!} - \frac{n'(x_0)(-\Delta x)^2}{2!} + \dots \quad (A1.3)$$

As can be seen, only the odd powers of the expansion, or the even derivatives of $n(x)$, remain. Therefore the fourth order approximation of $N(x_0)$ would be $2n(x_0)\Delta x + 2\frac{n''(x_0)(\Delta x)^3}{3!}$. In some instances (such as at or near the boundaries of materials) the fourth order is not sufficient and therefore we must include a term to approximate the remaining terms of the expansion. The correction term may consist of further components of the series or a multiple of the leading terms of several of the higher derivatives. The magnitude of the correction term is dependent on the level of accuracy that one requires and the energies used. If we compare this integral to Eq. (A1.1), we see that the integral across the detector now may be approximated by

$$F(x_0) \approx e^{-g(x_0)} \left\{ 1 + \frac{W^2}{3!} [g'(x_0)^2 - g''(x_0)] + C \right\}, \quad (A1.4)$$

where C is a correction for the remaining terms. Thus instead of performing M sets of evaluations per integral we have only to do one

in most cases.

As may be observed this approximation is only valid for a smoothly varying $g(x)$. If $g(x)$ differs greatly from $g(x_0)$ then the interval must be subdivided accordingly. As the phantom geometry has been restricted to include only cylinders, then $g(x)$ consists of the chord lengths through the respective elements multiplied by their appropriate linear attenuation coefficients. This gives the natural subdivisions of such an interval as the edges of the cylinders ($g'(x)$ becomes infinite), with the evaluation points as the centers of the subintervals. For two cylinders that have edges that overlap this would mean at most three subdivisions.

For the polyenergetic case Eq. (A1.1) now becomes

$$F(x_0) = \frac{\int_0^{E_{\max}} \Phi(E) E \int_{x_0-W}^{x_0+W} \exp(-\mu(x, E)) dx dE}{2W \int_0^{E_{\max}} \Phi(E) E dE} \quad (A1.5)$$

but $\int_0^{E_{\max}} \Phi(E) E dE$ is the constant Ψ_0 , thus we have

$$F(x_0) = \frac{1}{2W\Psi_0} \int_0^{E_{\max}} \Phi(E) E \int_{x_0-W}^{x_0+W} \exp(-\mu(x, E)) dx dE. \quad (A1.6)$$

The integral over the energies² can now be approximated by a sum in steps of 1 keV ($dE = 1 \text{ keV}$) giving

$$F(x_0) = \frac{1}{2W\Psi_0} \sum_{i=0}^{E_{\max}} E_i N(E_i) \int_{x_0-W}^{x_0+W} \exp(-\mu(x, E_i)) dx. \quad (A1.7)$$

where $N(E_i)$ is the number of photons in the spectrum of energy E_i . Applying Eq. (A1.4) we have in the polyenergetic case

$$F(x_0) = \frac{1}{\Psi_0} \sum_{i=0}^{E_{\max}} E_i N(E_i) e^{-g(x_0, E_i)} \left\{ 1 + \frac{W^2 [g'(x_0, E_i)^2 - g''(x_0, E_i)]}{3!} + C \right\} \quad (A1.8)$$

A1.3 GEOMETRY

The simulation programs were written to simulate the output of a first generation CT scanner for a perfectly collimated source and an intensity input uncontaminated with scatter and noise. The geometry is for a detector width of 0.10 cm, translational variation ($dx = 2W$) of 0.10 cm, and total translational length L and diameter of 31.0 cm. This relationship gives $N_x = 310$, and the number of translational measurements as 311. With the detector width of 0.10 cm, the maximum recorded frequency is therefore equal to 10 cm^{-1} and the number of angular variations³ N_ϕ is equal to (nearest integer) $(\frac{N_x \pi}{2}) = 487$. The phantom geometry is that of an outer cylinder of at most 10.5 cm radius and centered at the origin. This outer cylinder can have up to two inset cylinders of variable diameters, center positions (overlap of insets is not allowed) and composition.

For a first generation scanner in order to display a square image centered on the origin with a width corresponding to 21 cm in real space it is necessary that the data be taken from a circle having a diameter of approximately 30 cm. The reason for this can be seen in Fig. A1.2. Any pixel greater than a distance R equal to half the di-

agonal of the image square from the center of rotation is not always crossed by a ray path per angular increment and hence is underdetermined. For an image square 21 cm on a side, the diagonal is $21\sqrt{2}$ cm $\approx$ 29.7 cm. Making allowance to insure complete coverage of the pixels this gives $\approx$ 30 cm as the minimum diameter for a 21 x 21 cm² image.

Fig. A1.2 Image region of diagonal L requires data taken from a circle of radius $\approx$ 1/2 in order to compensate for underdetermined region.

A1.4 DATA EVALUATION

A1.4.1 Homogeneous phantom

For this discussion it is assumed that the phantom is composed of water (much of CT testing and calibration is done using water

phantoms) but the phantom can be composed of any material. Again, for simplicity, we will start with a monoenergetic source. For a homogeneous phantom (no insets) of radius R , centered at the origin of the scanner, the intersection points of a ray parallel to the rotated y axis and through x_0 and the phantom are given by $R^2 = x_0^2 + y^2$, for all angles. Hence, for a ray originating at $(x_0, \frac{L}{2})$ in a rotated frame we have the y positions of intersection as $\pm\sqrt{R^2 - x_0^2}$. This relationship gives the distance D traveled through the phantom by the ray (the chord length) as $2\sqrt{R^2 - x_0^2}$. The intensity recorded at the detector, relative to the no phantom intensity case (I_0) is given by Eq. (A1.4). The function $g(x_0)$ of Eq. (A1.1) is of the form $[\mu_{\text{air}}h(x_0) + \mu_{\text{water}}j(x_0)]$ (where $h(x_0) = L - D$, and $j(x_0) = D$). The first and second derivatives may be simplified ($h+j = L$, a constant); they become $\mu_1 j'(x_0)$ and $\mu_1 j''(x_0)$, respectively, where $\mu_1 = (\mu_{\text{water}} - \mu_{\text{air}})$. This gives the intensity $I(x_0)$, to a fourth order approximation plus the fourth order correction C , as

$$I(x_0) = N_0 e^{[-\mu_{\text{air}}L - \mu_1 j(x_0)]} \left\{ 2W + 2 \frac{W^3}{3!} [(\mu_1 j'(x_0))^2 - \mu_1 j''(x_0)] + C \right\}, \quad (\text{A1.9})$$

where N_0 is the number of input photons per detector width. The integral for I_0 is independent of x ($g_0(x_0) = \mu_{\text{air}}L$) and therefore $I_0(x_0) = 2WN_0 e^{[-\mu_{\text{air}}L]}$. Thus the intensity ratio is approximately equal to

$$F(x_0) = e^{[-\mu_1 j(x_0)]} \left\{ 1 + \frac{W^2}{3!} [(\mu_1 j'(x_0))^2 - \mu_1 j''(x_0)] + C \right\}, \quad (\text{A1.10})$$

Recalling that $j(x_0) = D = 2\sqrt{R^2 - x_0^2}$, this gives the intensity ratio as

$$F(x_0) = e^{[-2\mu_1 \sqrt{R^2 - x_0^2}]} \left\{ 1 + \frac{\mu_1 W^2 [2x_0^2 \mu_1 \sqrt{R^2 - x_0^2} + R^2]}{3(R^2 - x_0^2)^{3/2}} + C \right\} \quad (\text{A1.11})$$

where C is

$$\frac{2.25W^4\mu_1R^2}{4(R^2 - x_0^2)^{7/2}} \left\{ \frac{R^2 + 4x_0^2}{5} + \frac{W^2[R^4 + 12x_0^2R^2 + 8x_0^4]}{14(R^2 - x_0^2)^2} \right\}$$

and is equal to 2.25 times the sum of the 4th and 6th derivatives of $j(x)$, weighted by their respective expansion coefficients, and the coefficient 2.25 is found empirically. This approximation is valid so long as $|x_0|$ does not become equal to R (those cases when $|x_0| \rightarrow R$ will be discussed below). The output (monoenergetic) is therefore given by the equation

$$\begin{aligned} \ln(I_1^0) &= (\mu_a(L - D) + \mu_w(D)) - \mu_a L - \ln(\text{correction to first order approx-} \\ &\hspace{15em} \text{imation}) \\ &= \mu_1 D - \ln(\text{correction}) \\ &= 2\mu_1 \sqrt{R^2 - x_0^2} - \ln \left(1 + \frac{\mu_1 W^2 [2x_0^2 \mu_1 \sqrt{R^2 - x_0^2} + R^2]}{3(R^2 - x_0^2)^{3/2}} \right. \\ &\quad \left. + \frac{2.25W^4\mu_1R^2}{4(R^2 - x_0^2)^{7/2}} \left\{ \frac{R^2 + 4x_0^2}{5} + \frac{W^2[R^4 + 12x_0^2R^2 + 8x_0^4]}{14(R^2 - x_0^2)^2} \right\} \right) \end{aligned} \quad (A1.12)$$

for $|x_0| < R$, and zero for $|x_0| > R$.

At the boundaries the volume element described by the detector width may not be completely filled by the phantom. In this case to retain the use of the approximation, the detector is divided into two components. The components consist of that part lying completely outside the shadow of the phantom, and the part completely in its shadow. The sum of the intensities of the two detectors should give the intensity of the single larger detector. In the case of where the phantom disappears this should give the intensity ratio as though no phantom is present, thus providing a check of our approximation.

For the partial volume case we will consider a left edge (see Fig. A1.3), if we let W' equal half the distance of the detector face not in shadow and W'' equal half the distance in shadow Eq. (A1.2) becomes

$$N(x) = \int_{x_l - W'}^{x_l + W'} n_0(x) dx + \int_{x_r - W''}^{x_r + W''} n(x) dx \quad (A1.13)$$

where x_l = the center of the no shadow case and x_r = the center of the shadow case. Obviously the first term is equal to $2W'N_0e^{(-\mu_a L)}$, the second term when computed as above gives

$$N_0e^{(-\mu_a L - \mu_j(x_r))} \{ 2W'' \{ 1 + \frac{W''^2}{3!} [(\mu_j)'(x_r))^2 - \mu_j''(x_r)] + C \}. \quad (A1.14)$$

Therefore the intensity ratio $F(x)$ is given by

$$\frac{W'}{W} + \frac{W''e^{(-\mu_j(x_r))}}{W} \{ 1 + \frac{W''^2}{3!} [(\mu_j)'(x_r))^2 - \mu_j''(x_r)] + C \} \quad (A1.15)$$

But $\frac{W'}{W}$ and $\frac{W''}{W}$ are just f and $(1 - f)$ respectively, where f is the fraction of the detector's face not in shadow, i.e. the intensity ratio is given by

$$\frac{1}{I_0} = f + (1 - f)e^{(-\mu_j(x_r))} \{ 1 + \frac{W''^2}{3!} [(\mu_j)'(x_r))^2 - \mu_j''(x_r)] + C \} \quad (A1.16)$$

and in the case of $j(x_r) = 0$, the ratio reduces to $f + (1 - f) = 1$, as it should.

The treatment of the polyenergetic case is now straight forward. We apply Eq. (A1.8) over the spectrum of energies and sum the results,

$$F(x) \approx \frac{1}{\Psi_0} \sum_{i=0}^{E_{\max}} E_i N(E_i) [f + (1-f)e^{-\mu_1(E_i)j(x_0)} \{1 + \frac{W^2}{3!} [(\mu_1(E_i)j'(x_0))^2 - \mu_1(E_i)j''(x_0)] + C\}] \quad (A1.17)$$

Fig. A1.3 Fraction $(1 - f)$ of detector face is in the shadow of the phantom, the approximation for this fraction is taken using the center of the fraction. The fraction f is not in shadow and receives no approximation - $1/I_0$ of this fraction is one.

A1.4.2 Phantoms with Insets

We now introduce a cylindrical inset of material ξ into the homogeneous phantom. The inset is of radius R_2 and with its center at an angle θ with respect to the x axis and a positive distance Λ from the origin. The center of the inset is therefore $(x,y) = (\Lambda \cos(\theta), \Lambda \sin(\theta))$. If the axes are rotated by an angle ϕ then the center of the inset in the rotated frame is given by

$$\begin{aligned}
 x_c &= x \cos(\phi) + y \sin(\phi) \\
 &= A \cos(\theta) \cos(\phi) + A \sin(\phi) \sin(\theta) \\
 x_c &= A \cos(\theta - \phi) \quad (\text{A1.18})
 \end{aligned}$$

and

$$\begin{aligned}
 y_c &= -x \sin(\phi) + y \cos(\phi) \\
 &= -A \cos(\theta) \sin(\phi) + A \cos(\phi) \sin(\theta) \\
 y_c &= A \sin(\theta - \phi). \quad (\text{A1.19})
 \end{aligned}$$

By circular symmetry the projection of the inset onto the rotated x axis extends from $A \cos(\theta - \phi) - R_2$ to $A \cos(\theta - \phi) + R_2$. Therefore any ray path whose translational x is between $(A \cos(\theta - \phi) - R_2 - W)$ and $(A \cos(\theta - \phi) + R_2 + W)$ must cross the inset. As with the homogeneous phantom we will leave discussion of partial volumes until later and now only consider those x positions such that $|x - x_c| < R_2$. The equation for the edge of the inset is $R_2^2 = (x_0 - x_c)^2 + (y - y_c)^2$ hence the chord length across the inset is

$$D_2 = 2|y - y_c| = 2\sqrt{R_2^2 - (x_0 - x_c)^2}. \quad (\text{A1.20})$$

We now have $g(x_0) = \mu_a h(x_0) + \mu_w j(x_0) + \mu_\xi k(x_0)$ where $h(x_0)$ is as for the homogeneous phantom, $j(x_0) = D - D_2$, with $k(x_0) = D_2$. Again L is equal to $h + j + k$ and the derivatives may be simplified by removing the most troublesome function -- $j(x_0)$. This gives in the monoenergetic case

$$\begin{aligned}
 F(x_0) &= e^{[-\mu_1 D - \mu_2 k]} \left\{ 1 + \frac{W^2}{3!} [(-\mu_1 h'(x_0) + \mu_2 k'(x_0))^2 \right. \\
 &\quad \left. - (-\mu_1 h''(x_0) + \mu_2 k''(x_0))] + C'' \right\} \quad (\text{A1.21})
 \end{aligned}$$

where $\mu_2 = \mu_\xi - \mu_w$, and C'' is the correction term with respect to h and k .

If we now include another inset of material η with an attenuation coefficient μ_η the analysis follows the same path as that of the single inset. We now have x_{c1} and x_{c2} equal to $\Lambda_1 \cos(\theta_1 - \phi)$ and $\Lambda_2 \cos(\theta_2 - \phi)$ (Fig. A1.4), respectively and two inset chord lengths $D_2 = k(x)$ and $D_3 = l(x)$. Introducing the variables $\mu_3 = \mu_\eta - \mu_w$ and C'' , the correction term with respect to h , k and l we now have

$$F(x_0) = e^{(\mu_1 D - \mu_2 k - \mu_3 l)} \left\{ 1 + \frac{W^2}{3!} [(-\mu_1 h'(x_0) + \mu_2 k'(x_0) + \mu_3 l'(x_0))^2 - (-\mu_1 h''(x_0) + \mu_2 k''(x_0) + \mu_3 l''(x_0))] + C'' \right\} \quad (A1.22)$$

In the polycenergetic case we sum over the energies as with Eq. (A1.17).

Fig. A1.4 Geometry of the phantom with two insets. For source and detector position D is the phantom's outer cylinder chord length and D_3 is the inset chord length.

With insets we must deal with partial volumes in several forms (Fig. A1.5), that of the homogeneous phantom, the edge of one inset alone, the edge of one inset overlapping the other inset, and the edges of the two insets appearing in the same ray path. As with the homogeneous phantom the strip is subdivided by the tangents to edge(s) parallel to the ray, but now there can be up to three subdivisions: f_1 , f_2 , and $f_3 = 1 - (f_1 + f_2)$. The subdivided strips then fall into one of the categories already described. The decision table for this is given in Table A1.1. It should be noted that such a table is independent of the method of evaluation chosen as recognition of edge positions must be employed.

A1.5 EFFICIENCY

The accuracy of the image values are determined by the relative error in the computed intensity (input data is proportional to the $\ln(I)$) for both mono- and polyenergetic spectra. As the data is repeatedly manipulated, it was felt that the maximum acceptable image error due to data was 1×10^{-3} (corresponding to one CT number in the image). In the convolution the data is multiplied by the leading coefficient $\frac{2}{\pi N_\phi}$. If we take the weighting on the pixel to be maximal, i.e. 1 for all angles, then the effect of the backprojection is to multiply by N_ϕ , therefore the maximum acceptable relative error in the intensity is $(\frac{\pi}{2}) \times 10^{-3}$. If we set the maximum relative error at 1×10^{-3} we are well within limits. The data error is maximum near the edges ($\frac{1}{R^2 - x_0^2}$ increases as x_0 approaches R , but as the fraction

Fig. A1.5 Examples of partial volumes that occur with two insets.

Table A1.1 Decision Table for Phantom with Two Insets for a Fixed Angle.
Tests are read left to right.

Phantom			1 st Inset			2 nd Inset			Result
1*	2	3	4	5	6	7	8	9	**
F	F	F							A
T			†						B
F	F	T	†						B
F	T		F	F	F	F	F	F	C
F	T		T			F	F	F	D
F	T		T			T			E
F	T		T			F	T		F
F	T		T			F	F	T	E
F	T		F	T		F	F	F	G
F	T		F	T		T			F
F	T		F	T		F	T		H
F	T		F	T		F	F	T	F
F	T		F	F	T	F	F	F	D
F	T		F	F	T	T			E
F	T		F	F	T	F	T		F
F	T		F	F	T	F	F	T	E
F	T		F	F	F	T			D
F	T		F	F	F	F	T		G
F	T		F	F	F	F	F	T	D

† Overlap of inset edge and phantom edge is not allowed.

* Cases are as follows:

- phantom left edge + $\Delta x \geq x \geq$ phantom left edge - Δx
- phantom right edge - $\Delta x > x >$ phantom left edge + Δx
- phantom right edge + $\Delta x \geq x \geq$ phantom right edge - Δx
- 1st inset left edge + $\Delta x \geq x \geq$ 1st inset left edge - Δx
- 1st inset right edge - $\Delta x > x >$ 1st inset left edge + Δx
- 1st inset right edge + $\Delta x \geq x \geq$ 1st inset right edge - Δx
- 2nd inset left edge + $\Delta x \geq x \geq$ 2nd inset left edge - Δx
- 2nd inset right edge - $\Delta x > x >$ 2nd inset left edge + Δx
- 2nd inset right edge + $\Delta x \geq x \geq$ 2nd inset right edge - Δx

** Resulting computation: . .

- output equals 1
- apply Eq. A1.16
- apply Eq. A1.10
- subdivide and apply Eqs. A1.10 and A1.21 as needed
- subdivide and apply Eqs. A1.10, A1.21 and A1.22 as needed
- subdivide and apply Eqs. A1.21 and A1.22 as needed
- apply Eq. A1.21
- apply Eq. A1.22

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al component of the phantom observed within one detector width decreases it contributes less to the overall computed intensity) with the error decreasing as we move away from the edge. Study of the behavior near the edge of a homogeneous water phantoms of various radii and energies has empirically shown that for the absolute relative data error to be 1×10^{-3} or less for energies between 40 and 150 keV the coefficient of the correction term is 2.25.

Table A1.II An example of CPU cost for operations. Costs are for Fortran on a VAX/VMS Version V5.4 network (at Carleton University).

Operation	CPU Time μ s	Standard Deviation μ s
Negation	0.29	0.01
Addition	1.03	0.01
Subtraction	1.02	0.01
Multiplication	0.93	0.08
Division	1.93	0.02
Square	0.75	0.02
Square Root	15.18	0.09
Power (integer = 3)	16.25	0.05
Power (integer = 4)	17.85	0.09
Power (integer = 5)	18.29	0.06
Power (noninteger = 1.5)	48.11	0.07
Exponential	23.47	0.04
Logarithm	20.05	0.05

The number of computations in the expansion used is large. These computations are expensive when compared to that of one subdivision of quadrature methods such as Simpson's method or the Trapezoid rule. The expense of the expansion can be reduced by ob-

serving that the terms μ' , W^2 , R^2 , x_0^2 , $R^2 - x_0^2$, and $\sqrt{R^2 - x_0^2}$ that repeatedly appear need only be computed once and their values substituted into the equation. Referring to Table A1.II, using Fortran on a Vax/VMS Version V5.4 System as an example, we see that taking a square root, an exponentiation, performing a logarithm, and especially taking a power of a number are expensive when compared to an addition. The cost of the powers can be eliminated by judicious use of the square, the multiplication operators, and factoring out of common terms. If we now consider a homogeneous 20 cm diameter phantom, the expansion consists of 44 separate simple arithmetic operations equivalent to 42.63 additions, one square root (14.74 additions), and one exponential (polyenergetic case) equivalent to 22.79 additions, therefore there can be up to 80.16 addition equivalent operations per energy subdivision, in computing the intensity.

In applying a quadrature method such as the Trapezoid rule or Simpson's method each linear segment division evaluation consists of computing the coefficient (requiring 4.98 separate addition equivalent operations (assuming R^2 is fixed), and one square root) and taking the exponent and multiplying by the weighting factor, giving us a total of 43.41 addition equivalent operations per energy subdivision. The actual number of divisions of the line interval that are required to have a relative error of 1×10^{-3} (defined by the worst case, i.e. near the edge) that must be added together before taking the logarithm is 21 for the 142 Simpson's (second order) method, resulting in up to 932.47 total addition equivalent operations per energy; 30 subdivisions for the Trapezoid rule resulting in 1332.20 addition

equivalents per energy. Thus the expansion method is at least 11.63 times less computationally expensive. At best the Romberg method would require a minimum of three divisions of the line segment (the line segment located well away from any edge) which would mean 132.23 addition equivalent operations (aside from the convergence check) and therefore even in this case the presented method is at least 1.65 times more efficient.

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APPENDIX 2: CALCULATION OF POINT SPREAD FUNCTIONS USING THE MATRIX FORMULATION OF CT

A2.1 GENERAL EVALUATION OF $M^T M$

First we examine the form of the $M^T M$ matrix (for specific choices of M) in the hope of obtaining insight into its symmetries, since the H we seek must have the same symmetries. The final goal is to discover an H such that $M^T H M = I$ to a high degree of approximation, to eliminate the need for a matrix inversion. There will be tradeoffs to make between the size (and hence associated computational costs) of H and the accuracy we can achieve¹.

We begin our evaluation of $M^T M$ by looking at the matrix M . Each column of the matrix corresponds to a particular pixel. Each entry in a column is the weighting given to the pixel (the weights dependent on the interpolation routine used) for the detector position and angle corresponding to the row in which the entry is located. If we now assume that the filter H in Eq. (2.4) is simply the identity I , then for a given pixel (i,j) M^T operating on Y simply gives the weighted sum of the attenuation coefficient line integrals of the rays passing through the pixel -- the backprojection operator. The $M^T M$ matrix consists of the dot product of the i th and j th columns of the M matrix, thus the j th diagonal entry is just the sum of the squared weights of data points that contribute to the final j th pixel value. For an off diagonal (i,j) term the entry is the sum of the products of the weights of common contributors of the i th and j th pixels, therefore $M^T M$ is a

symmetric matrix.

We now consider computing the point spread function (PSF): starting with a unit vector X_j the bare backprojection computes the image of the PSF ($M^T Y_j$) to be $M^T M X_j$, so $M^T M$ can be recognized as the PSF operator (apart from normalization). If we let X_0 be the unit vector corresponding to the image origin then $X_j M^T M X_0$ is the PSF value of the pixel j for a source at the origin. Since we know from Fourier analysis that the PSF should ideally be of the form $\frac{1}{\pi r}$, we are led to compare these numbers with those computed from Fourier analysis.

A2.2 EVALUATION OF $M^T M$ FOR SPECIFIC INTERPOLATION SCHEMES

A2.2.1 Nearest Neighbor

For nearest neighbor we take the weighting in M to be 1 if for a given angle, the intersection of a pixel with a ray described by a detector is larger than neighboring rays and 0 otherwise; then the M matrix consists of zeros and ones, similar to Fig. 2.1. Using nearest neighbor the diagonal elements of $M^T M$ now are the number of data elements that contribute to the pixels (N_ϕ) and the off diagonal terms are the number of common elements. The obvious choice of scaling factor is to use the central diagonal entries N_ϕ . In this example we restrict ourselves to a first generation scanner geometry such that the detector width is the same as the pixel width. With such a ge-

ometry we take $N_\phi \Delta\phi = \pi$, and set the variables $N_d = 311$, $N_\phi = \frac{\pi N_d}{2} = 489$, and $N_x = N_y = 211$ ($-105 \leq x, y \leq 105$). Several computed values of the term in the nearest neighbor MTM matrix corresponding to (0,0) compared with $\frac{1}{\pi r}$ are given in Table A2.1.

Table A2.1 Comparison of number of rays that receive a weighting of 1 through a given pixel and the pixel at the origin using nearest neighbor weighting and the computed value using the expected Fourier PSF; N_ϕ (number of diagonal elements) = 489.

Point	Actual Number of Rays	$\frac{N_\phi}{\pi r}$ Computed Number of Rays	Difference
(0,0)	489	489	0
(0,1)	163	156	7
(0,2)	79	78	1
(0,3)	53	52	1
(0,4)	39	38	1
(0,5)	31	31	0
(0,6)	25	26	-1
(1,1)	112	110	2
(1,2)	70	70	0
(1,3)	49	49	0
(2,2)	55	55	0

As may be seen, as r becomes larger the difference quickly approaches zero, but for r near 0 the differences are relatively large ($\frac{7}{489} \cong 0.0143$). Some, but not all, of the magnitude of these differences can be attributed to 'noise' generated from use of the nearest neighbor weighting. The remaining discrepancy must be due to what is left out of the Fourier analysis. In the Fourier analysis we are assuming an infinite continuous data set, but in reality we are dealing

with the finite discrete case.

This analysis suggests that we may write $M^T M$ in the general case as $C(R_\pi + T)$ where R_π corresponds to the diagonal terms plus the $\frac{1}{\pi r}$ component of the off-diagonal terms, T contains the remaining values and C is a normalization constant dependent on M . The $(M^T M)^{-1}$ matrix then is equivalent to $\frac{1}{C}(R_\pi + T)^{-1}$, where in the continuous limit T goes to zero and Eq. (2.3) becomes $X = \frac{1}{C}(R_\pi^{-1} M^T)Y$. Since R_π depends solely on r , and not on the azimuthal angle, its corresponding filter H also is independent of angle, and hence has the form of a block-diagonal matrix, with all blocks identical. Also, since R_π is the same for each pixel taken as center, it is a banded matrix (constant diagonals), and so is the corresponding H (see Fig. A2.1); this allows the matrix multiplication by H to be replaced with a convolution. Note that T breaks both of these symmetries, as it depends on both the pixel position and azimuthal angle, so no convolution filtering (on detector position coordinate) can compensate for it. A similar breakdown, however, applies to any dependence on r alone, so a good strategy would be to extract as much as possible any r dependence, thus minimizing the effect of T .

As $\frac{1}{\pi r}$ does not accurately represent the off-diagonal terms, we must consider the geometric relationships of the pixels that give rise to these terms. In order to estimate the off-diagonal terms we first take the pixel size to be of unit length, and consider the behavior at the origin -- (0,0). Since all rays that pass through the pixel (0,0)

Fig. A2.1 An example of a banded block-diagonal matrix

must also pass through the (0,0) point exactly, then the number of rays which also go through the pixel (0,j) is easily calculated: a ray through (0,0) is deemed to pass through (0,j) if the perpendicular distance to the point (0,j) is within $\frac{1}{2}$ unit (the distance of the point projected onto the line of detectors and assuming that the detector width is the same as pixel width) (see Fig. A2.2 a). Hence the triangle OQP is a right triangle and $\sin(\phi_{MAX}) = \frac{1}{2j}$, for the extreme case. Since rays can have either positive or negative slopes, and the ray $\phi = 0$ is always present, then the number of rays is given exactly by

$$\begin{aligned} \# \text{ rays}\{ (0,0),(0,j) \} &= -1 + 2 \cdot \text{integer} \frac{\sin^{-1}(1/2j)}{\Delta\phi} \\ &\equiv \frac{2\sin^{-1}(1/2j)}{\Delta\phi} . \end{aligned} \quad (A2.1)$$

Fig. A2.2 Geometric relationship of lines through (0,0) and another pixel. a) same row b) general pixel location

For pixels in a general position (i,j) the approximate expression is still valid (to within ± 1), if j is replaced by $\sqrt{i^2+j^2} = r$ (see Fig.A2.2 b)(r is in terms of pixel widths and therefore dimensionless). The expression which results for (0,0) when normalizing by the diagonal term (N_ϕ) is :

$$R(0,P) \equiv \frac{2 \sin^{-1}(1/2r)}{N_\phi \Delta\phi},$$

as $N_\phi \Delta\phi = \pi$ this becomes

$$R(0,P) \equiv \frac{2}{\pi} (\sin^{-1}(1/2r)). \quad (A2.2)$$

The arcsine can be expanded in the power series

$$\sin^{-1}z = z + \frac{z^3}{2 \cdot 3} + \frac{3(z^5)}{2 \cdot 4 \cdot 5} + \dots$$

therefore this gives:

$$\begin{aligned} R(0,P) &\equiv \frac{2}{\pi} \left\{ \frac{1}{2r} + \frac{(1/2r)^3}{2 \cdot 3} + \frac{3(1/2r)^5}{2 \cdot 4 \cdot 5} + \dots \right\} \\ &= \frac{1}{\pi} \left\{ \frac{1}{r} + \frac{(1/r)^3}{24} + \frac{3(1/r)^5}{640} + \dots \right\} \end{aligned}$$

as the third term is so much smaller, we may ignore all but the first two terms giving :

$$R_2(0,P) = \frac{1}{\pi} \left(\frac{1}{r} + \frac{1}{24r^3} \right). \quad (\text{A2.3})$$

When Eqs. (A2.2) and (A2.3) are compared to $\frac{1}{\pi r}$ for the behavior about the origin in Table A2.II we see that the arcsine provides a very close match, but R_2 is a reasonable approximation. We conclude that the PSF for a nearest neighbor interpolation is of the form:

$$\frac{2}{\pi} (\sin^{-1}(1/2r)) \quad \text{for } r \geq 1.$$

Table A2.II Comparison of number of rays a pixel at a radius r from the origin and the pixel at the origin have in common predicted by the different formulae and that observed. Results are for $N\phi = 489$.

r	$\frac{N_0}{\pi r}$	$N_0 \cdot R_2(0,P)$	$N_0 \cdot R(0,P)$	Observed
1	155.65	162.14	163.00	163
2	77.827	78.637	78.661	79
3	51.885	52.125	52.128	53
4	38.913	39.015	39.015	39
5	31.131	31.183	31.183	31
6	25.942	25.972	25.972	25
7	22.236	22.255	22.255	23

A2.2.2 Linear Interpolation

When the weighting is taken to be that of linear interpolation the interpretation of the $M^T M$ matrix is not so simple. The entries of M are in this case the fractional weights ($0 \leq M_{ij} \leq 1$), and $M^T M$ consists of the inner products of the columns of M . As in the nearest neighbor case, the simplest entries to analyze are those around the origin; since lines through the (0,0) pixel go exactly through the point (0,0), the column for pixel (0,0) has entries which are either 0 or 1. Thus the inner product of the (0,0) column with the column for pixel (i,j) reduces to the sum of terms chosen from that column, corresponding to the lines passing through both (0,0) and (i,j) pixels. The contribution from such a line is $1 - d$, where $0 \leq d \leq 1$, and d is the distance from the line to the center of pixel (i,j).

The distance d is equal to $r \sin(\theta)$ where θ is the angle between the detector line and the line connecting the centers of the pixels, and r is the distance between the centers of the pixels and is again in units of pixel width. For the special case where one line goes exactly through both centers, we sum over all the lines traversing both pixels to give

$$N_\phi R(0,r) = -1 + 2 \sum_0^n (1 - r \sin(j \Delta \phi)) \quad (A2.4)$$

where the -1 compensates for the double counting of the center-center line. We compute n by noting that at the extreme limit of θ , $r \sin(\theta) = 1$, so

$$n = \frac{\sin^{-1}(1/r)}{\Delta \phi}$$

Therefore we have

$$\begin{aligned} N_\phi R(0,r) &= -1 + 2(n+1) - 2r \sum_0^n \sin(j\Delta\phi) \\ &= 1 + \frac{2\sin^{-1}(1/r)}{\Delta\phi} - 2r \sum_0^n \sin(j\Delta\phi) \end{aligned}$$

but $\sin(j\Delta\phi) = \frac{e^{ij\Delta\phi} - e^{-ij\Delta\phi}}{2i}$, therefore

$$\begin{aligned} \sum_0^n \sin(j\Delta\phi) &= \sum_0^n \frac{e^{ij\Delta\phi} - e^{-ij\Delta\phi}}{2i} \\ &= \frac{1}{2i} \left\{ \frac{1 - e^{i(n+1)\Delta\phi}}{1 - e^{i\Delta\phi}} - \frac{1 - e^{-i(n+1)\Delta\phi}}{1 - e^{-i\Delta\phi}} \right\} \\ &= \frac{1}{2i} \left\{ -\frac{e^{i\Delta\phi/2} + e^{-i\Delta\phi/2}}{e^{i\Delta\phi/2} - e^{-i\Delta\phi/2}} + \frac{e^{i(n+1/2)\Delta\phi} + e^{-i(n+1/2)\Delta\phi}}{e^{i\Delta\phi/2} - e^{-i\Delta\phi/2}} \right\} \\ &= \frac{1}{2i} \left\{ -\frac{2\cos(\Delta\phi/2)}{2i\sin(\Delta\phi/2)} + \frac{2\cos((n+1/2)\Delta\phi)}{2i\sin(\Delta\phi/2)} \right\} \\ &= \frac{1}{2} \left\{ \cot(\Delta\phi/2) - \frac{\cos((n+1/2)\Delta\phi)}{\sin(\Delta\phi/2)} \right\} \\ &= \frac{1}{2} \{ \cot(\Delta\phi/2)(1 - \cos(n\Delta\phi)) + \sin(n\Delta\phi) \} \end{aligned}$$

since $r\sin(n\Delta\phi) = 1$, and $\cos(n\Delta\phi) = \sqrt{1 - r^2}$

$$= \frac{1}{2} \{ \cot(\Delta\phi/2)(1 - \sqrt{1 - r^2}) + \frac{1}{r} \}.$$

Substituting this result into the above gives

$$N_\phi R(0,r) = \frac{2\sin^{-1}(1/r)}{\Delta\phi} - r \{ \cot(\Delta\phi/2)(1 - \sqrt{1 - r^2}) \}.$$

But $\Delta\phi$ is equal to $\frac{\pi}{N_\phi}$ and $\cot(z) = \frac{1}{z} - \frac{z^3}{3} - \frac{z^5}{45} + \dots$, therefore to a good approximation as N_ϕ is large $\cot(\pi/2N_\phi)$ is $\frac{2N_\phi}{\pi}$ and

$$N_\phi R(0,r) \approx \frac{2N_\phi}{\pi} \{ \sin^{-1}(1/r) - r(1 - \sqrt{1 - r^2}) \}.$$

Using the power series expansion of arcsine and of the square root ($r \geq 1$), this yields

$$R(0,r) \approx \frac{2}{\pi} \left\{ \left(\frac{1}{r} + \frac{1}{6r^3} + \dots \right) - r \left(\frac{1}{2r^2} + \frac{1}{8r^4} + \dots \right) \right\}$$

$$= \frac{1}{\pi} \left(\frac{1}{r} + \frac{1}{12r^3} + \dots \right),$$

$$\text{and } R_2(0,r) = \frac{1}{\pi} \left(\frac{1}{r} + \frac{1}{12r^3} \right). \quad (\text{A2.5})$$

Again we see the PSF contains a $\frac{1}{\pi r^3}$ term that is not predicted by the Fourier model. It should be noted that the coefficient of this term for linear interpolation is twice as large as that for nearest neighbor (Eq. (A2.3)). This is surprising as one might expect that a more sophisticated method would be somehow better. In Fig. A2.3 the difference between the calculated PSF and the theoretical PSF does not appear large, however, in the following section it will be shown that this leads to a significant effect.

Fig. A2.3 Comparison of the Fourier theoretical PSF and the calculated PSF for linear interpolation. The PSF is given in units of π , and the radius is in units of pixel width.

A2.3 CONSEQUENCES OF AN IMPERFECT POINT SPREAD FUNCTION

Since the standard theory² considers the unfiltered PSF function to be $\frac{1}{\pi r}$, and since $\mathcal{F}_2\{\frac{1}{\pi r}\} = \frac{1}{\pi \rho}$, the two dimensional filter is taken as $Q(\rho) = \pi \rho \Lambda(\rho)$ in the frequency domain, where $\Lambda(\rho)$ is an apodizing function, e.g. $\text{circ}(\frac{\rho}{\rho_{\max}})$ corresponding to the two dimensional form of the Ramachandran-Lakshminarayanan filter. This gives a post-filtered PSF of

$$\begin{aligned} \text{PSF}(r) &= \frac{1}{\pi r} ** q(r) \\ &= \mathcal{F}_2^{-1}[\mathcal{F}_2\{\frac{1}{\pi r}\}Q(\rho)] \\ &= \mathcal{F}_2^{-1}[(\frac{1}{\pi \rho}) \pi \rho \Lambda(\rho)] \\ &= a(r), \end{aligned} \quad (\text{A2.6})$$

where $a(r)$ is the apodizing function in real space. However we have seen that interpolations add a term $\approx (\frac{\alpha}{\pi r^3})$ to the unfiltered PSF, so using the same filter gives

$$\begin{aligned} \text{PSF}(r) &\approx (\frac{1}{\pi r} + \frac{\alpha}{\pi r^3}) ** q(r) \\ &= (\frac{1}{\pi r}) ** q(r) + (\frac{\alpha}{\pi r^3}) ** q(r) \\ &= a(r) + \alpha \mathcal{F}_2^{-1}[\mathcal{F}_2\{\frac{1}{\pi r^3}\} \pi \rho \Lambda(\rho)] \\ &= a(r) + \text{error term}. \end{aligned} \quad (\text{A2.7})$$

To evaluate the error term we apply the Central Slice Theorem which states that the two dimensional Fourier transform evaluated along a line through the origin is the one dimensional transform of the projection of the function onto the corresponding line in the dual space. Since $\frac{1}{\pi r^3}$ is radially symmetric we can arbitrarily choose any line in frequency space, e.g. $\eta = 0$. The corresponding projection is

$$\frac{1}{\pi} \int_{-\infty}^{\infty} (x^2 + y^2)^{-3/2} dy = \frac{2}{\pi} \int_0^{\infty} (x^2 + y^2)^{-3/2} dy = \left. \frac{2y}{\pi x^2 \sqrt{x^2 + y^2}} \right|_{y=0}^{\infty} = \frac{2}{\pi x^2}$$

so that equation (A2.7) becomes

$$a(r) + \alpha \mathcal{F}_2^{-1} \left[\mathcal{F}_{2_{\text{corr}}} \left\{ \frac{2}{\pi x^2} \right\} \pi \rho A(\rho) \right]$$

where $\mathcal{F}_{2_{\text{corr}}} \left\{ \frac{2}{\pi x^2} \right\}$ is the corresponding two dimensional form of $\mathcal{F}_1 \left\{ \frac{2}{\pi x^2} \right\}$.

We evaluate $\mathcal{F}_1 \left\{ \frac{2}{\pi x^2} \right\}$ by noting that $\mathcal{F}_1^{-1} \{ \text{sgn}(\xi) \} = \mathcal{P} \left(\frac{i}{\pi x} \right)$, where $\mathcal{P}$ is the Cauchy principal value, and using the derivative theorem of Fourier transforms we get

$$\mathcal{F}_1^{-1} \{ \text{sgn}(\xi) 2\pi i \xi \} = \mathcal{F}_1^{-1} \{ 2\pi i |\xi| \} = \frac{-i}{\pi x^2}$$

so

$$\mathcal{F}_1^{-1} \{ -4\pi |\xi| \} = \frac{2}{\pi x^2},$$

and therefore as the transform of a radially symmetric function is radially symmetric, we can replace $|\xi|$ with ρ to give the two dimensional form as

$$\mathcal{F}_2 \left\{ \frac{1}{\pi r^3} \right\} = -4\pi \rho. \quad (\text{A2.8})$$

Surprisingly this is the filter required (apart from normalization) in frequency space to compensate for the ideal backprojection, i.e. the first term in Eqs. (A2.3) and (A2.5). The standard filter of Eqs. (A2.6) and (A2.7) operating on an image $f(r, \theta)$ can thus be written as

$$\frac{-1}{4\pi r^3} ** a(r) ** f(r, \theta),$$

and applying the first two terms of the PSF of Eqs. (A2.3) and (A2.5) gives the resulting reconstructed image as

$$\hat{f}(r, \theta) = \left(\frac{1}{\pi r} + \frac{\alpha}{\pi r^3} \right) ** \left[\frac{-1}{4\pi r^3} ** a(r) ** f(r, \theta) \right] .$$

Distributing the convolution over addition and reducing using Eq. (A2.6) gives

$$\hat{f}(r, \theta) = a(r) ** f(r, \theta) + \frac{\alpha}{\pi r^3} ** \frac{-1}{4\pi r^3} ** a(r) ** f(r, \theta) .$$

In frequency space $\frac{1}{\pi r^3} ** \frac{-1}{4\pi r^3}$ becomes $(-4\pi\rho)(\pi\rho) = -4\pi^2\rho^2$. This value times a function in frequency space makes it equivalent to the transform of the Laplacian of its corresponding real space function. This can be seen as follows:

$$f(x, y) = \mathcal{F}_2^{-1}\{F(\xi, \eta)\}$$

then

$$\nabla^2 f(x, y) = \nabla^2 \mathcal{F}_2^{-1}\{F(\xi, \eta)\}$$

$$\begin{aligned} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \int_{-\infty}^{\infty} e^{-2\pi i(\xi x + \eta y)} F(\xi, \eta) d\xi d\eta \\ &= \int_{-\infty}^{\infty} e^{-2\pi i(\xi x + \eta y)} (-2\pi i)^2 (\xi^2 + \eta^2) F(\xi, \eta) d\xi d\eta \\ &= \mathcal{F}_2^{-1}\{-4\pi^2 \rho^2 F(\xi, \eta)\} . \end{aligned}$$

Therefore

$$\hat{f}(r, \theta) = a(r) ** f(r, \theta) + \alpha \nabla^2 \{a(r) ** f(r, \theta)\} . \quad (\text{A2.9})$$

The inclusion of this error term results in increased susceptibility to noise, oscillations at boundaries, and possible streaking between boundary edges as well as contour enhancement. To compensate for this it is necessary to soften the filter, i.e. to use one that is proportional to ρ near 0 but proportional to $\frac{1}{\rho}$ as it approaches the cutoff frequency as in Fig. 3.2. If we refer to Fig. A2.3 we see that a small difference in the PSF can have large effects in the behavior in the presence of large contours.

REFERENCES

- ¹H at best can only be approximated to the continuous case. The geometric limitations, having square pixels and data taken with respect to polar coordinates, restricts the accuracy of our evaluation of H.
- ²H.H. Barrett and W. Swindell, "Radiological Imaging, The Theory of Image Formation, Detection, and Processing", Academic Press, Inc., Toronto, vol. 2, Section 7.2 (1981).

APPENDIX3: THE BEAM HARDENING CORRECTION PROGRAM

[illegible]

```

      (ATTIMAGE(1-3,1))=-1.0000
      (OUTIMAGE(1-3,1))=-1.0000
04      ENDDO
05      ENDDO
06      OPEN(A,FILE='(XTABLES.ATT)WATERX.ATT',STATUS='OLD',
      &      FORM='UNFORMATTED',READONLY,IOSTAT=IOS)
      IF (IOS.NE.0) THEN
07          WRITE(FRW,*)IOS
          FORMAT(5X,' TROUBLE OPENING (XTABLES.ATT)WATERX.ATT,
      &      IOS= ',I3)
          GOTO 06
      ENDIF
      READ(A)ATT1
      CLOSE(A)

08      OPEN(B,FILE='(XTABLES.ATT)ALUMIN.ATT',STATUS='OLD',
      &      FORM='UNFORMATTED',READONLY,IOSTAT=IOS)
      IF (IOS.NE.0) THEN
09          WRITE(FRW,*)IOS
          FORMAT(5X,' TROUBLE OPENING (XTABLES.ATT)ALUMIN.ATT,
      &      IOS= ',I3)
          GOTO 08
      ENDIF
      READ(B)ATT2
      CLOSE(B)

10      OPEN(C,FILE='(XTABLES.SPEC)TU120AL40.SPC',
      &      STATUS='OLD',READONLY,IOSTAT=IOS)
      IF (IOS.NE.0) THEN
11          WRITE(FRW,*)IOS
          FORMAT(5X,' TROUBLE OPENING TU120AL40.SPC, IOS= ',I3)
          GOTO 10
      ENDIF
      READ(C,*)EMAX
      FORMAT(2X,I3)
      READ(C,*)
      NROW=EMAX/10
      IF ((EMAX/10)*10).LT.EMAX:NROW=NROW+1
      DO 05 I=1,NROW
12          READ(C,*) (SPECT(I,I),I=1,10)
          FORMAT(10F7,0)
04      ENDDO
      CLOSE(C)

-----
      K=1
      N=1
      DO 07 I=1,EMAX/5
          DO 06 J=1,5
              IF ((5*(I-1)+J).LT.EMAX) THEN
                  SPECT(I)=SPECT(N,K)+SPECT(I)
              ENDIF
              N=N+1
          ENDDO
          IF (N.EQ.11) THEN
              K=K+1
              N=1
          ENDIF
07      ENDDO

      DO 08 I=1,EMAX/5
          ATTN1(I)=ATT1(5*I-2)*0.10000/0.9992
          ATTN2(I)=ATT2(5*I-2)*0.10000/2.6000
08      ENDDO

      DO 09 K=1,EMAX/5
          IO=10*(5*K-2)*SPEC(F)
09      ENDDO

-----
200      WRITE(FRW,201)
201      FORMAT(5X,' ENTER PHANTOM FILE NAME ',/,S)
      READ(FRW,202)LENGTH,FILNAME
202      FORMAT(2,80A1)
      FILNAME(LENGTH+1)=0
      OPEN(A,FILE=FILNAME,FORM='UNFORMATTED',STATUS='OLD',
      &      READONLY,IOSTAT=IOS)
      IF (IOS.NE.0) THEN
203          WRITE(FRW,203)FILNAME,IOS
          FORMAT(5X,' TROUBLE OPENING ',80A1,', IOS = ',I3)
          GOTO 200
      ENDIF

      READ(A)MUMATF,DELTA,DELTAHI,UX,UFHI,'SPECTRUM,ENERGY,
      &      MATERIAL1,MATERIAL2
      DO 204 J=UX/2,UX/2
          READ(A) (AMPAL(I,I),I=1,UFHI)
204      ENDDO
      CLOSE(A)

```

READ IN WATER ATTENUATION TABLE

READ IN ALUMINUM ATTENUATION TABLE

READ IN EXPECTED SPECTRUM TABLE

REARRANGE SPECTRUM IN TERMS OF 5 KEV STEPS

COMPUTE MASS ATTENUATION SPECTRA FOR WATER AND ALUMINUM AND MULTIPLY BY CONVERSION FACTOR = 0.10

COMPUTE ESTIMATE OF IO

INPUT PHANTOM DATA FILE

```

205 WRITE (NFM,206)
206 FORMAT (5X,' ENTER OUTPUT FILE NAME',/,S)
      READ (FPP,202) LENGTH, FILNAME
      FILNAME (LENGTH+1) = 0
      OPEN (0, FILE = FILNAME, FORM = 'UNFORMATTED', STATUS = 'NEW',
6      IOSTAT = IOS)
      IF (IOS.NE.0) THEN
        WRITE (FPP,203) FILNAME, IOS
        GOTO 205
      ENDIF

      M=HX
      NIMX=DELTA X
      NIMY=DELTA Y
      FI=3.14159265
      CONST=-2.00/(FI*NFHI)
      NIMX=210
      NIMY=210

      A=0.6410256
      B=0.1730769
      C=0.0044103
      A2=0.7918106
      B2=0.1023259
      C2=0.0017667
      D2=0.0000021

      TABLE3(0,1)=2.40600E-5
      TABLE3(0,2)=0.0000
      TABLE3(0,3)=1.1426E-3
      TABLE3(0,4)=2.42900E-5
      TABLE3(1,1)=1.66429E-2
      TABLE3(1,2)=1.66198E-2
      TABLE3(1,3)=9.7879E-1
      TABLE3(1,4)=-1.4251E-2
      TABLE3(2,1)=1.809640E-2
      TABLE3(2,2)=1.45347E-3
      TABLE3(2,3)=0.99529
      TABLE3(2,4)=-5.2008E-2
      TABLE3(3,1)=2.11714E-2
      TABLE3(3,2)=3.07500E-3
      TABLE3(3,3)=1.0313
      TABLE3(3,4)=2.5740E-3
      TABLE3(4,1)=5.73685E-2
      TABLE3(4,2)=3.69453E-2
      TABLE3(4,3)=-0.23202
      TABLE3(4,4)=2.2442

      DO 207 I=0,M
        TABLE(I)=CONST/(4.*(FLOAT(I)**2)-1.)
207 ENDDO

      FHI=0.0000

      DO 208 I=0,NFHI
        TABLE2(1,I)=DCOS(FHI)
        TABLE2(2,I)=DSIN(FHI)
        FHI=FHI+DELTA FHI
208 ENDDO

      CALL RFI(LAMDA1, HX, NFHI, M, NIMX, NIMY, IMAGE)

      DO 22 L=1,NFHI
        DO 20 J=-HX/2+2, HX/2-2
          LAMDA3(J,L)=A*LAMDA1(J,L)+B*(LAMDA1(J+1,L)+
6          LAMDA1(J-1,L))+C*(LAMDA1(J+2,L)+LAMDA1(J-2,L))
20 ENDDO
        DO 21 J=-HY/2, HY/2
          LAMDA1(J,L)=0.0000
21 ENDDO
22 ENDDO

      IF (ABS(IMAGE(-NIMY/4,0)).GT.(1.0E-3)) THEN
        OFFSET=IMAGE(-NIMY/4,0)
      ELSE
        OFFSET=0.02046
      ENDIF

      DO 110 I=-NIMX/2, NIMX/2
        DO 109 J=-NIMY/2, NIMY/2
          OUTIMAGE(I-110, I-110)= (IMAGE(J,I)-OFFSET)/OFFSET
109 ENDDO
110 ENDDO

      DO 211 F=1,3
        CALL MAPPING(IMAGE, TABLE3, NIMX, NIMY, IMAGE1, IMAGE2)
        CALL PEPP0(IMAGE1, IMAGE2, HX, NFHI, NIMX, NIMY,
6        LAMDA21, LAMDA22)

```

OPEN OUTPUT FILE

SET UP PARAMETERS

DEFINE SMOOTHING WEIGHTS FOR FBI

DEFINE SMOOTHING WEIGHTS FOR NOISE CORRECTION

SET UP MAPPING TABLE FOR TABLE(X,Y)
X=POINT IN DUAL MATERIAL PLANE
Y=1 -- POINT'S MU(CO) INCH
Y=2 -- MU DISTANCE TO NEXT POINT
Y=3 -- WATER COEFFICIENT OF REFRACTION
Y=4 -- ALUMINUM COEFFICIENT OF REFRACTION

SET UP TABLE OF CONVOLUTION COEFFICIENTS

SET UP TABLE OF SINES AND COSINES IN DOUBLE PERIODIC

DO CONVOLUTION-INTERSECTION

DO FOR SMOOTHING OF INITIAL DATA

GENERATE IMAGE OFFSET FOR DILUTATION

CREATE WATER AND ALUMINUM DENSITY IMAGES
REPROJECT IMAGES TO FORM EFFECTIVE PAIR LENGTHS


```

DO 109 I=1,NH1
DO 110 J=-NY/2,NY/2
DUMMY=0.000
DO 111 L=1,NMAY/5
IF (ABS(LAMDA21(J,I)).GT.1.0E-10) THEN
COEF1=ATTN1(L)*LAMDA21(J,I)
ELSE
COEF1=0.00
ENDIF
IF (ABS(LAMDA22(J,I)).GT.1.0E-10) THEN
COEF2=ATTN2(L)*LAMDA22(J,I)
ELSE
COEF2=0.00
ENDIF
COEF=COEF1+COEF2
IF (COEF.LT.33).AND.(COEF.GT.-33) THEN
DUMMY=DUMMY+(5*L-2)*SPEC(L)*EXP(-COEF)
ENDIF
110 ENDDO
LAMDA2(J,I)=-LOG(DUMMY,10)
111 ENDDO
112 ENDDO
DO 122 L=1,NH1
DO 121 I=-NY/2+3,NY/2-3
LAMDA1(J,I)=A2*(LAMDA1(J,I)-LAMDA2(J,I))+B2*
(LAMDA1(J+1,I)+LAMDA3(J-1,I)-LAMDA2(J+1,I)-
LAMDA2(J-1,I))+C2*(LAMDA3(J+2,I)+LAMDA3(J-2,I)-
LAMDA2(J+2,I)-LAMDA2(J-2,I))+D2*(LAMDA3(J+3,I)+
LAMDA3(J-3,I)-LAMDA2(J+3,I)-LAMDA2(J-3,I))
121 ENDDO
122 ENDDO
CALL RFT(LAMDA1,NX,NH1,N,NH1X,NINY,EPROP)
DO 210 I=-NH1X/2,NH1X/2
DO 209 J=-NINY/2,NINY/2
IMAGE(J,I)=IMAGE(J,I)+EPROP(J,I)
209 ENDDO
210 ENDDO
211 ENDDO
-----
CALL MATTING(IMAGE,TABLE1,NH1X,NH1Y,IMAGE1,IMAGE2)
DO 315 I=-NH1X/2,NH1X/2
DO 314 J=-NH1Y/2,NH1Y/2
IMAGE1(J,I)=(IMAGE1(J,I)-0.9992)/0.9992
IMAGE2(J,I)=(IMAGE2(J,I)-0.9992)/0.9992
314 ENDDO
315 ENDDO
IF (ABS(IMAGE(-NH1X/4,0)).GT.(1.0E-3)) THEN
OFFSET=IMAGE(-NH1X,4,0)
ELSE
OFFSET=0.02046
ENDIF
DO 213 I=-NH1X/2,NH1X/2
DO 212 J=-NH1Y/2,NH1Y/2
OUTIMAGE(J-119,I+120)=(IMAGE(J,I)-OFFSET)/OFFSET
OUTIMAGE(J+120,I-119)=IMAGE1(J,I)
OUTIMAGE(J+120,I+120)=IMAGE2(J,I)
212 ENDDO
213 ENDDO
DO 218 I=-NH1Y/2,NH1Y/2
DO 217 J=-NH1X/2,NH1X/2
IF (OUTIMAGE(J,I).GT.32.7) THEN
FLAG1=.TRUE.
IF (OUTIMAGE(J,I).GT.327.0) FLAG2=.TRUE.
ENDIF
217 ENDDO
218 ENDDO
-----
WRITE(A)OUTBUFFER
IF (FLAG1) THEN
IF (FLAG2) THEN
DO 219 J=-239,240
WRITE(A)(NINT(10.*OUTIMAGE(J,I)),I=-239,240)
219 ENDDO
ELSE
DO 220 J=-239,240
WRITE(A)(NINT(100.*OUTIMAGE(J,I)),I=-239,240)
220 ENDDO
ENDIF
221 I=-239,240
WRITE(A)(NINT(1000.*OUTIMAGE(J,I)),I=-239,240)
221 ENDDO
ENDIF
CLOSE(A)
STOP
END

```

! CALCULATE REPROJECTED DATA SET

! SMOOTH DIFFERENCE OF PSF SMOOTHED INPUT
! DATA AND REPROJECTED DATA. RESULT
! IS EPROP DATA SET

! CREATE EPROP IMAGE

! ADD EPROP IMAGE TO INITIAL IMAGE

! CREATE FINAL WATER AND ALUMINUM DENSITY
! IMAGES
! NORMALIZE WATER AND ALUMINUM IMAGES WRT
! WATER'S DENSITY

! COMPUTE OUTPUT IMAGE OFFSET

! SET UP OUTPUT IMAGE FILE GEOMETRY

! RUN OVERFLOW CHECK

! WRITE OUTPUT FILE

```

SUBROUTINE MAPPING (IMAGE, TABLE, NIMX, NINY, IMAGE1, IMAGE2)
  REAL*4 TABLE(0:4,1:4),P
  REAL*8 IMAGE(-105:105,-105:105)
  REAL*8 IMAGE1(-105:105,-105:105)
  REAL*8 IMAGE2(-105:105,-105:105)

  INTEGER*2 NIMX,NINY

  DO 112 J=-NIMX/2,NIMX/2
    DO 111 I=-NINY/2,NINY/2
      IF (IMAGE(J,I)-TABLE(0,1)) 1,1,2
        IMAGE1(J,I)=TABLE(0,3)
        IMAGE2(J,I)=TABLE(0,4)
        GOTO 111
      IF (IMAGE(J,I)-TABLE(1,1)) 3,4,5
        P=(TABLE(1,1)-IMAGE(J,I))/TABLE(1,2)
        IMAGE1(J,I)=P*TABLE(0,3)+(1.000-P)*TABLE(1,3)
        IMAGE2(J,I)=P*TABLE(0,4)+(1.000-P)*TABLE(1,4)
        GOTO 111
      IF (IMAGE(J,I)-TABLE(1,3))
        IMAGE2(J,I)=TABLE(1,4)
        GOTO 111
      IF (IMAGE(J,I)-TABLE(2,1)) 6,7,8
        P=(TABLE(2,1)-IMAGE(J,I))/TABLE(2,2)
        IMAGE1(J,I)=P*TABLE(1,3)+(1.000-P)*TABLE(2,3)
        IMAGE2(J,I)=P*TABLE(1,4)+(1.000-P)*TABLE(2,4)
        GOTO 111
      IF (IMAGE(J,I)-TABLE(2,3))
        IMAGE2(J,I)=TABLE(2,4)
        GOTO 111
      IF (IMAGE(J,I)-TABLE(3,1)) 9,10,11
        P=(TABLE(3,1)-IMAGE(J,I))/TABLE(3,2)
        IMAGE1(J,I)=P*TABLE(2,3)+(1.000-P)*TABLE(3,3)
        IMAGE2(J,I)=P*TABLE(2,4)+(1.000-P)*TABLE(3,4)
        GOTO 111
      IF (IMAGE(J,I)-TABLE(3,3))
        IMAGE2(J,I)=TABLE(3,4)
        GOTO 111
      IF (IMAGE(J,I)-TABLE(4,1)) 12,13,13
        P=(TABLE(4,1)-IMAGE(J,I))/TABLE(4,2)
        IMAGE1(J,I)=P*TABLE(3,3)+(1.000-P)*TABLE(4,3)
        IMAGE2(J,I)=P*TABLE(3,4)+(1.000-P)*TABLE(4,4)
        GOTO 111
      IF (IMAGE(J,I)-TABLE(4,3))
        IMAGE2(J,I)=TABLE(4,4)
    ENDDO
  ENDDO
  RETURN
END

SUBROUTINE REFJ(L1,NX,NFYI,M,NIMX,NINY,IMG)
  REAL*4 L1(-155:155,4*7),I2(-155:155,4*7)
  REAL*8 IMG(-105:105,-105:105)
  REAL*8 FILTER,D,L,Q,R,S,AP,AN,AL,C1,C2

  REAL*8 X,DUMMY,M

  REAL*8 DELTAPHI,PHI,PI
  INTEGER*2 NX,NFYI,NIMX,NINY
  INTEGER*2 NUM,I,J,F,H,M

  SUBROUTINE TO CREATE WATER AND ALUMINUM
  EQUIVALENT DENSITY IMAGES
  R -- RATIO OF DISTANCE FROM POINT 1
  TO LOWER END OF INTERVAL IN M
  FOR TABLE ENERGY (50 KEV)

  IF IMAGE PIXEL VALUE LESS THAN 4 EQUAL
  TO AIR THEN DENSITY MAPS TAKE AIR
  EQUIVALENT VALUES
  ELSE
  IF IMAGE PIXEL VALUE LESS THAN VALUE FOR
  SWING POINT, THEN LINEARLY INTERPOLATE
  BETWEEN AIR AND SWING POINT
  ELSE
  IF EQUAL TO SWING PT TAKES ITS VALUE
  ELSE
  IF PIXEL VALUE LESS THAN THAT OF ADIPOSE,
  THEN INTERPOLATE BETWEEN SWING PT.
  AND ADIPOSE
  ELSE
  IF EQUAL TAKE ADIPOSE VALUES
  ELSE
  IF VALUE LESS THAN THAT OF POINT NUMBER
  OF MUSCLE, BLOOD, AND SOFT TISSUE
  THEN INTERPOLATE BETWEEN AIR, F
  AND MIXTURE
  ELSE
  IF EQUAL TAKE MIXTURE VALUES
  ELSE
  IF LESS THAN VALUE FOR BONE, THEN INTERPOLATE
  BETWEEN MIXTURE AND BONE
  ELSE
  TAKE VALUES OF BONE

  SUBROUTINE TO CONVOLVE AND REFORMAT
  DATA SETS
  L1 -- INPUT DATA SET
  L2 -- FILTERED DATA SET
  IMG -- OUTPUT IMAGE ARRAY
  FILTER -- CONVOLUTION DUMMY
  D -- DISTANCE OF INTERSECTION OF LINE
  AND BISECTING LINE WRT FHI
  (DISTANCES ARE IN PIXEL UNITS)
  L -- LONGEST PERPENDICULAR DISTANCE
  FROM BISECTING LINE TO A PIXEL
  CORNER
  S -- SHORTEST PERP. DISTANCE FROM
  BISECTING LINE TO A PIXEL CORNER
  Q -- DISTANCE FROM PIXEL CENTER TO
  FILE PAY'S LEFT EDGE
  P -- DISTANCE FROM PIXEL CENTER TO
  FILE PAY'S RIGHT EDGE
  AP -- WEIGHT OF PAY TO THE RIGHT OF
  PIXEL
  AN -- WEIGHT OF PAY CENTERED ON THE
  PIXEL
  AL -- WEIGHT OF PAY TO THE LEFT OF
  PIXEL
  C1,C2 -- COSINE AND SINE OF PHI
  X -- TRANSLATIONAL PIXEL POSITION IN
  ROTATED FRAME
  M -- VERTICAL WCT POSITION IN FILE
  + 1 POSITION, WRT ROTATED FRAME
  NUM -- NEAREST INTEGER COMPONENT OF X
  EQUIVALENT TO MIDDLE PAY'S POSITION

```

```

COMMON TABLE(0:10),TABLE2(2,0:407),DELTAFHI,PI
DO 100 I=-NIMY/2,NIMY/2
DO 100 J=-NIMX/2,NIMX/2
  INSG(J,I)=0.0000
ENDDO
100 ENDDO
FHI=0.000

DO 112 F=1,NPHI
DO 111 I=-NMX/2,NMX/2
  FILTER=0.0
DO 110 H=I-M,I+M
  IF (ABS(H).LT.NX/2) THEN
    FILTER=FILTER+TABLE(ABS(I-H))*L1(H,F)
  ENDIF
ENDDO
110 L2(I,F)=FILTER
ENDDO
111 ENDDO
112 ENDDO

DO 114 F=1,NPHI
  C1=TABLE2(1,F-1)
  C2=TABLE2(2,F-1)
  W=(NIMX+1)*C2
  X=-((NIMX+C1)/2+(NIMY+C2)/2.)

  IF (FHI.LT.PI/4) THEN
    D=1/C1
    L=(C1+C2)/2
    S=(C1-C2)/2
  ELSE
    IF (FHI.LT.PI/2) THEN
      D=1/C2
      L=(C1+C2)/2
      S=(C2-C1)/2
    ELSE
      IF (FHI.LT.3*PI/4) THEN
        D=1/C2
        L=(C2-C1)/2
        S=(C1+C2)/2
      ELSE
        D=-1/C1
        L=(C2-C1)/2
        S=-(C1+C2)/2
      ENDIF
    ENDIF
  ENDIF
  ENDDO
DO 115 I=-NIMX/2,NIMX/2
DO 1 J=-NIMY/2,NIMY/2
  NUM=NINT(X)
  Q=0.500-(X-NUM)
  R=1.-Q
  IF (S.LT.Q) THEN
    IF (Q.LT.L) THEN
      AR=(D*(L-Q)**2)-(2.0*(L-S))
    ELSE
      AR=0.000
    ENDIF
  ELSE
    AR=0.5-D*Q
  ENDIF
  IF (S.LT.R) THEN
    IF (R.LT.L) THEN
      AL=(D/(2.0*(L-S)))*(L-R)**2
    ELSE
      AL=0.000
    ENDIF
  ELSE
    AL=0.5-D*R
  ENDIF
  AM=1.00-(AL+AR)
  INSG(J,I)=INSG(J,I)+(AL*L2(NUM-1,F)+AM*L2(NUM,R)+
    AP*L2(NUM+1,F))
  X=X+C2
ENDDO
115 ENDDO
FHI=FHI+DELTAFHI
116 ENDDO
RETURN
END

SUBROUTINE REDDO(IMG1,IMG2,NX,NPHI,NIMX,NIMY,L21,L22)
REAL*4 L21(-155:155,407),L22(-155:155,407)
REAL*4 IMG1(-105:105,-105:105),IMG2(-105:105,-105:105)

```

LATIONAL POSITION

INITIALIZE IMAGE TO ZERO

INITIALIZE FHI

CONVOLVE INPUT DATA SET WITH FILTER

DO BACKPROJECTION-SUMMATION

COMPUTE W MPT FHI

COMPUTE X FOR LOWER LEFT HAND CORNER
POSITION OF IMAGE MPT ROTATED
FRAME

COMPUTE D,L,S MPT FHI

LOOP OVER IMAGE

FIND NEAREST INTEGER TO X

COMPUTE Q

COMPUTE R

COMPUTE AL, AM, AND AP MPT TO VALUES OF
L, S, Q, AND R

FIXIL VALUE INCREMENTED

X POSITION SHIFTED TO FIXIL TO FHI

X POSITION SHIFTE TO LEFT MOST COLUMN
AND ADVANCED ONE ROW
INCREMENT FHI

SUBROUTINE THAT PERFORMS INTERPOLATION OF SUBROUTINE REDJ, BUT INPUT
AND OUTPUT ARE IN TERMS OF WATER
AND ALUMINUM

L1,L2 == OUTPUT DATA SETS

IMG1,IMG2 == INPUT IMAGE SETS

```

REAL*8 L31(-155:155,407),L32(-155:155,407)
REAL*8 X,W,D,L,Q,P,F,AP,AM,AL
REAL*9 C1,C2,DELTA PHI,PHI,FI
INTEGER*2 NX,NPHI,NINX,NINY,EMAX
INTEGER*2 NUM,I,J,F,M,M

COMMON TABLE(0:310),TABLE2(2,0:407),DELTA PHI,FI

DO 103 I=1,NPHI
DO 107 J=-NX/2,NX/2
L31(J,I)=0.0000
L32(J,I)=0.0000
102 ENDDO
103 ENDDO
PHI=0.000

DO 106 F=1,NPHI
C1=TABLE2(1,F-1)
C2=TABLE2(2,F-1)
W=(NINY+1)*C2
X=-(NINX*C1/2.+NINY*C2./2.)
IF (PHI.LT.FI/4) THEN
D=1/C1
L=(C1+C2)/2
S=(C1-C2)/2
ELSE
IF (PHI.LT.FI/2) THEN
D=1/C2
L=(C1+C2)/2
S=(C2-C1)/2
ELSE
IF (PHI.LT.3*FI/4) THEN
D=1/C2
L=(C2-C1)/2
S=(C1+C2)/2
ELSE
D=1/C1
L=(C2-C1)/2
S=-(C1+C2)/2
ENDIF
ENDIF
DO 105 I=-NINX/2,NINX/2
DO 104 J=-NINY/2,NINY/2
NUM=NINT(X)
Q=0.500-(X-NUM)
F=1.-Q
IF (S.LT.Q) THEN
IF (Q.LT.L) THEN
AR=(D/(2.0*(L-S)))*(L-Q)**2
ELSE
AR=0.000
ENDIF
ELSE
AP=0.500-D*Q
ENDIF
IF (S.LT.P) THEN
IF (R.LT.L) THEN
AL=(D/(2.0*(L-S)))*(L-R)**2
ELSE
AL=0.000
ENDIF
ELSE
AL=0.500-D*P
ENDIF
AM=1.00-(AL+AR)
L31(NUM-1,F)=L31(NUM-1,F)+AL*INX1(J,I)

L31(NUM,F)=L31(NUM,F)+AM*INX1(J,I)
L31(NUM+1,F)=L31(NUM+1,F)+AR*INX1(J,I)

L32(NUM-1,F)=L32(NUM-1,F)+AL*INX2(J,I)

L32(NUM,F)=L32(NUM,F)+AM*INX2(J,I)
L32(NUM+1,F)=L32(NUM+1,F)+AP*INX2(J,I)
X=X+C2
104 ENDDO
X=X+C1-W
105 ENDDO
PHI=PHI+DELTA PHI
106 ENDDO

DO 108 F=1,NPHI
DO 107 I=-NX/2,NX/2
L21(I,F)=L31(I,F)
L22(I,F)=L32(I,F)
107 ENDDO
108 ENDDO
RETURN
END

```

END

24/12/92

FIN