**Representation Theory of the Lie algebras of
divergence zero and Hamiltonian vector fields on a
torus**

by

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A thesis submitted to the Faculty of Graduate and
Postdoctoral Affairs in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy
in
Mathematics

Carleton University
Ottawa, Ontario

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2010 *Mathematics Subject Classification.* 17B10, 17B66

Dedicated to Angela.

ABSTRACT. This thesis is about the representation theory of the Lie algebras of divergence zero and Hamiltonian vector fields on an N dimensional torus, denoted $\mathcal{S}_N$ and $\mathcal{H}_N$ respectively. It combines the results of three research papers written during the doctoral work of the author. First, conditions for irreducibility of a certain class of tensor modules for $\mathcal{S}_N$, are investigated. Next, a classification of indecomposable and irreducible $\mathcal{S}_N$ -modules, which have a compatible action by the commutative algebra of Laurent polynomials, and have a weight decomposition with finite dimensional weight spaces is presented. Such modules are referred to as category $\mathcal{J}$, and it is seen that the tensor modules considered above comprise many of the irreducible $\mathcal{S}_N$ -modules in category $\mathcal{J}$. Similar techniques are then applied to give a classification of indecomposable and irreducible category $\mathcal{J}$ modules for $\mathcal{H}_N$.

Acknowledgments

My sincere thanks to Professor Yuly Billig for his support throughout my life as a graduate student. Dr. Billig teaches with a great deal of care and dedication. He is a respected member of the School of Mathematics and Statistics at Carleton University for his outstanding research and for his ability to promote education in mathematics at all levels. I could not have asked for a better supervisor to guide me through writing this thesis.

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CHAPTER 1

Introduction

1. Background

An important source of infinite dimensional Lie algebras are the Lie algebras of vector fields. The derivations of the commutative algebra $R = \mathbb{C}[x_1, \dots, x_N]$ form the Lie algebra

$$\text{Span}_{\mathbb{C}} \left\{ f \frac{\partial}{\partial x_a} \mid f \in R, a \in \{1, \dots, N\} \right\},$$

with Lie bracket

$$\left[f \frac{\partial}{\partial x_a}, g \frac{\partial}{\partial x_b} \right] = f \frac{\partial g}{\partial x_a} \frac{\partial}{\partial x_b} - g \frac{\partial f}{\partial x_b} \frac{\partial}{\partial x_a},$$

and coincide with the Lie algebra of complex-valued polynomial vector fields on an affine space. This family of Lie algebras is denoted W_N and is called the *general series*. Interesting Lie subalgebras of W_N are defined using the Lie derivative $\mathcal{L}_X \omega$, of a particular differential form ω with respect to vector field X . The *special series* is given by

$$S_N = \text{Span}_{\mathbb{C}} \{ X \in W_N \mid \mathcal{L}_X(dx_1 \wedge \dots \wedge dx_N) = 0 \},$$

and corresponds to divergence zero vector fields. When N is even, $N = 2m$, the vector fields

$$H_N = \text{Span}_{\mathbb{C}} \left\{ X \in W_N \mid \mathcal{L}_X \left(\sum_{i=1}^m dx_i \wedge dx_{m+i} \right) = 0 \right\},$$

are called the *Hamiltonian series*. For $N = 2m+1$, let $\omega = dx_{2m+1} + \sum_{i=1}^m x_{m+i} dx_i - x_i dx_{m+i}$, then the *contact series* is defined by

$$K_N = \text{Span}_{\mathbb{C}} \{ X \in W_N \mid \mathcal{L}_X \omega = P\omega, P \in R_{2m+1} \}.$$

The four families, W_N , S_N , H_N , and K_N are called the Lie algebras of Cartan Type, and constitute an important class of simple infinite dimensional Lie algebras.

A natural progression is to consider the Lie algebra of derivations of Laurent polynomials in N variables. Let $A_N = \mathbb{C}[t_1^{\pm 1}, \dots, t_N^{\pm 1}]$, and denote its Lie algebra of derivations by $\mathcal{W}_N$. This has a geometric interpretation as the Lie algebra of vector fields on an N dimensional torus, by taking an angular coordinate system $\{\theta_1, \dots, \theta_N\}$ for the N -torus, and letting $t_j = e^{\sqrt{-1}\theta_j}$. The well-known Virasoro algebra is the universal central extension of $\mathcal{W}_1$.

A class of modules for $\mathcal{W}_N$ is investigated by Rao in [12], which the author refers to as Larsson's modules coming from [11]. They have also been studied by Rudakov in [14] and Shen in [15]. Let $V(\lambda)$ be a finite dimensional irreducible $\mathfrak{sl}_N$ -module of highest weight λ . Since $\mathfrak{gl}_N = \mathfrak{sl}_N \oplus \mathbb{C}I_N$, where I_N is the $N \times N$ identity matrix, $V(\lambda)$ can be made into a finite dimensional irreducible $\mathfrak{gl}_N$ -module $V(\lambda, b)$, by specifying the scalar action $b \in \mathbb{C}$ by I_N . The $\mathcal{W}_N$ -modules considered by Rao are constructed by taking the tensor product $A_N \otimes V(\lambda, b)$. These are generalized to include a larger class of representations by choosing a parameter $\sigma \in \mathbb{C}^N$, which appears in the action of $\mathcal{W}_N$. These tensor modules are denoted $F^\sigma(\lambda, b)$. In [12] Rao shows that the $F^\sigma(\lambda, b)$ are irreducible unless either $(\lambda, b) = (\omega_k, k)$ for $k \in \{1, \dots, N-1\}$ where ω_k is a fundamental dominant weight, or if $(\lambda, b) = (0, 0)$, or $(0, N)$ and $\sigma \in \mathbb{Z}^N$. In these special cases, irreducible quotients and submodules are determined.

This thesis is about the representation theory of the subalgebras $\mathcal{S}_N$ and $\mathcal{H}_N$ of $\mathcal{W}_N$. The Lie algebra $\mathcal{S}_N$ is the subalgebra of divergence zero (in angular coordinates) vector fields on an N dimensional torus, and $\mathcal{H}_N$ is the subalgebra of Hamiltonian vector fields on an N dimensional torus for N even. Both are defined using the relations for the subalgebras S_N and H_N of W_N above, by changing from coordinates x_j on affine space to coordinates θ_j on a torus. They will be described in more detail later on. The differential 1-form ω for the contact series K_N shown

above, is not well-defined on an N -torus under mapping $x_j \mapsto \theta_j$, and so an analogous subalgebra of $\mathcal{W}_N$ cannot be obtained in this way. Contact structures exist for 3 and 5 dimensional tori (see [6] Section 3.2.6), though it seems at the time of writing that contact structures for odd dimensional tori in general have not been found.

In [16] the tensor modules considered by Rao are restricted to $\mathcal{S}_N$ for $N \geq 2$. In this restriction the action by I_N does not appear, eliminating the need for scalar b , and thus finite dimensional irreducible $\mathfrak{sl}_N$ -modules $V(\lambda)$ are considered in place of $V(\lambda, b)$. Modules $F^\sigma(\lambda)$ are called *minuscule* when the highest weight λ is a fundamental dominant weight ω_k , or $\lambda = 0$. It is shown in [16] that these tensor modules are irreducible under conditions similar to Rao's. In particular non-minuscule modules are irreducible. Minuscule modules can be realized as the modules of differential k -forms on a torus. Tensor modules $F^\sigma(\omega_k)$ are isomorphic to Ω^k , the module (over Laurent polynomials) of differential k -forms on an N -torus. The modules of functions Ω^0 , and N -forms Ω^N , correspond to $F^\sigma(0)$, where $V(0)$ is a one dimensional $\mathfrak{sl}_N$ -module. It is then useful to consider the de Rham complex of differential forms,

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^N$$

(where d is the exterior derivative) since d yields a homomorphism between tensor modules. The kernels and images of d are submodules of the Ω^k and are used to determine submodules of the tensor modules.

A partial classification of irreducible $\mathcal{W}_N$ -modules is given by Rao in [13]. This paper investigates $\mathcal{W}_N$ -modules with finite dimensional weight spaces, which are also free modules of finite rank for the associative algebra A_N , and admit a compatible action by both A_N and $\mathcal{W}_N$. Rao proves that any irreducible module of this type is isomorphic to one of the tensor modules $F^\sigma(\lambda, b)$. The family of $\mathcal{W}_N$ -modules considered by Rao is formally defined in [3], where they are named “category $\mathcal{J}$ ”. Here Billig uses the techniques of polynomial modules (see [1] and

[5]) to recover the results of Rao and extends them to include a classification of all indecomposable modules. The result is the following.

THEOREM ([3] Theorem 3.3 (ii)). *Let $\lambda \in \mathbb{C}^N$ and $\mathcal{J}_\lambda$ the subcategory of $\mathcal{W}_N$ -modules in category $\mathcal{J}$ supported on $\lambda + \mathbb{Z}^N$, for $N \geq 1$. There is an equivalence of categories between the category of finite dimensional modules for*

$$W_N^+ = \text{Span}_{\mathbb{C}} \left\{ f \frac{\partial}{\partial x_a} \in W_N \mid f(0) = 0 \right\}.$$

and $\mathcal{J}_\lambda$. This equivalence maps V to $A_N \otimes V$ where V is a finite dimensional module for W_N^+ .

Category $\mathcal{J}$ may be considered for Lie algebras $\mathcal{S}_N$ and $\mathcal{H}_N$ with $N \geq 2$. Classifications of indecomposable $\mathcal{S}_N$ -modules and $\mathcal{H}_N$ -modules in category $\mathcal{J}$ are given in [4] and [17] respectively. The classification of irreducible modules in category $\mathcal{J}$ is presented separately in these papers as well. This was done using the general strategies of [3] employing the techniques of polynomial modules. For the case $N = 2$ it is shown directly that the action by any element of $\mathcal{S}_2$ may be interpolated by the action of a certain polynomial. This is extended to the general case by induction on N . Since $\mathcal{H}_2 \cong \mathcal{S}_2$, the basis of induction for polynomial action by Hamiltonian vector fields on a torus is the same as the divergence zero vector fields on a torus, however a separate induction is required to obtain polynomial action for any even N . Irreducible representations for the case $N = 2$ are studied in [10] by Jiang and Lin. The basis of induction in Chapter 3 makes use of a technique found in this paper in order to obtain a family of eigenvectors, and provides a crucial step in the classification.

The classification of $\mathcal{S}_N$ mimics that of $\mathcal{W}_N$ but has a twist. For even values of N , let $\mathfrak{h}_N$ be the $N + 1$ dimensional Heisenberg algebra. For any N let $\mathfrak{a}_N$ be an N dimensional abelian Lie algebra and let $S_N^+ = S_N \cap W_N^+$. The following is the main result of [4] and will be presented in detail later along with the action of $\mathcal{S}_N$.

THEOREM. *Let $\lambda \in \mathbb{C}^N$ and let $\mathcal{J}_\lambda$ be the subcategory of $\mathcal{S}_N$ -modules in category $\mathcal{J}$ supported on $\lambda + \mathbb{Z}^N$.*

- (a) *For $N = 2$ there is an equivalence of categories between the category of finite dimensional modules for $S_2^+ \oplus \mathfrak{h}_2$ and $\mathcal{J}_\lambda$. This equivalence maps V to $A_2 \otimes V$ where V is a finite dimensional module for $S_2^+ \oplus \mathfrak{h}_2$.*
- (b) *For $N \geq 3$, there is an equivalence of categories between the category of finite dimensional modules for $S_N^+ \oplus \mathfrak{a}_N$ and $\mathcal{J}_\lambda$. This equivalence maps V to $A_N \otimes V$ where V is a finite dimensional module for $S_N^+ \oplus \mathfrak{a}_N$.*

When $N \geq 3$, relations among the elements of $\mathcal{S}_N$ force a certain simplification of the action, and so the $N = 2$ case is exceptional with the Heisenberg algebra appearing in place of $\mathfrak{a}_N$. If irreducible modules from category $\mathcal{J}$ are considered, the module V in the theorem is irreducible, and the action by $\mathcal{S}_N$ simplifies. The central element of the Heisenberg algebra acts trivially, reducing its action to that of $\mathfrak{a}_2$. Then for $N \geq 2$, elements of $\mathfrak{a}_N$ are central and hence act as scalars. The action of S_N^+ for the simple modules reduces to $\mathfrak{sl}_N$. It turns out that the tensor modules $F^\sigma(\lambda)$ exhaust all irreducible $\mathcal{S}_N$ -modules in category $\mathcal{J}$ when $\mathfrak{a}_N$ acts by λ . Thus the choice of scalar action for the representation of $\mathfrak{a}_N$ yields a new larger class of irreducible modules for $\mathcal{S}_N$ than was considered in [16].

In the case of Hamiltonian vector fields let $H_N^+ = H_N \cap W_N^+$. Representations for the $N + 1$ dimensional Heisenberg algebra appears for all N . Below is the main result of [17] (details provided later).

THEOREM. *Let $\lambda \in \mathbb{C}^N$ and $\mathcal{J}_\lambda$ the subcategory of $\mathcal{H}_N$ -modules in category $\mathcal{J}$ supported on $\lambda + \mathbb{Z}^N$, where N is even and $N \geq 2$. There is an equivalence of categories between the category of finite dimensional modules for $H_N^+ \oplus \mathfrak{h}_N$ and $\mathcal{J}_\lambda$. This equivalence maps V to $A_N \otimes V$ where V is a finite dimensional module for $H_N^+ \oplus \mathfrak{h}_N$.*

This thesis is a presentation of the detailed results from [16], [4] and [17]. In the next section of the current chapter the basic theory of Lie algebras is reviewed

briefly in order to fix some notation and definitions. Also in this chapter, some more specific concepts and conventions will be covered, and these relate directly to the remaining chapters. Chapter 2 will cover the conditions for irreducibility of tensor modules for $\mathcal{S}_N$ of [16]. Chapters 3 and 4 will give the classification of category $\mathcal{J}$ modules for $\mathcal{S}_N$ and $\mathcal{H}_N$ respectively.

2. Basics of Lie Algebras

In order to fix some terminology, this section summarizes a basic knowledge of Lie algebras. For a proper treatise see [9].

A *Lie algebra* is a vector space L over a field F with a bilinear operation $[\cdot, \cdot] : L \times L \rightarrow L$, called the *Lie bracket*, such that $[x, x] = 0$ and the *Jacobi identity*, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, holds for all $x, y, z \in L$. Anticommutativity of the Lie bracket, $[x, y] = -[y, x]$, follows from these properties. Throughout this thesis all Lie algebras are considered over $\mathbb{C}$ (consult [9] for definitions involving a more general F).

Given a vector space V , the endomorphisms of V form a Lie algebra $\mathfrak{gl}(V)$ with Lie bracket given by the *commutator* $[x, y] = xy - yx$ for $x, y \in \text{End}(V)$, where xy is composition of endomorphisms. If $\dim(V) = N$ a choice of basis identifies $\mathfrak{gl}(V)$ with the Lie algebra of $N \times N$ matrices denoted $\mathfrak{gl}_N$. Its subalgebra of trace zero matrices is denoted $\mathfrak{sl}_N$. Let $E_{i,j}$ be the $N \times N$ matrix with a 1 in entry (i, j) and zeros elsewhere. For N even with $N = 2m$, the subalgebra $\mathfrak{sp}_N$ of $\mathfrak{gl}_N$ given by $\text{Span}_{\mathbb{C}} \{E_{i,j} - E_{m+j,m+i}, E_{i,m+j} + E_{j,m+i}, E_{m+i,j} + E_{m+j,i} \mid 1 \leq i, j \leq m\}$ is called the *symplectic algebra* and will be used in Chapter 4.

An *ideal* of L is a subspace I , such that $[x, y] \in I$ for all $x \in L, y \in I$. A Lie algebra is called *simple* if its only ideals are 0 and L , and if $[L, L] \neq 0$. Let $L^{(0)} = L$, and $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ for $i \in \{1, 2, 3, \dots\}$. A finite dimensional Lie algebra is called *solvable* if $L^{(n)} = 0$ for some n , and is called *semisimple* if L has no nonzero solvable ideals. It can be shown (over $\mathbb{C}$) that a finite dimensional semisimple Lie algebra L is a direct sum of its simple ideals.

If L and L' are Lie algebras (over the same field), a linear map $\varphi : L \rightarrow L'$ is a Lie algebra *homomorphism* if $\varphi[x, y] = [\varphi(x), \varphi(y)]$ for all $x, y \in L$, where the brackets on the left and right are understood to be those of L and L' respectively.

A *representation* of L is a Lie algebra homomorphism $\varphi : L \rightarrow \mathfrak{gl}(V)$. In this case V is called an L -*module* and φ defines an *action* of L on V given by $xv = \varphi(x)(v)$ for any $x \in L$, which satisfies the following: $(ax+by)v = a(xv)+b(yv)$, $x(au + bv) = a(xu) + b(xv)$, and $[x, y]v = x(yv) - y(xv)$, for all $a, b \in \mathbb{C}$, $u, v \in V$ and $x, y \in L$. An important example of a representation of L is its *adjoint* representation, which maps $x \in L$ to $\text{ad } x \in \mathfrak{gl}(L)$, where $\text{ad } x(y) = [x, y]$.

Finite dimensional L have a symmetric bilinear form $\kappa(x, y) = \text{Trace}(\text{ad } x \text{ ad } y)$ for $x, y \in L$, called the *Killing form*. It can be shown that L is semisimple if and only if the Killing form is nondegenerate, that is when $\{x \in L \mid \kappa(x, y) = 0 \text{ for all } y \in L\} = 0$.

A subspace W of an L -module V is an L -*submodule* if it is closed under the action of L . When V has exactly two submodules, namely itself and 0, it is called *simple* (or *irreducible*), and in this case its defining representation φ is *irreducible*. If V is the direct sum of irreducible submodules, then V is called *completely reducible*, and if V is nonzero and cannot be written as a direct sum of two nonzero submodules it is called *indecomposable*. The terms “completely reducible” and “indecomposable” also describe φ .

The finite dimensional representation theory of semisimple Lie algebras is well understood. In particular, the following theorem says that finite dimensional representations of a semisimple Lie algebra may be studied by its irreducible representations.

THEOREM 1.1 (Weyl's Theorem). *Let $\varphi : L \rightarrow \mathfrak{gl}(V)$ be a finite dimensional representation of a semisimple Lie algebra L . Then φ is completely reducible.*

A *Cartan subalgebra* H of a semisimple Lie algebra L is a maximal abelian subalgebra such that every element is diagonalizable in the adjoint representation. In a finite dimensional L -module V , H acts diagonally. That is, V decomposes into

the direct sum of subspaces

$$V_\lambda = \{v \in V | hv = \lambda(h)v \text{ for all } h \in H\},$$

where λ ranges over the dual space H^* of H . The $\lambda \in H^*$ with $V_\lambda \neq 0$ are called *weights*, and V_λ a *weight space*. The restriction of the Killing form to H is nondegenerate and so each $\lambda \in H^*$ can be identified with a unique element $t_\lambda \in H$ such that $\lambda(h) = \kappa(t_\lambda, h)$ for all $h \in H$. The form transfers to H^* by setting $(\lambda, \lambda') = \kappa(t_\lambda, t_{\lambda'})$ for any $\lambda, \lambda' \in H^*$.

The representation theory of $\mathfrak{sl}_2$ is extremely important for the general theory of semisimple Lie algebras. Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be a basis for $\mathfrak{sl}_2$. It can be shown that if V is an irreducible $\mathfrak{sl}_2$ -module of dimension $d + 1$, then V decomposes into the direct sum of weight spaces for h as $V = \bigoplus_{i=0}^d V_{d-2i}$ where $V_{d-2i} = \{v \in V | hv = (d - 2i)v\}$. In particular, the weights of V have integer values and the dimension of each weight space is 1. Furthermore $e : V_\lambda \rightarrow V_{\lambda+2}$ and $f : V_\lambda \rightarrow V_{\lambda-2}$, giving rise to the terms raising and lowering operators. Since V is finite dimensional, $ev = 0$ for any $v \in V_d$, and v (when nonzero) is called a highest weight vector for V with highest weight d .

The weight space decomposition in the case of the adjoint representation for a finite dimensional semisimple Lie algebra L is known as the *root space decomposition* of L , and is used in the classification of all finite dimensional simple (and hence semisimple) Lie algebras. More specifically, the root space decomposition of L is the direct sum $L = \bigoplus_{\alpha \in H^*} L_\alpha$, where $L_\alpha = \{x \in L | [h, x] = \alpha(h)x \text{ for all } h \in H\}$. The nonzero α with $L_\alpha \neq 0$ are called *roots*, and the set of all roots is called the *root system* of L with respect to H , denoted by Φ . Every root system Φ has a subset Π called a *base*, where each $\beta \in \Phi$ can be written as a sum $\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha$ such that the coefficients c_α are all nonnegative, or all nonpositive integers. This divides

Φ into two subsets, the positive roots Φ^+ and the negative roots Φ^- , respectively. The cardinality of the base is equal to the dimension of H .

If α is a root in Φ^+ then $-\alpha$ is also a root and lies in Φ^- . If $e_\alpha \in L_\alpha$ there is an element $f_\alpha \in L_{-\alpha}$ such that e_α, f_α and $h_\alpha = [e_\alpha, f_\alpha]$ span a subalgebra of L isomorphic to $\mathfrak{sl}_2$. In this case $\beta(h_\alpha) = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$ for all $\beta \in \Phi$. Let $\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$. Since L is a module for this Lie algebra under the adjoint action, the representation theory for $\mathfrak{sl}_2$ yields that each root space L_β is one dimensional, $\beta(h_\alpha) \in \mathbb{Z}$, and $e_\alpha : L_\beta \rightarrow L_{\beta+\alpha}, f_\alpha : L_\beta \rightarrow L_{\beta-\alpha}$. The set of roots of the form $\beta + i\alpha$ with $\alpha \neq \pm\beta$ and $i \in \mathbb{Z}$, is called the α -string through β . If $r, q \in \mathbb{Z}$ are the largest integers such that $\beta - r\alpha$ and $\beta + q\alpha$ are roots, then $\beta + i\alpha$ are roots for all $-r \leq i \leq q$, and $\langle \beta, \alpha \rangle = r - q$.

For $\alpha \in H^*$, let σ_α be the reflection in the hyperplane $\{\gamma \in H^* | (\gamma, \alpha) = 0\}$ in H^* . Then $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ for $\beta \in H^*$. The group generated by all reflections σ_α for $\alpha \in \Phi$ is called the *Weyl group* of Φ , denoted $\mathfrak{W}$, and it permutes the roots in Φ . Elements of $\mathfrak{W}$ induce automorphisms of L , and reveal useful properties of L and its representations.

Let $\Pi = \{\alpha_1, \dots, \alpha_N\}$ be a base of root system Φ for L . The unique elements $\omega_1, \dots, \omega_N$ of H^* , such that $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$, for Kronecker delta δ_{ij} , are called *fundamental dominant weights*. A weight λ is called *dominant integral* if it is a linear combination of the fundamental dominant weights, with nonnegative integer coefficients. A *highest weight module* is an L -module which is generated by a weight vector v , where $xv = 0$ for all $x \in L_\alpha$ with $\alpha \in \Phi^+$. Every finite dimensional irreducible representation for a semisimple Lie algebra L is a highest weight module, and is uniquely determined by its highest weight Λ which is dominant integral. This module is denoted by $V(\Lambda)$. Its highest weight space V_Λ is one dimensional. For any weight λ of $V(\Lambda)$ and $x \in L_\alpha$, $x : V_\lambda \rightarrow V_{\lambda+\alpha}$, and λ can be written $\lambda = \Lambda - \sum_{i=1}^N \gamma_i \alpha_i$ where the γ_i 's are nonnegative integers. As in the case of root strings, the set of all weights of the form $\lambda + i\alpha$, for $i \in \mathbb{Z}, \alpha \in \Phi$, forms the α -string through λ , and if $r, q \in \mathbb{Z}$ are the largest integers such that $\lambda - r\alpha$ and $\lambda + q\alpha$ are

weights, then $\lambda + i\alpha$ are weights for all $-r \leq i \leq q$, and $\langle \lambda, \alpha \rangle = r - q$. The set of weights of $V(\Lambda)$ is permuted by $\mathfrak{W}$, and $\dim V_\lambda = \dim V_{\sigma(\lambda)}$ for any $\sigma \in \mathfrak{W}$.

3. Preliminaries

Throughout this thesis $N \in \mathbb{N}$. Consider the column vector space $\mathbb{C}^N$ with standard basis $\{e_1, \dots, e_N\}$. Let $(\cdot|\cdot)$ be the symmetric bilinear form $(u|v) = u^T v \in \mathbb{C}$, where $u, v \in \mathbb{C}^N$ and u^T denotes matrix transpose. Denote the set of nonnegative integers by $\mathbb{Z}_{\geq 0}$. For $k \in \mathbb{Z}_{\geq 0}^N$, $|k| = k_1 + \dots + k_N$, $k! = k_1! \dots k_N!$ and $\binom{r}{k} = \frac{r!}{k!(r-k)!}$.

Let $A_N = \mathbb{C}[t_1^{\pm 1}, \dots, t_N^{\pm 1}]$ be the Laurent polynomials over $\mathbb{C}$. Elements of A_N will be presented with the multi-index notation $t^r = t_1^{r_1} \dots t_N^{r_N}$ for $r = (r_1, \dots, r_N) \in \mathbb{Z}^N$. Also, for $i \in \{1, \dots, N\}$, let $d_i = t_i \frac{\partial}{\partial t_i}$. The vector space of all of derivations of A_N , $\text{Der}(A_N)$, forms a Lie algebra called the Witt algebra denoted here by $\mathcal{W}_N$. Note that $\text{Der}(A_N) = \text{Span}_{\mathbb{C}} \{t^r d_i | i \in \{1, \dots, N\}, r \in \mathbb{Z}^N\}$. The Lie bracket in $\mathcal{W}_N$ is given by

$$[t^r d_i, t^s d_j] = s_i t^{r+s} d_j - r_j t^{r+s} d_i.$$

Elements of $\mathcal{W}_N$ have a $\mathbb{Z}^N$ -grading in variables $t_1, \dots, t_N$, and homogeneous elements in this grading shall be denoted $d(u, r) = \sum_{i=1}^N u_i t^r d_i$ for any $u \in \mathbb{C}^N, r \in \mathbb{Z}^N$. This notation for vector fields will be useful in Chapter 2. Its Lie bracket is given by

$$[d(u, r), d(v, s)] = d((u|s)v - (v|r)u, r + s)$$

for $u, v \in \mathbb{C}^N, r, s \in \mathbb{Z}^N$.

Geometrically, $\mathcal{W}_N$ may be interpreted as the Lie algebra of (complex-valued) polynomial vector fields on an N dimensional torus via the mapping $t_j = e^{\sqrt{-1}\theta_j}$ for all $j \in \{1, \dots, N\}$, where θ_j is the j th angular coordinate. This has an interesting subalgebra, the Lie algebra of divergence zero vector fields, denoted $\mathcal{S}_N$.

PROPOSITION 1.2. *Let $u \in \mathbb{C}^N, r \in \mathbb{Z}^N$. Then $d(u, r) \in \mathcal{S}_N$ if and only if $(u|r) = 0$.*

PROOF. Note that under the mapping $t_j = e^{\sqrt{-1}\theta_j}$, $\frac{\partial}{\partial\theta_j} = \frac{\partial t_j}{\partial\theta_j} \cdot \frac{\partial}{\partial t_j} = \sqrt{-1}t_j \frac{\partial}{\partial t_j} = \sqrt{-1}d_j$, so that $X = \sum_{j=1}^N f_j(t)d_j \in \mathcal{W}_N$ can be written $X = -\sqrt{-1}\sum_{j=1}^N f_j(t)\frac{\partial}{\partial t_j}$. The divergence of X with respect to the natural volume form in angular coordinates is then $-\sqrt{-1}\sum_{j=1}^N \frac{\partial f_j}{\partial\theta_j} = \sum_{j=1}^N t_j \frac{\partial f_j}{\partial t_j}$. Thus $\text{div}(d(u, r)) = \sum_{j=1}^N u_j r_j t^r = 0$ if and only if $(u|r) = 0$. $\square$

This proposition uses elements of $\mathcal{W}_N$ which are homogeneous in t , and by linear independence of the powers of t it may be applied to the general case. An element of $\mathcal{W}_N$ is in $\mathcal{S}_N$ if and only if its homogeneous components are in $\mathcal{S}_N$. Note that for $d(u, r), d(v, s) \in \mathcal{S}_N$, $((u|s)v - (v|r)u|r + s) = 0$, demonstrating that $\mathcal{S}_N$ is closed under the bracket of $\mathcal{W}_N$. Letting $d_{ab}(r) = r_b t^r d_a - r_a t^r d_b$, it follows that

$$\mathcal{S}_N = \text{Span}_{\mathbb{C}} \{d_a, d_{ab}(r) | a, b \in \{1, \dots, N\}, r \in \mathbb{Z}^N\}$$

and has a commutative Cartan subalgebra $\text{Span}_{\mathbb{C}}\{d_j | j \in \{1, \dots, N\}\}$. It will be useful to have the Lie bracket of $\mathcal{S}_N$ in terms of the elements $d_{ab}(r)$. For $r, s \in \mathbb{Z}^N$ and $a, b, p, q \in \{1, \dots, N\}$, $[d_a, d_{pq}(r)] = r_a d_{pq}(r)$, and

$$(1) \quad [d_{ab}(r), d_{pq}(s)]$$

$$= r_b s_p d_{aq}(r+s) - r_b s_q d_{ap}(r+s) - r_a s_p d_{bq}(r+s) + r_a s_q d_{bp}(r+s).$$

By definition $d_{ab}(0) = 0$, $d_{aa}(r) = 0$ and $d_{ba}(r) = -d_{ab}(r)$. When $N \geq 3$, $d_{ab}(r) = 0$ in the case $r_a = r_b = 0$, and in general,

$$r_p d_{ab}(r) + r_a d_{bp}(r) + r_b d_{pa}(r) = 0.$$

The other main object of study here is the Lie subalgebra of Hamiltonian vector fields $\mathcal{H}_N$. This is defined only when N is even, and so let $N = 2m$ in what follows, and in the context of Chapter 4. Hamiltonian vector fields on a torus are defined as all $X = \sum_{j=1}^N f_j d_j \in \mathcal{W}_N$ such that, $\mathcal{L}_X(\sum_{i=1}^m d\theta_i \wedge d\theta_{m+i}) = 0$. Since $d_j = \sqrt{-1}\frac{\partial}{\partial\theta_j}$, express X as $\sum_{j=1}^N f_j \frac{\partial}{\partial\theta_j}$ (dropping the factor of $\sqrt{-1}$ to ease notation). Then $0 = \mathcal{L}_X(\sum_{i=1}^m d\theta_i \wedge d\theta_{m+i}) = \sum_{i=1}^m (X(d\theta_i)) \wedge d\theta_{m+i} + d\theta_i \wedge (X(d\theta_{m+i}))$.

Acting by vector field X on each one form gives

$$0 = \sum_{i=1}^m \left(\sum_{j=1}^N \frac{\partial f_i}{\partial \theta_j} d\theta_j \right) \wedge d\theta_{m+i} + d\theta_i \wedge \left(\sum_{j=1}^N \frac{\partial f_{m+i}}{\partial \theta_j} d\theta_j \right).$$

Note that if $X = d_i$ the right side is zero. Grouping like terms shows that

$$\sum_{i=1}^m \sum_{j=1}^m \left(\frac{\partial f_j}{\partial \theta_i} + \frac{\partial f_{m+i}}{\partial \theta_{m+j}} \right) d\theta_i \wedge d\theta_{m+j} = 0,$$

$$\sum_{i=1}^m \sum_{j=1}^m \frac{\partial f_i}{\partial \theta_{m+j}} d\theta_{m+j} \wedge d\theta_{m+i} = 0,$$

$$\sum_{i=1}^m \sum_{j=1}^m \frac{\partial f_{m+i}}{\partial \theta_j} d\theta_i \wedge d\theta_j = 0,$$

and thus

$$\frac{\partial f_j}{\partial \theta_i} = -\frac{\partial f_{m+i}}{\partial \theta_{m+j}},$$

$$\frac{\partial f_i}{\partial \theta_{m+j}} = \frac{\partial f_j}{\partial \theta_{m+i}},$$

$$\frac{\partial f_{m+i}}{\partial \theta_j} = \frac{\partial f_{m+j}}{\partial \theta_i}.$$

These equations are satisfied for $f_i = \frac{\partial P}{\partial \theta_{m+i}}$ and $f_{m+i} = -\frac{\partial P}{\partial \theta_i}$ with $P \in A_N$.

The change of coordinates $t_j = e^{\sqrt{-1}\theta_j}$, gives $f_i = \sqrt{-1}t_{m+i} \frac{\partial P}{\partial t_{m+i}}$ and $f_{m+i} = -\sqrt{-1}t_i \frac{\partial P}{\partial t_i}$, and so vector fields of the form $X_P = \sum_{i=1}^m t_{m+i} \frac{\partial P}{\partial t_{m+i}} d_i - t_i \frac{\partial P}{\partial t_i} d_{m+i}$

are Hamiltonian. Along with vector fields d_i , for $i = 1, \dots, N$, these exhaust all

elements in $\mathcal{W}_N$ belonging to $\mathcal{H}_N$. Homogeneous elements are obtained by letting

$P = t^r$, for $r \in \mathbb{Z}^N$, and set $h(r) = \sum_{i=1}^m r_{m+i} t^r d_i - r_i t^r d_{m+i}$. So

$$\mathcal{H}_N = \text{Span}_{\mathbb{C}}\{d_i, h(r) | i \in \{1, \dots, N\}, r \in \mathbb{Z}^N\}$$

and has commutative Cartan subalgebra $\text{Span}_{\mathbb{C}}\{d_j | j \in \{1, \dots, N\}\}$. The Lie

bracket in terms of homogeneous elements is given by $[d_i, h(r)] = r_i h(r)$ and

$$[h(r), h(s)] = \sum_{i=1}^m (r_{m+i} s_i - r_i s_{m+i}) h(r+s).$$

Note that $h(0) = 0$, and $[h(r), h(-r)] = 0$.

Let V be a $\mathfrak{gl}_N$ -module. Consider the tensor product $A_N \otimes V$ with $\mathcal{W}_N$ -action defined by

$$(2) \quad t^r d_i . t^s \otimes v = s_i t^{r+s} \otimes v + t^{r+s} \otimes \sum_{j=1}^N r_j E_{ji} v.$$

To see that this defines a representation of $\mathcal{W}_N$ on $A_N \otimes V$ note that

$$\begin{aligned} [t^q d_i, t^r d_j] . t^s \otimes v &= (r_i t^{q+r} d_j - q_j t^{q+r} d_i) . t^s \otimes v \\ &= r_i s_j t^{q+r+s} \otimes v + r_i t^{q+r+s} \otimes \sum_{k=1}^N (q_k + r_k) E_{kj} v \\ &\quad - q_j s_i t^{q+r+s} \otimes v - q_j t^{q+r+s} \otimes \sum_{k=1}^N (q_k + r_k) E_{ki} v, \end{aligned}$$

while on the other hand

$$\begin{aligned} t^q d_i . (t^r d_j . t^s \otimes v) - t^r d_j . (t^q d_i . t^s \otimes v) &= t^q d_i . \left(s_j t^{r+s} \otimes v + t^{r+s} \otimes \sum_{k=1}^N r_k E_{kj} v \right) - t^r d_j . \left(s_i t^{q+s} \otimes v + t^{q+s} \otimes \sum_{k=1}^N q_k E_{ki} v \right) \\ &= s_j (r_i + s_i) t^{q+r+s} \otimes v + s_j t^{q+r+s} \otimes \sum_{k=1}^N q_k E_{ki} v + (r_i + s_i) t^{q+r+s} \otimes \sum_{k=1}^N r_k E_{kj} v \\ &\quad + t^{q+r+s} \otimes \sum_{\ell=1}^N \sum_{k=1}^N q_\ell r_k E_{\ell i} E_{kj} v - s_i (q_j + s_j) t^{q+r+s} \otimes v - s_i t^{q+r+s} \otimes \sum_{k=1}^N r_k E_{kj} v \\ &\quad - (q_j + s_j) t^{q+r+s} \otimes \sum_{k=1}^N q_k E_{ki} v - t^{q+r+s} \otimes \sum_{\ell=1}^N \sum_{k=1}^N r_\ell q_k E_{\ell j} E_{ki} v \\ &= s_j r_i t^{q+r+s} \otimes v - s_i q_j t^{q+r+s} \otimes v + r_i t^{q+r+s} \otimes \sum_{k=1}^N r_k E_{kj} v \\ &\quad - q_j t^{q+r+s} \otimes \sum_{k=1}^N q_k E_{ki} v + t^{q+r+s} \otimes \sum_{\ell=1}^N \sum_{k=1}^N q_\ell r_k [E_{\ell i}, E_{kj}] v \\ &= s_j r_i t^{q+r+s} \otimes v - s_i q_j t^{q+r+s} \otimes v + r_i t^{q+r+s} \otimes \sum_{k=1}^N r_k E_{kj} v \\ &\quad - q_j t^{q+r+s} \otimes \sum_{k=1}^N q_k E_{ki} v + t^{q+r+s} \otimes \sum_{\ell=1}^N r_i q_\ell E_{\ell j} v - t^{q+r+s} \otimes \sum_{k=1}^N r_k q_j E_{ki} v, \end{aligned}$$

as required.

The space $\bigwedge^k \mathbb{C}^N$ has a natural action by $\mathfrak{gl}_N$ given by

$$X.(v_1 \wedge \cdots \wedge v_k) = \sum_{\ell=1}^k v_1 \wedge \cdots \wedge X v_\ell \wedge \cdots \wedge v_k.$$

When restricted to $\mathfrak{sl}_N$, this is isomorphic to the irreducible highest weight module $V(\omega_k)$ (where ω_k is the fundamental dominant weight). Since $\mathfrak{gl}_N = \mathfrak{sl}_N \oplus \mathbb{C}I_N$, this space is viewed as the associated $\mathfrak{gl}_N$ -module where the identity matrix I_N acts by integer k . Let $\{e_1, \dots, e_N\}$ be the standard basis for $\mathbb{C}^N$. As an example, for indices $s_1, \dots, s_k$,

$$E_{ij}.(e_{s_1} \wedge \cdots \wedge e_{s_k}) = \sum_{\ell=1}^k e_{s_1} \wedge \cdots \wedge \delta_{js_\ell} e_i \wedge \cdots \wedge e_{s_k}.$$

The space Ω^k of differential k -forms on an N dimensional torus has a natural $\mathcal{W}_N$ -module structure where vector fields act on Ω^k via the Lie derivative. For example, the action of $t^r d_j \in \mathcal{W}_N$ on $t^q d\theta_{s_1} \wedge \cdots \wedge d\theta_{s_k} \in \Omega^k$, is given by

$$\begin{aligned} & t^r d_j.(t^q d\theta_{s_1} \wedge \cdots \wedge d\theta_{s_k}) \\ &= t^r d_j(t^q) d\theta_{s_1} \wedge \cdots \wedge d\theta_{s_k} + \sum_{\ell=1}^k t^q d\theta_{s_1} \wedge \cdots \wedge d(t^r d_j(\theta_{s_\ell})) \wedge \cdots \wedge d\theta_{s_k} \\ &= q_j t^{q+r} d\theta_{s_1} \wedge \cdots \wedge d\theta_{s_k} + t^{q+r} \sum_{i=1}^N r_i \left(\sum_{\ell=1}^k d\theta_{s_1} \wedge \cdots \wedge \delta_{js_\ell} d\theta_i \wedge \cdots \wedge d\theta_{s_k} \right). \end{aligned}$$

Mapping $d\theta_{s_1} \wedge \cdots \wedge d\theta_{s_k} \mapsto e_{s_1} \wedge \cdots \wedge e_{s_k}$ shows that the sum in parenthesis behaves like action of E_{ij} on the $\mathfrak{gl}_N$ -module shown above. It follows that Ω^k is isomorphic, as a $\mathcal{W}_N$ -module, to the tensor product $A_N \otimes V(\omega_k)$ with action given in (2).

The module structure given by vector fields acting via Lie derivative on k -forms leads naturally to consider a $\mathcal{W}_N$ -action on the more general spaces $A_N \otimes V(\lambda)$, where $V(\lambda)$ is the irreducible $\mathfrak{sl}_N$ -module of highest weight λ . In this case, to form the associated $\mathfrak{gl}_N$ -module, the action by I_N must be specified by a scalar $b \in \mathbb{C}$ when λ is not a fundamental dominant weight. This can be further extended to a larger family of representations by choosing $\sigma \in \mathbb{C}^N$ and considering $t^\sigma A_N \otimes V$.

Then the action in (2) becomes

$$t^r d_i t^{s+\sigma} \otimes v = (s_i + \sigma_i) t^{r+s+\sigma} \otimes v + t^{r+s+\sigma} \otimes \sum_{j=1}^N r_j E_{ji} v.$$

These representation are studied by Rao in [12] and their restriction to subalgebra $\mathcal{S}_N$ is considered in Chapter 2. To simplify notation the ever-present factor of t^σ is left out and such representations are denoted $F^\sigma(\lambda, b)$ as $\mathcal{W}_N$ -modules. Note that for $n \in \mathbb{Z}^N$ modules $F^\sigma(\lambda, b)$ and $F^{\sigma+n}(\lambda, b)$ are isomorphic via the map $t^s \otimes v \mapsto t^{s-n} \otimes v$.

The identification of $F^\sigma(\omega_k, k)$ with Ω_N^k yields the de Rham complex

$$A \xrightarrow{\psi_0} F^\sigma(\omega_1, 1) \xrightarrow{\psi_1} F^\sigma(\omega_2, 2) \xrightarrow{\psi_2} \dots \xrightarrow{\psi_{N-2}} F^\sigma(\omega_{N-1}, N-1)$$

where $\psi_i : F^\sigma(\omega_i, i) \rightarrow F^\sigma(\omega_{i+1}, i+1)$ are $\mathcal{W}_N$ -module homomorphisms defined by

$$t^s \otimes e_{s_1} \wedge \dots \wedge e_{s_k} \mapsto t^s \otimes (s + \sigma) \wedge e_{s_1} \wedge \dots \wedge e_{s_k}.$$

These homomorphisms are used to locate submodules of the $F^\sigma(\omega_k, k)$ which will appear in main theorem of Chapter 2. The image of ψ_i is the submodule

$$W_{i+1}^\sigma = \bigoplus_{n \in \mathbb{Z}^N} t^n \otimes (\mathbb{C}(n + \sigma) \wedge \mathbb{C}^N \wedge \dots \wedge \mathbb{C}^N)$$

of $F^\sigma(\omega_{i+1}, i+1)$. The kernel of ψ_i is W_i^σ when $\sigma \notin \mathbb{Z}^N$, however if $\sigma \in \mathbb{Z}^N$ its kernel is the larger submodule

$$\tilde{W}_i^\sigma = W_i^\sigma \oplus (t^{-\sigma} \otimes (\mathbb{C}^{N+1} \wedge \dots \wedge \mathbb{C}^{N+1})).$$

CHAPTER 2

Tensor Modules for Divergence Zero Vector Fields on a Torus

1. Weight Strings

The Lie algebra $\mathfrak{sl}_N$ is nontrivial for $N \geq 2$ and has root system of type A_{N-1} (not to be confused with Laurent polynomials mentioned above). Since this chapter makes frequent use of the root system, the index will be shifted from N with $N \geq 2$ to $N + 1$ with $N \geq 1$ for convenience of notation. This change of index will be in effect for this chapter only.

For $N \geq 1$ consider $\mathfrak{sl}_{N+1}$, the Lie algebra of all $(N + 1) \times (N + 1)$ traceless matrices with entries in $\mathbb{C}$, and fix a root system Φ_{A_N} , of type A_N , with positive roots $\Phi_{A_N}^+$ and simple roots $\Pi = \{\alpha_1, \dots, \alpha_N\}$. Let $V(\lambda)$ be a finite dimensional irreducible $\mathfrak{sl}_{N+1}$ -module of highest weight $\lambda = C_1\omega_1 + \dots + C_N\omega_N$, where $\langle \omega_i, \alpha_j \rangle \equiv \frac{2(\omega_i | \alpha_j)}{(\alpha_j | \alpha_j)} = \delta_{ij}$ and $C_i \in \mathbb{Z}_{\geq 0}$ for all $i, j \in \{1, \dots, N\}$. Any weight γ of $V(\lambda)$ can be written $\gamma = \lambda - \sum_{i=1}^N \gamma_i \alpha_i$ such that the γ_i 's are nonnegative integers. For convenience assign to each weight a *label*, defined as the N -tuple $(\langle \gamma, \alpha_1 \rangle, \dots, \langle \gamma, \alpha_N \rangle)$. Thus λ has label $(C_1, \dots, C_N)$. Note that here a weight of $V(\lambda)$ refers only those assigned to a nonzero weight space.

Let γ be a weight in $V(\lambda)$ and $\alpha \in \Phi_{A_N}$ (not necessarily simple). Then the set of weights of the form $\gamma + i\alpha$, for $i \in \mathbb{Z}$, is called the α -*string through* γ . let $r, q \in \mathbb{Z}_{\geq 0}$ be the largest integers such that $\gamma - r\alpha$ and $\gamma + q\alpha$ are weights of $V(\lambda)$. Then $\gamma + i\alpha$ is a weight for all $-r \leq i \leq q$; i.e. the α -string through γ is unbroken from $\gamma - r\alpha$ to $\gamma + q\alpha$. Furthermore it can be seen that $\langle \gamma, \alpha \rangle = r - q$. These unbroken sets of weights will be referred to as *weight strings*, and the number of

weights in this set its *length*. In this particular case the weight string is in the α direction.

The Weyl group for Φ_{A_N} is isomorphic to the symmetric group on $N + 1$ symbols, and so elements of the Weyl group will be denoted with cycle notation. The reflection through simple root α_i is the transposition $(i, i + 1)$, and the reflection through a root of the form $\alpha_i + \cdots + \alpha_{i+j}$ is the transposition $(i, i + j + 1)$, for $1 \leq i < i + j \leq N$.

LEMMA 2.1. *Let λ_ℓ denote the lowest weight of $V(\lambda)$, and write $\lambda_\ell = \lambda - \sum_{i=1}^N \kappa_i \alpha_i$, where $\kappa_i \in \mathbb{Z}_{\geq 0}$. Then for $1 \leq j \leq \lfloor \frac{N+1}{2} \rfloor$,*

$$\kappa_j = \kappa_{N+1-j} = \sum_{i=1}^N C_i + \sum_{i=2}^{N-1} C_i + \cdots + \sum_{i=j}^{N+1-j} C_i$$

PROOF. Consider the Weyl group element

$$w = \prod_{p=0}^{\lfloor \frac{N}{2} \rfloor} (1 + p, N + 1 - p).$$

Then $w(\alpha_i) = -\alpha_{N+1-i}$ for $1 \leq i \leq N$ and it follows that w maps the highest weight of an $\mathfrak{sl}_{N+1}$ -module to its lowest weight. The above transpositions are disjoint and hence commute. Let

$$\lambda_j = \left(\prod_{p=0}^{j-1} (1 + p, N + 1 - p) \right) (\lambda)$$

for $j = 1, \dots, \lfloor \frac{N}{2} \rfloor + 1$ and $\lambda_0 = \lambda$. Then

$$\begin{aligned} \lambda_1 &= (1, N + 1)(\lambda) \\ &= \lambda_0 - \langle \lambda_0, \alpha_1 + \cdots + \alpha_N \rangle (\alpha_1 + \cdots + \alpha_N) \\ &= \lambda_0 - \left(\sum_{i=1}^N C_i \right) (\alpha_1 + \cdots + \alpha_N). \end{aligned}$$

Proceed by induction on j . Assume $\lambda_j = \lambda_{j-1} - \sum_{i=j}^{N+1-j} C_i (\alpha_j + \cdots + \alpha_{N+1-j})$ for some $j \geq 1$. Then

$$\begin{aligned}
& \lambda_{j+1} \\
&= (1+j, N+1-j)(\lambda_j) \\
&= \lambda_j - \langle \lambda_j, \alpha_{j+1} + \cdots + \alpha_{N-j} \rangle (\alpha_{j+1} + \cdots + \alpha_{N-j}) \\
&= \lambda_j - \left\langle \lambda - \left(\sum_{i=1}^N C_i \right) (\alpha_1 + \cdots + \alpha_N) - \left(\sum_{i=2}^{N-1} C_i \right) (\alpha_2 + \cdots + \alpha_{N-1}) \right. \\
&\quad \left. - \cdots \right. \\
&\quad \left. - \left(\sum_{i=j}^{N+1-j} C_i \right) (\alpha_j + \cdots + \alpha_{N+1-j}), \alpha_{j+1} + \cdots + \alpha_{N-j} \right\rangle (\alpha_{j+1} + \cdots + \alpha_{N-j}) \\
&= \lambda_j - \langle \lambda, \alpha_{j+1} + \cdots + \alpha_{N-j} \rangle (\alpha_{j+1} + \cdots + \alpha_{N-j}) \\
&\quad - \left(\sum_{i=1}^N C_i \right) \langle \alpha_1 + \cdots + \alpha_N, \alpha_{j+1} + \cdots + \alpha_{N-j} \rangle (\alpha_{j+1} + \cdots + \alpha_{N-j}) \\
&\quad - \left(\sum_{i=2}^{N-1} C_i \right) \langle \alpha_2 + \cdots + \alpha_{N-1}, \alpha_{j+1} + \cdots + \alpha_{N-j} \rangle (\alpha_{j+1} + \cdots + \alpha_{N-j}) \\
&\quad - \cdots \\
&\quad - \left(\sum_{i=j}^{N+1-j} C_i \right) \langle \alpha_j + \cdots + \alpha_{N+1-j}, \alpha_{j+1} + \cdots + \alpha_{N-j} \rangle (\alpha_{j+1} + \cdots + \alpha_{N-j}) \\
&= \lambda_j - \left(\sum_{i=j+1}^{N-j} C_i \right) (\alpha_{j+1} + \cdots + \alpha_{N-j}),
\end{aligned}$$

since the angled bracket evaluates to zero in all but the second term. So by induction

$$w(\lambda) = \lambda_{\lfloor \frac{N}{2} \rfloor + 1} = \lambda - \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor + 1} \sum_{j=i}^{N+1-i} C_j \sum_{k=i}^{N+1-i} \alpha_k$$

with the final term being zero if N is even. Since $w(\lambda) = \lambda_\ell$, extracting coefficients of the α_i yields the result. $\square$

2. Tensor Modules

Let $F^\sigma(\lambda) = V(\lambda) \otimes \mathbb{C}[q_1^{\pm 1}, \dots, q_{N+1}^{\pm 1}]$ for $\sigma \in \mathbb{C}^{N+1}$. Homogeneous elements of $F^\sigma(\lambda)$ will be denoted $v(n) = v \otimes q^n$ for $v \in V(\lambda), n \in \mathbb{Z}^{N+1}$. Then $F^\sigma(\lambda)$ becomes a module for $\mathcal{S}_N$ via the action

$$d(u, r).v(n) = (u|n + \sigma)v(n + r) + (ru^T)v(n + r),$$

for $d(u, r) \in \mathcal{S}_N$. Note that ru^T is an $(N + 1) \times (N + 1)$ matrix with trace zero, as $(u|r) = 0$ by Proposition 1.2, and so the second term involves the $\mathfrak{sl}_{N+1}$ -module action. Elements of $\mathfrak{sl}_{N+1}$ are denoted in the usual way, with E_{ij} being the matrix with a 1 in the (i, j) position and zeros elsewhere.

The modules $V(\lambda)$ and $F^\sigma(\lambda)$ are called *minuscule* if $\lambda = \omega_i$ or $\lambda = 0$ for $i \in \{1, \dots, N\}$ (see [2] 2.11.15). Write $F^\sigma(\lambda) = \bigoplus_{n \in \mathbb{Z}^{N+1}} V(\lambda) \otimes q^n$ and let $v(n) \in V(\lambda) \otimes q^n$. Then $d(e_i, 0).v(n) = (n_i + \sigma_i)v(n)$ for all $i \in \{1, \dots, N + 1\}$, demonstrating that $V(\lambda) \otimes q^n$ is an eigenspace for each n , and thus $F^\sigma(\lambda)$ is a $\mathbb{Z}^N$ -graded module with respect to the action of the $d(e_i, 0)$. Any submodule M of $F^\sigma(\lambda)$ will inherit this gradation and so $M = \bigoplus_{n \in \mathbb{Z}^{N+1}} (V(\lambda) \otimes q^n) \cap M$. It follows that M contains nonzero elements of the form $v(n)$ (homogeneous elements).

PROPOSITION 2.2. *Let M be an $\mathcal{S}_N$ -submodule of $F^\sigma(\lambda)$. If $v(n) \in M$ then $(E_{ij}^k v)(n) \in M$ for any integer $k \geq 2$ and any $i \neq j \in \{1, \dots, N + 1\}$.*

PROOF. Note that $d(e_j, re_i) \in \mathcal{S}_N$ for any $r \in \mathbb{Z}$, as $(e_j|e_i) = 0$ for $i \neq j$. Fix an integer $k \geq 2$, and choose $r_1, \dots, r_k \in \mathbb{Z} \setminus \{0\}$ such that $\sum_{p=1}^k r_p = 0$. Then $d(e_j, re_i).v(n) = ((e_j|n + \sigma) + rE_{ij})v(n + re_i)$. Thus

$$\begin{aligned} d(e_i, r_1 e_j) \cdots d(e_i, r_k e_j).v(n) &= (K_0 + r_1 E_{ij}) \cdots (K_0 + r_k E_{ij})v\left(n + \sum_{p=1}^k r_p\right) \\ &= (K_0^k + K_1 E_{ij} + K_2 E_{ij}^2 + \cdots + K_k E_{ij}^k)v(n), \end{aligned}$$

where $K_0 = (e_j | n + \sigma)$, $K_1 = K_0^{k-1} \sum_{p=1}^k r_p = 0$, and K_k is nonzero. For the case $k = 2$, take $r_1 = 1, r_2 = -1$. Then $d(e_j, e_i).d(e_j, -e_i).v(n) = K_0^2 v(n) - E_{ij}^2 v(n) \in M$, and hence $E_{ij}^2 v(n) \in M$. The result follows by induction on k . $\square$

3. Existence of Weight Vectors

Recall that the root system of type $\mathfrak{sl}_{N+1}$ has highest root $\theta = \alpha_1 + \cdots + \alpha_N$. A weight string of maximal length in the θ direction will be called a *maximal θ -string*.

PROPOSITION 2.3. *Consider a highest weight $\mathfrak{sl}_{N+1}$ -module $V(\lambda)$. A θ -string through λ is a maximal θ -string, and the highest weight of any maximal θ -string has the form $\lambda - \sum_{i=2}^{N-1} \gamma_i \alpha_i$ for some non-negative integers γ_i .*

PROOF. Let ζ_λ be the θ -string through λ . Since λ is the highest weight of ζ_λ the lowest weight of ζ_λ is $\lambda - \langle \lambda, \theta \rangle \theta = \lambda - \left(\sum_{i=1}^N C_i \right) \theta$, and thus ζ_λ has length $1 + \sum_{i=1}^N C_i$. Let ζ_γ be the θ -string through some other weight γ , where γ is the highest weight of ζ_γ and $\gamma = \lambda - \sum_{j=1}^N \gamma_j \alpha_j$ for some nonnegative integers γ_j .

Suppose ζ_γ has length greater than that of ζ_λ , say $r + 1 + \sum_{i=1}^N C_i$ for some positive integer r . Then the lowest weight of ζ_γ is $\gamma - \left(r + \sum_{i=1}^N C_i \right) \theta = \lambda - \left(\sum_{j=1}^N \gamma_j \alpha_j \right) - \left(r + \sum_{i=1}^N C_i \right) \theta = \lambda - \sum_{j=1}^N \left(\gamma_j + r + \left(\sum_{i=1}^N C_i \right) \right) \alpha_j$. This contradicts Lemma 2.1 since a multiple of α_1 (or α_N) greater than $\sum_{i=1}^N C_i$ cannot be subtracted from λ (i.e. no such weight exists). Thus ζ_γ cannot have a length greater than that of ζ_λ , so ζ_λ has maximal length.

Suppose now that ζ_γ has the same (maximal) length as ζ_λ . Then the lowest weight of ζ_γ is $\lambda - \sum_{j=1}^N \left(\gamma_j + \left(\sum_{i=1}^N C_i \right) \right) \alpha_j$. This contradicts Lemma 2.1 in the same way as above unless both γ_1 and γ_N are zero. $\square$

COROLLARY 2.4. *Irreducible $\mathfrak{sl}_3$ -modules have exactly one maximal θ -string.*

PROOF. For the $N = 2$ case, λ is the only weight which meets the criteria of Proposition 2.3 forcing the only maximal θ -string to be the one through λ . $\square$

COROLLARY 2.5. *In a non-minuscule module the length of any maximal θ -string is at least 3.*

PROOF. As seen in the proof of Proposition 2.3, the length of the θ -string through λ is $1 + \langle \lambda, \theta \rangle = 1 + \sum_{i=1}^N C_i$. For a non-minuscle module $\sum_{i=1}^N C_i \geq 2$. $\square$

REMARK 2.6. *Let $V(\lambda)$ be a non-minuscle module and $N \geq 2$. The label assigned to λ , $(C_1, \dots, C_N)$, falls into at least one of the following three cases:*

- (i) $(C_1, \dots, C_{N-1})$ is a highest weight label for a non-minuscle $\mathfrak{sl}_N$ -module,
- (ii) $(C_2, \dots, C_N)$ is a highest weight label for a non-minuscle $\mathfrak{sl}_N$ -module, or
- (iii) $C_1 = C_N = 1$ and $C_2 = \dots = C_{N-1} = 0$.

Applying Lemma 2.1 to case (iii) shows that $\lambda_\ell = \lambda - 2\theta$ and hence it is the lowest weight for the maximal θ -string through λ . Suppose ζ_γ is a maximal θ -string through the weight $\gamma = \lambda - \sum_{i=1}^n \gamma_i \alpha_i$ for $\gamma_i \in \mathbb{Z}_{\geq 0}$. Then ζ_γ has lowest weight $\gamma - 2\theta = \lambda - \sum_{i=1}^n \gamma_i \alpha_i - 2\theta$. This forces each γ_i to be zero, and thus ζ_λ is the only maximal θ -string.

Let M be a nonzero submodule of $F^\sigma(\lambda)$. The goal for the remainder of this section is to show that if $F^\sigma(\lambda)$ is non-minuscle then $v_\lambda(n) \in M$ for all $n \in \mathbb{Z}^{N+1}$ where v_λ is a highest weight vector in $V(\lambda)$. Once it is established that $v_\lambda(n) \in M$, it will be used in the next section to generate $V(\lambda) \otimes q^n$ for each n , thus generating all of $F^\sigma(\lambda)$.

PROPOSITION 2.7. *Submodule M contains a vector $v(n)$, for some n , such that v has a nonzero component in V_λ , the highest weight space of $V(\lambda)$.*

PROOF. Let $v(m) \in M$ for some $m \in \mathbb{Z}^{N+1}$, and $v \in V(\lambda)$. Let $v = \sum_{\gamma \in \Lambda} v_\gamma$ be the decomposition of v into weight vectors where Λ is the set of weights of $V(\lambda)$. If $v_\lambda \neq 0$ there is nothing to prove, so suppose $v_\lambda = 0$. It follows from the definition of highest weight that there is some μ in Λ such that $E_{ij}v_\mu \in V_{\mu'} \setminus \{0\}$ and $v_{\mu'} = 0$ in the above decomposition, for some $i < j \in \{1, \dots, N+1\}$. Applying $d(e_j, e_i)$ to $v(m)$ yields $(e_j | m + \sigma)v(m + e_i) + E_{ij}v(m + e_i) = v'(m + e_i)$, where v' has a nonzero component in $V_{\mu'}$. If $\mu' \neq \lambda$ this process may be repeated by choosing another (i, j) -pair in the same way, until the highest weight space has been reached. $\square$

PROPOSITION 2.8. *Let $V(\lambda)$ be non-minuscule and suppose for some $n \in \mathbb{Z}^{N+1}$ that $v(n) \in M$ has nonzero component in V_λ . Then $v_\lambda(n) \in M$, where v_λ is a highest weight vector of $V(\lambda)$.*

PROOF. Suppose for some $v(n) \in M$ that v has a nonzero component v_λ which lies in the highest weight space of $V(\lambda)$. Such a vector exists by Proposition 2.7. Then $v = \sum_{\gamma \in \Lambda} v_\gamma$, where Λ is a set of weights of $V(\lambda)$ and $v_\lambda \neq 0$. By Proposition 2.3 there is a maximal θ -string of length $\langle \lambda, \theta \rangle + 1 \geq 3$ which has highest weight λ . Note that $E_{N+1,1}$ maps V_μ into $V_{\mu-\theta}$ for any weight μ of $V(\lambda)$. The goal in what follows is to annihilate those components different from v_λ . This will be accomplished by moving v back and forth along the θ -strings.

In the case $N = 1$, an irreducible $\mathfrak{sl}_2$ -module is a single θ -string. For $N = 2$, Corollary 2.4 says that irreducible $\mathfrak{sl}_3$ -modules have only one maximal θ -string. In both cases these strings necessarily have highest weight v_λ . By Proposition 2.2, $E_{N+1,1}^{\langle \lambda, \theta \rangle} v(n) \in M$ where $E_{N+1,1}^{\langle \lambda, \theta \rangle} v \in V_{\lambda - \langle \lambda, \theta \rangle \theta}$. All $v_\gamma \neq v_\lambda$ are annihilated because their weights either do not lie in the unique maximal θ -string, or if they do, they are of the form $\lambda - i\theta$ for $i \geq 1$. In either case $E_{N+1,1}^{\langle \lambda, \theta \rangle}(v_\gamma) = 0$, for $\gamma \neq \lambda$, due to maximal length. Applying $E_{1,N+1}^{\langle \lambda, \theta \rangle}$ to $E_{N+1,1}^{\langle \lambda, \theta \rangle} v(n)$ then yields a scalar multiple of $v_\lambda(n)$. Now proceed by induction on N with base case $N = 2$.

Assume now $N \geq 3$ and $v(n) \in M$ where $v = \sum_{\gamma \in \Lambda} v_\gamma$. Applying Proposition 2.2 yields $E_{N+1,1}^{\langle \lambda, \theta \rangle} v(n) = \sum_{\mu \in \Lambda_*} w_\mu(n) \in M$ where each $\mu \in \Lambda_*$ is given by $\mu = \gamma - \langle \lambda, \theta \rangle \theta$ for some $\gamma \in \Lambda$. By maximality of the length of a maximal θ -string, any $\mu \in \Lambda_*$ is a lowest weight in the θ -string which it belongs to; i.e. the only $\gamma \in \Lambda$ for which $E_{N+1,1}^{\langle \lambda, \theta \rangle} v_\gamma \neq 0$ are those which are the highest weight of their maximal θ -string. In particular $\lambda - \langle \lambda, \theta \rangle \theta \in \Lambda_*$. Then $v'(n) = E_{1,N+1}^{\langle \lambda, \theta \rangle} \left(\sum_{\mu \in \Lambda_*} w_\mu(n) \right) = \sum_{\gamma \in \Lambda_*} v'_\gamma(n) \in M$ where $\lambda \in \Lambda^*$ with $v'_\lambda \neq 0$ and $\Lambda^* \subset \Lambda$ are weights which occur as the highest weight in their respective maximal θ -string.

Let Λ_1 be the set of all weights in $V(\lambda)$ of the form $\lambda - \sum_{i=1}^{N-1} a_i \alpha_i$, and Λ_N be the set of all weights in $V(\lambda)$ of the form $\lambda - \sum_{i=2}^N b_i \alpha_i$, where $a_i, b_i \in \mathbb{Z}_{\geq 0}$. Let $V(\lambda)_1$ and $V(\lambda)_N$ be the span of all weight vectors with weights in Λ_1 and

Λ_N respectively. Then $V(\lambda)_1$ is an $\mathfrak{sl}_N$ -module, for the copy of $\mathfrak{sl}_N$ generated by $\{E_{ij} | 1 \leq i, j \leq N, i \neq j\} \subset \mathfrak{sl}_{N+1}$, with highest weight label $(C_1, \dots, C_{N-1})$. Similarly $V(\lambda)_N$ is an $\mathfrak{sl}_N$ -module, for the copy of $\mathfrak{sl}_N$ generated by $\{E_{ij} | 2 \leq i, j \leq N+1, i \neq j\} \subset \mathfrak{sl}_{N+1}$, and has highest weight label $(C_2, \dots, C_N)$. Modules $V(\lambda)_1$ and $V(\lambda)_N$ are irreducible since any critical vector they contain is also a critical vector for $\mathfrak{sl}_{N+1}$ in $V(\lambda)$ which is irreducible by assumption.

By Proposition 2.3 Λ^* is a subset of both Λ_1 and Λ_N . By Remark 2.6, either $V(\lambda)_1$ or $V(\lambda)_N$ is non-minuscule, or there is only one maximal θ -string. In the latter case $\Lambda^* = \{\lambda\}$ and so $v'(n)$ is a scalar multiple of $v_\lambda(n)$. Otherwise v' lies in a non-minuscule $\mathfrak{sl}_N$ -module W (either $V(\lambda)_1$ or $V(\lambda)_N$) and has nonzero component in V_λ . By induction $v_\lambda(n) \in W$ and thus $v_\lambda(n) \in M$. $\square$

The next lemma is an intermediate step towards showing that $v_\lambda(n) \in M$ for all $n \in \mathbb{Z}^{N+1}$.

LEMMA 2.9. *If $v_\lambda(n) \in M$ then $v_\lambda(n + re_1) \in M$ for any $r \in \mathbb{Z}$.*

PROOF. Let $v_\lambda(n) \in M$ for some $n \in \mathbb{Z}^{N+1}$. This result is clearly true for $r = 0$, so suppose $r \neq 0$. Then

$$d(e_{N+1}, re_1).v_\lambda(n) = (e_{N+1}|n + \sigma)v_\lambda(n + re_1) + rE_{1,N+1}v_\lambda(n + re_1),$$

where the second term is zero as v_λ is a highest weight vector, and if $(e_{N+1}|n + \sigma) \neq 0$ then $v_\lambda(n + re_1) \in M$. Suppose $(e_{N+1}|n + \sigma) = 0$. By Proposition 2.2 $E_{N+1,1}^2 v_\lambda(n) \in M$. This is a weight vector whose weight lies on the maximal θ -string through λ and is nonzero since the string has length at least 3. So

$$\begin{aligned} d(e_{N+1}, -re_1).E_{N+1,1}^2 v_\lambda(n) \\ = (e_{N+1}|n + \sigma)E_{N+1,1}^2 v_\lambda(n - re_1) - rE_{1,N+1}E_{N+1,1}^2 v_\lambda(n - re_1). \end{aligned}$$

The first term has coefficient zero and the second is a scalar multiple of $E_{N+1,1}v_\lambda(n - re_1)$. Then

$$\begin{aligned} d(e_{N+1}, 2re_1).E_{N+1,1}v_\lambda(n - re_1) \\ = (e_{N+1}|n - re_1 + \sigma)E_{N+1,1}v_\lambda(n + re_1) + 2rE_{1,N+1}E_{N+1,1}v_\lambda(n + re_1). \end{aligned}$$

Again the first term is zero and the second term is a nonzero scalar multiple of $v_\lambda(n + re_1)$. Thus $v_\lambda(n + re_1) \in M$ for all $r \in \mathbb{Z}$. $\square$

PROPOSITION 2.10. $v_\lambda(m) \in M$ for any $m \in \mathbb{Z}^{N+1}$.

PROOF. By Propositions 2.7 and 2.8 there exists a $v_\lambda(n) \in M$ for some $n \in \mathbb{Z}^{N+1}$. Using Lemma 2.9 assume that $(e_1|n + \sigma) \neq 0$. Then for any $r_{N+1} \in \mathbb{Z}$

$$\begin{aligned} d(e_1, r_{N+1}e_{N+1}).v_\lambda(n) \\ = (e_1|n + \sigma)v_\lambda(n + r_{N+1}e_{N+1}) + r_{N+1}E_{N+1,1}v_\lambda(n + r_{N+1}e_{N+1}). \end{aligned}$$

Since the first term above is nonzero it follows from Proposition 2.8 that $v_\lambda(n + r_{N+1}e_{N+1}) \in M$. Thus $v_\lambda(n + r_{N+1}e_{N+1}) \in M$ for all $r_{N+1} \in \mathbb{Z}$. Let $r = \sum_{i=1}^N r_i e_i \in \mathbb{Z}^{N+1}$. Then

$$\begin{aligned} d(e_{N+1}, r).v_\lambda(n + r_{N+1}e_{N+1}) \\ = (e_{N+1}|n + r_{N+1}e_{N+1} + \sigma)v_\lambda(n + r + r_{N+1}e_{N+1}) + \sum_{i=1}^N r_i E_{i,N+1}v_\lambda(n + r + r_{N+1}e_{N+1}), \end{aligned}$$

with the second term being zero by the highest weight property of v_λ . When $(e_{N+1}|n + r_{N+1}e_{N+1} + \sigma) \neq 0$ it follows by the arbitrary choice of r that $v_\lambda(n + r + r_{N+1}e_{N+1}) \in M$ for all $r \in \mathbb{Z}^N \times \{0\}$.

Suppose there exists $r_{N+1} \in \mathbb{Z}$ such that $(e_{N+1}|n + r_{N+1}e_{N+1} + \sigma) = 0$, say when $r_{N+1} = r_{N+1}^* \in \mathbb{Z}$. For any $r = \sum_{i=1}^N r_i e_i \in \mathbb{Z}^{N+1}$ and $\bar{r}_{N+1} \neq r_{N+1}^*$, the above shows that $v_\lambda(n + r + \bar{r}_{N+1}) \in M$. By Lemma 2.9 $v_\lambda(n + ae_1 + \sum_{i=2}^N r_i e_i +$

$\bar{r}_{N+1}e_{N+1}) \in M$ where $(e_1|n + ae_1 + \sum_{i=2}^N r_i e_i + \bar{r}_{N+1}e_{N+1} + \sigma) \neq 0$. Then

$$\begin{aligned} & d(e_1, (r_{N+1}^* - \bar{r}_{N+1})e_{N+1}).v_\lambda(n + ae_1 + \sum_{i=2}^N r_i e_i + \bar{r}_{N+1}e_{N+1}) \\ &= (e_1|n + ae_1 + \sum_{i=2}^N r_i e_i + \bar{r}_{N+1}e_{N+1} + \sigma)v_\lambda(n + ae_1 + \sum_{i=2}^N r_i e_i + r_{N+1}^*e_{N+1}) \end{aligned}$$

and so $v_\lambda(n + ae_1 + \sum_{i=2}^N r_i e_i + r_{N+1}^*e_{N+1}) \in M$. Apply Lemma 2.9 again to get that $v_\lambda(n + r + r_{N+1}^*e_{N+1}) \in M$. This along with the previous paragraph shows that $v_\lambda(m) \in M$ for any $m \in \mathbb{Z}^{N+1}$. $\square$

4. Generating the Module

THEOREM 2.11. *If $F^\sigma(\lambda)$ is non-minuscule then it is irreducible as an S_{N+1} -module.*

PROOF. From the theory of highest weight modules for semisimple Lie algebras, $V(\lambda)$ is spanned by v_λ and the vectors $y_1 \dots y_k v_\lambda$, where, for each $p \in \{1, \dots, k\}$, $y_p = E_{ij}$ for some (i, j) -pair with $i > j$. Let M be a nonzero submodule of $F^\sigma(\lambda)$. Proceed by induction on k to show that $y_1 \dots y_k v_\lambda(m) \in M$ for any $m \in \mathbb{Z}^{N+1}$. By Proposition 2.10, $v_\lambda(m) \in M$ for any $m \in \mathbb{Z}^{N+1}$, which is the basis for the induction.

Suppose for some $k \geq 0$ that $y_1 \dots y_k v_\lambda(m) \in M$ for any $m \in \mathbb{Z}^{N+1}$ and any $y_1, \dots, y_k$ as above. Then for any $i, j \in \{1, \dots, N+1\}$ with $i > j$

$$d(e_j, e_i).y_1 \dots y_k v_\lambda(m) = (e_j|m + \sigma)y_1 \dots y_k v_\lambda(m + e_i) + E_{ij}y_1 \dots y_k v_\lambda(m + e_i).$$

Since $y_1 \dots y_k v_\lambda(m + e_i) \in M$ by the induction hypothesis, $E_{ij}y_1 \dots y_k v_\lambda(m + e_i) \in M$. Because m is arbitrary, it follows that $E_{ij}y_1 \dots y_k v_\lambda(n) \in M$ for any $n \in \mathbb{Z}^{N+1}$. Thus $V(\lambda)(m) \subseteq M$ for all $m \in \mathbb{Z}^{N+1}$ and hence $M = F^\sigma(\lambda)$. $\square$

The k -fold wedge product $\bigwedge^k(\mathbb{C}^{N+1})$ is a highest weight $\mathfrak{sl}_{N+1}$ -module via the action

$$X(v_1 \wedge \dots \wedge v_k) = \sum_{p=1}^k v_1 \wedge \dots \wedge Xv_p \wedge \dots \wedge v_k$$

for any $X \in \mathfrak{sl}_{N+1}$ and $v_1, \dots, v_k \in \mathbb{C}^{N+1}$. It can be seen that $\bigwedge^k(\mathbb{C}^{N+1})$ has highest weight vector $e_1 \wedge \dots \wedge e_k$. It follows from $(E_{ii} - E_{i+1, i+1})(e_1 \wedge \dots \wedge e_k) = \delta_{ik}$ that $\bigwedge^k(\mathbb{C}^{N+1})$ has highest weight ω_k and so $\bigwedge^k(\mathbb{C}^{N+1}) \cong V(\omega_k)$ for $1 \leq k \leq N$, and $\bigwedge^0(\mathbb{C}^{N+1}) \cong \bigwedge^{N+1}(\mathbb{C}^{N+1}) \cong V(0)$. For convenience set $\omega_0 = \omega_{N+1} = 0$ so that $V(0) = V(\omega_0) = V(\omega_{N+1})$.

In [12] some $\mathcal{W}_{N+1}$ -submodules of $\bigwedge^k(\mathbb{C}^{N+1}) \otimes \mathbb{C}[q_1^{\pm 1}, \dots, q_{N+1}^{\pm 1}]$ are identified. As $\mathcal{S}_{N+1}$ -modules they are

$$W_k^\sigma = \bigoplus_{n \in \mathbb{Z}^{N+1}} (\mathbb{C}(n + \sigma) \wedge \mathbb{C}^{N+1} \wedge \dots \wedge \mathbb{C}^{N+1}) \otimes q^n,$$

and if $\sigma \in \mathbb{Z}^{N+1}$ there is a larger submodule

$$\tilde{W}_k^\sigma = W_k^\sigma \oplus (\mathbb{C}^{N+1} \wedge \dots \wedge \mathbb{C}^{N+1}) \otimes q^{-\sigma}.$$

Let $\tilde{W}_k^\sigma = W_k^\sigma$ in the case $\sigma \notin \mathbb{Z}^{N+1}$. For $k = 0$ this $\mathcal{S}_{N+1}$ -module is simply $\mathbb{C}[q_1^{\pm 1}, \dots, q_{N+1}^{\pm 1}]$, and will be denoted $F^\sigma(\omega_0)$. The module action reduces to

$$d(u, r).q^n = (u|n + \sigma)q^{n+r}.$$

If $\sigma \in \mathbb{Z}^{N+1}$ then $F^\sigma(\omega_0)$ has submodule $\tilde{W}_0^\sigma = \mathbb{C}q^{-\sigma}$.

LEMMA 2.12. *Let $N \geq 2$ and $1 \leq k \leq N - 1$. If M is an $\mathcal{S}_{N+1}$ -submodule of $\bigwedge^k(\mathbb{C}^{N+1}) \otimes \mathbb{C}[q_1^{\pm 1}, \dots, q_{N+1}^{\pm 1}]$ which properly contains $\tilde{W}_k^\sigma$, then M contains an element of the form $u(n)$, where u is a weight vector in $\bigwedge^k(\mathbb{C}^{N+1})$, and $n + \sigma \neq 0$.*

PROOF. Since M properly contains $\tilde{W}_k^\sigma$, it contains a nonzero vector $v(n) \in F^\sigma(\omega_k) \setminus \tilde{W}_k^\sigma$ where $n + \sigma \neq 0$. Thus $(e_x|n + \sigma) \neq 0$ for some $x \in \{1, \dots, N + 1\}$ and so $\{n + \sigma, e_1, \dots, e_{x-1}, e_{x+1}, \dots, e_{N+1}\}$ is a basis for $\mathbb{C}^{N+1}$. Write

$$v = \sum_{\substack{p_1, \dots, p_k=1 \\ p_1, \dots, p_k \neq x}}^{N+1} \gamma_{p_1, \dots, p_k} e_{p_1} \wedge \dots \wedge e_{p_k}$$

(modulo $\tilde{W}_k^\sigma$) and assume that the coefficients satisfy $\gamma_{p_1, \dots, p_k} = \text{sgn}(\pi) \gamma_{\pi(p_1, \dots, p_k)}$ for any permutation π . Since $v \neq 0$, at least one of the above coefficients is nonzero,

say $\gamma_{i_1, \dots, i_k}$. Assume indices $i_1, \dots, i_k, x, y$ are distinct. Note that in the right hand side of

$$\begin{aligned} & d(e_x, -e_y).d(e_{i_1}, e_y).v(n) \\ &= (e_x|n+\sigma)(e_{i_1}|n+\sigma)v(n) - (e_{i_1}|n+\sigma)E_{yx}v(n) + (e_x|n+\sigma)E_{yi_1}v(n) - E_{yx}E_{yi_1}v(n), \end{aligned}$$

the first term is in M , while the second and fourth terms are both zero since e_x does not appear in v . Thus the third term, call it $v^{(1)}(n)$, is in M and recall $(e_x|n+\sigma) \neq 0$. Then,

$$\begin{aligned} v^{(1)}(n) &= (e_x|n+\sigma)E_{yi_1}v(n) \\ &= (e_x|n+\sigma) \sum_{\substack{p_2, \dots, p_k=1 \\ p_2, \dots, p_k \neq x}}^{N+1} \gamma_{i_1, p_2, \dots, p_k} e_y \wedge e_{p_2} \wedge \dots \wedge e_{p_k}(n) \\ &\quad + (e_x|n+\sigma) \sum_{\substack{p_1, p_3, \dots, p_k=1 \\ p_1, p_3, \dots, p_k \neq x}}^{N+1} \gamma_{p_1, i_1, p_3, \dots, p_k} e_{p_1} \wedge e_y \wedge e_{p_3} \wedge \dots \wedge e_{p_k}(n) \\ &\quad + \dots \\ &\quad + (e_x|n+\sigma) \sum_{\substack{p_1, \dots, p_{k-1}=1 \\ p_1, \dots, p_{k-1} \neq x}}^{N+1} \gamma_{p_1, \dots, p_{k-1}, i_1} e_{p_1} \wedge \dots \wedge e_{p_{k-1}} \wedge e_y(n). \end{aligned}$$

Relabeling indices, the above can be written

$$(e_x|n+\sigma) \sum_{\substack{p_2, \dots, p_k=1 \\ p_2, \dots, p_k \neq x}}^{N+1} \Gamma e_y \wedge e_{p_2} \wedge \dots \wedge e_{p_k}(n),$$

with $\Gamma = \gamma_{i_1, p_2, \dots, p_k} + (-1)\gamma_{p_2, i_1, p_3, \dots, p_k} + \dots + (-1)^{k-1}\gamma_{p_2, \dots, p_k, i_1}$ which follows from the alternating property of the wedge product. By the assumption on the coefficients, $\gamma_{p_2, \dots, p_s, i_1, p_{s+1}, \dots, p_k} = (-1)^s \gamma_{i_1, p_2, \dots, p_k}$ for $2 \leq s \leq k$, and so

$$v^{(1)}(n) = k(e_x|n+\sigma) \sum_{\substack{p_2, \dots, p_k=1 \\ p_2, \dots, p_k \neq x}}^{N+1} \gamma_{i_1, p_2, \dots, p_k} e_y \wedge e_{p_2} \wedge \dots \wedge e_{p_k}(n).$$

Note that the coefficients in this summation are the same as those from v which have first index equal to i_1 . Doing this process again, by applying $d(e_x, -e_{i_1})d(e_{i_2}, e_{i_1})$ to $v^{(1)}(n)$ in the first step, will yield

$$v^{(2)}(n) = k^2(e_x|n + \sigma)^2 \sum_{\substack{p_3, \dots, p_k=1 \\ p_3, \dots, p_k \neq x}}^{N+1} \gamma_{i_1, i_2, p_3, \dots, p_k} e_y \wedge e_{i_1} \wedge e_{p_3} \wedge \dots \wedge e_{p_k}(n).$$

Repeating (with suitable choices of indices) another $k - 2$ times yields

$$k!(e_x|n + \sigma)^k \gamma_{i_1, \dots, i_k} e_y \wedge e_{i_1} \wedge \dots \wedge e_{i_{k-1}}(n)$$

which is nonzero because both $(e_x|n + \sigma) \neq 0$ and $\gamma_{i_1, \dots, i_k} \neq 0$. The wedge product $e_y \wedge e_{i_1} \wedge \dots \wedge e_{i_{k-1}}$ is a weight vector in $\bigwedge^k(\mathbb{C}^{N+1})$ and so the proof is complete. $\square$

THEOREM 2.13. *Let $N \geq 1$ and $0 \leq k \leq N$.*

- (a) *If $\lambda \neq 0, \omega_k$ then $F^\sigma(\lambda)$ is an irreducible $\mathcal{S}_{N+1}$ -module.*
- (b) *W_k^σ and $F^\sigma(\omega_k)/\tilde{W}_k^\sigma$ are irreducible $\mathcal{S}_{N+1}$ -modules and*

$$F^\sigma(\omega_k)/\tilde{W}_k^\sigma \cong W_{k+1}^\sigma.$$

PROOF. Part (a) follows from Theorem 2.11.

For $0 \leq k \leq N$, the linear map $\psi_k : F^\sigma(\omega_k) \rightarrow F^\sigma(\omega_{k+1})$ defined by

$$v_1 \wedge \dots \wedge v_k(n) \mapsto (n + \sigma) \wedge v_1 \wedge \dots \wedge v_k(n)$$

is an $\mathcal{S}_{N+1}$ -module homomorphism, where $\ker(\psi_k) = \tilde{W}_k^\sigma$ and $\text{Im}(\psi_k) = W_{k+1}^\sigma$.

Thus

$$F^\sigma(\omega_k)/\tilde{W}_k^\sigma \cong W_{k+1}^\sigma.$$

Lemma 2.12 will be used to show that $F^\sigma(\omega_k)/\tilde{W}_k^\sigma$ is irreducible (and hence that W_{k+1}^σ is irreducible) for $1 \leq k \leq N - 1$ with $N \neq 2$. The remaining cases are $F^\sigma(\omega_0)/\tilde{W}_0^\sigma$ and $F^\sigma(\omega_N)/\tilde{W}_N^\sigma$ for any $N \geq 1$. The $F^\sigma(\omega_0)/\tilde{W}_0^\sigma$ case is handled first by showing directly that W_1^σ is irreducible.

Any nonzero submodule M_1 of $W_1^\sigma = \bigoplus_{n \in \mathbb{Z}^{N+1}} \mathbb{C}(n + \sigma) \otimes q^n$, contains a homogeneous vector $(n + \sigma) \otimes q^n$ for some $n \in \mathbb{Z}^{N+1}$ where $(n + \sigma) \neq 0$. Choose e_x so that $(e_x | n + \sigma) \neq 0$ and $r \in \mathbb{Z}^{N+1}$ with $(e_x | r) = 0$. Then $d(e_x, r) \cdot (n + \sigma) \otimes q^n = (e_x | n + \sigma)(n + r + \sigma) \otimes q^{n+r}$ and so $(n + r + \sigma) \otimes q^{n+r} \in M_1$ for any $r \in \{e_x\}^\perp \cap \mathbb{Z}^{N+1}$. Let $e_j \neq e_x$, and $r_x \in \mathbb{Z}$. Then

$$\begin{aligned} d(e_j - e_x, r_x(e_x + e_j)) \cdot (n + r - r_x e_j + \sigma) \otimes q^{n+r-r_x e_j} \\ = (e_j - e_x | n + r - r_x e_j + \sigma)(n + r + r_x e_x + \sigma) \otimes q^{n+r+r_x e_x}. \end{aligned}$$

If $(e_j - e_x | n + r - r_x e_j + \sigma) \neq 0$ this implies $(m + \sigma) \otimes q^m \in M_1$ for any $m \in \mathbb{Z}^{N+1}$. Otherwise if $(e_j - e_x | n + r - r_x e_j + \sigma) = 0$ then consider instead

$$\begin{aligned} d(e_j + e_x, r_x(e_x - e_j)) \cdot (n + r + r_x e_j + \sigma) \otimes q^{n+r+r_x e_j} \\ = (e_j + e_x | n + r + r_x e_j + \sigma)(n + r + r_x e_x + \sigma) \otimes q^{n+r+r_x e_x} \end{aligned}$$

so that $(e_j + e_x | n + r + r_x e_j + \sigma) \neq 0$ and again this yields that $(m + \sigma) \otimes q^m \in M_1$ for any $m \in \mathbb{Z}^{N+1}$. Hence W_1^σ is irreducible.

It remains to show that $F^\sigma(\omega_k)/\tilde{W}_k^\sigma$ is irreducible for $1 \leq k \leq N$. Let M be a submodule of $F^\sigma(\omega_k)$ which properly contains $\tilde{W}_k^\sigma$.

For $N \geq 2$ with $1 \leq k \leq N - 1$, Lemma 2.12 shows that M contains a vector of the form $e_y \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}(n)$, with n such that $(e_x | n + \sigma) \neq 0$ for some $x \in \{1, \dots, N + 1\}$. For any $r \in \{e_x\}^\perp$,

$$d(e_x, r) \cdot e_y \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}(n) = (e_x | n + \sigma) e_y \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}(n + r).$$

Thus $e_y \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}(n + r) \in M$ for any $r \in \{e_x\}^\perp$. So for $a \notin \{x, y, i_1, \dots, i_{k-1}\}$,

$$\begin{aligned} d(e_x - e_a, r_x(e_x + e_a)) \cdot e_y \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}(n + r - r_x e_a) \\ = (e_x - e_a | n + r - r_x e_a + \sigma) e_y \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}(n + r + r_x e_x). \end{aligned}$$

If $(e_x - e_a|n + r - r_x e_a + \sigma) \neq 0$ this yields that $e_y \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}(m) \in M$ for all $m \in \mathbb{Z}^{N+1}$. If $(e_x - e_a|n + r - r_x e_a + \sigma) = 0$ apply instead $d(e_x + e_a, r_x(e_x - e_a))$ to $e_y \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}(n + r + r_x e_a)$ and obtain the same result.

Using the fact that $e_y \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}(m) \in M$ for all $m \in \mathbb{Z}^{N+1}$, any vector of the form $e_{b_1} \wedge e_{b_2} \wedge \cdots \wedge e_{b_k}(m)$ can be obtained by doing the following. To replace $e_j \in \{e_y, e_{i_1}, \dots, e_{i_{k-1}}\}$ with e_a , $a \notin \{y, i_1, \dots, i_{k-1}\}$ in $e_y \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}(m)$, note that

$$\begin{aligned} & d(e_j, e_a).e_y \wedge e_{i_1} \wedge \cdots \wedge e_j \wedge \cdots \wedge e_{i_{k-1}}(m - e_a) \\ &= (e_j|m - e_a + \sigma)e_y \wedge e_{i_1} \wedge \cdots \wedge e_j \wedge \cdots \wedge e_{i_{k-1}}(m) + e_y \wedge e_{i_1} \wedge \cdots \wedge e_a \wedge \cdots \wedge e_{i_{k-1}}(m). \end{aligned}$$

The first term on the right hand side is already in M which implies $e_y \wedge e_{i_1} \wedge \cdots \wedge e_a \wedge \cdots \wedge e_{i_{k-1}}(m) \in M$. This process of swapping out e_j 's can be repeated until the desired vector is obtained. Vectors of the form $e_{b_1} \wedge e_{b_2} \wedge \cdots \wedge e_{b_k}(m)$ form a basis for $V(\omega_k)(m)$, and thus M must be all of $F^\sigma(\omega_k)$. It follows that $F^\sigma(\omega_k)/\tilde{W}_k^\sigma$ is irreducible.

Consider the case $k = N$ for $N \geq 1$. M contains a nonzero vector $v(n)$, where $v(n) \notin \tilde{W}_N^\sigma$ such that $n + \sigma \neq 0$. Suppose $(e_x|n + \sigma) \neq 0$ for some $x \in \{1, \dots, N + 1\}$, and so $\{n + \sigma, e_1, \dots, e_{x-1}, e_{x+1}, \dots, e_{N+1}\}$ is a basis for $\mathbb{C}^{N+1}$. From the definition of $\tilde{W}_N^\sigma$ it follows that $v(n)$ is a nonzero scalar multiple of $e_1 \wedge \cdots \wedge e_{x-1} \wedge e_{x+1} \wedge \cdots \wedge e_{N+1}(n)$ plus some vector in $\tilde{W}_N^\sigma$. Assume without loss of generality that $v(n) = e_1 \wedge \cdots \wedge e_{x-1} \wedge e_{x+1} \wedge \cdots \wedge e_{N+1}(n)$. Note that

$$V(\omega_N)(n) = \mathbb{C}v(n) \oplus (n + \sigma) \wedge \left(\bigwedge^{N-1} \mathbb{C}^{N+1} \right) (n)$$

and so $V(\omega_N)(n) \subset M$. For any $r \in \{e_x\}^\perp \cap \mathbb{Z}^{N+1}$, $d(e_x, r).v(n) = (e_x|n + \sigma)v(n + r)$, thus $v(n + r) \in M$. Since $(e_x|n + r + \sigma) = (e_x|n + \sigma) \neq 0$ it follows that

$$V(\omega_N)(n + r) = \mathbb{C}v(n + r) \oplus (n + r + \sigma) \wedge \left(\bigwedge^{N-1} \mathbb{C}^{N+1} \right) (n + r)$$

and so $V(\omega_N)(n+r) \subset M$ for any $r \in \{e_x\}^\perp \cap \mathbb{Z}^{N+1}$. It remains to show that $V(\omega_N)(n+r+r_x e_x) \subset M$ for any $r_x \in \mathbb{Z}$.

Let $j > x$, $r_x \in \mathbb{Z} \setminus \{0\}$, and $r \in \{e_x\}^\perp \cap \mathbb{Z}^{N+1}$. Then

$$d(e_j, r_x e_x).v_{\omega_N}(n+r) = (e_j|n+r+\sigma)v_{\omega_N}(n+r+r_x e_x),$$

where v_{ω_N} is a highest weight vector in $V(\omega_N)$. If $(e_j|n+r+\sigma) \neq 0$ then $v_{\omega_N}(n+r+r_x e_x) \in M$, and by the proof of Lemma 2.11 $V(\omega_N)(n+r+r_x e_x) \subset M$. Let v_ℓ be a lowest weight vector of $V(\omega_N)$, and $j < x$, then

$$d(e_j, r_x e_x).v_\ell(n+r) = (e_i|n+r+\sigma)v_\ell(n+r+r_x e_x).$$

If $(e_i|n+r+\sigma) \neq 0$ then $v_\ell(n+r+r_x e_x) \in M$. Again the proof of Lemma 2.11 shows how to generate all of $V(\omega_N)(n+r+r_x e_x)$, only this time the $y_p = E_{ij}$ where $i < j$ are applied to lowest weight vector $v_\ell(n+r+r_x e_x)$. Suppose for some $r \in \{e_x\}^\perp \cap \mathbb{Z}^{N+1}$, say $r = r_0$, that $(e_j|n+r_0+\sigma) = 0$ for all $j \neq x$; i.e. $(n+r_0+\sigma) = K e_x \neq 0$. Then for $j \neq x$

$$\begin{aligned} d(e_j - e_x, r_x(e_j + e_x)).v(n+r_0-r_x e_j) &= K v(n+r_0+r_x e_x) \\ &+ r_x e_1 \wedge \cdots \wedge e_{j-1} \wedge e_x \wedge e_{j+1} \wedge \cdots \wedge e_{x-1} \wedge e_{x+1} \wedge \cdots \wedge e_{N+1}(n+r_0+r_x e_x). \end{aligned}$$

Since $(n+r_0+r_x e_x+\sigma)$ is a scalar multiple of e_x , the second term above is in $\tilde{W}_N^\sigma$, and thus $v(n+r_0+r_x e_x) \in M$. If $(e_x|n+r_0+r_x e_x+\sigma) \neq 0$ then

$$\begin{aligned} V(\omega_N)(n+r_0+r_x e_x) \\ = \mathbb{C}v(n+r_0+r_x e_x) \oplus (n+r_0+r_x e_x+\sigma) \wedge \left(\bigwedge^{N-1} \mathbb{C}^{N+1} \right) (n+r_0+r_x e_x) \end{aligned}$$

and so $V(\omega_N)(n+r_0+r_x e_x) \subset M$. If $(e_x|n+r_0+r_x e_x+\sigma) = 0$, then $(n+r_0+r_x e_x+\sigma) = 0$, in which case $V(\omega_N)(n+r_0+r_x e_x) \subset \tilde{W}_N^\sigma \subset M$. Thus $V(\omega_N)(m) \subset M$ for all $m \in \mathbb{Z}^{N+1}$ and so $M = F^\sigma(\omega_N)$. $\square$

CHAPTER 3

Category $\mathcal{J}$ for Divergence Zero Vector Fields on a Torus

1. Category $\mathcal{J}$ for $\mathcal{S}_N$

A family of modules for $\mathcal{W}_N$ called category $\mathcal{J}$ was defined in [3]. An analogous category of modules for $\mathcal{S}_N$ is defined as follows:

DEFINITION 3.1. *Let $N > 1$. An $\mathcal{S}_N$ -module J belongs to category $\mathcal{J}$ if the following properties hold:*

- (J1) *The action of d_a on J is diagonalizable for all $a \in \{1, \dots, N\}$.*
- (J2) *Module J is a free A_N -module of finite rank.*
- (J3) *For any $X \in \mathcal{S}_N, f \in A_N$ and $u \in J$, $X(fu) = (X(f))u + f(Xu)$.*

The morphisms for category $\mathcal{J}$ are module homomorphisms preserving the action of both A_N and $\mathcal{S}_N$. A submodule of any $J \in \mathcal{J}$ must be invariant under the actions of both A_N and $\mathcal{S}_N$. Classifying the modules of category $\mathcal{J}$ is the goal of this chapter. From property (J2) it follows that any module in $\mathcal{J}$ is a finite direct sum of indecomposable modules and hence it suffices to examine indecomposable modules $J \in \mathcal{J}$. Property (J1) yields the weight decomposition, $J = \bigoplus_{\lambda \in \mathbb{C}^N} J_\lambda$, with respect to the action of the d_a , where $J_\lambda = \{u \in J \mid d_a(u) = \lambda_a u\}$. The set of all λ such that $J_\lambda \neq 0$ is called the *support* of J .

For $u \in J_\lambda$,

$$d_a(d_{bc}(r)u) = d_{bc}(r)d_a u + [d_a, d_{bc}(r)]u = (\lambda_a + r_a)d_{bc}(r)u$$

and thus $d_{bc}(r)J_\lambda \subset J_{\lambda+r}$. Similarly by (J3) $t^r J_\lambda \subset J_{\lambda+r}$. These two relations partition the weights of J into $\mathbb{Z}^N$ -cosets of $\mathbb{C}^N$, and decompose J into a direct sum

of submodules, each corresponding to a distinct coset. Thus if J is indecomposable its set of weights is one such coset $\lambda + \mathbb{Z}^N$ for $\lambda \in \mathbb{C}^N$ and $J = \bigoplus_{r \in \mathbb{Z}^N} J_{\lambda+r}$. For a fixed $\lambda \in \mathbb{C}^N$, $\mathcal{J}_\lambda$ shall denote the subcategory of $\mathcal{J}$ supported on $\lambda + \mathbb{Z}^N$ (i.e. the subcategory of $\mathcal{J}$ consisting of all $\mathcal{S}_N$ -modules whose weights are of the form $\lambda + \mathbb{Z}^N$). From now on assume $J \in \mathcal{J}_\lambda$.

Let $U = J_\lambda$. The invertible map $t^r : U \rightarrow J_{\lambda+r}$ identifies all weight spaces with U and since J is a free module for the associative algebra A_N it follows that any basis for U is also basis for J viewed as a free A_N -module. Furthermore the finite rank condition of property (J2) implies that U must be finite dimensional. Then as vector spaces, $J \cong A_N \otimes U$ (this may also be viewed as an isomorphism of free A_N -modules). Homogeneous elements of J will be denoted $t^s \otimes v$, for $s \in \mathbb{Z}^N, v \in U$.

2. Polynomial Action

The map $d_{ab}(r) : 1 \otimes U \rightarrow t^r \otimes U$ induces an endomorphism $D_{ab}(r) : U \rightarrow U$ defined by

$$D_{ab}(r)u = (t^{-r} \circ d_{ab}(r))u$$

for $u \in U$. Combining this with (J3) yields

$$(3) \quad d_{ab}(r)(t^s \otimes v) = (r_b s_a - s_b r_a) t^{r+s} \otimes v + t^{r+s} \otimes D_{ab}(r)v,$$

and so the action of $d_{ab}(r)$ on J is determined by that of $D_{ab}(r)$ on U .

The key to proving the main result is to show that $D_{ab}(r)$ acts on U by an $\text{End}(U)$ -valued polynomial in r when $N \geq 3$. In this case it is said that $D_{ab}(r)$ has *polynomial action*. That is,

$$D_{ab}(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P_{ab}^{(k)},$$

where the $P_{ab}^{(k)} \in \text{End}(U)$ do not depend on r , and the sum is finite. The factor of $k!$ is there for convenience. In the case $N = 2$ a slight modification needs to be

made and the corresponding expansion has the form

$$D_{ab}(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^2} \frac{r^k}{k!} P_{ab}^{(k)} - \delta_{r,0} P_{ab}^{(0)},$$

where $\delta_{r,0}$ is the Kronecker delta.

Since $D_{ab}(r) = t^{-r} d_{ab}(r)$, the Lie bracket for $D_{ab}(r)$ follows from that of $d_{ab}(r)$.

$$\begin{aligned} & [D_{ab}(r), D_{cd}(s)] \\ &= [t^{-r} d_{ab}(r), t^{-s} d_{cd}(s)] \\ &= t^{-r} (d_{ab}(r)(t^{-s})) d_{cd}(s) - t^{-s} (d_{cd}(s)(t^{-r})) d_{ab}(r) + t^{-r-s} [d_{ab}(r), d_{cd}(s)] \\ &= t^{-r} (-r_b s_a + r_a s_b) t^{r-s} d_{cd}(s) - t^{-s} (-r_c s_d + r_d s_c) t^{s-r} d_{ab}(r) \\ &\quad + t^{-r-s} (r_b s_c d_{ad}(r+s) - r_b s_d d_{ac}(r+s) - r_a s_c d_{bd}(r+s) + r_a s_d d_{bc}(r+s)) \\ &= (r_a s_b - r_b s_a) D_{cd}(s) + (r_c s_d - r_d s_c) D_{ab}(r) \\ &\quad + r_b s_c D_{ad}(r+s) - r_b s_d D_{ac}(r+s) - r_a s_c D_{bd}(r+s) + r_a s_d D_{bc}(r+s). \end{aligned}$$

This has special case

$$(4) \quad [D_{ab}(r), D_{ab}(s)] = (r_a s_b - r_b s_a) (D_{ab}(r) + D_{ab}(s) - D_{ab}(r+s)).$$

Note that for $N \geq 3$

$$r_c D_{ab}(r) + r_a D_{bc}(r) + r_b D_{ca}(r) = 0.$$

For a function f whose domain is $\mathbb{Z}^N$ the *difference derivative* in direction $r \in \mathbb{Z}^N$, denoted by $\partial_r f$, is defined as

$$\partial_r f(s) = f(s+r) - f(s).$$

Higher order derivatives are obtained by iteration and thus

$$(5) \quad \partial_r^m f(s) = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} f(s+ir).$$

To simplify notation let $\partial_a = \partial_{e_a}$. Applying the above twice yields

$$(6) \quad \partial_a^m \partial_b^n f(s) = \sum_{i=0}^m \sum_{j=0}^n (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} f(s + ie_a + je_b).$$

A technique for finding eigenvectors was found in [10] and provides a key step to proving the result here (cf. Lemma 4 in [10]).

LEMMA 3.2. *Let $m, n \in \mathbb{Z}_{\geq 0}$, and $a, b \in \{1, \dots, N\}$. Then for $m \geq 1$*

$$[D_{ab}(-e_b), [D_{ab}(-e_a), \partial_a^m \partial_b^n D_{ab}(e_a)]] = -n(m+1) \partial_a^m \partial_b^n D_{ab}(e_a).$$

PROOF. First apply the formula for the difference derivatives,

$$\begin{aligned} & [D_{ab}(-e_b), [D_{ab}(-e_a), \partial_a^m \partial_b^n D_{ab}(e_a)]] \\ &= \sum_{i=0}^m \sum_{j=0}^n (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} [D_{ab}(-e_b), [D_{ab}(-e_a), D_{ab}((i+1)e_a + je_b)]], \end{aligned}$$

then evaluate the Lie bracket for $D_{ab}(r)$,

$$\begin{aligned} &= - \sum_{i=0}^m \sum_{j=0}^n (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} j [D_{ab}(-e_b), D_{ab}(-e_a)] \\ &\quad - \sum_{i=0}^m \sum_{j=0}^n (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} j [D_{ab}(-e_b), D_{ab}((i+1)e_a + je_b)] \\ &\quad + \sum_{i=0}^m \sum_{j=0}^n (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} j [D_{ab}(-e_b), D_{ab}(ie_a + je_b)]. \end{aligned}$$

Simplifying the binomial coefficients and evaluating the Lie bracket yields,

$$\begin{aligned} &= -n \sum_{j=1}^n (-1)^{n-j} \binom{n-1}{j-1} \left(\sum_{i=0}^m (-1)^{m-i} \binom{m}{i} \right) [D_{ab}(-e_b), D_{ab}(-e_a)] \\ &\quad - n \sum_{i=0}^m \sum_{j=1}^n (-1)^{m+n-i-j} \binom{m}{i} \binom{n-1}{j-1} \\ &\quad \times (i+1)(D_{ab}(-e_b) + D_{ab}((i+1)e_a + je_b) - D_{ab}((i+1)e_a + (j-1)e_b)) \\ &\quad + n \sum_{i=0}^m \sum_{j=1}^n (-1)^{m+n-i-j} \binom{m}{i} \binom{n-1}{j-1} \end{aligned}$$

$$\times i(D_{ab}(-e_b) + D_{ab}(ie_a + je_b) - D_{ab}(ie_a + (j-1)e_b)).$$

The first term vanishes because the sum in parentheses is zero for $m \geq 1$. For a similar reason the terms involving $D_{ab}(-e_b)$ will vanish. A change of summation index causes a sign change leaving,

$$\begin{aligned} & n \sum_{i=0}^m \sum_{j=0}^{n-1} (-1)^{m+n-i-j} \binom{m}{i} \binom{n-1}{j} \\ & \times (D_{ab}((i+1)e_a + (j+1)e_b) - D_{ab}((i+1)e_a + je_b)) \\ & - mn \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (-1)^{m+n-i-j} \binom{m-1}{i} \binom{n-1}{j} \\ & \times (D_{ab}((i+2)e_a + (j+1)e_b) - D_{ab}((i+2)e_a + je_b)) \\ & + mn \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (-1)^{m+n-i-j} \binom{m-1}{i} \binom{n-1}{j} \\ & \times (D_{ab}((i+1)e_a + (j+1)e_b) - D_{ab}((i+1)e_a + je_b)). \end{aligned}$$

Applying the definition of the difference derivative combines terms to give

$$\begin{aligned} & n \sum_{i=0}^m \sum_{j=0}^{n-1} (-1)^{m+n-i-j} \binom{m}{i} \binom{n-1}{j} \partial_b D_{ab}((i+1)e_a + je_b) \\ & - mn \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (-1)^{m+n-i-j} \binom{m-1}{i} \binom{n-1}{j} \\ & \times (\partial_b D_{ab}((i+2)e_a + je_b) - \partial_b D_{ab}((i+1)e_a + je_b)) \\ & = -n \partial_a^m \partial_b^{n-1} (\partial_b D_{ab}(e_a)) \\ & - mn \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (-1)^{m+n-i-j} \binom{m-1}{i} \binom{n-1}{j} \partial_a \partial_b D_{ab}((i+1)e_a + je_b) \\ & = -n \partial_a^m \partial_b^n D_{ab}(e_a) - mn \partial_a^{m-1} \partial_b^{n-1} (\partial_a \partial_b D_{ab}(e_a)) \\ & = -n(m+1) \partial_a^m \partial_b^n D_{ab}(e_a). \end{aligned}$$

□

The lemma above shows that for various m and n , $\partial_a^m \partial_b^n D_{ab}(e_a)$ are eigenvectors for $\text{ad}(D_{ab}(-e_b))\text{ad}(D_{ab}(-e_a))$ and yield infinitely many distinct eigenvalues. This fact will be used to show that $\partial_a^m \partial_b^n D_{ab}(e_a) = 0$ for large enough values of m and n .

The following lemma was proven in [3] and is presented here without proof.

LEMMA 3.3 ([3] Lemma 3.4). *Let $\mathfrak{L}$ be a Lie algebra with nonzero elements $y, y_1, y_2, \dots$ with the property that*

$$[y, y_i] = \alpha_i y_i$$

for $i = 1, 2, \dots$, and $\alpha_i \in \mathbb{C}$. Then for a finite dimensional representation (U, ρ) of $\mathfrak{L}$, there are at most $(\dim U)^2 - \dim U + 1$ distinct eigenvalues for which $\rho(y_i) \neq 0$.

Now consider the case $N = 2$ where $\mathcal{S}_2 = \text{Span}_{\mathbb{C}} \{d_1(r), d_2(r), d_{12}(r) | r \in \mathbb{Z}^2\}$. Combining the two lemmas above shows that there exists $K \in \mathbb{N}$ such that $\partial_1^{m+1} \partial_2^n D_{12}(e_1) = 0$ and $\partial_1^m \partial_2^{n+1} D_{12}(e_2) = 0$ for all $m + n > K$. This fact along with the following lemmas will show that $D_{12}(r)$ is a polynomial plus a delta function.

LEMMA 3.4. *If $\partial_1^m \partial_2^n f(r) = 0$ for all $m, n \geq 0$ then $f(r + ie_1 + je_2) = 0$ for all $i, j \geq 0$.*

PROOF. Using (5) it follows by induction that if $\partial_a^m f(r) = 0$ for all $m \geq 0$ then $f(r + ie_a) = 0$ for all $i \geq 0$.

Suppose $\partial_1^m \partial_2^n f(r) = 0$ for all $m, n \geq 0$, and let $g_n(r) = \partial_2^n f(r)$ so that by assumption, for each $n \geq 0$, $\partial_1^m g_n(r) = 0$ for all $m \geq 0$. By the first part of the proof this implies that $g_n(r + ie_1) = 0$ for all $i \geq 0$. So for any $i \geq 0$, $\partial_2^n f(r + ie_1) = 0$ for all $n \geq 0$, which implies that $f(r + ie_1 + je_s) = 0$ for all $i, j \geq 0$. $\square$

For $K + 1$ ordered pairs (x_i, a_i) , $i = 0, \dots, K$ with distinct x_i , there exists a unique interpolating polynomial $P(X)$ of degree at most K , such that $P(x_i) = a_i$. This can be extended to functions of two variables in the following way. (The following two lemmas are standard results.)

LEMMA 3.5. *Given $\frac{(K+1)(K+2)}{2}$ triples (x_i, y_j, a_{ij}) for $0 \leq i + j \leq K$, with distinct x_i and distinct y_j , there exists a unique polynomial $P(X, Y)$ of degree at most K such that $P(x_i, y_j) = a_{ij}$.*

PROOF. For $K = 0$ the constant function $P(X, Y) = a_{00}$ is the unique polynomial of degree 0 through (x_0, y_0, a_{00}) . Proceed by induction on K . The univariate case yields a unique polynomial $R(X)$ of degree at most K such that $R(x_i) = a_{i0}$ for all $i \in \{0, \dots, K\}$. For $i \geq 0$ and $j \geq 1$ let

$$b_{ij} = \frac{a_{ij} - R(x_i)}{y_j - y_0}.$$

By induction there is a unique interpolating polynomial $Q(X, Y)$ of degree at most $K - 1$ such that $Q(x_i, y_j) = b_{ij}$ for the $\frac{K(K+1)}{2}$ triples (x_i, y_j, b_{ij}) where $1 \leq i + j \leq K$, and $j \geq 1$. Polynomial $P(X, Y) = R(X) + (Y - y_0)Q(X, Y)$ is of degree at most K and $P(x_i, y_j) = a_{ij}$ for $0 \leq i + j \leq K$.

Suppose $T(X, Y)$ is a polynomial of degree at most K and $T(x_i, y_j) = a_{ij}$ for $0 \leq i + j \leq K$. Since the decomposition $T(X, Y) = F(X) + (Y - y_0)G(X, Y)$ is unique for polynomials F and G , it must be that $F(X) = R(X)$ and $G(X, Y) = Q(X, Y)$. Hence $P(X, Y)$ is unique. $\square$

LEMMA 3.6. *Let $S = S_1 \times \dots \times S_N \subset \mathbb{C}^N$, where each S_i is a set with $K + 1$ elements, and let F and G be polynomials of degree at most K in N variables, $X_1, \dots, X_N$, that agree on S . Then $F = G$.*

PROOF. Use induction on N where the case $N = 1$ is well known (the case $N = 2$ follows from Lemma 3.5). Let $a \in S_1$ and divide $F(X)$ and $G(X)$ by $(X_1 - a)$ to get $F(X) = (X_1 - a)P(X) + R(X_2, \dots, X_N)$ and $G(X) = (X_1 - a)Q(X) + T(X_2, \dots, X_N)$. Then R and T are of degree at most K , and agree on $S_2 \times \dots \times S_N$. By induction $R = T$ and so $P(x) = Q(x)$ for all $x \in S' = (S_1 \setminus \{a\}) \times S_2 \times \dots \times S_N$. Then $P = Q$ also by induction, since P and Q are of degree at most $K - 1$ and S' contains a cube of size K . Therefore $F = G$. $\square$

LEMMA 3.7. *Let $r \in \mathbb{Z}^2$. Suppose $\partial_1^m \partial_2^n f(r) = 0$ for all $m + n > K$, for some $K \in \mathbb{N}$. Let $p(t)$ be the bivariate interpolating polynomial of degree at most K such that $p(r + ie_1 + je_2) = f(r + ie_1 + je_2)$ for $0 \leq i + j \leq K$. Then $f(s) = p(s)$ for all $s_1 \geq r_1, s_2 \geq r_2$.*

PROOF. Let $h(t) = f(t) - p(t)$. Then (6) implies that $\partial_1^m \partial_2^n h(r) = 0$ for $m + n \leq K$, because $h(r + ie_1 + je_2) = 0$ for $i + j \leq K$. When $m + n > K$, $\partial_1^m \partial_2^n f(r) = 0$ by assumption and $\partial_1^m \partial_2^n p(r) = 0$ since it is a polynomial of degree at most K . Thus $\partial_1^m \partial_2^n h(r) = 0$ for $m + n \geq 0$ and so by Lemma 3.4, $f(r + ie_1 + je_2) = p(r + ie_1 + je_2)$ for all $i, j \geq 0$. $\square$

PROPOSITION 3.8. *Let $N = 2$ and let $J = A_2 \otimes U$ be a module in category $\mathcal{J}$. Then $D_{12}(r)$ acts on U by*

$$D_{12}(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^2} \frac{r^k}{k!} P_{12}^{(k)} - \delta_{r,0} P_{12}^{(0)}$$

for all $r \in \mathbb{Z}^2$, where $P_{12}^{(k)} \in \text{End}(U)$ does not depend on r , and the summation is finite.

PROOF. It follows from Lemmas 3.2 and 3.3 that there exists a $K \in \mathbb{N}$ such that $\partial_1^{m+1} \partial_2^n D_{12}(e_1) = 0$ and $\partial_1^m \partial_2^{n+1} D_{12}(e_2) = 0$ for $n + m > K$. By Lemma 3.7 there exist $\text{End}(U)$ -valued polynomials P_1 and P_2 such that

$$P_1(r) = \partial_1 D_{12}(r) \text{ and } P_2(s) = \partial_2 D_{12}(s),$$

for all $r = r_1 e_1 + r_2 e_2$ and $s = s_1 e_1 + s_2 e_2$ with $r_1, s_2 \geq 1$ and $r_2, s_1 \geq 0$. Taking polynomial difference antiderivatives $\bar{P}_1(r)$ and $\bar{P}_2(s)$ respectively, yields

$$D_{12}(r) = \bar{P}_1(r) + g_1(r_2) \text{ and } D_{12}(s) = \bar{P}_2(s) + g_2(s_1),$$

where $\bar{P}_1$ and $\bar{P}_2$ are polynomials, and g_1 and g_2 are functions of r_2 and s_1 respectively. Then $\bar{P}_1(r) + g_1(r_2) = \bar{P}_2(r) + g_2(r_1)$ for $r_1, r_2 \geq 1$, so

$$g_2(r_1) - g_1(r_2) = \bar{P}_1(r) - \bar{P}_2(r).$$

Taking the m th difference derivative in e_1 where $m > K$ gives

$$\partial_1^m g_2(r_1) = \partial_1^m (\bar{P}_1(r) - \bar{P}_2(r)) = 0$$

which implies that g_2 is a polynomial in r_1 . Similarly g_1 is a polynomial in r_2 . Thus $D_{12}(r) = \bar{P}_1(r) + g_1(r_2)$ and $D_{12}(r) = \bar{P}_2(r) + g_2(r_1)$, are polynomials that agree on $\mathcal{R}_1 = \{(i, j) \in \mathbb{Z}^2 | i, j \geq 1\}$, and hence must be equal by Lemma 3.6. Therefore $D_{12}(r)$ is an $\text{End}(U)$ -valued polynomial $Q_1(r)$ on $\mathcal{R}_1$. It remains to show that $D_{12}(r)$ acts by a polynomial $P(r)$ on all of $\mathbb{Z}^2$ except at the origin.

Let $\mathcal{L}$ be the Lie algebra with basis elements $D_{12}(r)$ for $r \in \mathbb{Z}^2$ and Lie bracket given by (4). Consider the automorphisms φ_1 and φ_2 of $\mathcal{L}$, where $\varphi_1(D_{12}(r_1, r_2)) = -D_{12}(-r_1, r_2)$, $\varphi_2(D_{12}(r_1, r_2)) = -D_{12}(r_1, -r_2)$, and their composition $\varphi_2 \circ \varphi_1$ where $\varphi_2 \circ \varphi_1(D_{12}(r_1, r_2)) = D_{12}(-r_1, -r_2)$. What was proven for $\mathcal{L}$ is also true for its image under these automorphisms. Thus there are $\text{End}(U)$ -valued polynomials Q_2, Q_3 , and Q_4 such that $D_{12}(r) = Q_2(r)$ for $r \in \mathcal{R}_2 = \{(-i, j) \in \mathbb{Z}^2 | i, j \geq 1\}$, $D_{12}(r) = Q_3(r)$ for $r \in \mathcal{R}_3 = \{(-i, -j) \in \mathbb{Z}^2 | i, j \geq 1\}$, and $D_{12}(r) = Q_4(r)$ for $r \in \mathcal{R}_4 = \{(i, -j) \in \mathbb{Z}^2 | i, j \geq 1\}$.

Let $\sigma : \mathcal{L} \rightarrow \mathcal{L}$ be the automorphism defined by $\sigma(D_{12}(r_1, r_2)) = -D_{12}(-r_1 + r_2, r_2)$. When applied to $\mathcal{R}_1$, this implies the existence of an $\text{End}(U)$ -valued polynomial P such that $D_{12}(r) = P(r)$ on the region $\mathcal{R}_5 = \{(i, j) \in \mathbb{Z}^2 | j \geq 1, i \leq j\}$. Lemma 3.6 may be applied to the intersection of $\mathcal{R}_1$ and $\mathcal{R}_5$ which says that $Q_1 = P$. Applied again to the intersection of $\mathcal{R}_2$ and $\mathcal{R}_5$ yields that $Q_2 = P$. Thus $D_{12}(r)$ acts by $\text{End}(U)$ -valued polynomial $P(r)$ for $r \in \{(i, j) \in \mathbb{Z}^2 | j \geq 1\}$.

Similar techniques may be applied to connect this region with $\mathcal{R}_3$ and $\mathcal{R}_4$. The result that follows is that $D_{12}(r)$ acts by $\text{End}(U)$ -valued polynomial $P(r)$ for $r \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.

To indicate that the polynomial obtained above is specific to the operator $D_{12}(r)$, write P_{12} instead of P . Decompose $P_{12}(r)$ into powers of r as

$$P_{12}(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^2} \frac{r^k}{k!} P_{12}^{(k)},$$

where the sum is finite, the $P_{12}^{(k)} \in \text{End}(U)$ do not depend on r , and the factor of $k!$ is there for convenience. Note that $P_{12}(0,0) = P_{12}^{(0,0)}$, however $D_{12}(0,0) = 0$ by definition and so it must act by zero. To avoid a contradiction at the origin a delta function is added so that

$$D_{12}(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^2} \frac{r^k}{k!} P_{12}^{(k)} - \delta_{r,0} P_{12}^{(0)},$$

which is now valid for all $r \in \mathbb{Z}^2$. $\square$

LEMMA 3.9. *Let $N \geq 2$ and $J = A_N \otimes U$ be a module in category $\mathcal{J}$. Then for $a, b \in \{1, \dots, N\}$, $a \neq b$, $D_{ab}(r)$ acts on U by*

$$D_{ab}(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P_{ab}^{(k)}$$

for $r \in \mathbb{Z}^N$ with $(r_a, r_b) \neq (0,0)$, where the $P_{ab}^{(k)} \in \text{End}(U)$ do not depend on r , and the summation is finite.

PROOF. It follows from Proposition 3.8 that for any $a, b \in \{1, \dots, N\}$, $a \neq b$, the operators $D_{ab}(r_a e_a + r_b e_b)$ act by polynomial in r_a, r_b , when $(r_a, r_b) \neq (0,0)$, since for these values the delta function vanishes.

The result will be proven by induction on N , with the induction hypothesis that $D_{12}(r_1 e_1 + \dots + r_{N-1} e_{N-1})$ acts as a polynomial for $(r_1, r_2) \neq (0,0)$. For convenience this is stated for $D_{12}(r)$, though it holds for any $D_{ab}(r)$ by a change of indices. The basis of induction $N = 2$ follows from Proposition 3.8.

Assume until otherwise stated that $r_i \neq 0$ for all $i \in \{1, \dots, N\}$. Consider

$$\begin{aligned} & [D_{1N}(r_N e_N), D_{12}(r_1 e_1 + \dots + r_{N-1} e_{N-1})] \\ &= -r_1 r_N D_{12}(r_1 e_1 + \dots + r_{N-1} e_{N-1}) + r_1 r_N D_{12}(r_1 e_1 + \dots + r_N e_N). \end{aligned}$$

Both $D_{1N}(r_N e_N)$, and $D_{12}(r_1 e_1 + \cdots + r_{N-1} e_{N-1})$ have polynomial action by the induction hypothesis. Rearrange to get

$$\begin{aligned} & r_1 r_N D_{12}(r_1 e_1 + \cdots + r_N e_N) \\ &= [D_{1N}(r_N e_N), D_{12}(r_1 e_1 + \cdots + r_{N-1} e_{N-1})] + r_1 r_N D_{12}(r_1 e_1 + \cdots + r_{N-1} e_{N-1}). \end{aligned}$$

The right hand side is a polynomial in $r_1, \dots, r_N$ and thus $r_1 r_N D_{12}(r) = P(r)$, for some $\text{End}(U)$ -valued polynomial P , and $r = r_1 e_1 + \cdots + r_N e_N$. Symmetry in indices 1 and 2 yields that $r_2 r_N D_{12}(r) = Q(r)$, for some $\text{End}(U)$ -valued polynomial Q . Thus,

$$r_2 P(r) = r_1 Q(r).$$

Unique factorization of a polynomial into irreducible factors implies that P factors as $P(r) = r_1 \bar{P}(r)$ and so $r_1 r_N D_{12}(r) = r_1 \bar{P}(r)$. Since $r_1 \neq 0$, division of polynomials gives that $r_N D_{12}(r) = \bar{P}(r)$. Thus, $r_N D_{12}(r)$ has polynomial action, or more generally, $r_a D_{bc}(r)$ has polynomial action for $a \neq b, c$.

Fix $s_N \neq 0$ and consider

$$\begin{aligned} & [D_{1N}(r_1 e_1 + s_N e_N), D_{2N}(r_2 e_2 + r_3 e_3 + \cdots + (r_N - s_N) e_N)] \\ &= r_1 (r_N - s_N) D_{2N}(r_2 e_2 + r_3 e_3 + \cdots + (r_N - s_N) e_N) \\ &\quad - s_N r_2 D_{1N}(r_1 e_1 + s_N e_N) - r_1 (r_N - s_N) D_{2N}(r_1 e_1 + \cdots + r_N e_N) \\ &\quad - s_N (r_N - s_N) D_{12}(r_1 e_1 + \cdots + r_N e_N) + s_N r_2 D_{1N}(r_1 e_1 + \cdots + r_N e_N). \end{aligned}$$

Let $r = r_1 e_1 + \cdots + r_N e_N$, and isolate the D_{12} term to get

$$\begin{aligned} & s_N (r_N - s_N) D_{12}(r) = r_1 (r_N - s_N) D_{2N}(r_2 e_2 + r_3 e_3 + \cdots + (r_N - s_N) e_N) \\ &\quad - [D_{1N}(r_1 e_1 + s_N e_N), D_{2N}(r_2 e_2 + r_3 e_3 + \cdots + (r_N - s_N) e_N)] \\ &\quad - s_N r_2 D_{1N}(r_1 e_1 + s_N e_N) - r_1 (r_N - s_N) D_{2N}(r) + s_N r_2 D_{1N}(r). \end{aligned}$$

The first three terms on the right hand side have polynomial action by induction.

The last two terms are of the form $r_a D_{bc}(r)$, as is $r_N D_{12}(r)$ on the left hand side,

and hence these act as polynomials by the previous step. Since both left hand side $s_N(r_N - s_N)D_{12}(r)$, and $s_N r_N D_{12}(r)$ have polynomial action, so does their difference $-s_N^2 D_{12}(r)$. Because s_N is a nonzero constant this implies that $D_{12}(r)$ has polynomial action. Again by considering a change of indices, this proves that $D_{ab}(r)$ has polynomial action on the region $\mathcal{R}_0 = \bigcap_i \{r_i \neq 0\}$. It remains to show that $D_{ab}(r)$ has polynomial action for $(r_a, r_b) \neq (0, 0)$.

Let $r, s \in \mathbb{Z}^N$ where s is constant. Rearranging the bracket formula gives

$$(s_a r_b - r_a s_b)D_{ab}(r) = (s_a r_b - r_a s_b)(D_{ab}(s) + D_{ab}(r - s)) - [D_{ab}(s), D_{ab}(r - s)].$$

On the right hand side $D_{ab}(s)$ is constant in r and, by what was just shown, $D_{ab}(r - s)$ has polynomial action in the region $\mathcal{R}_s = \bigcap_i \{r_i \neq s_i\}$. Thus there is an $\text{End}(U)$ -valued polynomial T such that $(s_a r_b - r_a s_b)D_{ab}(r) = T(r)$ for $r \in \mathcal{R}_s$. Similarly for $s' \in \mathbb{Z}^N$, there is a polynomial T' such that $(s'_a r_b - r_a s'_b)D_{ab}(r) = T'(r)$ for $r \in \mathcal{R}_{s'}$. Then

$$(s'_a r_b - r_a s'_b)T(r) = (s_a r_b - r_a s_b)T'(r)$$

for $r \in \mathcal{R}_s \cap \mathcal{R}_{s'}$, which implies that $(s_a r_b - r_a s_b)$ is an irreducible factor of $T(r)$. So $(s_a r_b - r_a s_b)D_{ab}(r) = (s_a r_b - r_a s_b)\bar{T}(r)$, for polynomial $\bar{T}$, and when $s_a r_b - r_a s_b \neq 0$, $D_{ab}(r) = \bar{T}(r)$. Thus $D_{ab}(r)$ has polynomial action on the region $\mathcal{R}_s \cap \{s_a r_b \neq s_b r_a\}$. The union of these regions is $\bigcup_s (\mathcal{R}_s \cap \{s_a r_b \neq s_b r_a\}) = \{(r_a, r_b) \neq (0, 0)\}$. Since these regions are defined by deleting a finite number of hyperplanes from $\mathbb{Z}^N$, the intersection of any two contains a cube of arbitrary size. So any two polynomials that agree on the intersection must be equal. Therefore $D_{ab}(r)$ acts by an $\text{End}(U)$ -valued polynomial P_{ab} on the region $\{(r_a, r_b) \neq (0, 0)\}$. The polynomial P_{ab} can be decomposed into a finite sum in powers of r so that

$$D_{ab}(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P_{ab}^{(k)}.$$

for all $r \in \mathbb{Z}^N$ with $\{(r_a, r_b) \neq (0, 0)\}$. □

PROPOSITION 3.10. *Let $N \geq 3$ and $J = A_N \otimes U$ a module in category $\mathcal{J}$. Then for $a, b \in \{1, \dots, N\}$, $a \neq b$, $D_{ab}(r)$ acts on U by*

$$D_{ab}(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P_{ab}^{(k)}$$

for all $r \in \mathbb{Z}^N$, where the $P_{ab}^{(k)} \in \text{End}(U)$ do not depend on r , and the summation is finite. In addition $P_{ab}^{(k)} = 0$ when $k_a = k_b = 0$.

PROOF. Since $D_{ab}(r) = 0$ when $r_a = r_b = 0$ by definition, it follows from Lemma 3.9 that $D_{ab}(r)$ may be expressed as

$$D_{ab}(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P_{ab}^{(k)} - \delta_{r_a, 0} \delta_{r_b, 0} \sum_{\substack{k \in \mathbb{Z}_{\geq 0}^N \\ k_a = k_b = 0}} \frac{r^k}{k!} P_{ab}^{(k)}$$

which holds for all $r \in \mathbb{Z}^N$. Now substitute the expression above into the equation $r_c D_{ab}(r) + r_a D_{bc}(r) + r_b D_{ca}(r) = 0$ on the region $r_i \neq 0$ for all $i \in \{1, \dots, N\}$. The terms with delta functions vanish leaving

$$(7) \quad r_c \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P_{ab}^{(k)} + r_a \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P_{bc}^{(k)} + r_b \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P_{ca}^{(k)} = 0.$$

Extracting the coefficient of $r_c r^k$ for $k \in \mathbb{Z}_{\geq 0}^N$ with $k_a = k_b = 0$ yields that

$$\frac{1}{k!} P_{ab}^{(k)} = 0.$$

This shows that the terms with delta functions are not necessary, and therefore

$$D_{ab}(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P_{ab}^{(k)}$$

for all $r \in \mathbb{Z}^N$. □

3. Classification

Consider the Lie algebra of derivations of polynomials in N variables,

$$\text{Der}(\mathbb{C}[x_1, \dots, x_N]) = \text{Span}_{\mathbb{C}} \left\{ x^k \frac{\partial}{\partial x_a} \mid a \in \{1, \dots, N\}, k \in \mathbb{Z}_{\geq 0}^N \right\},$$

and its subalgebra consisting of divergence zero elements,

$$S_N = \text{Span}_{\mathbb{C}} \{ S_{ab}(k) \mid a, b \in \{1, \dots, N\}, k \in \mathbb{Z}_{\geq 0}^N \},$$

where $S_{ab}(k) = k_b x^{k-e_b} \frac{\partial}{\partial x_a} - k_a x^{k-e_a} \frac{\partial}{\partial x_b}$. Its Lie bracket is given by

$$\begin{aligned} [S_{ab}(q), S_{cd}(k)] &= q_b k_c S_{ad}(q+k-e_b-e_c) - q_b k_d S_{ac}(q+k-e_b-e_d) \\ &\quad - q_a k_c S_{bd}(q+k-e_a-e_c) + q_a k_d S_{bc}(q+k-e_a-e_d). \end{aligned}$$

Note that $S_{ab}(e_a) = -\frac{\partial}{\partial x_b}$ and $S_{ab}(e_b) = \frac{\partial}{\partial x_a}$.

For $n \in \mathbb{N}$ let $\mathfrak{L}_n = \text{Span}_{\mathbb{C}} \{ S_{ab}(k) \mid a, b \in \{1, \dots, N\}, |k| = n+2 \}$ so that $S_N = \bigoplus_{i=-1}^{\infty} \mathfrak{L}_i$. The bracket above shows that $[\mathfrak{L}_i, \mathfrak{L}_j] \subset \mathfrak{L}_{i+j}$.

LEMMA 3.11. *In the grading $S_N = \bigoplus_{i=-1}^{\infty} \mathfrak{L}_i$, the component $\mathfrak{L}_0$ is isomorphic to $\mathfrak{sl}_N$ and each $\mathfrak{L}_i$ is an irreducible $\mathfrak{sl}_N$ -module.*

PROOF. To see that $\mathfrak{L}_0$ is isomorphic to $\mathfrak{sl}_N$, identify $x_a \frac{\partial}{\partial x_b}$ with E_{ab} and $x_a \frac{\partial}{\partial x_a} - x_b \frac{\partial}{\partial x_b}$ with elements $E_{aa} - E_{bb}$ of the Cartan subalgebra. Each $\mathfrak{L}_i$ is an $\mathfrak{sl}_N$ -module via the adjoint action of $\mathfrak{L}_0$.

By Weyl's Theorem on complete reducibility and the fact that every finite dimensional simple $\mathfrak{sl}_N$ -module is a highest weight module, it suffices to show that each $\mathfrak{L}_i$ has a unique highest weight vector. In other words the goal is to show that for each i there exists a unique (up to scalar) $v \in \mathfrak{L}_i$ such that $\left[x_a \frac{\partial}{\partial x_b}, v \right] = 0$ for all a, b with $a < b$.

An arbitrary member of $\mathfrak{L}_n$ can be expressed as $\sum_{|m|=n} u_m$ with $u_m = \sum_{j=1}^N C_j x^{m+e_j} \frac{\partial}{\partial x_j}$ where $\sum_{j=1}^N C_j(m_j+1) = 0$, since it has divergence zero. Since $\left[x_a \frac{\partial}{\partial x_a} - x_b \frac{\partial}{\partial x_b}, u_m \right] = (m_a - m_b)u_m$ for all $a, b \in \{1, \dots, N\}$, weight vectors of $\mathfrak{L}_n$ must have the form u_m for some fixed m .

Let u_m be a highest weight vector for $\mathfrak{L}_n$. Since x may only have nonnegative exponents, two cases arise; either $m_j = -1$ for a single index j and $m_i \geq 0$ otherwise, or else all entries of m are nonnegative. The former forces all coefficients

C_i to be zero except for C_j , and hence $u_m = Cx^k \frac{\partial}{\partial x_j}$ with $k_j = 0$. In the latter $u_m = \sum_{j=1}^N C_j x^{m+e_j} \frac{\partial}{\partial x_j}$ with $m_j + 1 > 0$ for each j .

Suppose $u_m = Cx^k \frac{\partial}{\partial x_j}$ with $k_j = 0$. Then $\left[x_a \frac{\partial}{\partial x_b}, u_m \right] = Ck_b x^{k+e_a-e_b} \frac{\partial}{\partial x_j} - \delta_{aj} Cx^{k+e_j} \frac{\partial}{\partial x_b}$. Since $1 \leq a < b \leq N$ it follows that the only u_m of this form annihilated by all raising operators $x_a \frac{\partial}{\partial x_b}$ is $u_m = Cx_1^{n+1} \frac{\partial}{\partial x_N}$.

It remains to show that no vectors of the form $u_m = \sum_{j=1}^N C_j x^{m+e_j} \frac{\partial}{\partial x_j}$ with $m_j + 1 > 0$ for each j , are highest weight vectors. Suppose for $a < b$ that $\left[x_a \frac{\partial}{\partial x_b}, \sum_{j=1}^N C_j x^{m+e_j} \frac{\partial}{\partial x_j} \right] = 0$. The coefficient of $\frac{\partial}{\partial x_b}$ on the right hand side is $(C_b(m_b + 1) - C_a)x^{m+e_a} = 0$. Letting $b = N$ and varying a shows that $C_a = C_N(m_N + 1)$ for $a = 1, \dots, N-1$. Plugging this into the expression for the divergence of u gives

$$\sum_{j=1}^{N-1} C_N(m_N + 1)(m_j + 1) + C_N(m_N + 1) = C_N(m_N + 1) \left(\sum_{j=1}^{N-1} (m_j + 1) + 1 \right),$$

which is zero only when $C_N = 0$ since each $(m_j + 1)$ was assumed to be positive.

Then $C_N = 0$ implies $C_a = 0$ for $a = 1, \dots, N-1$, and thus $u_m = 0$. $\square$

The action of the $P_{ab}^{(k)}$ with $|k| > 1$ will be shown to define a representation of the subalgebra

$$S_N^+ = \text{Span}_{\mathbb{C}} \{ S_{ab}(k) \mid a, b \in \{1, \dots, N\}, k \in \mathbb{Z}_{\geq 0}^N, |k| > 1 \}.$$

PROPOSITION 3.12. *The map $\rho(S_{ab}(k)) = P_{ab}^{(k)} \in \text{End}(U)$ for $|k| > 1$ is a finite dimensional representation of S_N^+ on U for $N \geq 2$.*

PROOF. Using Lemma 3.9 the Lie bracket of $D_{ab}(r)$ with $D_{cd}(s)$ with $r, s \neq 0$ may be expressed

$$[D_{ab}(r), D_{cd}(s)] = \sum_{j, k \in \mathbb{Z}_{\geq 0}^N} \frac{r^j s^k}{j!k!} [P_{ab}^{(j)}, P_{cd}^{(k)}],$$

where the left hand side may be computed

$$\begin{aligned}
(8) \quad & (r_a s_b - r_b s_a) D_{cd}(s) + (r_c s_d - r_d s_c) D_{ab}(r) + r_b s_c D_{ad}(r+s) \\
& - r_b s_d D_{ac}(r+s) - r_a s_c D_{bd}(r+s) + r_a s_d D_{bc}(r+s) \\
& = (r_a s_b - r_b s_a) \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{s^k}{k!} P_{cd}^{(k)} + (r_c s_d - r_d s_c) \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P_{ab}^{(k)} + r_b s_c \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{(r+s)^k}{k!} P_{ad}^{(k)} \\
& - r_b s_d \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{(r+s)^k}{k!} P_{ac}^{(k)} - r_a s_c \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{(r+s)^k}{k!} P_{bd}^{(k)} + r_a s_d \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{(r+s)^k}{k!} P_{bc}^{(k)}.
\end{aligned}$$

Thus $[P_{ab}^{(j)}, P_{cd}^{(k)}]$ is obtained by extracting the coefficient of $\frac{r^j s^k}{j! k!}$ in the expression above. Then for any $j, k \in \mathbb{Z}_{\geq 0}^N$ with $|j|, |k| > 1$, the bracket is given by

$$\begin{aligned}
(9) \quad & [P_{ab}^{(j)}, P_{cd}^{(k)}] = \\
& j_b k_c P_{ad}^{(j+k-e_b-e_c)} - j_b k_d P_{ac}^{(j+k-e_b-e_d)} - j_a k_c P_{bd}^{(j+k-e_a-e_c)} + j_a k_d P_{bc}^{(j+k-e_a-e_d)}.
\end{aligned}$$

Note that the expression on the right will differ if either $|j| \leq 1$ or $|k| \leq 1$. The equation above shows that $\rho(S_{ab}(k)) = P_{ab}^{(k)}$ preserves the Lie bracket of S_N^+ and is therefore a finite dimensional representation on U . $\square$

Since $D_{ab}(r) = -D_{ba}(r)$ it follows that $P_{ab}^{(k)} = -P_{ba}^{(k)}$ for any $k \in \mathbb{Z}_{\geq 0}^N$. A linear relationship for the $P_{ab}^{(k)}$ is seen in (7), and extracting the coefficient on r^k with $k = e_b + e_c$ gives that $P_{ab}^{(e_b)} = P_{ac}^{(e_c)}$. For $N \geq 3$, $P_{ab}^{(0)} = 0$ and $P_{ab}^{(e_i)} = 0$ for $i \neq a, b$.

Consider the Lie algebra spanned by $\left\{ P_{ab}^{(k)} \mid a, b \in \{1, \dots, N\}, k \in \mathbb{Z}_{\geq 0}^N \right\}$. As was noted above, the expression in (9) is valid only when $|j|, |k| > 1$. The remaining brackets are obtained by extracting the coefficient of $\frac{r^j s^k}{j! k!}$ in (8) for appropriate values of j and k . Doing so yields that $[P_{ab}^{(j)}, P_{cd}^{(k)}] = 0$ when either j or k is zero. If $|j| > 1, |k| = 1$ or $|j| = 1, |k| > 1$ the terms on the right hand side of (8) vanish using the relationship in (7) when $N \geq 3$ or cancel directly when $N = 2$. When both $|j| = 1$ and $|k| = 1$ the right hand side of (8) has only terms $P_{ab}^{(0)}$ (for some a and b). In the case $N = 2$, $[P_{ab}^{(e_a)}, P_{ab}^{(e_b)}] = P_{ab}^{(0)}$, and $[P_{ab}^{(e_a)}, P_{ab}^{(e_a)}] = 0$, however when $N \geq 3$, $P_{ab}^{(0)} = 0$, and so $[P_{ab}^{(j)}, P_{cd}^{(k)}] = 0$ when $|j| = 1$ and $|k| = 1$. Thus

for $N \geq 3$ the subset of elements $P_{ab}^{(e_i)}$ spans an abelian algebra with generators $\left\{ P_{12}^{(e_2)}, P_{i1}^{(e_1)} \mid i = 2, \dots, N \right\}$, and in the case $N = 2$ there is a Heisenberg algebra spanned by $\left\{ P_{12}^{(e_1)}, P_{12}^{(e_2)}, P_{12}^{(0)} \right\}$.

Let $\mathfrak{a}_N = \text{Span}_{\mathbb{C}}\{C_i \mid i = 1, \dots, N\}$ be an N dimensional abelian Lie algebra, and $\mathfrak{h} = \text{Span}_{\mathbb{C}}\{X, Y, Z\}$ a three dimensional Heisenberg algebra with central element $Z = [X, Y]$. For $N \geq 3$ the map $\rho(C_a) = P_{ab}^{(e_b)}$ is a finite dimensional representation of $\mathfrak{a}_N$ on U . When $N = 2$ the map $\rho(X) = P_{12}^{(e_2)}, \rho(Y) = P_{21}^{(e_1)}$, and $\rho(Z) = P_{12}^{(0)}$ is a finite dimensional representation of $\mathfrak{h}$ on U . The following theorem considers Lie algebras $S_2^+ \oplus \mathfrak{h}$ and $S_N^+ \oplus \mathfrak{a}_N$. In either case the bracket of $\mathfrak{h}$ or $\mathfrak{a}_N$ with S_N^+ is zero.

Since $[\rho(S_{ab}(e_a + e_b)), \rho(S_{ab}(ne_a))] = n\rho(S_{ab}(ne_a))$ for $n \geq 0$, Lemma 3.3 implies that for some $k_0 \geq 0$, $\rho(S_{ab}(ke_a))$ acts as zero on U for all $k \geq k_0$. The irreducibility in Lemma 3.11 ensures that all of $\mathfrak{L}_k$ acts as zero. So for some $k_0 \geq 0$, $\mathfrak{L}_k$ acts trivially on U for all $k \geq k_0$.

THEOREM 3.13. *Let $\lambda \in \mathbb{C}^N$ and let $\mathcal{J}_\lambda$ be the subcategory of modules in $\mathcal{J}$ supported on $\lambda + \mathbb{Z}^N$.*

- (a) *For $N = 2$ there is an equivalence of categories between the category of finite dimensional modules for $S_2^+ \oplus \mathfrak{h}$ and $\mathcal{J}_\lambda$. This equivalence maps U to $A_2 \otimes U$ where U is a finite dimensional module for $S_2^+ \oplus \mathfrak{h}$. The action of S_2 on $A_2 \otimes U$ is given by $d_a(t^s \otimes u) = (s_a + \lambda_a)t^s \otimes u$ and for $r \neq 0$,*

$$(10) \quad d_{12}(r)(t^s \otimes u) = (r_2 s_1 - r_1 s_2) t^{r+s} \otimes u \\ + t^{r+s} \otimes \sum_{\substack{k \in \mathbb{Z}_{\geq 0}^2 \\ |k| > 1}} \frac{r^k}{k!} \rho(S_{12}(k)) u + t^{r+s} \otimes (r_1 \rho(X) - r_2 \rho(Y) + \rho(Z)) u.$$

- (b) *For $N \geq 3$, there is an equivalence of categories between the category of finite dimensional modules for $S_N^+ \oplus \mathfrak{a}_N$ and $\mathcal{J}_\lambda$. This equivalence maps U to $A_N \otimes U$ where U is a finite dimensional module for $S_N^+ \oplus \mathfrak{a}_N$. The*

action of $\mathcal{S}_N$ on $A_N \otimes U$ is given by $d_a(t^s \otimes u) = (s_a + \lambda_a)t^s \otimes u$ and

$$(11) \quad d_{ab}(r)(t^s \otimes u) = (r_b s_a - r_a s_b) t^{r+s} \otimes u \\ + t^{r+s} \otimes \sum_{\substack{k \in \mathbb{Z}_{\geq 0}^N \\ |k| > 1}} \frac{r^k}{k!} \rho(S_{ab}(k)) u + t^{r+s} \otimes (r_b \rho(C_a) - r_a \rho(C_b)) u.$$

PROOF. Let J be a module in $\mathcal{J}_\lambda$. As was noted at the end of Section 1, the module J may be identified with $A_N \otimes U$ where U is the weight space J_λ . Then (J1) with (J3) yields $d_a(t^s \otimes u) = (s_a + \lambda_a)t^s \otimes u$ for $u \in U$.

Section 2 showed that the action of $d_{ab}(r) \in \mathcal{S}_N$ on J is determined by its restriction to U and is given by an $\text{End}(U)$ -valued polynomial in r . When $r \neq 0$,

$$d_{ab}(r)(t^s \otimes u) = (r_b s_a - s_b r_a) t^{r+s} \otimes u + t^{r+s} \otimes \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P_{ab}^{(k)} u,$$

and for $r = 0$, $d_{ab}(r) = 0$ and thus acts trivially. Proposition 3.12 and the remarks that follow show that U is a finite dimensional $S_2^+ \oplus \mathfrak{h}$ -module when $N = 2$ and U is a finite dimensional $S_N^+ \oplus \mathfrak{a}_N$ -module when $N \geq 3$. The actions in (10) and (11) follow.

Conversely let U be a finite dimensional module for $S_2^+ \oplus \mathfrak{h}$. Identify the elements of $S_2^+ \oplus \mathfrak{h}$ with the $P_{12}^{(k)}$ as above and let

$$D_{12}(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^2} \frac{r^k}{k!} P_{12}^{(k)}.$$

This sum is finite due to the discussion just before the theorem. The Lie bracket of $S_2^+ \oplus \mathfrak{h}$ yields the commutator relations for the $D_{12}(r)$ operators via equation (8). The Lie bracket of the $D_{12}(r)$ along with the action of $d_{12}(r)$ given in (3) recovers the commutator relations in $\mathcal{S}_2$. Thus $A_2 \otimes U$ is an $\mathcal{S}_2$ -module.

The fact that a finite dimensional $S_N^+ \oplus \mathfrak{a}_N$ -module U yields a finite dimensional module $A_N \otimes U$ for $\mathcal{S}_N$ follows in a similar fashion for the $N \geq 3$ case. $\square$

4. Irreducible Tensor Modules

This section considers simple modules from category $\mathcal{J}$. Note that in a finite dimensional irreducible representation of the Heisenberg algebra, the central element must act by zero. Hence $P_{12}^{(0)} = 0$ in the case $N = 2$ and so the Heisenberg algebra $\mathfrak{h}$ used above gets replaced with the two dimensional abelian algebra $\mathfrak{a}_2$. A further simplification found in irreducible modules is that the action of S_N^+ becomes the action of $\mathfrak{sl}_N$, its degree zero component from the grading in Lemma 3.11. The following will be used to show this (cf. [7] Lemma 2.4 and [8] Lemma 9.13).

LEMMA 3.14. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$ with solvable radical $\text{Rad}(\mathfrak{g})$. Then $[\mathfrak{g}, \text{Rad}(\mathfrak{g})]$ acts trivially on any finite dimensional irreducible $\mathfrak{g}$ -module.*

As noted above, there exists k_0 such that $\mathfrak{L}_k$ acts as zero for all $k \geq k_0$, and so the ideal $I = \bigoplus_{k \geq k_0} \mathfrak{L}_k$ must also act trivially. To apply Lemma 3.14 consider the finite dimensional Lie algebra $\mathfrak{g} = S_N^+/I \oplus \mathfrak{a}_N$ and its action on U . Since $[\mathfrak{L}_n, \mathfrak{L}_m] \subset \mathfrak{L}_{n+m}$ it follows that $\text{Rad}(\mathfrak{g}) = (\bigoplus_{n > 0} \mathfrak{L}_n)/I \oplus \mathfrak{a}_N$ and hence $[\mathfrak{g}, \text{Rad}(\mathfrak{g})] = (\bigoplus_{n > 0} \mathfrak{L}_n)/I$ acts trivially. Therefore the ideal $\bigoplus_{n > 0} \mathfrak{L}_n$ of $S_N^+ \oplus \mathfrak{a}_N$ acts trivially on a simple module from category $\mathcal{J}$.

THEOREM 3.15. *Let $\lambda \in \mathbb{C}^N$ and let $\mathcal{J}_\lambda$ be the subcategory of modules in $\mathcal{J}$ supported on $\lambda + \mathbb{Z}^N$. For $N \geq 2$ there is a one-to-one correspondence between the finite dimensional irreducible modules for $\mathfrak{sl}_N \oplus \mathfrak{a}_N$ and the irreducible modules in $\mathcal{J}_\lambda$. This correspondence maps a finite dimensional irreducible module V for $\mathfrak{sl}_N \oplus \mathfrak{a}_N$ to $A_N \otimes V$. The action of $\mathcal{S}_N$ on $A_N \otimes V$ is given by $d_a(t^s \otimes u) = (s_a + \lambda_a)t^s \otimes u$ and*

$$\begin{aligned} d_{ab}(r)(t^s \otimes u) &= (r_b(s_a + \mu_a) - r_a(s_b + \mu_b))t^{r+s} \otimes u \\ &\quad + t^{r+s} \otimes \sum_{\substack{i=1 \\ i \neq a}}^N r_i r_a \varphi(E_{ia})u - \sum_{\substack{i=1 \\ i \neq b}}^N r_i r_b \varphi(E_{ib})u + r_a r_b \varphi(E_{aa} - E_{bb})u, \end{aligned}$$

where $\mu_a, \mu_b \in \mathbb{C}$ are the action of $C_a, C_b \in \mathfrak{a}_N$, and φ is a representation of $\mathfrak{sl}_N$.

PROOF. The correspondence is given in Theorem 4.6. By Lemma 3.14 and the discussion above, the ideal $I = \text{Span}\{S_{ab}(j) \mid a, b \in \{1, \dots, N\}, |j| > 2\}$ acts trivially on V . Then $S_N^+/I \cong \mathfrak{sl}_N$ and so the action of $\mathcal{L}_0$ in (11) is represented by elements of $\mathfrak{sl}_N$ as seen in Lemma 3.11. By Schur's Lemma, elements of $\mathfrak{a}_N$ act by scalars and so $\rho(C_a)$ and $\rho(C_b)$ become μ_a and μ_b respectively in (11). $\square$

Theorem 3.15 shows that the irreducible tensor modules considered in Chapter 2 exhaust all irreducible $\mathcal{S}_N$ -modules in category $\mathcal{J}$ when $\mu = \lambda$.

CHAPTER 4

Category $\mathcal{J}$ for Hamiltonian Vector Fields on a Torus

1. Category $\mathcal{J}$ for $\mathcal{H}_N$

Recall the definition of the Lie algebra of Hamiltonian vector fields on an N dimensional torus from Chapter 1 Section 3. Category $\mathcal{J}$ for $\mathcal{H}_N$ is defined as follows:

DEFINITION 4.1. *Let $N \geq 2$ with N even. An $\mathcal{H}_N$ -module J belongs to category $\mathcal{J}$ if the following properties hold:*

- (J1) *The action of d_i on J is diagonalizable for all $i \in \{1, \dots, N\}$.*
- (J2) *Module J is a free A_N -module of finite rank.*
- (J3) *For any $X \in \mathcal{H}_N$, $f \in A_N$ and $v \in J$, $X(fv) = (Xf)v + f(Xv)$.*

Any submodule of $J \in \mathcal{J}$ must be invariant under the actions of both A_N and $\mathcal{H}_N$. By (J2) any module in $\mathcal{J}$ is a finite direct sum of indecomposable modules, hence the goal is to classify indecomposable modules $J \in \mathcal{J}$. Property (J1) decomposes J into weight spaces, $J = \bigoplus_{\lambda \in \mathbb{C}^N} J_\lambda$ where $J_\lambda = \{v \in J | d_i v = \lambda_i v\}$. Note that $h(r)J_\lambda \subset J_{\lambda+r}$ since,

$$d_i(h(r)v) = h(r)d_i v + [d_i, h(r)]v = (\lambda_i + r_i)h(r)v$$

for $v \in J_\lambda$. Similarly by (J3) $t^r J_\lambda \subset J_{\lambda+r}$. These two relations partition the weights of J into $\mathbb{Z}^N$ -cosets of $\mathbb{C}^N$, and decompose J into a direct sum of submodules, each corresponding to a distinct coset. Thus if J is indecomposable its set of weights is one such coset $\lambda + \mathbb{Z}^N$ for $\lambda \in \mathbb{C}^N$ and $J = \bigoplus_{r \in \mathbb{Z}^N} J_{\lambda+r}$. Denote by $\mathcal{J}_\lambda$, the

subcategory of $\mathcal{J}$ supported on $\lambda + \mathbb{Z}^N$ for a fixed $\lambda \in \mathbb{C}^N$ and from now on assume that $J \in \mathcal{J}_\lambda$.

Let $V = J_\lambda$. The invertible map $t^r : V \rightarrow J_{\lambda+r}$ identifies all weight spaces with V and since J is a free module for the associative algebra A_N it follows that any basis for V is also basis for J viewed as a free A_N -module. The finite rank condition of (J2) implies that V must be finite dimensional. This yields that $J \cong A_N \otimes V$ as vector spaces. Homogeneous elements of J will be denoted $t^s \otimes v$, for $s \in \mathbb{Z}^N, v \in V$.

The action of $h(r)$ on J maps $1 \otimes V \rightarrow t^r \otimes V$. This induces an endomorphism $H(r)$ on V given by

$$H(r)v = (t^{-r} \circ h(r))v.$$

The action in (J3) can then be expressed as

$$(12) \quad h(r)(t^s \otimes v) = \sum_{i=1}^m (r_{m+i}s_i - r_i s_{m+i}) t^{r+s} \otimes v + t^{r+s} \otimes H(r)v.$$

Thus the action of $H(r)$ on V determines the action of $h(r)$ on J .

The Lie bracket for the $H(r)$ terms is given by

$$\begin{aligned} & [H(r), H(s)] \\ &= [t^{-r}h(r), t^{-s}h(s)] \\ &= t^{-r}(h(r)(t^{-s}))h(s) - t^{-s}(h(s)(t^{-r}))h(r) + t^{-r-s}[h(r), h(s)] \\ &= t^{-r} \left(\sum_{i=1}^m r_{m+i}(-s_i) - r_i(-s_{m+i}) \right) t^{r-s}h(s) \\ &\quad - t^{-s} \left(\sum_{i=1}^m s_{m+i}(-r_i) - s_i(-r_{m+i}) \right) t^{s-r}h(r) \\ &\quad + t^{-r-s} \sum_{i=1}^m (r_{m+i}s_i - r_i s_{m+i}) h(r+s) \\ &= \sum_{i=1}^m (r_i s_{m+i} - r_{m+i} s_i) (H(r) + H(s) - H(r+s)). \end{aligned}$$

2. Polynomial Action

The concept of a polynomial module was introduced in [1] and then generalized in [5]. Polynomial modules were used in [3] for classifying category $\mathcal{J}$ modules for $\mathcal{W}_N$. This technique was applied in the previous chapter to the $\mathcal{S}_N$ -modules and will be used again here. The main idea is to show that $H(r)$ acts on V by an $\text{End}(V)$ -valued polynomial in r with constant term zero. In this case it is said that $H(r)$ has *polynomial action*. Specifically,

PROPOSITION 4.2. *Let $N \geq 2$ (with N even) and let $J = A_N \otimes V$ be an $\mathcal{H}_N$ -module in category $\mathcal{J}$. Then $H(r)$ acts on V by*

$$H(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P^{(k)} - \delta_{r,0} P^{(0)},$$

for all $r \in \mathbb{Z}^N$, where $P^{(r)} \in \text{End}(V)$ and do not depend on r , and the summation is finite.

The Kronecker delta function is there to ensure that $H(0) = 0$. This will be proven by induction on N with the base case $N = 2$ already shown Lemma 3.8 since $\mathcal{H}_2 \cong \mathcal{S}_2$. Stated in the context of Hamiltonian vector fields this is:

LEMMA 4.3. *Let $N = 2$ and $J = A_2 \otimes V$ be a module for $\mathcal{H}_2$ in category $\mathcal{J}$. Then $H(r)$ acts on V by*

$$H(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^2} \frac{r^k}{k!} P^{(k)} - \delta_{r,0} P^{(0)},$$

for all $r \in \mathbb{Z}^2$, where $P^{(r)} \in \text{End}(V)$ and do not depend on r , and the summation is finite.

Recall Lemma 3.6, that two multivariate polynomials are equal if they agree on a cube of arbitrary size. This was used repeatedly in Chapter 3, and it will be used in the proof below.

PROOF OF PROPOSITION 4.2. Proceed by induction on $N = 2m$ with basis of induction given by Lemma 4.3. Let $\tilde{r}_i = r - r_i e_i - r_{m+i} e_{m+i}$ for $i \in \{1, \dots, m\}$. By the induction hypothesis, $H(\tilde{r}_i)$ acts by an $\text{End}(V)$ -valued polynomial in r for $\tilde{r}_i \neq 0$. Let $k \in \{1, \dots, m\}$ with $k \neq i$. Then $H(e_i + e_k)$ and $H(e_i + e_{m+k})$ also act by an $\text{End}(V)$ -valued polynomials in r since $e_i + e_k$ and $e_i + e_{m+k}$ are constant. Then

$$[H(\tilde{r}_i), H(e_i + e_k)] = -r_{m+k}(H(\tilde{r}_i) + H(e_i + e_k) - H(\tilde{r}_i + e_i + e_k)),$$

and

$$[H(\tilde{r}_i), H(e_i + e_{m+k})] = r_k(H(\tilde{r}_i) + H(e_i + e_{m+k}) - H(\tilde{r}_i + e_i + e_{m+k})).$$

Rearranging these equations gives

$$r_{m+k}H(\tilde{r}_i + e_i + e_k) = r_{m+k}(H(\tilde{r}_i) + H(e_i + e_k)) + [H(\tilde{r}_i), H(e_i + e_k)],$$

and

$$r_kH(\tilde{r}_i + e_i + e_{m+k}) = r_k(H(\tilde{r}_i) + H(e_i + e_{m+k})) - [H(\tilde{r}_i), H(e_i + e_{m+k})],$$

so that the terms on the right side act by an $\text{End}(V)$ -valued polynomial in r for $\tilde{r}_i \neq 0$. Thus $r_{m+k}H(\tilde{r}_i + e_i + e_k)$ and $r_kH(\tilde{r}_i + e_i + e_{m+k})$ have polynomial action when $\tilde{r}_i \neq 0$, and hence for all $r \in \mathbb{Z}^N$. It can be shown in a similar fashion that both $r_{m+k}H(\tilde{r}_i + e_{m+i} + e_k)$ and $r_kH(\tilde{r}_i + e_{m+i} + e_{m+k})$ have polynomial action for all r . This holds for all $i \neq k$, and in particular it follows that $r_iH(r_i e_i + r_{m+i} e_{m+i} + e_{m+i} + e_{m+k})$ has polynomial action for all r .

Now consider the Lie bracket

$$\begin{aligned} & [r_{m+k}H(\tilde{r}_i + e_i + e_k), r_iH(r_i e_i + r_{m+i} e_{m+i} + e_{m+i} + e_{m+k})] \\ &= r_{m+k}r_i(r_{m+i} + 1)(r_k + 1)(H(\tilde{r}_i + e_i + e_k) \\ & \quad + H(r_i e_i + r_{m+i} e_{m+i} + e_{m+i} + e_{m+k}) - H(r + e_i + e_{m+i} + e_k + e_{m+k})). \end{aligned}$$

Isolating the term

$$r_{m+k}r_i(r_{m+i}+1)(r_k+1)H(r+e_i+e_{m+i}+e_k+e_{m+k})$$

on the right side shows that it must act by an $\text{End}(V)$ -valued polynomial in r , call it Q_1 , for all $r \in \mathbb{Z}^N$ since all other terms do. Similarly form the Lie brackets

$$\begin{aligned} &[r_k H(\tilde{r}_i + e_i + e_{m+k}), r_i H(r_i e_i + r_{m+i} e_{m+i} + e_{m+i} + e_k)], \\ &[r_{m+k} H(\tilde{r}_i + e_{m+i} + e_k), r_{m+i} H(r_i e_i + r_{m+i} e_{m+i} + e_i + e_{m+k})], \\ &[r_k H(\tilde{r}_i + e_{m+i} + e_{m+k}), r_{m+i} H(r_i e_i + r_{m+i} e_{m+i} + e_i + e_k)] \end{aligned}$$

to show that

$$\begin{aligned} &r_k r_i (r_{m+i} + 1)(r_{m+k} + 1)H(r + e_i + e_{m+i} + e_k + e_{m+k}), \\ &r_{m+k} r_{m+i} (r_i + 1)(r_k + 1)H(r + e_i + e_{m+i} + e_k + e_{m+k}), \\ &r_k r_{m+i} (r_i + 1)(r_{m+k} + 1)H(r + e_i + e_{m+i} + e_k + e_{m+k}) \end{aligned}$$

act by polynomials Q_2, Q_3 , and Q_4 respectively, for all $r \in \mathbb{Z}^N$.

Note that

$$r_k(r_{m+k} + 1)Q_1 = r_{m+k}(r_k + 1)Q_2$$

which implies that both r_{m+k} and $r_k + 1$ are factors of Q_1 since they do not divide $r_k(r_{m+k} + 1)$ (and both r_k and $r_{m+k} + 1$ are factors of Q_2). Comparing Q_1 and Q_3 ,

$$r_{m+i}(r_i + 1)Q_1 = r_i(r_{m+i} + 1)Q_3,$$

and so both r_i and $r_{m+i} + 1$ are factors of Q_1 since they do not divide $r_{m+i}(r_i + 1)$.

Thus

$$Q_1 = r_{m+k}r_i(r_{m+i} + 1)(r_k + 1)P_1$$

for some polynomial P_1 , and hence $H(r+e_i+e_{m+i}+e_k+e_{m+k}) = P_1$ for $r_i, r_{m+k} \neq 0$ and $r_{m+i}, r_k \neq -1$. By continuing this process of equating the Q_i with appropriate linear factors, it can be shown that $H(r+e_i+e_{m+i}+e_k+e_{m+k})$ acts by a polynomial

P_2 on the region $\{r_i, r_k \neq 0\} \cap \{r_{m+i}, r_{m+k} \neq -1\}$, P_3 on the region $\{r_{m+i}, r_{m+k} \neq 0\} \cap \{r_i, r_k \neq -1\}$, and P_4 on the region $\{r_{m+i}, r_k \neq 0\} \cap \{r_i, r_{m+k} \neq -1\}$.

Since these four regions are obtained by deleting a finite number of hyperplanes from $\mathbb{Z}^N$, their intersection contains a cube of arbitrary size. Since P_1, P_2, P_3 and P_4 agree on this cube they must be equal, and so $H(r + e_i + e_{m+i} + e_k + e_{m+k})$ has polynomial action on the union of the four regions, which is $\{(r_i, r_{m+i}) \neq (0, 0), (-1, -1)\} \cap \{(r_k, r_{m+k}) \neq (0, 0), (-1, -1)\}$. Shifting coordinates, this implies that $H(r)$ acts by polynomial P on the region

$$\mathcal{R} = \{(r_i, r_{m+i}) \neq (0, 0), (1, 1)\} \cap \{(r_k, r_{m+k}) \neq (0, 0), (1, 1)\}.$$

What remains is to extend this region to all $r \in \mathbb{Z}^N \setminus \{0\}$.

Let $\mathcal{L}$ be the Lie algebra spanned by elements $H(r)$ for $r \in \mathbb{Z}^N$, and consider the automorphisms $\sigma_i : \mathcal{L} \rightarrow \mathcal{L}$ given by

$$\sigma_i(H(r)) = H(r_1, \dots, -r_i, \dots, -r_{m+i}, \dots, r_N),$$

for $i \in \{1, \dots, m\}$. The region $\mathcal{R}$ may be expanded by considering automorphisms σ_i of $\mathcal{L}$ since whatever was proven for $\mathcal{L}$ will also be true of its image under σ_i . Thus $\sigma_i(H(r))$ acts by polynomial $\bar{P}$ on $\mathcal{R}$ which implies that $H(r)$ acts by $\bar{P}$ on $\{(r_i, r_{m+i}) \neq (0, 0), (-1, -1)\} \cap \{(r_k, r_{m+k}) \neq (0, 0), (1, 1)\}$. Lemma 3.6 may be applied to the intersection of these regions, showing that $P = \bar{P}$ there, and hence $H(r)$ acts by P on their union $\mathcal{R}' = \{(r_i, r_{m+i}) \neq (0, 0)\} \cap \{(r_k, r_{m+k}) \neq (0, 0), (1, 1)\}$. In the same way, applying σ_k to $\mathcal{R}$ yields that $H(r)$ acts by P on the region $\mathcal{R}_{i,k} = \{(r_i, r_{m+i}) \neq (0, 0)\} \cap \{(r_k, r_{m+k}) \neq (0, 0)\}$.

Now apply Lemma 3.6 to the intersection of regions $\mathcal{R}_{i,k}$ for various k to show that $H(r)$ acts by P on

$$\mathcal{R}_i = \bigcup_{\substack{k=1 \\ k \neq i}}^m \mathcal{R}_{i,k} = \{(r_i, r_{m+i}) \neq (0, 0)\} \cap \{\tilde{r}_i \neq 0\}.$$

Applying the same lemma again to these regions shows that $H(r)$ acts by P on

$$\bigcup_{i=1}^m \mathcal{R}_i = \bigcap_{i=1}^m \{\tilde{r}_i \neq 0\}.$$

Fix $s \in \mathbb{Z}^N$ and let $\bar{s}_k = s_k e_k + s_{m+k} e_{m+k}$ for $k \in \{1, \dots, m\}$. Further assume that $\bar{s}_k \neq 0$. Then

$$[H(\bar{s}_k), H(r - \bar{s}_k)] = (s_k r_{m+k} - s_{m+k} r_k)(H(\bar{s}_k) + H(r - \bar{s}_k) - H(r))$$

and so

$$\begin{aligned} (s_k r_{m+k} - s_{m+k} r_k)H(r) &= \\ &= (s_k r_{m+k} - s_{m+k} r_k)(H(\bar{s}_k) + H(r - \bar{s}_k)) - [H(\bar{s}_k), H(r - \bar{s}_k)]. \end{aligned}$$

From what was proven above, all operators on the right side of this equation have polynomial action on $\bigcap_{i=1}^m \{\tilde{r}_i \neq \bar{s}_k\}$, and hence $(s_k r_{m+k} - s_{m+k} r_k)H(r)$ acts by polynomial T on this region. Similarly for $s' \in \mathbb{Z}^N$, $(s'_k r_{m+k} - s'_{m+k} r_k)H(r)$ acts by polynomial T' on $\bigcap_{i=1}^m \{\tilde{r}_i \neq \bar{s}'_k\}$ and on the intersection of these regions

$$(s'_k r_{m+k} - s'_{m+k} r_k)T = (s_k r_{m+k} - s_{m+k} r_k)T',$$

implying that $(s_k r_{m+k} - s_{m+k} r_k)$ is a factor of T . Thus $H(r)$ has polynomial action when $(s_k r_{m+k} - s_{m+k} r_k) \neq 0$ and hence has polynomial action on the region

$$\mathcal{T}_{s,k} = \{s_k r_{m+k} \neq s_{m+k} r_k\} \cap \left(\bigcap_{i=1}^m \{\tilde{r}_i \neq \bar{s}_k\} \right).$$

Again use Lemma 3.6 to patch together the regions which $H(r)$ acts by polynomial. First

$$\left(\bigcap_{i=1}^m \{\tilde{r}_i \neq 0\} \right) \cup \mathcal{T}_{s,k} = \{(r_k, r_{m+k}) \neq (0, 0)\} \cap \{r \neq \bar{s}_k\}.$$

Then

$$\bigcup_{s \in \mathbb{Z}^N} \left(\bigcap_{i=1}^m \{\tilde{r}_i \neq 0\} \right) \cup \mathcal{T}_{s,k} = \{(r_k, r_{m+k}) \neq (0, 0)\},$$

and

$$\bigcup_{k=1}^m \{(r_k, r_{m+k}) \neq (0, 0)\} = \{r \neq 0\}.$$

Thus $H(r)$ acts by $\text{End}(V)$ -valued polynomial in r for all $r \neq 0$. To include $r = 0$ and to accommodate the fact that $H(0) = 0$, a delta function is added, and so $H(r)$ may be expressed

$$H(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P^{(k)} - \delta_{r,0} P^{(0)},$$

for all $r \in \mathbb{Z}^N$, where $P^{(r)} \in \text{End}(V)$, and the sum is finite. $\square$

3. Classification

The result of the previous section is that the action of the $H(r)$ operators are now in terms of the $P^{(k)}$, $k \in \mathbb{Z}_{\geq 0}^N$. Following the idea of Chapter 3, the Lie brackets between the operators $P^{(k)}$ are computed below and show that these operators realize a representation of a familiar Lie algebra. This will yield a classification of the modules in category $\mathcal{J}$.

To compute the Lie bracket $[P^{(j)}, P^{(k)}]$ use Proposition 4.2 and consider a region where $r, s \neq 0$ so that

$$(13) \quad [H(r), H(s)] = \sum_{j \in \mathbb{Z}_{\geq 0}^N} \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^j s^k}{j! k!} [P^{(j)}, P^{(k)}].$$

The left side of this evaluates to

$$\begin{aligned} & \sum_{i=1}^m (r_i s_{m+i} - r_{m+i} s_i) (H(r) + H(s) - H(r+s)) \\ &= \sum_{i=1}^m (r_i s_{m+i} - r_{m+i} s_i) \left(\sum_{j \in \mathbb{Z}_{\geq 0}^N} \frac{r^j}{j!} P^{(j)} + \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{s^k}{k!} P^{(k)} - \sum_{\ell \in \mathbb{Z}_{\geq 0}^N} \frac{(r+s)^\ell}{\ell!} P^{(\ell)} \right). \end{aligned}$$

Then $[P^{(j)}, P^{(k)}]$ is obtained by extracting the coefficient of $\frac{r^j s^k}{j! k!}$ above for various j and k . For $|j|, |k| > 1$

$$[P^{(j)}, P^{(k)}] = \sum_{i=1}^m (j_{m+i} k_i - j_i k_{m+i}) P^{(j+k-e_i-e_{m+i})}.$$

If $|j| \leq 1$ then $[P^{(j)}, P^{(k)}] = 0$ for all k with the exception that $[P^{(e_i)}, P^{(e_{m+i})}] = P^{(0)}$ (and the usual anticommutativity property holds).

Consider the Lie algebra of derivations of polynomials in N variables,

$$\text{Der}(\mathbb{C}[x_1, \dots, x_N]) = \text{Span}_{\mathbb{C}} \left\{ x^r \frac{\partial}{\partial x_a} \middle| a \in \{1, \dots, N\}, r \in \mathbb{Z}_{\geq 0}^N \right\}.$$

This has as a subalgebra the Hamiltonian Lie algebra given by

$$\mathfrak{H}_N = \text{Span}_{\mathbb{C}} \left\{ X(r) = \sum_{i=1}^m r_{m+i} x^{r-e_{m+i}} \frac{\partial}{\partial x_i} - r_i x^{r-e_i} \frac{\partial}{\partial x_{m+i}} \middle| r \in \mathbb{Z}_{\geq 0}^N \right\},$$

with Lie bracket

$$[X(r), X(s)] = \sum_{i=1}^m (r_{m+i} s_i - r_i s_{m+i}) X(r + s - e_i - e_{m+i}).$$

(The notation $\mathfrak{H}_N$ is used in place of H_N from the introduction so as not to confuse it with operators $H(r)$).

Let $\mathfrak{H}_N^+ = \text{Span}_{\mathbb{C}} \{X(r) \in \mathfrak{H}_N \mid |r| > 1\}$ and let $\mathfrak{h}_N$ be the $N+1$ dimensional Heisenberg algebra $\text{Span}_{\mathbb{C}} \{p^i, q^i, \mathfrak{c} \mid i \in \{1, \dots, m\}\}$ with $[p^i, q^i] = \mathfrak{c}$ and all other brackets zero. The bracket for the $P^{(k)}$ operators found above demonstrates the following.

PROPOSITION 4.4. *The map $\rho : \mathfrak{H}_N^+ \oplus \mathfrak{h}_N \rightarrow \text{End}(V)$ given by*

$$\rho(X(r)) = P^{(r)} \text{ for } |r| > 1,$$

$$\rho(p^i) = P^{(e_i)},$$

$$\rho(q^i) = P^{(e_{m+i})},$$

$$\rho(\mathfrak{c}) = P^{(0)},$$

is a finite dimensional representation of $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$ on V for $N = 2m \geq 2$.

PROPOSITION 4.5. *Let $\mathfrak{L}_n = \text{Span}_{\mathbb{C}} \{X(r) \in \mathfrak{H}_N \mid |r| = n+2\}$. In the grading $\mathfrak{H}_N = \bigoplus_{n=-1}^{\infty} \mathfrak{L}_n$, component $\mathfrak{L}_0$ is isomorphic to the symplectic algebra $\mathfrak{sp}_N$ and each $\mathfrak{L}_n$ is an irreducible $\mathfrak{sp}_N$ -module.*

PROOF. Let E_{ab} be the matrix with a 1 in entry (a, b) and zeros elsewhere. An isomorphism between $\mathfrak{L}_0$ and $\mathfrak{sp}_N$ is obtained by identifying basis elements

$$x_i \frac{\partial}{\partial x_j} - x_{m+j} \frac{\partial}{\partial x_{m+i}}, x_i \frac{\partial}{\partial x_{m+j}} + x_j \frac{\partial}{\partial x_{m+i}}, x_{m+i} \frac{\partial}{\partial x_j} + x_{m+j} \frac{\partial}{\partial x_i},$$

of $\mathfrak{L}_0$ with $E_{i,j} - E_{m+j,m+i}, E_{i,m+j} + E_{j,m+i}, E_{m+i,j} + E_{m+j,i}$ respectively from $\mathfrak{sp}_N$, where $i, j \in \{1, \dots, m\}$. Then each $\mathfrak{L}_n$ is an $\mathfrak{sp}_N$ -module via the adjoint action of $\mathfrak{L}_0$. The Cartan subalgebra of $\mathfrak{sp}_N$ is spanned by diagonal elements $E_{i,i} - E_{m+i,m+i}$, and raising operators (positive root spaces) are spanned by $E_{i,m+i}, E_{i,j} - E_{m+j,m+i}, E_{i,m+j} + E_{j,m+i}$ for $i < j$.

Every finite dimensional simple $\mathfrak{sp}_N$ -module is a highest weight module, and so by Weyl's Theorem on complete reducibility it suffices to show that there exists a unique (up to scalar) highest weight vector in each $\mathfrak{L}_i$. This requires showing that there is a unique $v \in \mathfrak{L}_n$ which is annihilated by the adjoint action of all raising operators. Elements $X(r)$ of $\mathfrak{L}_n$ are weight vectors as $\left[x_i \frac{\partial}{\partial x_i} - x_{m+i} \frac{\partial}{\partial x_{m+i}}, X(r) \right] = (r_i - r_{m+i})X(r)$ for all i , and any scalar multiple of $X(ne_1) = -x_1^n \frac{\partial}{\partial x_{m+1}}$ is a highest weight vector of $\mathfrak{L}_n$ since for any $i < j$,

$$\begin{aligned} \left[x_i \frac{\partial}{\partial x_{m+i}}, x_1^n \frac{\partial}{\partial x_{m+1}} \right] &= 0, \\ \left[x_i \frac{\partial}{\partial x_j} - x_{m+j} \frac{\partial}{\partial x_{m+i}}, x_1^n \frac{\partial}{\partial x_{m+1}} \right] &= 0, \\ \left[x_i \frac{\partial}{\partial x_{m+j}} + x_j \frac{\partial}{\partial x_{m+i}}, x_1^n \frac{\partial}{\partial x_{m+1}} \right] &= 0. \end{aligned}$$

It remains to show that there are no other highest weight vectors.

A weight vector $u \in \mathfrak{L}_n$ of weight λ can be expressed as a finite sum $u = \sum_{r \in I} C_r X(r)$ where $X(r)$ has weight λ for each $r \in I$ and $C_r \in \mathbb{C}^N$. The task of finding highest weight vectors amounts to finding all such u with $[X(s), u] = 0$, for $s = 2e_i, e_i + e_{m+j}$ and $e_i + e_j$ with $i < j$. By linear independence

$$0 = [X(2e_i), u] = \sum_{r \in I} C_r (-2r_{m+i}) X(r + e_i - e_{m+i})$$

implies that $C_r = 0$ whenever $r_{m+i} \neq 0$ for some i . Since u is a weight vector this means that $u = X(r)$ for some r with $r_{m+1} = \cdots = r_{2m} = 0$. Then

$$0 = [X(e_i + e_{m+j}), u] = r_j X(r + e_i - e_j)$$

implies $r_j = 0$ for all i, j with $i < j$. This leaves only the case $r = ne_1$ and thus $X(ne_1)$ is the unique highest weight vector of $\mathfrak{L}_n$ up to scalar. $\square$

Notice that $[X(e_i + e_{m+i}), X(ne_i)] = nX(ne_i)$ for $n \geq 0$, suggesting that in the representation of $\mathfrak{sp}_N$ on $\mathfrak{H}_N^+$, there are infinitely many distinct eigenvalues. However the Lemma 3.3 shows that this cannot be the case, meaning that for the representation ρ of Proposition 4.4, there exists k_0 such that $\rho(X(ke_i)) = 0$ for all $k \geq k_0$. The fact that each $\mathfrak{L}_k$ is an irreducible $\mathfrak{sp}_N$ -module implies that all of $\mathfrak{L}_k$ acts trivially on V for $k \geq k_0$.

With this the groundwork is laid for the main result.

THEOREM 4.6. *Let $\lambda \in \mathbb{C}^N$ and let $\mathcal{J}_\lambda$ be the subcategory of modules in $\mathcal{J}$ supported on $\lambda + \mathbb{Z}^N$, where $N = 2m \geq 2$. There is an equivalence of categories between the category of finite dimensional modules for $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$ and $\mathcal{J}_\lambda$. This equivalence maps V to $A_N \otimes V$ where V is a finite dimensional module for $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$. The action of $\mathcal{H}_N$ on $A_N \otimes V$ is given by $d_a(t^s \otimes v) = (s_a + \lambda_a)t^s \otimes v$, and for $r \neq 0$,*

$$(14) \quad h(r)(t^s \otimes v) = \sum_{i=1}^m (r_{m+i}s_i - r_i s_{m+i}) t^{r+s} \otimes v + t^{r+s} \otimes \rho(\mathfrak{c})v \\ + t^{r+s} \otimes \sum_{i=1}^m (r_i \rho(p^i) + r_{m+i} \rho(q^i)) v + t^{r+s} \otimes \sum_{\substack{k \in \mathbb{Z}_{\geq 0}^N \\ |k| > 1}} \frac{r^k}{k!} \rho(X(k))v.$$

PROOF. Following the definition of category $\mathcal{J}$, it was noted that each module in $\mathcal{J}_\lambda$ may be identified with $A_N \otimes V$, where V is any weight space. Combining

(J1) and (J3) gives the action for d_i , and applying Proposition 4.2 to (12) yields

$$h(r)(t^s \otimes v) = \sum_{i=1}^m (r_{m+i}s_i - r_i s_{m+i}) t^{r+s} \otimes v + t^{r+s} \otimes \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P^{(k)} v,$$

for $r \neq 0$, and $h(0) = 0$ otherwise. Proposition 4.4 shows that V is a finite dimensional $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$ -module, and (14) is obtained from the expression above.

On the other hand suppose V is a finite dimensional module for $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$. Identify the elements of $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$ with the $P^{(k)}$ as in Proposition 4.4 and let $H(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P^{(k)}$. This sum has only finitely many nonzero terms because in the decomposition $\mathfrak{H}_N^+ = \bigoplus_{k=0}^{\infty} \mathfrak{L}_k$, there exists k_0 such that $\mathfrak{L}_k$ acts trivially on V for all $k \geq k_0$. The Lie bracket for $H(r)$ terms is obtained from (13), and defining the action of $h(r)$ on $A_N \otimes V$ by (12) recovers the commutator relations for $\mathcal{H}_N$. Thus $A_N \otimes V$ is a finite dimensional $\mathcal{H}_N$ -module. $\square$

4. Irreducible Representations

In a way similar to the divergence zero case of Chapter 3, the action of $\mathcal{H}_N$ simplifies considerably when only simple modules from category $\mathcal{J}$ are considered. The central element of the Heisenberg algebra acts trivially in a finite dimensional irreducible representation, and so an N dimensional abelian algebra $\mathfrak{a}_N = \text{Span}_{\mathbb{C}} \{C_i | i \in \{1, \dots, N\}\}$ replaces $\mathfrak{h}_N$. Lemma 3.14 shows that the action of $\mathfrak{H}_N^+$ reduces to that of $\mathfrak{sp}_N$. To apply this lemma let $\mathfrak{g} = \mathfrak{H}_N^+/I \oplus \mathfrak{a}_N$, where I is the ideal $I = \bigoplus_{k \geq k_0} \mathfrak{L}_k$ which acts trivially on V for some k_0 . Then $\text{Rad}(\mathfrak{g}) = (\bigoplus_{k > 0} \mathfrak{L}_k)/I \oplus \mathfrak{a}_N$ since $[\mathfrak{L}_n, \mathfrak{L}_k] \subset \mathfrak{L}_{n+k}$. So $[\mathfrak{g}, \text{Rad}(\mathfrak{g})] = (\bigoplus_{k > 0} \mathfrak{L}_k)/I$ acts trivially by Lemma 3.14, and hence the ideal $\bigoplus_{k > 0} \mathfrak{L}_k$ must act trivially on V .

THEOREM 4.7. *Let $\lambda \in \mathbb{C}^N$ and let $\mathcal{J}_\lambda$ be the subcategory of modules in $\mathcal{J}$ supported on $\lambda + \mathbb{Z}^N$. For $N = 2m \geq 2$, there is a one-to-one correspondence between the finite dimensional irreducible modules for $\mathfrak{sp}_N \oplus \mathfrak{a}_N$ and the irreducible modules in $\mathcal{J}_\lambda$. This correspondence maps a finite dimensional irreducible module V for $\mathfrak{sp}_N \oplus \mathfrak{a}_N$ to $A_N \otimes V$. The action of $\mathcal{H}_N$ on $A_N \otimes V$ is given by $d_a(t^s \otimes u) =$*

$(s_a + \lambda_a)t^s \otimes v$ and for all r

$$\begin{aligned}
h(r)(t^s \otimes v) &= \sum_{i=1}^m (r_{m+i}(s_i + \mu_{m+i}) - r_i(s_{m+i} + \mu_i))t^{r+s} \otimes v \\
&+ t^{r+s} \otimes \sum_{i=1}^m (r_{m+i}^2 \varphi(E_{m+i,i}) + r_i r_{m+i} \varphi(E_{i,i} - E_{m+i,m+i}) - r_i^2 \varphi(E_{i,m+i})) v \\
&+ t^{r+s} \otimes \sum_{\substack{i,j=1 \\ i < j}}^m (r_{m+i} r_{m+j} \varphi(E_{m+j,i} + E_{m+i,j}) + r_i r_{m+j} \varphi(E_{i,j} - E_{m+i,m+j}) \\
&\quad - r_i r_j \varphi(E_{i,m+j} + E_{j,m+i})) v,
\end{aligned}$$

where μ_i is the action of $C_i \in \mathfrak{a}_N$, and φ is an irreducible representation of $\mathfrak{sp}_N$.

PROOF. Since $\bigoplus_{k>0} \mathfrak{L}_k$ acts trivially on V , the action of $\mathfrak{H}_N^+$ reduces to that of $\mathfrak{L}_0 \cong \mathfrak{sp}_N$. As mentioned above, the Heisenberg algebra is replaced by abelian algebra $\mathfrak{a}_N$, and by Schur's Lemma its elements C_i act by scalars μ_i in an irreducible representation. Thus the action in (14) reduces to that above. Theorem 4.6 gives the correspondence between these sets of modules. $\square$

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