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Application of the Boundary Element Method to Plates on Elastic Foundations

by

CAIFENG HU

**B. Eng., Zhejiang University, China
M. Sc., Zhejiang University, China**

**A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfilment of the requirement
for the degree of
Doctor of Philosophy
in Civil and Environmental Engineering***

**Carleton University
Ottawa, Ontario, Canada K1S 5B6
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to Plates on Elastic Foundations

submitted by
CAIFENG HU, B. Eng., M.Sc.

in partial fulfilment of the requirement for
the degree of Doctor of Philosophy

Name of Chair, Department of Civil and
Environment Engineering

Thesis Supervisor

External Examiner

Carleton University
Ottawa, Ontario, Canada
June, 1994

Abstract

The boundary element method has emerged in the last two decades as a powerful numerical computational tool for the solution of plate bending problems. This thesis develops a general approach for the boundary element analysis of the interaction between elastic thin plates and elastic foundations. Foundation models studied include the elastic half-space, the Winkler foundation and edge supports. Corner lift-off and edge singularity conditions are considered.

In the analysis of plates on elastic foundations, the plate and the foundation are treated as two separate elastic bodies in unilateral contact at the interface. The behaviour of the elastic plate under both the external loads and the contact pressure is modelled by the direct boundary integral equation formulation. The contact pressure between the plate and the foundation is modelled by contact elements. Boussinesq's solution is used to determine the flexibility matrix of the elastic half-space. The displacement compatibility conditions between the plate and the foundation are enforced at contact element nodes.

In the case of plates unilaterally edge supported, the contact pressure between the plate and the elastic edge supports are treated as the plate boundary variables, which are included in the boundary integral equation directly. Thus, no contact elements are required for this particular case of elastic foundation.

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Chapter 1

INTRODUCTION

1.1 General Remarks

The elastic analysis of plates resting on elastic foundations (or plate-foundation interaction) occupies a prominent place in both structural mechanics and engineering practice, and many investigations have been carried out on this subject. The solution of this problem has potential application to the analysis and design of structural foundations, flexible pavement systems, and floating structures[1].

The behaviour of plate-foundation systems is a complex phenomenon. In order to simplify the problem, the plate-foundation system is idealized as a thin elastic plate resting on a linearly elastic foundation. A number of models have been developed for describing the behaviour of the linearly elastic foundations. The three common types of elastic foundation models are Winkler, two-parameter and elastic half-space foundations.

The Winkler model is the oldest (1867) and simplest elastic foundation model. In this model it is assumed that the displacement at any point on the surface of the foundation is directly proportional to the foundation surface pressure acting at that point and is independent of pressure applied at other locations.

Two-parameter models are defined by two independent elastic constants, one of which describes the interaction among the separated springs. Various ways are used to develop the connection among the springs in the two-parameter models[1]; for example, an elastic membrane in the Filonenko-Borodich model, an elastic thin plate in the Hetenyi model and an elastic shear layer in the Pasternak model.

The elastic half-space model describes the foundation as an isotropic, homogeneous and continuous half-space. The stress and displacement in an isotropic elastic half-space due to the action of a concentrated normal force on its surface was first given by Boussinesq.

When an elastic foundation is modelled as a Winkler or two-parameter supporting medium, the interaction problem between the plate and the elastic foundation can be governed by one differential equation, where the plate deflection is the unknown function. In the case of the elastic half-space model, more than one governing equation are required to represent the interaction problem between the plate and the elastic foundation.

Solution techniques employed to solve the interaction problem of a plate and an elastic foundation can be (i) analytical methods, (ii) approximate analytical techniques, and (iii) numerical methods. The analytical treatments of this problem are generally restricted to the cases in which plates have simple geometry and are subjected to simple loading[1,2]. Numerical methods become necessary for a general purpose approach, such as solving problems involving plates of complex shapes, separation between the plate and the foundation, and the influence of neighbouring structures.

A finite element study for the interaction problem of plates on elastic foundations was first done by Cheung and Zienkiewicz[3], followed by many related investigations. A review on the applications of the finite element techniques in the interaction analysis of plates on elastic foundations was given in reference[4].

The boundary element method (BEM) or boundary integral equation method has emerged in the last two decades as a powerful numerical computational tool in many areas of engineering science, and in particular for the solution of plate bending related problems[5,6,7]. There are two types of boundary element methods: the "Direct" method in which the primary unknowns in the final system of discrete

equations are the unspecified boundary variables with concrete physical meaning; and the "Indirect" method in which the primary unknowns are not physical quantities, but rather fictitious boundary variables which by superposition force the real boundary conditions to be satisfied. The boundary element method starts with an integral equation formulation of the problem and then proceeds to obtaining an approximate solution by solving the integral equations numerically. In general the boundary element method has some distinct advantages over the finite element methods, for instance requiring only a surface discretization instead of a discretization of both the interior and the surface of the body and being a continuous solution throughout the interior of the body. The earliest boundary element solutions to the plate foundation interaction problem were given by Katsikadelis and Armenakas[8,9] for the Winkler foundation model by using the direct boundary element method. Up to now, all previous studies on the boundary element analysis of plate elastic foundation interaction problems employed Winkler or Pasternak foundations as the elastic foundation models.

This thesis is concerned with providing a general purpose approach of the direct boundary element method to the interaction problems of finite size thin elastic plates on elastic foundations incorporating elastic half space model, Winkler model and elastic spring supports around the plate edges.

1.2 Previous Work

1.2.1 Mathematical Background

The formulation of the direct boundary element method for the interaction analysis of plates on elastic foundations can be developed by utilizing Green's second formula and the Rayleigh-Green identity which is derived from Green's second formula.

Green's second formula or the Rayleigh-Green identity, which reads

$$\int_{\Omega} (u \nabla^2 v - v \nabla^2 u) d\Omega = \oint_s \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds \quad (1.1)$$

and

$$\int_{\Omega} (u \nabla^4 v - v \nabla^4 u) d\Omega = \oint_s \left(\nabla^2 u \frac{\partial v}{\partial n} - v \frac{\partial \nabla^2 u}{\partial n} + u \frac{\partial \nabla^2 v}{\partial n} - \nabla^2 v \frac{\partial u}{\partial n} \right) ds, \quad (1.2)$$

respectively, transforms the integral over a domain Ω into the integral along the closed boundary s which bounds the domain Ω . The quantities $u = u(x, y, z)$ and $v = v(x, y, z)$ in equations (1.1) and (1.2) are two independent functions which are continuously differentiable up to order four inside the domain and order three on the boundary. The quantity n is the outward normal direction of the boundary s ; ∇^2 is the Laplacian operator and $\nabla^4 = \nabla^2 \nabla^2$ is the biharmonic operator.

Green's second formula[10] and the Rayleigh-Green identity are valid in general for three dimensional problems. Since the plate bending problems are two dimensional, the domain Ω and the boundary s refer to the plate plane geometry and the closed plate edges when equations (1.1) and (1.2) are used to develop the boundary integral equations for plate bending problems. Functions u and v denote the plate deflection solution to be found and a known fundamental solution, respectively. The fundamental solution used most often in thin plate bending is given in the classic text by Timoshenko and Woinowsky-Krieger[21]. This provides

the displacement in a plate of infinite extent subject to a unit concentrated load. By applying partial integration[11], the boundary integral in the right hand side of equation (1.2) can be further derived as integral about the plate boundary values with direct physical meanings, such as plate deflections, normal rotations, normal moments (M_n), equivalent shear forces (V_n) and corner force T . Thus, for a plate with N_c corners, the Rayleigh-Green identity has its alternative form of

$$\begin{aligned} \int_{\Omega} (u \nabla^4 v - v \nabla^4 u) d\Omega = \frac{1}{D} \oint_s \left[\frac{\partial u}{\partial n} M_n(v) + u V_n(v) \right. \\ \left. - M_n(u) \frac{\partial v}{\partial n} - V_n(u) v \right] ds + \frac{1}{D} \sum_{i=1}^{N_c} (u T_i(v) - v T_i(u)) \end{aligned} \quad (1.3)$$

in which D refers to the flexural rigidity of the plate.

1.2.2 Plates on Winkler Foundations

Several researchers, such as Katsikadelis and Armenakas[8,9], Costa and Brebbia[12,13], Bezzine[14], de León and París[15], as well as Sapountzakis and Katsikadelis[16], have employed the boundary element methods for the interaction problems between plates and Winkler type foundations. A majority of this work has dealt with the case of plates in bonded contact with their supports. The solution provided by Sapountzakis and Katsikadelis[16] was concerned with the case of unilateral contact between a plate and a Winkler type foundation.

For an elastic thin plate resting on a Winkler foundation and subjected to a transverse load $q(x, y)$, its deflection $w(x, y)$ is governed by the differential equation

$$D \nabla^4 w + k_f w = q(x, y) \quad (1.4)$$

where coefficient k_f is the stiffness of the Winkler foundation in units of pressure per length. There were basically three approaches developed in the previous work for solving the problems of a plate on a Winkler foundation with BEM, depending

on which fundamental solution is used to establish the integral representation. Each of them has some distinct features.

The first approach of the boundary integral method for plates on Winkler foundations was developed by Katsikadelis and Armenakas[8,9] as well as Costa and Brebbia[12,13]. These researchers used the fundamental solution,

$$w^*(r) = -\frac{l^2}{2\pi D} kci\left(\frac{r}{l}\right) \quad (1.5)$$

with

$$l^4 = \frac{D}{k_f}; \quad r = |P - Q|$$

to construct the boundary integral equations, in which $kci(r/l)$ is a Kelvin function. Physically, this fundamental solution refers to the deflection solution at point Q of an infinite plate resting on a Winkler foundation and subjected to a unit force load at "source" point P . This fundamental solution satisfies the differential equation,

$$D\nabla^4 w^* + k_f w^* = \delta(r) \quad (1.6)$$

where $\delta(r)$ is the Dirac delta function.

Substituting equations (1.4) and (1.6) into the Rayleigh-Green identity (equation (1.2) used by Katsikadelis and Armenakas and equation (1.3) used by Costa and Brebbia), where functions u and v were, respectively, replaced by the plate deflection w and the fundamental solution w^* , leads to the following boundary integral equation for plates supported on Winkler foundations,

$$\begin{aligned} Cw(P) = \int_{\Omega} w^* q d\Omega + D \oint_s \left(\nabla^2 w \frac{\partial w^*}{\partial n} - w^* \frac{\partial \nabla^2 w}{\partial n} \right. \\ \left. + w \frac{\partial \nabla^2 w^*}{\partial n} - \nabla^2 w^* \frac{\partial w}{\partial n} \right) ds, \end{aligned} \quad (1.7)$$

in which C is a coefficient depending on the location of the point P .

Though the domain integral is involved in the above equation, the unknown functions are contained only in the integrands of the line integral along the plate boundary. Therefore, only the plate boundary is required to be discretized for the numerical solutions. Since the fundamental solution used in the aforementioned approach is dependent on the Winkler foundation stiffness, the integral formulation developed cannot solve the problems of: (a) plates supported on nonhomogeneous Winkler foundations where $k_f = k_f(x, y)$ and (b) part area of plates lifting away from the foundations after loading. Because of the complicated expression of the fundamental solution, the integral involved can only be evaluated by numerical quadrature.

The second approach of the boundary element method for plates on Winkler foundations was presented by Bezine (1988)[14]. He used the following function,

$$w^*(r) = \frac{1}{8\pi} r^2 \ln(r); \quad r = |P - Q|, \quad (1.8)$$

as the fundamental solution to establish the integral representation. The function expressed by equation (1.8) is the singular solution of an infinite plate subjected to a unit force load and without being supported on any foundation. This fundamental solution satisfies the following differential equation,

$$D\nabla^4 w^* = \delta(r). \quad (1.9)$$

Substitution of the field governing equation (1.4) and equation (1.9) into the Rayleigh Green identity of equation (1.3) results in the integral equation for plates on Winkler foundations,

$$\begin{aligned}
Cw(P) = & \oint_s \left[\frac{\partial w}{\partial n} M_n(w^*) + w V_n(w^*) - M_n(w) \frac{\partial w^*}{\partial n} - V_n(w) w^* \right] ds \\
& + \sum_{i=1}^{N_c} (w T_i(w^*) - w^* T_i(w)) + \int_{\Omega} w^* q d\Omega - \int_{\Omega} w^* k_f w d\Omega
\end{aligned} \tag{1.10}$$

In the above integral equation, the unknown function w is involved in the integrand of the domain integral such as

$$\int_{\Omega} w^* k_f w d\Omega$$

Thus, both the plate boundary and the plate domain are needed to be discretized in the numerical solution. The rectangular plate domain was divided into finite panels in Bezine's paper. The value of the unknown deflection w was defined at the center point of each panel and was constant over each panel. The surface integrations over each panel were performed by a Gauss-Legendre method with 6×6 integration points. Though Bezine presented the numerical results of rectangular plates in bonded contact with Winkler foundations, the formulation he provided can be used to solve the plate on an unilateral Winkler foundation by introducing an iteration procedure.

Sapountzakis and Katsikadelis[16] have used this second approach to solve the unilaterally supported plates on "Winkler" type elastic foundations. The subgrade model consisted of closely spaced independent springs. The foundation reaction might depend linearly or nonlinearly on the deflection. The initial gap between the plate and the foundation surface was also considered. The domain discretization was performed using Gauss quadrature and the plate deflections at the Gauss integration points were chosen as primary unknowns. The two-term acceleration method was employed in the iteration procedure, which started with an assumed vector of internal plate deflections at the Gauss integration points.

The third approach of BEM for plates on Winkler foundations was developed

by de León and París[15] by using the singular solution ψ^* of the Poisson equation,

$$\nabla^2 \psi^* = \delta(r) \quad (1.11)$$

as the fundamental solution. In their formulation, the plate governing equation (1.4) was first decomposed into two second order differential equations as

$$\nabla^2 M = (q - k_f w) \quad (1.12a)$$

and

$$D\nabla^2 w = M. \quad (1.12b)$$

where $M = M(x, y)$ is an intermediate variable related to the plate bending moments. Then, these two second order differential equations were transformed into the integral equations by applying Green's second formula to equations (1.12a) and (1.12b). Thus,

$$CM(P) = \int_{\Omega} \psi^* q d\Omega - \int_{\Omega} \psi^* k_f w d\Omega + \oint_s (M \frac{\partial \psi^*}{\partial n} - \psi^* \frac{\partial M}{\partial n}) ds \quad (1.13a)$$

and

$$Cw(P) = \frac{1}{D} \int_{\Omega} \psi^* M d\Omega + \oint_s (w \frac{\partial \psi^*}{\partial n} - \psi^* \frac{\partial w}{\partial n}) ds \quad (1.13b)$$

The reader will note that two domain integrals involving unknown integrands such as

$$\int_{\Omega} \psi^* M d\Omega \quad \text{and} \quad \int_{\Omega} \psi^* w d\Omega$$

exist in the integral equations (1.13a) and (1.13b). These two domain integrals, along with the domain integral about the known external load $q(x, y)$, were further transformed into line integrals along the plate boundary in the following procedure.

(1) First a set of generic functions ξ_j is taken which verifies Poisson's equation,

$$\nabla^2 \xi_j = f_j \quad (1.14)$$

where ξ_j and f_j are known functions. The functions f_j must be chosen to be simple enough to allow particular solutions ξ_j to be found.

(2) Then equation (1.14) is transformed into the integral equation by using Green's second formula,

$$C\xi_j = \int_{\Omega} \psi^* f_j d\Omega + \oint_s (\xi_j \frac{\partial \psi^*}{\partial n} - \psi^* \frac{\partial \xi_j}{\partial n}) ds. \quad (1.15)$$

(3) The physical values q , w and M , which are defined over the domain, are expressed approximately as a linear combination of f_j functions.

$$q = \sum_{j=1}^L \alpha_j f_j \quad (1.16a)$$

$$w = \sum_{j=1}^L \beta_j f_j \quad (1.16b)$$

$$M = \sum_{j=1}^L \gamma_j f_j \quad (1.16c)$$

where coefficients $\{\alpha_j\}$, $\{\beta_j\}$ and $\{\gamma_j\}$ of order L are determined by collocation, forcing equations (1.16a,b,c) at L points. Therefore, the values of w and M at the collocation points have to be treated as primary unknowns in the numerical solution.

(4) The domain integrals become

$$\int_{\Omega} \psi^* q d\Omega = \sum_{j=1}^L \alpha_j [C\xi_j - \oint_s (\xi_j \frac{\partial \psi^*}{\partial n} - \psi^* \frac{\partial \xi_j}{\partial n}) ds], \quad (1.17a)$$

$$\int_{\Omega} \psi^* k w d\Omega = k \sum_{j=1}^L \beta_j [C\xi_j - \oint_s (\xi_j \frac{\partial \psi^*}{\partial n} - \psi^* \frac{\partial \xi_j}{\partial n}) ds] \quad (1.17b)$$

and

$$\int_{\Omega} \psi^* M d\Omega = \sum_{j=1}^L \gamma_j [C\xi_j - \oint_s (\xi_j \frac{\partial \psi^*}{\partial n} - \psi^* \frac{\partial \xi_j}{\partial n}) ds]. \quad (1.17c)$$

In the third approach, though the only integrals involved are the line integral along the plate boundary, some collocation points located within the plate domain are needed to determine the coefficients in equations (1.16a-c).

Among the three approaches of BEM solutions for plates on Winkler type foundations, the first one is considered to be a pure boundary element method in which the boundary values alone are chosen as the primary unknowns in the numerical solutions. The second and third approaches may be classified as "domain boundary element method" in which both the boundary values and some domain values are chosen as the primary unknowns.

1.2.3 Plates on Pasternak Foundations

The boundary element methods for the interaction problems between elastic plates and Pasternak foundations have been first developed by Balas et al.[17] for a circular plate under a concentrated central load, later by Puttonen and Varpasuo [18], as well as Katsikadelis and Kallivokas[19,20] for more general cases.

The interaction problem between a thin elastic plate and a Pasternak foundation is governed by the following differential equation

$$D\nabla^4 w - G_p \nabla^2 w + k_f w = q(x, y) \quad (1.18)$$

In this case, the foundation reaction between the plate and the foundation is given as

$$p(x, y) = -G_p \nabla^2 w(x, y) + k_f w(x, y) \quad (1.19)$$

where G_p is the shear modulus of the elastic shear layer.

By using the method of weighted residuals, Puttonen and Varpasuo[18] derived the following integral equation for a plate with smooth boundary,

$$Cw(P) = \int_s w^* V_n ds + \int_s \Theta_n^* M_n ds - \int_s (M_n^* - G_p w^*) \Theta_n ds - \int_s (V_n^* + G_p \Theta_n^*) w ds + \int_\Omega q w^* d\Omega \quad (1.20)$$

in which w^* is the weighted function or the fundamental solution. They chose

$$w^*(r) = \frac{a^2}{2\pi D} \int_0^\infty \frac{t}{t^4 + \frac{a^2}{D}(G_p t^2 + k_f a^2)} J_0\left(t \frac{r}{a}\right) dt \quad (1.21)$$

as the fundamental solution, in which J_0 is the zero-order Bessel function of first kind and a is a constant with the dimension of length. The fundamental solution of equation (1.21) is the singular solution of an infinite plate resting on Pasternak foundation and it is obtained by using the Hankel transform. It satisfies the differential equation

$$D\nabla^4 w^* - G_p \nabla^2 w^* + k_f w^* = \delta(r) \quad (1.22)$$

The boundary values, w , Θ_n , M_n and V_n , involved in the integral equation (1.20) are plate edge deflection, normal rotation, normal moment and equivalent shear force. Two of them are specified for the plate edge conditions of free, simply supported and clamped.

Numerical approximations in four different phases were performed in this BEM solution. First, the two unknown boundary functions were expressed approximately as polynomials within each boundary element. Second the values of the Bessel functions were calculated at the integration points of the Fourier-Bessel integral. Then, the Fourier-Bessel integrals were evaluated and finally, the approximations of the line or domain integral were obtained. The sample problems solved in reference[18] were a circular plate subjected to a boundary shear force and an edge rotation, and the bottom slab of a nuclear reactor building for outer shock wave load.

Puttonen and Varpasuo[18] also studied the indirect boundary element method solution of the problem for the case of the circular plate loaded by a boundary shear force and an edge rotation. They concluded that the direct BEM method was more accurate than the indirect one.

By using the Rayleigh-Green identity (equation (1.2)) combined with Green's second formula, Katsikadelis and Kallivokas[19,20] presented the integral equation,

$$Cw(P) = \int_{\Omega} qw^* d\Omega + D \oint_s \left(\nabla^2 w \frac{\partial w^*}{\partial n} - w^* \frac{\partial \nabla^2 w}{\partial n} + w \frac{\partial \nabla^2 w^*}{\partial n} - \nabla^2 w^* \frac{\partial w}{\partial n} - \frac{G_p}{D} w \frac{\partial w^*}{\partial n} + \frac{G_p}{D} w^* \frac{\partial w}{\partial n} \right) ds \quad (1.23)$$

for the boundary element analysis of the problems of plates resting on Pasternak foundations. (If the second form of the Rayleigh-Green identity of equation (1.3) had been used, instead of equation (1.2), the integral representative of the problem would have been the same as integral equation (1.20).)

The fundamental solution w^* used by Katsikadelis and Kallivokas[19,20] also satisfies the differential equation (1.22), but it has the expression of

$$w^*(r) = \frac{l^2}{4D \sin 2\theta} \operatorname{Re}[H_0^{(1)}(\beta \rho)] \quad (1.24)$$

where

$$l^2 = (D/k_f)^{\frac{1}{3}}, \quad \rho = r/l$$

$$\beta = \cos\theta + i\sin\theta, \quad 2\theta = \arctan(-\sqrt{1/\mu - 1})$$

and $\operatorname{Re}[H_0^{(1)}(\beta \rho)]$ denotes the real part of the zero order Hankel function of the first kind. This fundamental solution is valid only for the case of $\mu < 1$, where $\mu = G_p/4k_f D$.

It is clear that two of the four boundary values, w , $\frac{\partial w}{\partial n}$, $\nabla^2 w$ and $\frac{\partial \nabla^2 w}{\partial n}$, involved in the integral equation (1.23) have no direct physical meaning. In the case of

a plate with clamped edges, the boundary values w and $\frac{\partial w}{\partial n}$ are equal to zero. Two unknowns remain for each boundary point. It is well known that discretized boundary integral equations can be used to solve the plate boundary values with two unknowns associated with each boundary node. The details of this will be clear in the next chapter. However, for the case of plates with free edges, four of the boundary values in equation (1.23) are not specified, so that four unknown nodal values are associated with each boundary node. Thus, the use of discretized boundary integral equations alone is not sufficient to solve the plate boundary values.

In the numerical solution provided by Katsikadelis and Kallivokas for plates with free edges[20], these writers combined the discretized integral equations with two additional differential equations which were derived from the boundary conditions relating to the plate edge normal moment and equivalent shear force at each boundary node. The differential equations were solved by using the finite difference method.

Both BEM solutions for plates on Pasternak foundations, presented by Puttonen and Varpasuo [18] and Katsikadelis and Kallivokas[19,20], covered the problems of plates on Winkler type foundations by using a zero value of the shear modulus G_p in the fundamental solutions and in the integral equations.

1.3 Problems of Plates on Elastic half-space

If the foundation is modelled as an elastic half-space, the interaction problem between the plate and the foundation is governed by the following two equations,

$$D\nabla^4 w(x, y) + p(x, y) = q(x, y) \quad (1.25)$$

and

$$w(x, y) = \frac{(1 - \nu_s)}{2\pi G_s} \int_{\Omega} \frac{p(\xi, \eta)}{\sqrt{(\xi - x)^2 + (\eta - y)^2}} d\xi d\eta \quad (1.26)$$

where G_s and ν_s are the shear modulus and Poisson's ratio of the elastic half space, and $p(\xi, \eta)$ is the foundation reaction or the contact pressure between the plate and the foundation.

The singular solution for an infinite plate resting on an elastic half space and subjected to a concentrated unit force load exists and it is given as [21]

$$w^*(r) = \frac{l_0^2}{2\pi D} \int_0^\infty \frac{J_0(\frac{\lambda r}{l_0})}{1 + \lambda^3} d\lambda \quad (1.27)$$

where J_0 denotes the Bessel function of zero order and $l_0^3 = 2D(1 - \nu_s^2)/E_s$. This singular solution satisfies the governing equation of

$$D\nabla^4 w^* + p^* = \delta(r) \quad (1.28)$$

where p^* is the foundation reaction pressure and its distribution at any point is given as

$$p^*(r) = \frac{1}{2\pi l_0^2} \int_0^\infty \frac{J_0(\frac{\lambda r}{l_0})\lambda}{1 + \lambda^3} d\lambda \quad (1.29)$$

By substituting the governing equations (1.25) and (1.28) into the Rayleigh Green identity (equation (1.2)) where functions u and v are replaced by w and w^* respectively, we have the following integral representative for the interaction problem between a plate and an elastic foundation,

$$\begin{aligned} Cw(P) = & D \oint_s \left(\nabla^2 w \frac{\partial w^*}{\partial n} - w^* \frac{\partial \nabla^2 w}{\partial n} + w \frac{\partial \nabla^2 w^*}{\partial n} - \nabla^2 w^* \frac{\partial w}{\partial n} \right) ds \\ & + \int_\Omega w^* q d\Omega + \int_\Omega (wp^* - w^* p) d\Omega \end{aligned} \quad (1.30)$$

in which the domain integral

$$\int_\Omega (wp^* - w^* p) d\Omega$$

contains the unknown functions w and p in the integrand.

In the case of Winkler type foundation, this domain integral term vanishes,

$$\begin{aligned}\int_{\Omega}(wp^* - w^*p)d\Omega &= \int_{\Omega}(wk_fw^* - w^*k_fw)d\Omega \\ &= 0\end{aligned}\tag{1.31}$$

In the case of the Pasternak foundation model, the domain integral term can be transformed into the line integral along the plate boundary.

$$\begin{aligned}\int_{\Omega}(wp^* - w^*p)d\Omega &= \int_{\Omega}[w(-G_p\nabla^2w^* + k_fw^*) - w^*(-G_p\nabla^2w + k_fw)]d\Omega \\ &= G_p \int_{\Omega}(w^*\nabla^2w - w\nabla^2w^*)d\Omega \\ &= G_p \oint_s(w^*\frac{\partial w}{\partial n} - w\frac{\partial w^*}{\partial n})ds\end{aligned}\tag{1.32}$$

In the case of elastic half-space, the domain integral cannot be further simplified because of the integral relationship between the plate deflections and the foundation reactions. Thus, domain discretization has to be used in the BEM solution even when the singular solution of the problem, i.e. an infinite plate supported on elastic half-space foundation, is employed to construct the integral equation. An alternative BEM formulation for the interaction analysis of plates and elastic half-space foundations is proposed in this thesis.

1.4 Objectives

The first objective of the thesis is to develop the numerical procedure based on the direct boundary element method to analyze the interaction problem between an elastic thin plate and an elastic half-space. In the proposed solution, the plate and the foundation are treated as two separate bodies and they are in unilateral contact at the interface. The behaviour of the plate subjected to both the external load and the contact pressure at the interface is presented by the direct boundary element method according to the plate theory. The response of the elastic foundation under

the surface pressure is presented by the Boussinesq's solution. The BEM equations and the Boussinesq's equation are coupled together via the compatibility conditions of both the displacements and the traction in the contact areas. The contact region is treated as an unknown which is to be determined by iteration. The interaction between a plate and a Winkler foundation is treated as a special case in the proposed solution method.

The second objective of the thesis is the boundary element analysis of thin plates unilaterally and elastically edge supported. The support system in this case consists of discrete elastic springs along the plate edges. The purpose of the unilateral contact analysis between plates and the edge supports is to determine the extent of the plate corner lift-off and nonlinear effects on plate internal actions due to the loss of the contact between the plates and its edge supports.

The third objective of the thesis is to develop the direct boundary element method to analyze a plate with elastic edge beam boundary conditions. All of the published papers with regard to the boundary element analysis of plate bending problems dealt with plates with free, simply supported or clamped boundary conditions.

1.5 Thesis Overview

Chapter 1 gives the general background of plate foundation interaction problems, reviews literature with regard to the boundary element analysis of plates on elastic foundations, points out problems associated with plates on elastic half space foundations, presents the objectives of the thesis and summarizes the thesis structure.

Chapter 2 gives the integral formulation for plates on elastic half space, presents the fundamental solution used in the formulation and discusses loadings exerted directly on plates.

Chapter 3 presents the numerical implementation details on the contact area discretization and plate boundary discretization, as well as the iteration procedure involved, compares the proposed method with the finite element method and discusses the suitability of the present method for rigid plates on elastic half-space foundations.

Chapter 4 carries out evaluations of both line integrations along boundary elements and surface integrations over contact elements, and discusses the accuracy of surface integrations evaluated by using Gauss quadrature.

Chapter 5 discusses the effect with regard to the location of boundary source points on numerical result accuracy, provides numerical results of plates on Winkler foundations with various boundary conditions and results of plates on elastic half-space foundation subjected to various loadings.

Chapter 6 presents formulation for plates with edge beam boundary conditions. Two numerical models are developed and a set of numerical results are presented to validate the proposed models.

Chapter 7 discusses the need for solving problems of unilaterally edge supported plates, reviews literature with regard to various methods for this particular problem, gives boundary element formulation for plate on elastic and unilateral edge supports and presents some numerical results.

Chapter 8 summarizes the present work, presents conclusions and gives some recommendations for future work.

Chapter 2

INTEGRAL FORMULATION FOR PLATES ON ELASTIC HALF-SPACE

2.1 Governing Equations

An elastic thin plate in unilateral and frictionless contact with an elastic half space foundation is subjected to both the external load $q(x, y)$ applied at the top surface and the unknown contact pressure $p(x, y)$ at the bottom surface, as shown in Fig.2.1. The plate deflection function $w(x, y)$ is governed by the equation

$$D\nabla^4 w = q(x, y) - p(x, y) \quad \text{in } \Omega, \quad (2.1)$$

with

$$p(x, y) \begin{cases} > 0 & \text{if } (x, y) \in \Omega_c \\ = 0 & \text{if } (x, y) \notin \Omega_c \end{cases}$$

where Ω refers to the domain of the plate; Ω_c is the contact region; and D is the flexural rigidity of the plate defined as $D = Eh^3/12(1 - \nu^2)$ in which E , h and ν are, respectively, the Young's modulus, thickness and Poisson's ratio of the plate.

The response of the elastic half-space under the distributed contact pressure $p(x, y)$ is obtained by employing the superposition method to Boussinesq's solution[1,22],

$$w_s(P) = \frac{(1 - \nu_s)}{2\pi G_s} \int_{\Omega_c} \frac{p(x, y)}{\sqrt{(x - x_P)^2 + (y - y_P)^2}} dx dy \quad (2.2)$$

where $w_s(P)$ is the surface deflection of the half-space at point P ; and G_s and ν_s are, respectively, the shear modulus and Poisson's ratio of the elastic half-space.

Fig. 2.1 Plate on elastic half-space

In the case of loads applied directly on the foundation surface and outside the plate domain, the surface deflection of the elastic half-space consists of two parts. One is due to the contact pressure beneath the plate and the other is due to the loads exerted outside the plate domain. Thus,

$$\begin{aligned}
w_s(P) &= \frac{(1-\nu_s)}{2\pi G_s} \int_{\Omega_c} \frac{p(x,y)}{\sqrt{(x-x_P)^2 + (y-y_P)^2}} dx dy \\
&+ \frac{(1-\nu_s)}{2\pi G_s} \int_{A_e} \frac{p_e(x,y)}{\sqrt{(x-x_P)^2 + (y-y_P)^2}} dx dy \\
&= \frac{(1-\nu_s)}{2\pi G_s} \int_{\Omega_c} \frac{p(x,y)}{\sqrt{(x-x_P)^2 + (y-y_P)^2}} dx dy + w_{se}(P)
\end{aligned} \tag{2.3}$$

in which $p_e(x,y)$ refers to the distributed load applied outside the plate domain over an area A_e , and $w_{se}(P)$ denotes the foundation surface deflection at point P due to the known $p_e(x,y)$.

The displacement and stress at any location within the half-space due to the surface loads $p(x,y)$ and $p_e(x,y)$ can be calculated by using the formulation provided in reference[22].

From thin plate bending theory[21], the rotations, moments and shear forces within the plate are obtained by differentiating the plate deflection function according to the following definitions:

$$\Theta_x = \frac{\partial w}{\partial x} \tag{2.4a}$$

$$\Theta_y = \frac{\partial w}{\partial y} \tag{2.4b}$$

$$M_{xx} = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \tag{2.4c}$$

$$M_{yy} = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \tag{2.4d}$$

$$\begin{aligned}
 M_{xy} &= -M_{yx} \\
 &= -D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y}
 \end{aligned}
 \tag{2.4e}$$

$$Q_x = D \frac{\partial}{\partial x} (\nabla^2 w) \tag{2.4f}$$

$$Q_y = D \frac{\partial}{\partial y} (\nabla^2 w), \tag{2.4g}$$

and their positive directions are illustrated in Fig. 2.2.

Along the plate boundary, there are four edge variables (Fig. 2.3): the deflection W , the normal slope Θ_n , the normal moment M_n , and the equivalent shear force V_n . There may be some corner forces T at plate corners. These plate boundary variables are defined as

$$W = w \tag{2.5a}$$

$$\Theta_n = \frac{\partial w}{\partial n} \tag{2.5b}$$

$$M_n = -D \left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2} \right) \tag{2.5c}$$

$$\begin{aligned}
 V_n &= Q_n - \frac{\partial M_{nt}}{\partial t} \\
 &= D \frac{\partial}{\partial n} \left[\frac{\partial^2 w}{\partial n^2} + (2 - \nu) \frac{\partial^2 w}{\partial t^2} \right]
 \end{aligned}
 \tag{2.5d}$$

where n and t are local moving Cartesian coordinates which are normal and parallel, respectively, to the tangent of boundary points.

At a corner point where the normal to the boundary is discontinuous, the twisting moment is discontinuous. A corner force (Fig. 2.3), T , resulting from this

Fig. 2.2 Internal actions in a plate element

Fig. 2.3 Plate boundary variables

discontinuity is given as

$$T = M_{nt}^- - M_{nt}^+ \quad (2.6a)$$

where M_{nt}^- and M_{nt}^+ are the twisting moments at both sides of a corner and the twisting moment on the boundary is defined as

$$M_{nt} = -D(1 - \nu) \frac{\partial^2 w}{\partial n \partial t} \quad (2.6b)$$

2.2 Boundary Conditions

Along the plate edge with boundary conditions of free, simply supported or clamped, two of the four boundary variables, W , Θ_n , M_n and V_n defined in equations (2.5a)–(2.5d), are equal to zero. They are:

- a) $M_n = 0$ and $V_n = 0$ for free edges;
- b) $W = 0$ and $M_n = 0$ for simply supported; and
- c) $W = 0$ and $\Theta_n = 0$ for clamped.

When plate edges have monolithic elastic beams, all four boundary variables have non-zero values, but only two of them are independent. The treatment of the elastic beam supported boundary conditions will be discussed in Chapter 6 in detail.

At a corner point, however, the unknowns may number more than two due to the discontinuous normal, as shown in Fig.2.4. After considering the compatibility at a corner point, it can be theoretically justified to consider only two independent unknown variables at each corner point. According to reference[23], these two independent variables are:

- a) V_n^+ and M_n^+ for the free-clamped corner;
- b) V_n and M_n for the clamped-clamped corner assuming that $V_n = V_n^- = V_n^+$ and $M_n = M_n^- = M_n^+$;

Fig.2.4 Unknowns at corners

- c) V_n and M_n^- for the clamped–simply supported corner after assuming $V_n^- = V_n^+$ and considering the compatibility condition of

$$T = -M_{nt}^+ = (1 - \nu)M_n^- \cos(\alpha) \sin(\alpha)$$

- d) V_n and T for the corner of simply supported–simply supported assuming that $V_n = V_n^- = V_n^+$; and
- e) W and Θ^b for the free–free corner where the normal rotations at both sides are expressed as[23],

$$\Theta_n^- = \Theta^b \cos \frac{\alpha}{2} - \frac{1}{2 \cos \frac{\alpha}{2}} \left[\left(\frac{\partial W}{\partial s} \right)^- + \left(\frac{\partial W}{\partial s} \right)^+ \right] \sin \frac{\alpha}{2} \quad (2.7)$$

$$\Theta_n^+ = \Theta^b \cos \frac{\alpha}{2} + \frac{1}{2 \cos \frac{\alpha}{2}} \left[\left(\frac{\partial W}{\partial s} \right)^- + \left(\frac{\partial W}{\partial s} \right)^+ \right] \sin \frac{\alpha}{2} \quad (2.8)$$

in which Θ^b is the rotation in the bisector direction of the corner point and angle α is indicated in Fig.2.4.

For the free–simply supported corner (Fig.2.4(f)), T and Θ_n^+ , instead of V_n^+ and Θ_n^+ (assuming $T = 0$) in reference[23], are chosen here as the independent nodal variables assuming that $V_n^+ = 0$ and imposing the compatibility of rotation $\Theta_n^- = \Theta_n^+ \cos(\alpha)$.

At the edge point where there is a change of boundary conditions, the number of unknown boundary quantities is, as in the corner point, also greater than two. This type of edge point is treated in a manner similar to the corner points.

2.3 Integral Representation

The boundary integral representation for the analysis of a thin plate on an elastic foundation can be developed either by using the Rayleigh–Green identity or directly from Maxwell-Betti's theorem, similar to that found in a number of published papers, for example, Bezine[11], Hartmann and Zontementel[24], Stern[25],

and Abdel-Akher and Hartley[26,27] for the plate bending problems. In the present case, the plate is subjected to both the external load $q(x, y)$ and the contact pressure $p(x, y)$, and the second form of the Rayleigh-Green identity,

$$\begin{aligned} \int_{\Omega} (u \nabla^4 v - v \nabla^4 u) d\Omega = \frac{1}{D} \oint_s \left(\frac{\partial u}{\partial n} M_n(v) + u V_n(v) \right. \\ \left. - M_n(u) \frac{\partial v}{\partial n} - V_n(u) v \right) ds + \frac{1}{D} \sum_{i=1}^{N_c} (u T_i(v) - v T_i(u)) \end{aligned} \quad (1.3)$$

is employed here to establish the integral representation for plates on elastic foundations.

Replacing the function u in equation (1.3) by the plate deflection w which is governed by equation (2.1), and the function v by a fundamental solution w^* which is the singular solution of equation,

$$D \nabla^4 w^*(P, Q) = \delta(P, Q), \quad (1.9)$$

we get

$$\begin{aligned} \int_{\Omega} w \delta(P, Q) d\Omega - \int_{\Omega} w^* (q - p) d\Omega \\ = \oint_s (\Theta_n M_n^* + W V_n^* - M_n \Theta_n^* - V_n W^*) ds + \sum_{i=1}^{N_c} (W T_i^* - W^* T_i) \end{aligned} \quad (2.9)$$

where N_c is the number of plate corners; and W , Θ_n , M_n , V_n and T_i are, respectively, the deflection, normal rotation, normal moment, equivalent shear force and the corner force along the boundary s of the plate directed as shown in Fig. 2.5(a). The quantities W^* , Θ_n^* , M_n^* , V_n^* and T_i^* are, respectively, the deflection, normal rotation, normal moment, equivalent shear force and the corner force of the fundamental solution along the curve s^* shown in Fig. 2.5(b) identical in shape with the boundary s of the real plate. By definition[21], they are

(a) given plate problem

(b) fundamental solution

Fig.2.5 Two companion states

$$W^*(P, Q) = w^*(P, Q) \quad (2.10a)$$

$$\Theta_n^*(P, Q) = \frac{\partial w^*}{\partial n_Q} \quad (2.10b)$$

$$M_n^*(P, Q) = -D \left(\frac{\partial^2 w^*}{\partial n_Q^2} + \nu \frac{\partial^2 w^*}{\partial t_Q^2} \right) \quad (2.10c)$$

$$V_n^*(P, Q) = D \frac{\partial}{\partial n_Q} \left[\frac{\partial^2 w^*}{\partial n_Q^2} + (2 - \nu) \frac{\partial^2 w^*}{\partial t_Q^2} \right] \quad (2.10d)$$

$$T_i^*(P, Q) = (M_{nt}^*)_i^- - (M_{nt}^*)_i^+ \quad (2.10e)$$

with

$$M_{nt}^*(P, Q) = -D(1 - \nu) \frac{\partial^2 w^*}{\partial n_Q \partial t_Q} \quad (2.10f)$$

in which operators $\frac{\partial}{\partial n_Q}$ and $\frac{\partial}{\partial t_Q}$ denote the directional derivatives with respect to directions n and t at point Q .

Applying the filtering property of the function δ results in

$$\int_{\Omega} w \delta(P, Q) d\Omega = C w(P) \quad (2.11)$$

where $w(P)$ is the plate deflection at point P and the coefficient C has the following definition:

$$C = \begin{cases} 1, & \text{point } P \text{ is inside the plate domain;} \\ 0, & \text{point } P \text{ is outside the plate domain;} \\ \frac{\alpha}{2\pi}, & \text{point } P \text{ is on the boundary of the plate.} \end{cases} \quad (2.12)$$

in which α in radians refers to the plate internal angle along the boundary. For example, $\alpha = \frac{\pi}{2}$ when point P is on a plate corner point with right angle and $\alpha = \pi$ if point P is not located on the corner points along the plate boundary.

Thus, the integral equation for plate on elastic foundation becomes

$$\begin{aligned}
 Cw(P) = & \int_{\Omega} qw^* d\Omega - \int_{\Omega_c} pw^* d\Omega \\
 & + \oint_s (\Theta_n M_n^* + W V_n^* - M_n \Theta_n^* - V_n W^*) ds + \sum_{i=1}^{N_c} (W T_i^* - W^* T_i)
 \end{aligned} \tag{2.13}$$

The second coupled integral equation required for plate bending problems is obtained by taking a derivative of equation (2.13) with respect to direction ξ , therefore,

$$\begin{aligned}
 C \frac{\partial w(P)}{\partial \xi} = & \int_{\Omega} q \frac{\partial w^*}{\partial \xi} d\Omega - \int_{\Omega_c} p \frac{\partial w^*}{\partial \xi} d\Omega \\
 & + \oint_s \left(\Theta_n \frac{\partial M_n^*}{\partial \xi} + W \frac{\partial V_n^*}{\partial \xi} - M_n \frac{\partial \Theta_n^*}{\partial \xi} - V_n \frac{\partial W^*}{\partial \xi} \right) ds \\
 & + \sum_{i=1}^{N_c} \left(W \frac{\partial T_i^*}{\partial \xi} - \frac{\partial W^*}{\partial \xi} T_i \right)
 \end{aligned} \tag{2.14}$$

in which $\frac{\partial w(P)}{\partial \xi}$ refers to the rotation of the plate in direction ξ at point P .

The integral equations (2.3), (2.13) and (2.14) are the three basic equations in the BEM solution for the plate elastic half-space interaction analysis. The primary unknowns involved in the problem are the plate boundary functions and the contact pressure function as well as the contact region. The numerical solution for these unknowns is presented in Chapter 3.

2.4 Plate Internal Actions

Once the plate boundary functions, the contact pressure and the contact region are determined, the plate deflection and rotation at any location can be calculated, respectively, from equation (2.13) and (2.14) with $C = 1$. In this case the source point P is located where the deflection and rotation are to be found.

The bending moments within the plate at point P are defined by the following equation,

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}_P = -D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu) \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}_P \quad (2.15)$$

The curvature vector in equation (2.15) can be obtained by taking derivatives of equation (2.13) as follows:

$$\begin{aligned} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}_P &= \int_{\Omega} q \begin{Bmatrix} \frac{\partial^2 w^*}{\partial x^2} \\ \frac{\partial^2 w^*}{\partial y^2} \\ \frac{\partial^2 w^*}{\partial x \partial y} \end{Bmatrix} d\Omega - \int_{\Omega_c} p \begin{Bmatrix} \frac{\partial^2 w^*}{\partial x^2} \\ \frac{\partial^2 w^*}{\partial y^2} \\ \frac{\partial^2 w^*}{\partial x \partial y} \end{Bmatrix} d\Omega \\ &+ \oint_s (\Theta_n \begin{Bmatrix} \frac{\partial^2 M_n^*}{\partial x^2} \\ \frac{\partial^2 M_n^*}{\partial y^2} \\ \frac{\partial^2 M_n^*}{\partial x \partial y} \end{Bmatrix} + W \begin{Bmatrix} \frac{\partial^2 V_n^*}{\partial x^2} \\ \frac{\partial^2 V_n^*}{\partial y^2} \\ \frac{\partial^2 V_n^*}{\partial x \partial y} \end{Bmatrix} - M_n \begin{Bmatrix} \frac{\partial^2 \Theta_n^*}{\partial x^2} \\ \frac{\partial^2 \Theta_n^*}{\partial y^2} \\ \frac{\partial^2 \Theta_n^*}{\partial x \partial y} \end{Bmatrix} - V_n \begin{Bmatrix} \frac{\partial^2 W^*}{\partial x^2} \\ \frac{\partial^2 W^*}{\partial y^2} \\ \frac{\partial^2 W^*}{\partial x \partial y} \end{Bmatrix}) ds \\ &+ \sum_{i=1}^{N_c} (W \begin{Bmatrix} \frac{\partial^2 T_i^*}{\partial x^2} \\ \frac{\partial^2 T_i^*}{\partial y^2} \\ \frac{\partial^2 T_i^*}{\partial x \partial y} \end{Bmatrix} - \begin{Bmatrix} \frac{\partial^2 W^*}{\partial x^2} \\ \frac{\partial^2 W^*}{\partial y^2} \\ \frac{\partial^2 W^*}{\partial x \partial y} \end{Bmatrix} T_i) \end{aligned} \quad (2.16)$$

The plate shear forces at point P are defined as,

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix}_P = D \begin{Bmatrix} \frac{\partial \nabla^2 w}{\partial x} \\ \frac{\partial \nabla^2 w}{\partial y} \end{Bmatrix}_P \quad (2.17)$$

in which

$$\begin{aligned}
\left\{ \frac{\partial \nabla^2 w}{\partial x} \right\}_P &= \int_{\Omega} q \left\{ \frac{\partial \nabla^2 w^*}{\partial x} \right\} d\Omega - \int_{\Omega_c} p \left\{ \frac{\partial \nabla^2 w^*}{\partial y} \right\} d\Omega \\
&+ \oint_s \left(\Theta_n \left\{ \frac{\partial \nabla^2 M_n^*}{\partial x} \right\} + W \left\{ \frac{\partial \nabla^2 V_n^*}{\partial r} \right\} - M_n \left\{ \frac{\partial \nabla^2 \Theta_n^*}{\partial r} \right\} - V_n \left\{ \frac{\partial \nabla^2 W^*}{\partial y} \right\} \right) ds \\
&+ \sum_{i=1}^{N_c} \left(W \left\{ \frac{\partial \nabla^2 T_i^*}{\partial x} \right\} - \left\{ \frac{\partial \nabla^2 W^*}{\partial x} \right\} T_i \right)
\end{aligned} \tag{2.18}$$

2.5 The Fundamental Solution

The fundamental solution w^* used in the integral equations is given by the expression

$$w^*(P, Q) = \frac{r^2}{8\pi D} \ln(r) \tag{2.19}$$

This expression gives the transverse displacement of an infinite plate at **field point** Q due to a unit concentrated force applied at **source point** P [21]. The variable r is the distance between these two points.

In the local coordinate system (n, t) as shown in Fig.2.6, the variable distance r has the expression of

$$\begin{aligned}
r &= \sqrt{(n_Q - n_P)^2 + (t_Q - t_P)^2} \\
&= \sqrt{c^2 + z^2}
\end{aligned} \tag{2.20}$$

in which

$$z = t_Q - t_P, \quad c = n_Q - n_P \tag{2.21}$$

where (n_P, t_P) and (n_Q, t_Q) refer, respectively, to the coordinate values of the source point P and the field point Q in local coordinate system (n, t) .

Fig.2.6 Boundary local coordinate system

Taking partial derivatives of the fundamental solution w^* with respect to n and t , the fundamental solution related quantities in equations (2.13) (for calculating the plate deflection) are obtained in terms of r , e and z as,

$$\Theta_n^* = \frac{e}{8\pi D} [1 + 2\ln(r)] \quad (2.22a)$$

$$M_n^* = -\frac{1}{8\pi} [(1 + 3\nu) + 2(1 + \nu)\ln(r) + 2(1 - \nu)\frac{e^2}{r^2}] \quad (2.22b)$$

$$V_n^* = \frac{1}{8\pi} [2(1 + \nu)\frac{e}{r^2} + 4(1 - \nu)\frac{e^3}{r^4}] \quad (2.22c)$$

$$M_{nt}^* = -\frac{(1-\nu)ez}{4\pi r^2} \quad (2.22d)$$

According to the standard formula for directional derivatives,

$$\frac{\partial f}{\partial \xi} = [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{\partial f}{\partial n} \\ \frac{\partial f}{\partial t} \end{array} \right\} \quad (2.23)$$

those fundamental solution related quantities in equation (2.14) (for calculating the plate rotation) have the following expressions,

$$\frac{\partial W^*}{\partial \xi} = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} e + 2eln(r) \\ z + 2zln(r) \end{array} \right\} \quad (2.24a)$$

$$\frac{\partial \Theta_n^*}{\partial \xi} = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} 1 + 2\frac{e^2}{r^2} + 2ln(r) \\ 2\frac{ez}{r^2} \end{array} \right\} \quad (2.24b)$$

$$\frac{\partial M_n^*}{\partial \xi} = \frac{1}{4\pi} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} (1+\nu)\frac{e}{r^2} + 2(1-\nu)\frac{ez^2}{r^4} \\ (1+\nu)\frac{z}{r^2} - 2(1-\nu)\frac{e^2z}{r^4} \end{array} \right\} \quad (2.24c)$$

$$\frac{\partial V_n^*}{\partial \xi} = \frac{1}{4\pi} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} 4(2\nu-1)\frac{e^2}{r^4} + 8(1-\nu)\frac{e^4}{r^6} - (1+\nu)\frac{1}{r^2} \\ 2(1+\nu)\frac{ez}{r^4} + 8(1-\nu)\frac{e^3z}{r^6} \end{array} \right\} \quad (2.24d)$$

$$\frac{\partial M_{nt}^*}{\partial \xi} = -\frac{(1-\nu)}{4\pi} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{2e^2z}{r^4} - \frac{z}{r^2} \\ \frac{2ez^2}{r^4} - \frac{e}{r^2} \end{array} \right\} \quad (2.24e)$$

in which ξ_n and ξ_t refer to the direction cosines for the direction ξ in the (n, t) coordinate system.

Similarly, the fundamental solution related quantities in equation (2.16) (for calculating the plate moments) and in equation (2.18) (for calculating the plate shear forces) can also be expressed in terms of variables r , c and z , which are presented in Appendix A.

2.6 Loads Applied Directly at the Plate

The load included in integral equations (2.13), (2.14), (2.16) and (2.18) is a distributed one, $q(x, y)$. A plate may be subjected to a number of distributed loads, concentrated forces and moments, or to a combination of these. If that is the case, the load related terms in these integral equations have to be justified.

The load related term in integral equation (2.13) is

$$\int_{\Omega} q w^* d\Omega, \quad (2.25)$$

of which the physical meaning is the virtual work done by the distributed load $q(x, y)$ over the virtual displacement w^* .

When the plate is subjected to N_f concentrated forces and moments, the virtual work done by these loads is equal to

$$\sum_{i=1}^{N_f} (f_i w^*(P, Q_i) + m_i \frac{\partial w^*(P, Q_i)}{\partial \eta_i}) \quad (2.26)$$

where f_i and m_i are the magnitude of the i^{th} concentrated force and moment applied at point Q_i , and η_i is the direction associated with the i^{th} moment.

In the case that a plate is subjected to the combination of the distributed pressure and a number of concentrated forces and moments, the total virtual work done by these loads is

$$\int_{\Omega} qw^* d\Omega + \sum_{i=1}^{N_f} (f_i w^*(P, Q_i) + m_i \frac{\partial w^*(P, Q_i)}{\partial \eta_i}), \quad (2.27)$$

and the load related term in integral equation (2.13) has to be replaced by expression (2.27).

After justifying the load related term, integral equation (2.13) becomes

$$\begin{aligned} Cw(P) = & \int_{\Omega} qw^* d\Omega + \sum_{i=1}^{N_f} (f_i w^*(P, Q_i) + m_i \frac{\partial w^*(P, Q_i)}{\partial \eta_i}) - \int_{\Omega_c} pw^* d\Omega \\ & + \oint_s (\Theta_n M_n^* + W V_n^* - M_n \Theta_n^* - V_n W^*) ds + \sum_{i=1}^{N_c} (W T_i^* - W^* T_i) \end{aligned} \quad (2.28)$$

Expression (2.27) is used to justify integral equation (2.13), replacing the load related term when the plate is subjected to the combination of a distributed load and a set of concentrated forces and moments. Similar justification has to be done for integral equations (2.14), (2.16) and (2.18), and the load related terms in equations (2.14), (2.16) and (2.18) are replaced, respectively, by expressions,

$$\int_{\Omega} q \frac{\partial w^*}{\partial \xi} d\Omega + \sum_{i=1}^{N_f} (f_i \frac{\partial w^*(P, Q_i)}{\partial \xi} + m_i \frac{\partial^2 w^*(P, Q_i)}{\partial \xi \partial \eta_i}), \quad (2.29)$$

$$\int_{\Omega} q \left\{ \begin{array}{c} \frac{\partial^2 w^*}{\partial x^2} \\ \frac{\partial^2 w^*}{\partial y^2} \\ \frac{\partial^2 w^*}{\partial x \partial y} \end{array} \right\} d\Omega + \sum_{i=1}^{N_f} (f_i \left\{ \begin{array}{c} \frac{\partial^2 w^*(P, Q_i)}{\partial x^2} \\ \frac{\partial^2 w^*(P, Q_i)}{\partial y^2} \\ \frac{\partial^2 w^*(P, Q_i)}{\partial x \partial y} \end{array} \right\} + m_i \frac{\partial}{\partial \eta_i} \left\{ \begin{array}{c} \frac{\partial^2 w^*(P, Q_i)}{\partial x^2} \\ \frac{\partial^2 w^*(P, Q_i)}{\partial y^2} \\ \frac{\partial^2 w^*(P, Q_i)}{\partial x \partial y} \end{array} \right\}) \quad (2.30)$$

and

$$\int_{\Omega} q \left\{ \begin{array}{c} \frac{\partial \nabla^2 w^*}{\partial x} \\ \frac{\partial \nabla^2 w^*}{\partial y} \end{array} \right\} d\Omega + \sum_{i=1}^{N_f} (f_i \left\{ \begin{array}{c} \frac{\partial \nabla^2 w^*(P, Q_i)}{\partial x} \\ \frac{\partial \nabla^2 w^*(P, Q_i)}{\partial y} \end{array} \right\} + m_i \frac{\partial}{\partial \eta_i} \left\{ \begin{array}{c} \frac{\partial \nabla^2 w^*(P, Q_i)}{\partial x} \\ \frac{\partial \nabla^2 w^*(P, Q_i)}{\partial y} \end{array} \right\}). \quad (2.31)$$

Chapter 3

NUMERICAL IMPLEMENTATION

3.1 Contact Region Discretization

As mentioned earlier, integral equations (2.3), (2.13) and (2.14) are the three basic equations in the BEM analysis of plate elastic half-space foundation interaction problems. These three equations can be solved numerically by discretizing the plate boundary and the contact area in a manner similar to that illustrated in Fig 3.1.

The contact region, Ω_c , between the plate and the foundation is divided into a number of subdomains which are called here **contact elements**. Each of them may contain several nodes. Within each contact element the distribution of the contact pressure is interpolated through its nodal values as,

$$p(x, y) = \sum_{j=1}^{M_i} N_{ij}(x, y) p_{ij} \quad (3.1)$$

where M_i is the number of contact nodes of the i^{th} element; p_{ij} is the contact pressure value at node j within element i ; and $N_{ij}(x, y)$ is the shape function which has unit value at node j and zero values at others.

Two types of contact elements are studied herein: single node constant element and six node triangular element as shown in Fig. 3.2. Within each constant contact element, the contact pressure is assumed to be constant and to be equal to the centroid nodal value, thus,

$$N_{i1}(x, y) = 1 \quad (3.2)$$

Within each six node triangular contact element, the contact pressure is approximated as a quadratic function and it is continuous across the adjacent elements. In the global coordinate system (x, y) , the shape functions for the six node contact element have the expression of

$$\begin{Bmatrix} N_{i1} \\ N_{i2} \\ N_{i3} \\ N_{i4} \\ N_{i5} \\ N_{i6} \end{Bmatrix} = \frac{1}{2a} \begin{bmatrix} (\frac{a_1^2}{a} - a_1) & (\frac{2a_1b_1}{a} - b_1) & (\frac{2a_1c_1}{a} - c_1) & \frac{b_1^2}{a} & \frac{2b_1c_1}{a} & \frac{c_1^2}{a} \\ (\frac{a_2^2}{a} - a_2) & (\frac{2a_2b_2}{a} - b_2) & (\frac{2a_2c_2}{a} - c_2) & \frac{b_2^2}{a} & \frac{2b_2c_2}{a} & \frac{c_2^2}{a} \\ (\frac{a_3^2}{a} - a_3) & (\frac{2a_3b_3}{a} - b_3) & (\frac{2a_3c_3}{a} - c_3) & \frac{b_3^2}{a} & \frac{2b_3c_3}{a} & \frac{c_3^2}{a} \\ \frac{2a_1a_2}{a} & \frac{2(a_1b_2+a_2b_1)}{a} & \frac{2(a_1c_2+a_2c_1)}{a} & \frac{2b_1b_2}{a} & \frac{2(b_1c_2+c_1b_2)}{a} & \frac{2c_1c_2}{a} \\ \frac{2a_2a_3}{a} & \frac{2(a_2b_3+a_3b_2)}{a} & \frac{2(a_2c_3+a_3c_2)}{a} & \frac{2b_2b_3}{a} & \frac{2(b_2c_3+c_2b_3)}{a} & \frac{2c_2c_3}{a} \\ \frac{2a_3a_1}{a} & \frac{2(a_3b_1+a_1b_3)}{a} & \frac{2(a_3c_1+a_1c_3)}{a} & \frac{2b_3b_1}{a} & \frac{2(b_3c_1+c_3b_1)}{a} & \frac{2c_3c_1}{a} \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{Bmatrix} \quad (3.3)$$

in which

$$\begin{aligned} a_i &= x_j y_k - x_k y_j \\ b_i &= y_j - y_k \\ c_i &= -x_j + x_k \end{aligned} \quad (3.4)$$

$$(i = 1, 2, 3; \quad j = 2, 3, 1; \quad k = 3, 2, 1)$$

and

$$2a = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \quad (3.5)$$

$$= x_1 y_2 + x_2 y_3 + x_3 y_1 - x_1 y_3 - x_2 y_1 - x_3 y_2$$

where (x_i, y_i) , $(i = 1, 2, 3)$, refer to the three corner node coordinates as shown in Fig.3.2(b).

After the contact region discretization, the integral equation (2.3) becomes

Fig. 3.1 Plate boundary and contact area discretizations

Fig. 3.2 Contact elements

$$w_s(P) = \frac{(1 - \nu_s)}{2\pi G_s} \sum_{i=1}^{M_e} \sum_{j=1}^{M_i} \int_{\Omega_{ci}} \frac{N_{ij}(x, y)}{\sqrt{(x - x_P)^2 + (y - y_P)^2}} dx dy \quad (3.6)$$

$$+ w_{se}(P)$$

in which M_e is the number of contact elements.

Applying equation (3.6) to all of the contact element nodes, say M , results in the following matrix equation,

$$\{w_s\} = [F]\{Q\} + \{w_{se}\} \quad (3.7)$$

where vector $\{w_s\}_M$ contains the total foundation surface deflections at contact element nodes; vector $\{w_{se}\}_M$ which is known contains the foundation surface deflections at contact element nodes due to the load applied outside the plate domain; vector $\{Q\}_M$ contains nodal contact pressures; and matrix $[F]_{M \times M}$ is the flexibility matrix of the half-space, made up of Boussinesq's coefficients.

Because of the contact area discretization, the domain integral terms related to the contact pressure $p(x, y)$ in integral equations (2.13) and (2.14) are expressed as

$$\begin{aligned} \int_{\Omega_c} p w^* d\Omega &= \sum_{i=1}^{M_e} \sum_{j=1}^{M_i} \left(\int_{\Omega_{ci}} N_{ij} w^* d\Omega \right) p_{ij} \\ &= \{B_1^*\}^T \{Q\} \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} \int_{\Omega_c} p \frac{\partial w^*}{\partial \xi} d\Omega &= \sum_{i=1}^{M_e} \sum_{j=1}^{M_i} \left(\int_{\Omega_{ci}} N_{ij} \frac{\partial w^*}{\partial \xi} d\Omega \right) p_{ij} \\ &= \{B_2^*\}^T \{Q\} \end{aligned} \quad (3.9)$$

where $\{B_1^*\}_M$ and $\{B_2^*\}_M$ are vectors of coefficients resulting from corresponding domain integrations.

3.2 Plate Boundary Discretization

The plate boundary s is divided into a number of segments which are called **boundary elements**. Each boundary element comprises several nodes, say N_i , as shown in Fig.3.3 each having as primary unknowns the nodal values, such as the nodal deflection and normal rotation for the free plate edge. Lagrangian interpolation is employed to approximate the continuous distribution of the boundary variables within each boundary element, such as,

$$W(s) = \sum_{j=1}^{N_i} L_{ij}(s) W_{ij} \quad (3.10)$$

$$\Theta_n(s) = \sum_{j=1}^{N_i} L_{ij}(s) \Theta_{nij} \quad (3.11)$$

$$M_n(s) = \sum_{j=1}^{N_i} L_{ij}(s) M_{nij} \quad (3.12)$$

$$V_n(s) = \sum_{j=1}^{N_i} L_{ij}(s) V_{nij} \quad (3.13)$$

in which W_{ij} , Θ_{nij} , M_{nij} and V_{nij} are the boundary values at the j^{th} node within the i^{th} boundary element, and $L_{ij}(s)$ is the shape function having expression of

$$L_{ij}(s) = \frac{\prod_{k=1, k \neq j}^{N_i} (s - s_{ik})}{\prod_{k=1, k \neq j}^{N_i} (s_{ij} - s_{ik})} \quad (3.14)$$

where s_{ik} denotes the coordinate value of node k of element i along the plate boundary s .

After the boundary discretization, the boundary integral equations (2.13) and (2.14) become,

$$\begin{aligned} Cw(P) = & \sum_{i=1}^{N_e} \sum_{j=1}^{N_i} [W_{ij} \int_{s_i} L_{ij} V_n^* ds + \Theta_{nij} \int_{s_i} L_{ij} M_n^* ds \\ & - M_{nij} \int_{s_i} L_{ij} \Theta_n^* ds - V_{nij} \int_{s_i} L_{ij} W^* ds] \\ & + \sum_{i=1}^{N_e} (WT_i^* - W^* T_i) + \int_{\Omega} q w^* d\Omega \\ & - \{B_1^*\}^T \{Q\} \end{aligned} \quad (3.15)$$

and

$$\begin{aligned}
C \frac{\partial w(P)}{\partial \xi} = & \sum_{i=1}^{N_e} \sum_{j=1}^{N_i} [W_{ij} \int_{s_i} L_{ij} \frac{\partial V_n^*}{\partial \xi} ds + \Theta_{nij} \int_{s_i} L_{ij} \frac{\partial M_n^*}{\partial \xi} ds \\
& - M_{nij} \int_{s_i} L_{ij} \frac{\partial \Theta_n^*}{\partial \xi} ds - V_{nij} \int_{s_i} L_{ij} \frac{\partial W^*}{\partial \xi} ds] \\
& + \sum_{i=1}^{N_e} (W \frac{\partial T_i^*}{\partial \xi} - \frac{\partial W^*}{\partial \xi} T_i) + \int_{\Omega} q \frac{\partial w^*}{\partial \xi} d\Omega \\
& - \{ B_2^* \}^T \{ Q \}
\end{aligned} \tag{3.16}$$

where N_e is the number of the boundary elements.

After imposing the plate boundary conditions, the discretized integral equations (3.15) and (3.16) become two linear algebraic equations which contain $2N + M$ unknowns associated with the N boundary nodes and M contact element nodes, respectively. Written in matrix form, these two linear algebraic equations are formed as

$$Cw(P) + \{ A_1^* \}^T \{ X \} + \{ B_1^* \}^T \{ Q \} = y_1^* \tag{3.17}$$

and

$$C \frac{\partial w(P)}{\partial \xi} + \{ A_2^* \}^T \{ X \} + \{ B_2^* \}^T \{ Q \} = y_2^* \tag{3.18}$$

where vector $\{ X \}$ contains $2N$ unknown boundary nodal variables and N is the number of total boundary nodes; $\{ A_1^* \}_{2N}$ and $\{ A_2^* \}_{2N}$ are vectors of coefficients computed from line integrations corresponding to boundary values in the vector $\{ X \}$; and y_1^* and y_2^* are computed values from the load related terms which are described in Section 2.6.

Fig. 3.3 Boundary element model

3.3 Solution for Boundary Variables and Contact Pressure

There are $(2N + M)$ unknown variables involved in equations (3.17) and (3.18). Thus, $(2N + M)$ independent linear algebraic equations are required in order to solve the problem.

For any source point P , equations (3.17) and (3.18) are two independent linear algebraic equations. By positioning N source points at the N boundary nodes on the plate edge and applying equations (3.17) and (3.18), a set of $2N$ simultaneous

equations is obtained. Similar to the analytical procedures described in Abdel Akher[23], the source points are located a very small distance from each boundary node, orthogonal to and just outside the boundary of the plate with $C = 0$. These source points associated with the corner boundary nodes are located along the corner bisector directions. If the source point is situated directly on the boundary node (with $C = 0.5$) the resulting integral is singular. In a sense the approach here is a practical way of obtaining the limiting value of the integral. The method of positioning boundary source points outside the plate domain to avoid dealing with singular integrals was also used in some existing publications of BEM analysis of plate bending related problems, such as by Tottenham[28] in the BEM analysis of plates and shells, by Puttonen and Varpasuo[18] in the BEM analysis of plates on Winkler and two-parameter foundations and by Hu[29] in the BEM analysis of eigenvalue problems of thin plates, and it has proven to be a simple, reliable and practical technique to handle the singular integrals.

In matrix form, these $2N$ equations are given by

$$[A]\{X\} + [B]\{Q\} = \{Y\}, \quad (3.19)$$

where matrices $[A]_{2N \times 2N}$ and $[B]_{2N \times M}$ are coefficient matrices resulting from the boundary integrals and domain integrals related to the boundary unknowns and contact pressures; and vector $\{Y\}_{2N}$ contains the integral results of the applied loads.

Now positioning M source points at the M contact element nodes and applying equation (3.17) only, with $C = 1$ in this case, we have a set of M simultaneous equations,

$$\{w\} + [A1]\{X\} + [B1]\{Q\} = \{Y1\}, \quad (3.20)$$

in which vector $\{w\}$ contains plate deflections at M contact nodes; matrices $[A1]$ of order $M \times 2N$ and $[B1]$ of order $M \times M$ are coefficient matrices similar to

matrices $[A]$ and $[B]$ in equation (3.19); and vector $\{Y1\}_M$ contains the integral results of the applied loads.

Within the contact region Ω_c , the plate deflection and foundation surface deflection are compatible. Therefore

$$\{w\} = \{w_s\} \quad (3.21)$$

while,

$$\{w\} = [F]\{Q\} + \{w_{se}\} \quad (3.7)$$

Substituting equation (3.7) into equation (3.20) and then combining it with equation (3.19) leads to

$$\begin{bmatrix} A & B \\ A1 & B1 + F \end{bmatrix} \begin{Bmatrix} X \\ Q \end{Bmatrix} = \begin{Bmatrix} Y \\ Y1 - w_{se} \end{Bmatrix} \quad (3.22)$$

In the case of plates supported on Winkler foundations, the corresponding foundation flexibility matrix in equation (3.22) is replaced by

$$[F] = \frac{1}{k_f} [I] \quad (3.23)$$

in which k_f is the Winkler stiffness and $[I]$ the unit matrix of order $M \times M$.

For an assumed or a specified contact region, equation (3.22) is solved for the boundary nodal unknowns $\{X\}$ and the nodal contact pressures $\{Q\}$,

$$\begin{Bmatrix} X \\ Q \end{Bmatrix} = \begin{bmatrix} A & B \\ A1 & B1 + F \end{bmatrix}^{-1} \begin{Bmatrix} Y \\ Y1 - w_{se} \end{Bmatrix} \quad (3.24)$$

However, the contact region, which is a prior unknown itself in the contact analysis of plates on elastic foundations, has to be determined by an iterative process.

3.4 Receding Type Contact

Dundurs and Stippes[30] classified the contact problems between two linear elastic bodies into three types, namely receding, stationary and advancing, depending on how the contact surfaces in the loaded configuration compare with those established by geometries in the unloaded states. If the contact surface under the loading is contained within the contact surface in the unloaded configuration, the contact is defined as receding. The receding type contact has been proved having the following properties[22,30]:

- (i) the contact area changes discontinuously from its initial to its loaded shape and size on application of the first increment of load;
- (ii) if the load increases in magnitude, but does not in disposition, the contact area does not change in shape and size; and
- (iii) the displacement, strain and stresses increase in direct proportion to the load.

The contact problem between elastic plates and elastic foundations studied here belongs to the receding type, since the loaded contact area is completely contained within the initial unloaded area, which is the plate plane geometry. Thus, the increment method is not necessary here because the contact region is independent of the loading level. However, the contact areas under the loaded configuration do depend on the relative stiffness of the plate to the foundation, the plate plane geometry, loading configuration, and the Poisson's ratio values.

The fact that the contact area between a plate and a foundation is independent of the loading level can also be explained as follows. Each iterative cycle consists of solving the problem solution (equation (3.22)) over an assumed contact region and then checking it with the compatibility conditions. Since both the plate and the

(a) Too larger contact area assumed

(b) Correct contact area assumed

Fig. 3.4 Distribution of contact pressure $p(x,y)$ over assumed contact areas

foundation are linear elastic bodies, the contact pressure solved over each assumed contact area is a linear function of the external load. Fig.3.4 presents distributions of the contact pressure over two assumed contact regions, where case (a) corresponds to an assumed too large contact area so that part of the contact pressure is in tension and case (b) refers to the correct contact area being assumed. As the load increases from q to $2q$, the value of the contact pressure increases in the same proportion, and the location of the boundary c , as indicated in Fig.3.4, which bounds the compressive contact pressure from the tensile contact pressure, remains the same. This implies that the shape and the size of the area with compressive contact pressure is independent of the loading level in each iterative cycle, including the iterative cycle where the correct contact region happens to be chosen as shown in Fig.3.4(b). Therefore, it can be concluded that the contact area between the plate and the elastic foundation is independent of the loading level if there is no initial gap existing which is the case in the present study.

3.5 Iteration Procedure

The compatibility conditions between the plate and the foundation are written as

$$w(x, y) = w_s(x, y), \quad p(x, y) > 0 \quad \text{for } (x, y) \in \Omega_c \quad (3.25a)$$

$$w(x, y) > w_s(x, y), \quad p(x, y) = 0 \quad \text{for } (x, y) \notin \Omega_c \quad (3.25b)$$

with w and w_s denoting the plate deflection and the foundation surface deflection, respectively. The first geometrical compatibility condition has been imposed in equation (3.21) at the contact nodes.

The iteration process consists of searching the contact region Ω_c with only compressive contact pressure inside and without geometrical incompatibilities outside, and starts with the assumption that the whole plate keeps in contact with the

foundation. It follows the steps: (1) assume a contact region Ω_c ; (2) solve equation (3.22) for the given contact region; (3) check elements of vector $\{Q\}$; (4) eliminate the negative value elements of $\{Q\}$ which imply tension nodal contact pressure; (5) re-solve equation (3.22) for the modified contact region; and (6) check the second displacement compatibility (equation (3.25b)) and if necessary choose a new modified contact region.

The uniqueness of the contact area solution is guaranteed by assuming the plate plane geometry as the initial contact region. In the contact area solution solved by using constant contact elements, the boundary of the correct contact area is between the centroid nodes of the outer element in contact and its adjacent element, as indicated in Fig.3.5. Convergence of the contact area solution can be achieved by contact element mesh refinement.

Using the six node quadratic contact elements, there are three possible contact area solutions as shown in Fig. 3.6, where solution (c) contains some tensile contact pressures which are relatively small values. These tensile contact pressures may cause geometrical incompatibilities outside the contact region. Therefore, when the six node triangular contact elements are used to find out the contact areas, and the separation between the plate and the foundation takes place, the second geometrical compatibility condition (equation (3.25b)) should be checked for those element nodes outside the contact region. If equation (3.25b) is violated, the problem is resolved over a new modified contact region.

3.6 A Comparison of the Proposed Solution with the Finite Element Method

A finite element study of plates on elastic half-space was first introduced by Cheung and Zienkiewicz[3], followed by a great number of related investigations. In the finite element analyses, both the plate and the surface of the foundation

Fig. 3.5 Contact solution with constant elements

Fig. 3.6 Three possible solutions with quadratic elements

Fig. 3.7 Finite element idealization of plate-elastic foundation system

are discretized, as indicated in Fig.3.7. The plate is represented by a number of plate bending elements of various types. In order to approximate the distribution of the contact pressure, the surface of the elastic half space in contact with the plate is divided into subdomains with the same panels as the plate bending elements. The contact pressure over each panel (or plate bending element) is modelled in several ways. In Cheung and Zienkiewicz[3], and Rajapakse and Selvadurai[4], the distribution of the contact pressure was approximated as discrete concentrated forces acting at corner nodes of the rectangular element. In Cheung and Nag[31], the reaction was assumed constant over each plate element. In Svec and McNeice[32] and Svec and Gladwell[33], the distribution was approximated as cubic polynomials over each ten-noded triangular plate element. In each case the stiffness matrix of the half-space foundation is calculated by employing the Boussinesq solution.

Compared with the finite element analysis of plates on half space, the proposed method has the following features:

- (i) In terms of the plate itself, the dimensionality of the problem is reduced from two to one. As indicated in Fig.3.1, the plate is subjected to boundary discretization only. The domain division is applied to the foundation surface to approximate the contact pressure distribution.
- (ii) The integral solutions for the plate deflection, rotation, moment, and shear force, represented respectively by equations (2.13), (2.14), (2.16) and (2.8), themselves are statements of exact solutions. Errors involved in the present method are due to the numerical approximations of plate boundary variables and the contact pressure. All of the integrals here are evaluated analytically, the details of which are in Chapter 3. The plate internal stress resultants have the same accuracy as the plate deflections.
- (iii) The solutions for the plate internal actions are continuous throughout the interior body of the plate.

3.7 Rigid Plates on Elastic Half-space

For the analysis of plates on elastic half-space foundations, the plate is often considered to be rigid when the relative stiffness of the foundation to the plate approaches a very small value[1,34]. In the case of rigid plates on elastic half-space, the contact pressure between the plate and the foundation becomes singular in the plate edge areas, as shown in Fig. 3.8. This might give rise to questions of validity of the modelling procedure.

In the present numerical method, the singular contact pressure is approximated as a step-wise function (i.e. constant contact elements used.) The present method satisfies all structural analysis requirements. First, the equilibrium condition of a rigid plate subjected to both the external load and the contact pressure is satisfied in the present formulation. As mentioned earlier in Section 2.3, the integral equations for plate bending problems can be developed by using the Rayleigh-Green identity or directly from Maxwell-Betti's theorem, and they are identical. From the viewpoint of Maxwell-Betti's theorem, the equilibrium condition of a plate under both applied loads and contact pressure is automatically satisfied in the integral equations. Secondly, the displacement compatibility conditions between the rigid plate and the elastic half-space are satisfied in each contact node, and finally the plate boundary deflection and rotation are approximated as polynomials in the present boundary element model, which covers the case where a plate can go through rigid body displacement and rotations only. Therefore, the present method can be used to solve rigid plates on elastic half-space foundations, since both the force equilibrium and the displacement compatibility are satisfied in the formulation. Numerical results of rigid plates on elastic half-space foundations will be presented in Chapter 5.

The present procedure is a relatively inefficient numerical method for solving the contact problem between a rigid plate and elastic half-space. It involves bound-

ary discretization to approximate the distribution of the plate boundary deflection and normal rotations. In the case of rigid plates on elastic foundations, the plate deflection and rotation in any direction at any point is explicitly related to the plate central displacement and two rotations along axis x and axis y . Thus, the boundary discretization involved in the present method becomes unnecessary. If the contact analysis between rigid plates and elastic half-space foundations is the objective of the research, the present procedure should be modified accordingly, such as discarding the plate boundary discretization and using a singular contact element[22]. However, it is the intention here to perform a more general purpose contact analysis of rigid plates on elastic half-space.

Fig. 3.8 A rigid plate on elastic half-space

Chapter 4

ANALYTICAL EVALUATION OF INTEGRATIONS

4.1 Introduction

Three types of integrations need to be evaluated in the BEM analysis of plates on elastic half-space, and they are: (a) line integrations involving the product of a shape function and a fundamental solution quantity along each boundary element, (b) surface integrations relating to the product of a shape function and a fundamental solution over each contact element area, and (c) surface integrations of Boussinesq's solution. All of these integrations can be evaluated analytically and they are presented in the following sections.

4.2 Boundary Integrations

All of the line integrals, or boundary integrals, given in Chapters 2 and 3 have the form of

$$IBG = \int_{s_i} L_{ij}(s)G(e, z)ds \quad (4.1)$$

where $G(e, z)$ represents the fundamental solution quantities along the plate boundary profile, such as W^* , Θ_n^* ... etc, which are given in Appendix A, and $L_{ij}(s)$ is the Lagrange shape function which is given in equation (3.14). These shape functions can also be expressed as power series,

$$L_{ij}(s) = \sum_{k=0}^{N_i-1} A_{ik}(z)^k \quad (4.2)$$

where N_i is the number of nodes within the i^{th} boundary element and the constant coefficients A_{ik} are determined by the nodal coordinates.

Fig. 4.1 Straight boundary element i

For a straight boundary profile of element i as shown in Fig.4.1, coordinate s is consistent with the coordinate z while the variable e in the integral is constant. Thus, the line integral along the boundary element i becomes

$$IB_{ij} = \sum_{k=0}^{N_i-1} A_{ik} \int_{z_{i1}}^{z_{iN_i}} (z)^k G(e, z) dz \quad (4.3)$$

in which z_{i1} and z_{iN_i} refer to the coordinate values of the first and last nodes of boundary element i , as indicated in Fig.4.1.

These types of line integrals expressed in equation (4.3) were carried out analytically in closed form by Abdel-Akher and Hartley[26,27] in the course of the BEM analysis of plate bending problems. They compared analytical integration results with the numerical results by Gauss quadrature and concluded that the potential benefits of the analytical evaluation of integration are a more efficient computer program, avoidance of numerical instability, and generally better accuracy.

Abdel-Akher and Hartley[26,27] have summarized all boundary integrals expressed by equation (4.3) as linear combinations of the following five basic forms,

$$\mathbf{I}^k = \int (z)^k \ln(r) dz \quad (4.4a)$$

$$\mathbf{J}^k = \int \frac{(z)^k}{r^2} dz \quad (4.4b)$$

$$\mathbf{K}^k = \int \frac{e(z)^k}{r^4} dz \quad (4.4c)$$

$$\mathbf{L}^k = \int \frac{e^3(z)^k}{r^6} dz \quad (4.4d)$$

$$\mathbf{M}^k = \int \frac{e^5(z)^k}{r^8} dz \quad (4.4e)$$

with

$$r = \sqrt{e^2 + z^2},$$

and they were carried out as

$$\mathbf{I}^k = \frac{1}{k+1} [(z)^{k+1} \ln(r) - \mathbf{J}^{k+2}] \quad (4.5a)$$

$$\mathbf{J}^k = \begin{cases} \left[\frac{1}{k-1} z^{k-1} - e^2 \mathbf{J}^{k-2} \right] & k \geq 3 \\ \left[z - e \tan^{-1}\left(\frac{z}{e}\right) \right] & k=2 \\ \ln(r) & k=1 \\ \frac{1}{e} \tan^{-1}\left(\frac{z}{e}\right) & k=0 \end{cases} \quad (4.5b)$$

$$\mathbf{K}^k = \begin{cases} \left[\frac{e}{2} \left[-\frac{z^{k-1}}{r^2} + (k-1) \mathbf{J}^{k-2} \right] \right] & k \geq 3 \\ \left[\frac{1}{2} \left[-\frac{ez}{r^2} + \tan^{-1}\left(\frac{z}{e}\right) \right] \right] & k=2 \\ -\frac{e}{2r^2} & k=1 \\ \left[\frac{1}{2e^2} \left[\frac{ez}{r^2} + \tan^{-1}\left(\frac{z}{e}\right) \right] \right] & k=0 \end{cases} \quad (4.5c)$$

$$\mathbf{L}^k = \begin{cases} \left[\frac{e^2}{4} \left[-\frac{ez^{k-1}}{r^4} + (k-1) \mathbf{K}^{k-2} \right] \right] & k \geq 3 \\ \left[\frac{1}{8} \left[-\frac{2e^3 z}{r^4} + \frac{ez}{r^2} + \tan^{-1}\left(\frac{z}{e}\right) \right] \right] & k=2 \\ -\frac{e^3}{4r^4} & k=1 \\ \left[\frac{1}{4} \left[\frac{ez}{r^4} + 3\mathbf{K}^0 \right] \right] & k=0 \end{cases} \quad (4.5d)$$

$$\mathbf{M}^k = \begin{cases} \left[\frac{e^2}{6} \left[-\frac{e^3 z^{k-1}}{r^6} + (k-1) \mathbf{L}^{k-2} \right] \right] & k \geq 2 \\ -\frac{e^5}{6r^6} & k=1 \\ \left[\frac{1}{6} \left[\frac{e^3 z}{r^6} + 5\mathbf{L}^0 \right] \right] & k=0 \end{cases} \quad (4.5e)$$

The boundary integrals related to integral equations (3.15) and (3.16) (for calculating plate deflections and rotations) are

$$IBW^* = \sum_{k=0}^{N_i-1} A_{ik} \int_{z_{i1}}^{z_i N_i} (z)^k W^* dz \quad (4.6a)$$

$$IB\Theta_n^* = \sum_{k=0}^{N_i-1} A_{ik} \int_{z_{i1}}^{z_i N_i} (z)^k \Theta_n^* dz \quad (4.6b)$$

$$IBM_n^* = \sum_{k=0}^{N_i-1} A_{ik} \int_{z_{i1}}^{z_i N_i} (z)^k M_n^* dz \quad (4.6c)$$

$$IBV_n^* = \sum_{k=0}^{N_i-1} A_{ik} \int_{z_{i1}}^{z_i N_i} (z)^k V_n^* dz \quad (4.6d)$$

and

$$IB \frac{\partial W^*}{\partial \xi} = \sum_{k=0}^{N_i-1} A_{ik} \int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial W^*}{\partial \xi} dz \quad (4.7a)$$

$$IB \frac{\partial \Theta_n^*}{\partial \xi} = \sum_{k=0}^{N_i-1} A_{ik} \int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial \Theta_n^*}{\partial \xi} dz \quad (4.7b)$$

$$IB \frac{\partial M_n^*}{\partial \xi} = \sum_{k=0}^{N_i-1} A_{ik} \int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial M_n^*}{\partial \xi} dz \quad (4.7c)$$

$$IB \frac{\partial V_n^*}{\partial \xi} = \sum_{k=0}^{N_i-1} A_{ik} \int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial V_n^*}{\partial \xi} dz \quad (4.7d)$$

in which

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k W^* dz = \frac{1}{8\pi D} [\mathbf{I}^{k+2} + e^2 \mathbf{I}^k] \Big|_{z_{i1}}^{z_{iN_i}} \quad (4.8a)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \Theta_n^* dz = \frac{e}{8\pi D} \left[\frac{z^{k+1}}{k+1} + 2\mathbf{I}^k \right] \Big|_{z_{i1}}^{z_{iN_i}} \quad (4.8b)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k M_n^* dz = -\frac{1}{8\pi} \left[(1+3\nu) \frac{z^{k+1}}{k+1} + 2(1+\nu) \mathbf{I}^k + 2(1-\nu) e^2 \mathbf{J}^k \right] \Big|_{z_{i1}}^{z_{iN_i}} \quad (4.8c)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k V_n^* dz = \frac{1}{8\pi} [2(1+\nu) e \mathbf{J}^k + 4(1-\nu) e^2 \mathbf{K}^k] \Big|_{z_{i1}}^{z_{iN_i}} \quad (4.8d)$$

and

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial W^*}{\partial \xi} dz = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \frac{e z^{k+1}}{k+1} + 2e \mathbf{I}^k \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (4.9a)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial \Theta_n^*}{\partial \xi} dz = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{z^{k+1}}{k+1} + 2e^2 \mathbf{J}^k + 2\mathbf{I}^k \\ 2e \mathbf{J}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (4.9b)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial M_n^*}{\partial \xi} dz = \frac{1}{4\pi} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} (1+\nu)e \mathbf{J}^k + 2(1-\nu) \mathbf{K}^{k+2} \\ (1+\nu) \mathbf{J}^{k+1} - 2(1-\nu)e \mathbf{K}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (4.9c)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial V_n^*}{\partial \xi} dz = \frac{1}{4\pi} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} 4(2\nu-1)e \mathbf{K}^k + 8(1-\nu)e \mathbf{L}^k - (1+\nu) \mathbf{J}^k \\ 2(1+\nu) \mathbf{K}^{k+1} + 8(1-\nu) \mathbf{L}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (4.9d)$$

Equations (4.8a)–(4.9d) are reproduced from reference[23]. Similarly, the boundary integrals related to integral equations (2.16) and (2.18) for calculating the moments and shear forces, respectively, can also be expressed in terms of the five basic integral forms (equation (4.4a)–(4.4e)), and they are presented in Appendix B.

4.3 Surface Integration

All of the surface integrations about the product of a shape function and a fundamental solution over each contact element area have the form of

$$IDG = \int_{\Omega_{ci}} N_{ij}(x, y) G(P, Q) d\Omega(Q) \quad (4.10)$$

where $G(P, Q)$ refers to a fundamental solution quantity or a vector which is the function of the distance between the source point P and the field point Q within the area Ω_{ci} as indicated in Fig.4.2. The source point P lies either within the region Ω_{ci} or outside.

For the triangular region Ω_{ci} as shown in Fig.4.2, its area can be found from the areas of three constituent triangles as

Fig. 4.2 Area of contact element i

Fig. 4.3 Local coordinate system of triangle area i

$$A = A_{P12} + A_{P23} - A_{P13} \quad (4.11)$$

The subscripts Pij refer to the vertices of each triangle beginning with the common vertex P and in counterclockwise fashion around the triangles. In a right-handed coordinate system (x, y) , the value of the area of a triangle determined by its vertices coordinates is positive if the vertices are numbered counterclockwise along the triangle; otherwise, it is negative. For example, $A_{P13} = -A_{P31}$. Therefore,

$$A = A_{P12} + A_{P23} + A_{P31} \quad (4.12)$$

The surface integration over the region Ω_{ci} can be evaluated as the sum of the three integrations over three triangles, i.e.

$$\begin{aligned} IDG &= \sum_{i=1}^3 \int_{\Delta_i} N_{ij}(x, y) G(P, Q) dx dy \\ &= \sum_{i=1}^3 IDG_i \end{aligned} \quad (4.13)$$

where Δ_i refers to the i^{th} triangle ($i=1,2,3$). The surface integral over Δ_i is carried out in a local coordinate system (n_i, t_i) as shown in Fig.4.3, where n_i and t_i are axes normal and parallel to the base ij of triangle Pij . The original point of this local coordinate system is located at the source point P . Similarly, a surface integral over an N -sided polygonal region is the sum of N integrals over N triangles.

From equation (3.3), the shape function N_{ij} in equation (4.13) has the expression of

$$N_{ij}(x, y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 \quad (4.14)$$

in the global coordinate system. Substituting the coordinate transformation formulas

$$x = x_P + n_i \cos \alpha - t_i \sin \alpha \quad (4.15a)$$

$$y = y_P + n_i \sin \alpha + t_i \cos \alpha \quad (4.15b)$$

into equation (4.14) yields

$$N_{ij}(n_i, t_i) = b_1 + b_2 n_i + b_3 t_i + b_4 n_i t_i + b_5 n_i^2 + b_6 t_i^2 \quad (4.16)$$

where x_P and y_P are coordinates of point P in the global coordinate system (x, y) ; and α is the angle between axes x and n_i . The constant coefficients in this equation are:

$$b_1 = a_1 + a_2 x_P + a_3 y_P + a_4 x_P^2 + a_5 x_P y_P + a_6 y_P^2 \quad (4.17a)$$

$$b_2 = a_2 \cos \alpha + a_3 \sin \alpha + 2a_4 x_P \cos \alpha + a_5 (y_P \cos \alpha + x_P \sin \alpha) + 2a_6 y_P \sin \alpha \quad (4.17b)$$

$$b_3 = -a_2 \sin \alpha + a_3 \cos \alpha - 2a_4 x_P \sin \alpha + a_5 (x_P \cos \alpha - y_P \sin \alpha) + 2a_6 y_P \cos \alpha \quad (4.17c)$$

$$b_4 = (2a_6 - 2a_4) \cos \alpha \sin \alpha + a_5 (\cos^2 \alpha - \sin^2 \alpha) \quad (4.17d)$$

$$b_5 = a_4 \cos^2 \alpha + a_5 \sin \alpha \cos \alpha + a_6 \sin^2 \alpha \quad (4.17e)$$

$$b_6 = a_4 \sin^2 \alpha - a_5 \sin \alpha \cos \alpha + a_6 \cos^2 \alpha \quad (4.17f)$$

The surface integration over the i^{th} triangle now becomes

$$\begin{aligned} IDG_i &= \int_{\Delta_i} (b_1 + b_2 n_i + b_3 t_i + b_4 n_i t_i + b_5 n_i^2 + b_6 t_i^2) G(P, Q) dn_i dt_i \\ &= \int_0^{E_i} \int_{\frac{Z_{i1} n_i}{E_i}}^{\frac{Z_{i2} n_i}{E_i}} (b_1 + b_2 n_i + b_3 t_i + b_4 n_i t_i + b_5 n_i^2 + b_6 t_i^2) G(P, Q) dt_i dn_i \\ &= (b_1 IDG_i^1 + b_2 IDG_i^n + b_3 IDG_i^t + b_4 IDG_i^{nt} + b_5 IDG_i^{n^2} + b_6 IDG_i^{t^2}) \end{aligned} \quad (4.18)$$

in which

$$IDG_i^1 = \int_0^{E_i} \int_{\frac{Z_{i1} n_i}{E_i}}^{\frac{Z_{i2} n_i}{E_i}} G(n_i, t_i) dt_i dn_i \quad (4.19a)$$

$$IDG_i^n = \int_0^{E_i} \int_{\frac{Z_{i1}n_i}{E_i}}^{\frac{Z_{i2}n_i}{E_i}} n_i G(P, Q) dt_i dn_i \quad (4.19b)$$

$$IDG_i^t = \int_0^{E_i} \int_{\frac{Z_{i1}n_i}{E_i}}^{\frac{Z_{i2}n_i}{E_i}} t_i G(P, Q) dt_i dn_i \quad (4.19c)$$

$$IDG_i^{nt} = \int_0^{E_i} \int_{\frac{Z_{i1}n_i}{E_i}}^{\frac{Z_{i2}n_i}{E_i}} n_i t_i G(P, Q) dt_i dn_i \quad (4.19d)$$

$$IDG_i^{n^2} = \int_0^{E_i} \int_{\frac{Z_{i1}n_i}{E_i}}^{\frac{Z_{i2}n_i}{E_i}} n_i^2 G(P, Q) dt_i dn_i \quad (4.19e)$$

$$IDG_i^{t^2} = \int_0^{E_i} \int_{\frac{Z_{i1}n_i}{E_i}}^{\frac{Z_{i2}n_i}{E_i}} t_i^2 G(P, Q) dt_i dn_i \quad (4.19f)$$

and distances E_i , Z_{i1} and Z_{i2} are illustrated in Fig.4.2.

For calculating the plate deflection $W(P)$ expressed in equation (3.15), the corresponding $G(P, Q)$ in the above double integrals is

$$G(P, Q) = w^*(P, Q) = \frac{r_i^2}{8\pi D} \ln(r_i), \quad r_i^2 = n_i^2 + t_i^2 \quad (4.20)$$

By using the software *Mathematica*[35] or any mathematical handbook, these double integrals in equations (4.19a)-(4.19f) can be carried out as follows:

$$ID(w^*)_i^1 = \frac{1}{8\pi D} \left[\frac{E_i Z_i}{12} (Z_i^2 + 3E_i^2) \ln(R_i) - \frac{7}{144} E_i Z_i^3 - \frac{33}{144} E_i^3 Z_i + \frac{E_i^4}{6} \tan^{-1} \frac{Z_i}{E_i} \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.21a)$$

$$ID(w^*)_i^n = \frac{1}{8\pi D} \left[\frac{E_i^2}{15} (Z_i (Z_i^2 + 3E_i^2) \ln(R_i) - \frac{13}{5} E_i^2 Z_i - \frac{8}{15} Z_i^3 + 2E_i^3 \tan^{-1} \frac{Z_i}{E_i}) \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.21b)$$

$$ID(w^*)_i^t = \frac{1}{8\pi D} \left[\frac{E_i}{400} (20R_i^4 \ln(R_i) - 18E_i^2 Z_i^2 - 9Z_i^4) \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.21c)$$

$$ID(w^*)_i^{nt} = \frac{1}{8\pi D} \left[\frac{E_i^2}{24} R_i^4 \ln(R_i) - \frac{5}{288} E_i^2 R_i^4 \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.21d)$$

$$ID(w^*)_i^{n^2} = \frac{1}{8\pi D} \left[\frac{E_i^3}{6} \left((E_i^2 Z_i + \frac{Z_i^3}{3}) \ln(R_i) - \frac{5}{6} E_i^2 Z_i - \frac{1}{6} Z_i^3 + \frac{2}{3} E_i^3 \tan^{-1} \frac{Z_i}{E_i} \right) \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.21e)$$

$$ID(w^*)_i^{t^2} = \frac{1}{8\pi D} \left[\frac{E_i}{90} (5E_i^2 Z_i^3 + 3Z_i^5) \ln(R_i) - \frac{1}{60} E_i^3 Z_i^3 - \frac{11}{900} E_i Z_i^5 + \frac{1}{45} E_i^5 Z_i - \frac{1}{45} E_i^6 \tan^{-1} \frac{Z_i}{E_i} \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.21f)$$

where

$$R_i^2 = Z_i^2 + E_i^2$$

For calculating the plate rotations $\frac{\partial w}{\partial \xi}$ which are expressed in equation (3.16), the corresponding $G(P, Q)$ becomes

$$G(P, Q) = \frac{\partial w^*(P, Q)}{\partial \xi} = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \begin{Bmatrix} n_i + 2n_i \ln(r_i) \\ t_i + 2t_i \ln(r_i) \end{Bmatrix} \quad (4.22)$$

in which ξ_n and ξ_t are the directional cosines of direction ξ with respect to axes n_i and t_i . In this case, the associated double integrations are carried out as,

$$ID\left(\frac{\partial w^*}{\partial \xi}\right)_i^t = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \begin{Bmatrix} \frac{E_i^2}{9} [6Z_i \ln(R_i) - 5Z_i + 6E_i \tan^{-1} \frac{Z_i}{E_i}] \\ \frac{E_i}{3} (R_i^2 \ln(R_i) - \frac{1}{3} Z_i^2) \end{Bmatrix} \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.23a)$$

$$ID\left(\frac{\partial w^*}{\partial \xi}\right)_i^n = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \begin{Bmatrix} \frac{E_i^3}{8} [1Z_i \ln(R_i) - 3Z_i + 4E_i \tan^{-1} \frac{Z_i}{E_i}] \\ \frac{E_i^2}{16} [4R_i^2 \ln(R_i) - Z_i^2] \end{Bmatrix} \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.23b)$$

$$ID\left(\frac{\partial w^*}{\partial \xi}\right)_i^t = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{E_t^2}{16} [4R_i^2 \ln(R_i) - Z_i^2] \\ \frac{E_t}{12} [12Z_i^3 \ln(R_i) - Z_i^3 + 12E_i^2 Z_i - 12E_i^3 \tan^{-1} \frac{Z_i}{E_i}] \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.23c)$$

$$ID\left(\frac{\partial w^*}{\partial \xi}\right)_i^{nt} = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{E_t^3}{25} [5R_i^2 \ln(R_i) - Z_i^2] \\ \frac{2E_t^2}{15} [Z_i^3 \ln(R_i) - \frac{1}{30} Z_i^3 + E_i^2 Z_i - E_i^3 \tan^{-1} \frac{Z_i}{E_i}] \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.23d)$$

$$ID\left(\frac{\partial w^*}{\partial \xi}\right)_i^{n2} = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{2E_t^4}{5} [Z_i \ln(R_i) - \frac{7Z_i}{10} + E_i \tan^{-1} \frac{Z_i}{E_i}] \\ \frac{E_t^3}{5} [R_i^2 \ln(R_i) - \frac{1}{5} Z_i^2] \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.23e)$$

$$ID\left(\frac{\partial w^*}{\partial \xi}\right)_i^{t2} = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{2E_t^2}{15} [Z_i^3 \ln(R_i) - \frac{1}{30} Z_i^3 + E_i^2 Z_i - E_i^3 \tan^{-1} \frac{Z_i}{E_i}] \\ \frac{E_t R_i^2}{10} (Z_i^2 - E_i^2) \ln(R_i) + \frac{E_t^3 Z_i^2}{20} + \frac{1}{200} E_t Z_i^4 \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.23f)$$

The double integrations required for calculating the plate moments and shear forces in equations (2.16) and (2.18) can be obtained by replacing $G(P, Q)$ in equations (4.19a)–(4.19f) by its corresponding form of

$$\left\{ \begin{array}{c} \frac{\partial^2 w^*}{\partial x^2} \\ \frac{\partial^2 w^*}{\partial y^2} \\ \frac{\partial^2 w^*}{\partial x \partial y} \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} \frac{\partial \nabla^2 w^*}{\partial x} \\ \frac{\partial \nabla^2 w^*}{\partial y} \end{array} \right\},$$

respectively. These double integrations are presented in Appendix C.

Although the surface integrations are evaluated for a quadratic function of $N_{ij}(x, y)$, surface integration about higher polynomial functions can be carried out in the same way.

4.3.1 Accuracy of Gauss Quadrature

In this section, Gauss quadrature is used to evaluate the surface integration,

$$IDG = \int_{\Omega_{ci}} N_{ij}(x, y) G(P, Q) d\Omega(Q), \quad (4.10)$$

over a square area as shown in Fig.4.4. The fundamental solution quantity $G(P, Q)$ represents the following four particular functions,

$$G(P, Q) = w^* = \frac{1}{8\pi D} r^2 \ln(r) \quad (4.24a)$$

$$G(P, Q) = \frac{\partial w^*}{\partial x_P} = -\frac{1}{8\pi D} [(x - x_P) + 2(x - x_P) \ln(r)] \quad (4.24b)$$

$$G(P, Q) = \frac{\partial^2 w^*}{\partial x_P^2} = \frac{1}{8\pi D} \left[1 + \frac{2(x - x_P)^2}{r^2} + 2 \ln(r) \right] \quad (4.24c)$$

$$G(P, Q) = \frac{\partial \nabla^2 w^*}{\partial x_P} = -\frac{1}{8\pi D} \frac{4(x - x_P)}{r^2} \quad (4.24d)$$

with

$$r = \sqrt{(x - x_P)^2 + (y - y_P)^2}$$

where r denotes the distance between points P and Q and (x_P, y_P) are the coordinate values of point P . The distributions of these four functions over the square integral area are illustrated in Fig.4.5 for the case of $(x_P = 0.0, y_P = 0.0)$. At the area near the source point P , functions $\frac{\partial^2 w^*}{\partial x_P^2}$ and $\frac{\partial \nabla^2 w^*}{\partial x_P}$ show singular behaviour of order $O(\ln(r))$ and $O(\frac{1}{r})$, respectively.

For the sake of simplicity, let $N_{ij}(x, y) = 1$. Substituting the expression of equations (4.24a-d) into equation (4.10) leads to

$$IDw^* = \frac{1}{8\pi D} \int_{-1}^1 \int_{-1}^1 r^2 \ln(r) dx dy \quad (4.25a)$$

Fig. 4.4 Square integration area

$$IDw_{,x}^* = -\frac{1}{8\pi D} \int_{-1}^1 \int_{-1}^1 [(x - x_P) + 2(x - x_P)\ln(r)] dx dy \quad (4.25b)$$

$$Dw_{,xx}^* = \frac{1}{8\pi D} \int_{-1}^1 \int_{-1}^1 [1 + \frac{2(x - x_P)^2}{r^2} + 2\ln(r)] dx dy \quad (4.25c)$$

$$ID\nabla^2 w_{,x}^* = -\frac{1}{8\pi D} \int_{-1}^1 \int_{-1}^1 \frac{4(x - x_P)}{r^2} dx dy \quad (4.25d)$$

Gauss quadrature results of the above four integrals are presented in Table 4.1 for various Gauss point meshes and for three different locations of the source point. The exact values are also given in Table 4.1 for comparison, which are obtained by the formulas developed in Section 4.3. It is discovered from Table 4.1 that when the source point is located at the corner of the integration area (case(1)) very accurate Gauss quadrature results are obtained for all four integrals. When the source point P is located within the integration area (case(2) and case (3)), the numerical integration results for integrals about $\frac{\partial^2 w^*}{\partial x_P^2}$ and $\frac{\partial \nabla^2 w^*}{\partial x_P}$ are unstable,

(a)

(b)

(c)

(d)

Fig. 4.5 Distributions of (a) $8\pi D w^*$, (b) $-8\pi D \frac{\partial w^*}{\partial x}$,
 (c) $8\pi D \frac{\partial^2 w^*}{\partial x^2}$ and (d) $-8\pi D \frac{\partial \nabla^2 w^2}{\partial x}$

Table 4.1 Some Gauss quadrature results ($\times 8\pi D$)

Gauss points mxm	IDw^*	$-IDw^*_{,x}$	$IDw^*_{,xx}$	$-ID \nabla^2 w^*_{,x}$
case (1) $x_P = -1.0$ $y_P = -1.0$				
2x2	6.424081	8.057714	10.537020	9.000000
3x3	6.390569	8.094173	10.582220	9.062693
4x4	6.390307	8.096663	10.593630	9.071047
5x5	6.390269	8.097182	10.597540	9.070121
6x6	6.390260	8.097330	10.599160	9.067829
8x8	6.390256	8.097401	10.600320	9.063963
Exact values	6.390256	8.097419	10.600951	9.055774
case (2) $x_P = 0.0$ $y_P = 0.0$				
2x2	-0.540620	0.000000	6.378139	0.000000
3x3	-0.016303	0.000000	$-\infty$	∞
4x4	-0.266415	0.000000	5.419504	0.000000
5x5	-0.119752	0.000000	$-\infty$	∞
6x6	-0.253890	0.000000	5.223642	0.000000
8x8	-0.251814	0.000000	5.152482	0.000000
Exact values	-0.250828	0.0	5.055774	0.0
case (3) $x_P = -0.22$ $y_P = 0.0$				
2x2	-0.387388	1.383331	5.120749	0.381318
3x3	-0.088759	0.894661	5.707774	14.167110
4x4	-0.136544	1.170816	5.177168	1.301553
5x5	-0.126977	1.095372	5.673252	5.418303
6x6	-0.128580	1.133066	5.108461	2.418025
8x8	-0.127943	1.124812	5.181093	3.217552
Exact values	-0.127641	1.127524	5.264255	2.778798

especially the latter one because of its stronger singularity. The infinite values in Table 4.1 are due to the coincidence of a Gauss point and the singular point P .

4.4 Explicit Boussinesq's Solution

Recalling the Boussinesq's solution over the discretized domain,

$$w_s(P) = \frac{(1 - \nu_s)}{2\pi G_s} \sum_{i=1}^{M_e} \sum_{j=1}^{M_i} \int_{\Omega_{ci}} \frac{N_{ij}(x, y)}{\sqrt{(x - x_P)^2 + (y - y_P)^2}} dx dy \quad (3.6)$$

$$+ w_{se}(P)$$

Thus, the surface integrations that need to be evaluated have the form of

$$I = \int_{\Omega_{ci}} \frac{N_{ij}(x, y)}{\sqrt{(x - x_P)^2 + (y - y_P)^2}} dx dy \quad (4.26)$$

Svec and Gladwell[36] presented the explicit Boussinesq's solution for a cubic polynomial distributed $N_{ij}(x, y)$ over a triangular area by using complex arithmetic to express the integrals. In the present work, the surface integration in equation (4.26) over an N -sided polygonal region is carried out analytically by using the same technique just described in the previous section.

First, the integration over an N -sided plane is replaced by the sum of N integrations over N triangular regions,

$$I = \sum_{i=1}^N \int_{\Delta_i} \frac{N_{ij}(x, y)}{\sqrt{(x - x_P)^2 + (y - y_P)^2}} dx dy \quad (4.27)$$

$$= \sum_{i=1}^N I_i$$

Then, the integrand of the surface integration is transformed into a local coordinate system (n_i, t_i) as shown in Fig.4.3,

$$\begin{aligned}
I_i &= \int_{\Delta_i} \frac{N_{ij}(n_i, t_i)}{\sqrt{n_i^2 + t_i^2}} dn_i dt_i \\
&= \int_0^{E_i} \int_{\frac{Z_{i1}n_i}{E_i}}^{\frac{Z_{i2}n_i}{E_i}} (b_1 + b_2 n_i + b_3 t_i + b_4 n_i t_i + b_5 n_i^2 + b_6 t_i^2) \frac{1}{\sqrt{n_i^2 + t_i^2}} dt_i dn_i \quad (4.28) \\
&= (b_1 I_i^1 + b_2 I_i^n + b_3 I_i^t + b_4 I_i^{nt} + b_5 G_i^{n^2} + b_6 I_i^{t^2})
\end{aligned}$$

in which the coefficients, b_i , and values, E_i , Z_{i1} and Z_{i2} , have already been defined in section 4.3.

Finally, again using *Mathematica*, each of the double integrals in equation (2.28) are carried out as,

$$I_i^1 = E_i \ln(Z_i + \sqrt{E_i^2 + Z_i^2}) \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.27a)$$

$$I_i^n = \frac{1}{2} E_i^2 \ln(Z_i + \sqrt{E_i^2 + Z_i^2}) \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.27b)$$

$$I_i^t = \frac{1}{2} E_i \sqrt{E_i^2 + Z_i^2} \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.27c)$$

$$I_i^{nt} = \frac{1}{3} E_i^2 (\sqrt{E_i^2 + Z_i^2}) \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.27d)$$

$$I_i^{n^2} = \frac{1}{3} E_i^3 \ln(Z_i + \sqrt{E_i^2 + Z_i^2}) \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.27e)$$

$$I_i^{t^2} = \frac{1}{6} E_i [Z_i \sqrt{E_i^2 + Z_i^2} - E_i^2 \ln(Z_i + \sqrt{E_i^2 + Z_i^2})] \Big|_{Z_{i1}}^{Z_{i2}} \quad (4.27f)$$

Chapter 5

NUMERICAL RESULTS

5.1 Introduction

This chapter presents numerical results of elastic thin plates resting on unilateral elastic foundations, using the formulation developed in the previous chapters. In the case of a Winkler type foundation model, results are given for plates with various boundary conditions and under various loadings. For plates on elastic half-space foundations, the results presented in this chapter are for plates with free edge boundary conditions. Results of plates with the boundary condition of attached edge beams will be given in Chapter 6.

There are three parameters which are selected on a case by case basis having effects on the accuracy of numerical results. They are the number of boundary elements, the number of contact elements and the location of boundary source points outside the plate domain.

The effect with regard to the number of boundary elements on the accuracy of numerical results can be found in many publications relating to the boundary element analysis of plate bending problems. For example, the convergence due to the increase of boundary elements (h-method) and the increase of boundary element order (p-method) was well demonstrated by Abdel-Akher in his thesis[23] for plates with various boundary conditions and plane geometries. Therefore, it is not necessary to go into details here. However, in order to illustrate the fact that sufficient accuracy can easily be achieved, a simply supported square plate, subjected to a uniform pressure load and without being supported on any elastic

foundation, is solved by modelling the plate boundary into various element meshes. Plate central point deflections and moments are given in Table 5.1. It is seen that errors almost vanish when each plate edge is modelled as one 4th order boundary element or three 2nd order boundary elements.

The effect with regard to the number of contact elements on the resulting accuracy will be demonstrated later in the course of presenting some results for plates on Winkler or elastic half-space foundations. The effect with regard to the location of boundary source points is examined in the next section.

Table 5.1 Results of simply supported square plate under uniform load ($\nu = 0.3$)

no. of elements per edge	$\frac{wD}{qa^4}$	$\frac{M_{xx}}{qa^2}$
1 linear	0.0017277	0.030204
2 linear	0.0037755	0.045636
3 linear	0.0039770	0.047114
4 linear	0.0040164	0.047452
1 2nd order	0.0040971	0.048089
1 3rd order	0.0040628	0.047839
1 4th order	0.0040628	0.047887
2 2nd order	0.0040583	0.047833
3 2nd order	0.0040618	0.047881
analytical results	0.00406	0.04788

5.2 Location of Boundary Source Points

As mentioned in Section 3.3, the set of $2N$ linear algebraic equations is formed by applying the discretized integral equations (3.17) and (3.18) to N source points along the plate edges. These N source points are located outside the plate domain at a very short distance, say ϵ , from their corresponding boundary nodes as shown in Fig. 5.1.

Those source points associated with the non-corner boundary nodes are positioned along the normal direction of the boundary and the others are positioned along the corner bisector directions. If the boundary source points are placed directly on the boundary nodes, i.e. $\epsilon = 0$, the line integrals along the plate boundary in equations (3.17) and (3.18) will contain singular kernels having order of $\ln(\epsilon)$, $1/\epsilon$ and $1/\epsilon^2$, respectively, according to the formulation presented in Appendix A.

In order to illustrate the effect with regard to the boundary source location on the accuracy of numerical solutions, a uniformly loaded square plate with free edge boundary conditions and being supported on a Winkler foundation is solved here by positioning the boundary source points at various distances from the plate domain. The plate central point deflections, central point moments and the normal rotations at the mid-edge point corresponding to various values of ϵ are presented in Table 5.2.

The results given in Table 5.2 are calculated by modelling each edge of the square plate with two 2nd order boundary elements, modelling the whole contact area with four constant contact elements and using the following input data: plate stiffness $D = 69230$, plate edge length $a = 10$, poisson's ratio $\nu = 0.3$, uniform load $q = 200$ and Winkler stiffness $k_f = 200$. This simple problem has an exact solution of unit value for deflection and zero values for rotation and moment over the whole plate. It is shown in Table 5.2 that errors are so small that they can be negligible

Fig.5.1 Boundary nodes and their corresponding source points

Table 5.2 Results of uniformly loaded free edge square plate on Winkler foundation

$\epsilon =$	central deflection w	central moments M_{xx}	mid edge rotation θ_n
10a	1.0031	10.327	0.006205
5a	1.0001	-0.008731	0.000261
4a	1.0000	0.024352	-0.000051
3a	1.0000	-0.000685	0.000001
a	1.0000	0.33×10^{-7}	-0.23×10^{-9}
a/10	1.0000	0.42×10^{-11}	-0.28×10^{-14}
a/100	1.0000	0.12×10^{-10}	0.74×10^{-14}
a/10000	1.0000	0.42×10^{-10}	0.84×10^{-14}
exact results	1.0	0.0	0.0

when the value of ϵ is within the range from $a/10000$ to a where a is the square plate's side length.

Very accurate numerical results can be obtained by using so small a value of ϵ because all the line integrations, among which some contain singular kernels like $\ln(\epsilon)$, $1/\epsilon$ and $1/\epsilon^2$, are evaluated analytically. Abdel-Akher[23] has demonstrated that ϵ could be as small as $s_n/10000$ in which s_n referred to the average spacing between the boundary nodes. The value of ϵ used to calculate all examples in this thesis will be within the range from $a/100$ to $a/1000$.

5.3 Contact Nodes on Plate Boundary

When the six-node triangular contact elements are used to model the contact area between the plate and the elastic foundation, some of the contact element nodes are located on the plate boundary, as shown in Fig.5.2.

Fig.5.2 Contact nodes and boundary contact node source points

In the present solution procedure, integral equations for calculating plate deflections at these contact nodes need to be formed which are part of the M simultaneous equations (3.20). If these integral equations are formed by positioning source points directly at the contact nodes which are located on the plate boundary, singular integration problems arise again for the same reasons described in the above section.

In order to avoid dealing with this singular integration problem, the source points corresponding to plate boundary contact nodes are positioned at a very short distance inside the plate domain, say $\hat{\epsilon}$, as shown in Fig.5.2. The value of $\hat{\epsilon}$ is taken as $a/10000$ where ' a ' refers to the plate side length. This treatment results in the plate deflection at a plate boundary contact node approximately equal to the plate deflection at another point which is adjacent to the contact node with very short distance $\hat{\epsilon}$. This technique has been used by Abdel-Alker[23] in dealing with point supports along the plate boundary.

5.4 Plates on Winkler Foundations

The examples given in this section are square plates with various boundary conditions and in unilateral contact with Winkler foundations. The results are presented for varying values of relative stiffness defined as

$$k = \frac{k_f a^4}{D} \quad (5.1)$$

in which a is the side length of the square plate.

Each side of the plate boundary is modelled using a 9-noded, or 8th order, boundary element as shown in Fig.5.3(a). The plate-foundation initial contact area is divided into a uniform mesh, typically 10×10 constant contact elements, as shown in Fig. 5.3(b), in the examples in this section. In all cases in which separation actually occurs, the iteration is completed after 3 or 4 cycles.

(a) boundary discretization

(b) 10 x 10 constant elements

(c) 32 quadratic elements

Fig.5.3 Boundary and contact element meshes

Example 5.1. Plate with simple edge supports

The numerical results of a simply supported plate subjected to a uniformly distributed load are presented here to provide a comparison with an available analytical solution[21]. The centerline deflections, downward positive, are given in Fig.5.4(a) for a range of relative stiffness values. The corresponding moments M_{xx} along the plate centerline are given in Fig. 5.4(b). These numerical results are calculated by using the constant contact element mesh of 10×10 shown in Fig.5.3(b), and they agree with the analytical solution very well. The differences for both deflections and moments are less than 1% and are not discernible given the scale of the figures.

In order to illustrate the effect with regard to the number of contact elements and the order of contact elements on the numerical result accuracy, this simply supported plate is solved by using different contact element meshes, such as 5×5 constant elements, 8×8 constant elements and 32 six-noded quadratic contact elements (Fig.5.3(c)). For the specified value of relative foundation stiffness $k = 1280$, the plate deflections and moments along the x axis are presented in Table 5.3, along with the results based on a mesh of 10×10 constant elements. The analytical solution of this plate is also given in Table 5.3. An indication of convergence with the refinement of the contact element mesh can be seen in Table 5.3. It is also seen from Table 5.3 that six-node quadratic contact elements model this problem more effectively than constant contact elements do.

Example 5.2. Plate with clamped edge supports

Deflections and moments along the centerline of a clamped square plate under a uniform distributed load are given in Figs. 5.5(a) and 5.5(b), which are calculated again by using a contact element mesh of 10×10 . Results for the same problem given by Ng[37] are also provided for comparison. The largest difference of 10% between the moments applies for the largest value of $k = 3200$.

Table 5.3 Results of simply supported plate on Winkler foundation via various domain division ($k=1280$, $\nu=0.15$)

Dellection $\frac{w(x,y)D}{qa^4} \times 10^{-3}$										
(x,y)	analytical solution	BEM								
		5x5 const.	error (%)	8x8 const.	error (%)	10x10 const.	error (%)	32 quad.	error (%)	
(0,0,0,0)	0.8852	0.9082	2.5	0.8941	1.0	0.8908	0.6	0.8862	0.1	
(1,0,0,0)	0.8549	0.8772	2.6	0.8636	1.0	0.8604	0.6	0.8560	0.1	
(2,0,0,0)	0.7592	0.7786	2.6	0.7667	1.0	0.7640	0.6	0.7600	0.1	
(3,0,0,0)	0.5857	0.6006	2.5	0.5914	1.0	0.5893	0.6	0.5863	0.1	
(4,0,0,0)	0.3267	0.3346	2.4	0.3299	1.0	0.3287	0.6	0.3270	0.1	
Moment $\frac{M_{xx}}{qa^2}$										
(0,0,0,0)	0.00684	0.00699	2.2	0.00693	1.3	0.00689	0.7	0.00685	0.1	
(1,0,0,0)	0.00735	0.00761	3.5	0.00742	1.0	0.00740	0.7	0.00736	0.1	
(2,0,0,0)	0.00854	0.00870	1.9	0.00861	0.8	0.00860	0.7	0.00855	0.1	
(3,0,0,0)	0.00931	0.00960	3.1	0.00941	1.1	0.00937	0.6	0.00932	0.1	
(4,0,0,0)	0.00755	0.00764	1.2	0.00768	1.7	0.00760	0.7	0.00754	-0.1	

Example 5.3. Plate with free edges

For a plate with free edge under a concentrated load P at the center, the centerline deflections are plotted in Fig. 5.6(a) for several relative stiffness values. Some of the deflections are plotted as negative values showing that liftoff occurs. The corresponding contact pressure distribution along the plate centerline is given in Fig. 5.6(b), where the contact pressure for the relative stiffness $k = 86.4$ and $k = 328.5$ is plotted for comparison with published FEM results[3]. As the plate stiffness increases, the contact pressure becomes increasingly uniform; the case of $k = 86.4$ was treated as a rigid plate in reference[3]. Following separation, the geometry of the foundation contact area is almost a circle. The shaded areas, plotted in Fig. 5.7, illustrate the contact region for the four relative stiffness values.

Example 5.4. Plate subject to forces and moments

A simple mat foundation is shown in Fig. 5.8(a). The slab thicknesses are 0.3 m, 0.5 m and 0.7 m, corresponding respectively to k values of 3200, 691 and 252. The slab boundary is divided into four eighth order boundary elements and again the contact domain is divided into 10×10 constant elements.

The deflections are given in an overall sense in Fig. 5.8(b) for the slab thickness of 0.3 m. In this figure the maximum deflection is 0.0108 m at the corner and the central deflection is -0.0011 m, where the negative again implies loss of contact. The corresponding contact pressure distribution is shown in Fig. 5.8(c) along axes of $y = 0.0$, $y = -1.5$, $y = -2.5$, $y = -3.5$ and $y = -5.0$. The contact pressure at the corner point is 4.5 times the average value of 50 KN/m^2 .

For the range of footing thickness, Fig. 5.9 shows the distribution of the contact pressure along the edge of the slab. This slab keeps full contact with the foundation for the thickness of $h = 0.5$ m and $h = 0.7$ m. When $h = 0.7$ m, the contact pressure at the corner point is about 2.6 times that of the central point. Applying

conventional design methods (where the mat foundation is assumed rigid), the foundation reaction for this example is equal to 50 kN/m^2 and uniformly distributed over the bottom surface of the footing for any slab thickness.

Fig.5.4(a) Deflections along axis x of uniformly loaded simply supported plate

Fig.5.4(b) Moments M_{xx} along axis x of uniformly loaded simply supported plate

Fig.5.5(a) Deflections along axis x of uniformly loaded clamped plate

Fig.5.5(b) Moments M_{xx} along axis x of uniformly loaded clamped plate

Fig.5.6(a) Deflections along axis x of centrally loaded free edge plate

Fig.5.6(b) Contact pressure along axis x of centrally loaded free edge plate

Fig.5.7 Contact areas of centrally loaded free edge plate

Fig.5.8(a) A mat foundation

Fig.5.8(b) Deformation of mat foundation

Fig.5.8(c) Contact pressure along various axes of mat foundation
($h = 0.3 \text{ m}$, $k = 3200$)

Fig.5.9 Contact pressure along the edge of mat foundation

5.5 Plates on Elastic Half-space

The examples given in this section are limited to square plates on an elastic half-space. The software that was developed to carry out these analyses, however, is capable of analyzing general polygonal plate shapes, subject to general distributions of loading. All of the plate examples in this section have free edge boundary conditions.

In the presentation of the results of square plates, a foundation to plate relative stiffness parameter,

$$\gamma = \frac{\pi E_s a^3}{8D(1 - \nu_s^2)} \quad (5.2)$$

is used, which was defined by Gorbunov-Possadov and Serebrjanyi[34] more generally for rectangular plates supported on elastic half-space. The symbol 'a' in equation (5.2) refers to the side length of a square plate.

The smaller the relative stiffness parameter γ is, the stiffer the plate. It was concluded in reference[34] that the plate could be considered rigid when the value of the relative stiffness parameter satisfies $\gamma \leq 8$ (in the case of a square plate). The displacement, again modified here for a rigid square plate, given by Gorbunov Possadov and Serebrjanyi was

$$w_r = \frac{0.913(1 - \nu_s^2)P_o}{aE_s} \quad (5.3)$$

for plates under symmetrical loading, in which P_o is the magnitude of the total vertical loads.

In the case of a rigid plate under eccentric loadings, Gorbunov-Possadov and

Serebrjanyi gave the following two solutions,

$$tg\phi_x = \frac{8(1 - \nu_s^2)K_1M_x}{a^3} \quad (5.4a)$$

and

$$tg\phi_y = \frac{8(1 - \nu_s^2)K_2M_y}{a^3} \quad (5.4b)$$

to calculate the rigid body rotations, ϕ_x and ϕ_y , of a rectangular plate. Symbols M_x and M_y in equations (5.4a) and (5.4b) were denoted as the external moment loads acting directly along the symmetrical axes x and y or the resultant couples along axes x and y . In the case of an eccentrically loaded square rigid plate, coefficients K_1 and K_2 in equations (5.4a) and (5.4b) were approximately equal to 0.5, which were interpreted from figures given in reference[34].

In the present numerical study associated with square plates, each edge of the square plate is modelled as one fifth order boundary element shown in Fig.5.10(a). Unless specifically mentioned, the initial contact area is divided uniformly into an 11×11 mesh of constant elements, shown as mesh 1 in Fig.5.10(b). Only a quarter of the domain division is plotted in Fig. 5.10(b) due to symmetry. The values of the Poisson's ratio for both the plate and the elastic foundation are taken as 0.15.

Example 5.5. Plate subjected to a uniform distributed load q

The deflections for the uniformly distributed load case in Fig. 5.11(a) are normalized by dividing these by the rigid plate displacement, given in equation (5.3). These are computed along the x axis of the square plate and are given for various relative stiffness parameters. The plate displacement approaches a straight line for the lowest value of $\gamma = 3.016$, corresponding to the case of near rigid plate.

The distributions of the moments M_{xx} and M_{yy} along the x axis of the plate subjected to uniform loading are shown in Figs. 5.11(b) and 5.11(c), for the same

range of relative stiffness parameters. The quantity P_0 is the total vertical load given as qa^2 . The contact pressure distributions along the x axis of the square plate are illustrated in Fig. 5.11(d), where, for clarity, straight lines are shown connecting the nodal values. The square plate maintains full contact with the half-space in this case.

Example 5.6. Plate subjected to a concentrated load P at center

In Figs. 5.12(a) through 5.12(d) the results are given for deflections, moments and contact pressure for a square plate subjected to a concentrated load applied at the plate center. The values given are plate centerline values for a range of relative stiffnesses from $\gamma = 0.03$ to 3016. In Fig. 5.12(a) the displacement difference of the central point displacement and the edge midpoint displacement is 0.09% of the plate center value at $\gamma = 0.03$. For comparison the difference is 18.5% for $\gamma = 8$. For the central concentrated load, the case of $\gamma = 0.03$ is considered representative of the rigid plate case.

It can be seen in Fig. 5.12(d) that, when the relative stiffness of the foundation to plate approaches $\gamma = 300$, the centrally loaded plate loses contact at the edge, and contact pressure over the plate center area increases substantially. The approximate contact regions associated with $\gamma = 301.6$ and $\gamma = 3016$ are shown in Fig. 5.13.

In Fig. 5.14 the contact pressure along the centerline of the centrally loaded rigid plate ($\gamma = 0.03$) is given for different contact element meshes. Here mesh 2 consists of 11×11 constant contact elements with the geometry shown in Fig. 5.10(b); and mesh 3 consists of 72 quadratic contact elements, again shown in Fig. 5.10(b). The singular nature of the contact pressure at the edge of the plate is evident in Fig. 5.14, especially clearly for mesh 2 and mesh 3. A lack of smoothness near the plate edge for mesh 3 is also shown. This latter mathematical behaviour is due to the inability of the quadratic polynomial to model the singularity in the contact pressure near the edge of the rigid plate on the elastic half-space. The unsmooth

distribution of the contact pressure results in a stiffer foundation and smaller rigid plate displacement (as shown in Table 5.4). Thus, it can be concluded that the discontinuous step function approximation of the contact pressure is more suitable for a near rigid plate resting on a half-space than the continuous polynomials of order two.

Table 5.4 Numerical factor α for displacement of rigid square plate

$\alpha = \frac{a E_s w_r}{(1 - \nu_s^2) P_0}$		
present study	mesh 1	0.8961
	mesh 2	0.8836
	mesh 3	0.8449
Gorbunov-Possadov & Serebrjanyi		0.913
Rajapakse & Selvadurai (FEM)		0.869

A comparison of the contact pressure along the centerline of the centrally loaded plate with other solutions, given by Gorbunov-Possadov and Serebrjanyi[34] and Rajapakse and Selvadurai[4], is shown in Fig. 5.15. Gorbunov-Possadov and Serebrjanyi expressed the distribution of contact pressure as a double power series up to order eight over the whole square plate in their solutions. The results presented by Rajapakse and Selvadurai were based on the application of the rectangular 'heterosis' Mindlin plate bending elements. It is seen that the singularity of the contact pressure near the plate edge is well represented by the present method. A comparison of the corresponding moments M_{xx} along the centerline of the plate is given in Fig. 5.16. In addition, a comparison of the rigid displacements of a centrally

loaded square plate ($\gamma = 0.03$) is presented in Table 5.4.

Example 5.7. Plate subjected to a linearly distributed load

In this example, the square plate is subjected to a linearly distributed load. The deflections along the plate centerline (axis x) are shown in Fig.5.17(a) for various relative stiffness of elastic half-space to plate. The linearly distributed load, from zero value at one side to q value at the other side along direction x and constant along direction y , is also shown in Fig. 5.17(a). The distribution of the plate deflection along x axis is near a straight line when $\gamma = 3.016$, i.e. the plate becomes relatively rigid.

The associated rigid body rotation of this eccentrically loaded square plate along the axis y calculated from the shape of plate deflection is,

$$tg\phi_x = \frac{w(\frac{a}{2}, 0) - w(-\frac{a}{2}, 0)}{a} = 0.12258 \quad (5.5)$$

According to the formula (equation (5.4a)) given by Gorbunov-Possadov and Serebrjanyi, the rigid body rotation along axis y is computed as $tg\phi_x = 0.1164$. Comparing this number with the result ($tg\phi_x = 0.12258$) from the present method, the error is 5.3%.

The moments M_{xx} and M_{yy} along axis x of the plate under linear distributed load are given in Figs.5.17(b) and 5.17(c), where P_o refers to the total vertical load ($P_o = qa^2/2$). Fig. 5.17(d) gives the distribution of the contact pressure along axis x . It is seen from Fig.5. 17(d) that the singularity behaviour of the contact pressure does not exit at the plate edge where the applied load is zero.

Example 5.8. Plate with a neighbouring concentrated load

In this example, the square plate with dimension ($a \times a$) is subjected to a

uniformly distributed load q and the foundation is subjected to a concentrated load P_e which is applied outside of the plate domain at the foundation surface, as shown in Fig. 5.18. The distance between the plate center point and the concentrated load acting point is denoted as d . The foundation relative stiffness parameter is set at $\gamma = 30.16$.

The first set of given numerical results are for the location of concentrated load P_e fixed at $d = 0.7a$ while the value of P_e varying from $P_e = 0.05qa^2$ to $P_e = qa^2$, in which qa^2 is the total load value applied directly on the square plate. Figs. 5.19(a)–5.19(d) give, respectively, the plate deflections, bending moments M_{xx} , bending moments M_{yy} and contact pressure along the x axis of the square plate. For comparison, the results of this uniformly loaded square plate without any neighbouring load are also presented in Figs. 5.19(a)–5.19(d) and they corresponds to the case of $P_e = 0$. The plate deflections plotted in Fig. 5.19(a) are normalized by dividing these by the rigid plate displacement given in equation (5.3). It is seen from Figs. 5.19(a)–5.19(d) that plate deflections, moments and contact pressure redistribute as the neighbouring load increases. The plate loses contact with the elastic foundation partly in the cases of $P_e = 0.5qa^2$ and $P_e = qa^2$, and the corresponding contact areas are plotted in Fig. 5.20(a) and (b), where the shaded areas refer to the contact regions. The moments M_{xx} along the x axis decrease as P_e increases, while the corresponding moments M_{yy} increase quite significantly.

The second set of numerical results of this example are for $P_e = qa^2$, while the acting point of P_e moves from $d = 0.6a$ to $d = 3a$. The distributions of plate deflections, of plate moments M_{xx} and M_{yy} and of the contact pressure along x axis are presented in Figs. 5.21(a)–5.21(d), respectively. Again, the results associated with the plate without any neighbouring loads are presented for comparison, which correspond to the case of $d = \infty$ in Figs. 5.21(a)–5.21(d). Influence of the neighbouring load on the plate behaviour vanishes as the acting point of P_e moves far away from the plate domain. The contact area associated with the case of $d = 0.6a$

is plotted in Fig. 5.20(c).

(a) boundary division

(b) contact element division

Fig.5.10 Boundary and contact element divisions

Fig.5.11(a) Deflections along axis x of the uniformly loaded plate

Fig.5.11(b) Moments M_{xx} along axis x of the uniformly loaded plate

Fig.5.11(c) Moments M_{yy} along axis x of the uniformly loaded plate

Fig.5.11(d) Contact pressure along axis x of the uniformly loaded plate

Fig.5.12(a) Deflections along axis x of the centrally loaded plate

Fig.5.12(b) Moments M_{xx} along axis x of the centrally loaded plate

Fig.5.12(c) Moments M_{yy} along axis x of the centrally loaded plate

Fig.5.12(d) Contact pressure along axis x of the centrally loaded plate

(a) $\gamma = 3016$

(b) $\gamma = 301.6$

Fig.5.13 Approximate contact regions of the centrally loaded plate

Fig.5.14 Contact pressure along axis x of the centrally loaded rigid plate versus different contact divisions ($\gamma = 0.03$)

Fig.5.15 Comparison of contact pressure along axis x of centrally loaded rigid plate

Fig.5.16 Comparison of moments M_{xx} along axis x of centrally loaded rigid plate

Fig.5.17(a) Deflections along axis x of the plate under linearly distributed load

Fig.5.17(b) Moments M_{xx} along axis x of the plate under linearly distributed load

Fig.5.17(c) Moments M_{yy} along axis x of the plate under linearly distributed load

Fig.5.17(d) Contact pressure along axis x of the plate under linearly distributed load

Fig. 5.18 Square plate with neighbouring load ($\gamma = 30.16$)

Fig. 5.19(a) Deflections along axis x of plate

Fig. 5.19(b) Moments M_{xx} along axis x of plate

Fig. 5.19(c) Moments M_{yy} along axis x of plate

Fig. 5.19(d) Contact pressure along axis x of plate

(a) $d = 0.7a$, $P_0 = 0.5qa^2$

(b) $d = 0.7a$, $P_0 = qa^2$

(c) $d = 0.6a$, $P_0 = qa^2$

Fig. 5.20 Approximated contact areas

Fig.5.21(a) Deflections along axis x of plate

Fig. 5.21(b) Moments M_{xx} along axis x of plate

Fig.5.21(c) Moments M_{yy} along axis x of plate

Fig.5.21(d) Contact pressure along axis x of plate

Chapter 6

ELASTIC ANALYSIS OF PLATES WITH EDGE BEAMS

6.1 Introduction

This chapter gives a direct boundary element procedure for the bending analysis of thin elastic plates with boundary conditions of elastic beam supports. Two models are provided to deal with the elastic boundary condition. One is a continuous edge beam boundary condition model called here a "boundary condition" type, in which the edge beam displaces in a fully compatible mode with the plate. The other is called the "attached beam element" type in which the edge beams are connected to the plate at discrete points. The advantages and disadvantages of both methods go beyond mathematical sophistication when singularities are involved in the problems, and these issues also are described.

6.2 Boundary Condition Equations

There are four quantities associated with each point on the plate edge. In plate analysis these are identified as: lateral displacement; edge normal rotation; edge normal moment; and equivalent edge shear. For a given mathematical plate problem, two edge quantities are unspecified or independent and two are specified. For the plate with a built-in edge beam on the boundary, as shown in Fig. 1, all four boundary quantities are non-zero in general, but again there are only two independent conditions. The normal moment and the equivalent edge shear force along the plate are equal to the distributed torque and force acting on the attached

Fig. 6.1 Plate with edge beam

beam. According to elastic beam theory, the boundary conditions at the beam supported edge for the plate are

$$M_n = -GJ \frac{\partial^2 \Theta_n}{\partial s^2} \quad (6.1)$$

and

$$V_n = EI \frac{\partial^4 W}{\partial s^4} \quad (6.2)$$

where W and Θ_n are the deflection and normal rotation along the plate boundary; EI and GJ are the beam flexural and torsional stiffnesses, respectively; and s is the local coordinate along the plate boundary. The combinations of GJ and EI ranging from values of 0 to ∞ can, of course, represent the more standard boundary conditions free, simply supported and clamped.

6.3 Continuous Edge Beam "Boundary Condition" Model

Each boundary element having a boundary condition defined as "edge beam" comprises several nodes, say N_i , as shown in Fig. 3.3, each having as primary unknowns the values of deflection and normal rotation. Lagrangian interpolation is employed to approximate the continuous distribution of boundary deflection and normal rotation along the element. Applying Lagrange's N_i point formula,

$$W(s) = \sum_{j=1}^{N_i} L_{ij}(s)W_{ij} \quad (6.3)$$

and

$$\Theta_n(s) = \sum_{j=1}^{N_i} L_{ij}(s)\Theta_{nij} \quad (6.4)$$

in which W_{ij} and Θ_{nij} are the nodal values as shown in Fig. 3.3; and $L_{ij}(s)$ has the expression of equation (3.14)

Applying the boundary conditions (equations (6.1) and (6.2)) of elastic beam support with the continuous boundary function replaced by the Lagrange approximation, the normal moment and equivalent shear along the boundary element are given by,

$$M_n(s) = -GJ \sum_{j=1}^{N_i} \frac{\partial^2 L_{ij}(s)}{\partial s^2} \Theta_{nij} \quad (6.5)$$

and

$$V_n(s) = EI \sum_{j=1}^{N_i} \frac{\partial^4 L_{ij}(s)}{\partial s^4} W_{ij} \quad (6.6)$$

From equation (6.6) it is apparent that at least a quartic order of boundary element is required in order to have non-zero distributed shear forces transmitted between the plate and edge beam. The use of a quartic order of interpolation will give rise to a numerical model in which the distributed equivalent edge shear has constant value over the lengths of each element.

After the boundary element modelling, the discretized integral equations (3.15) and (3.16) are modified for a plate with N_{eb} boundary elements having boundary condition of edge beams,

$$\begin{aligned} Cw(P) = & \sum_{i=1}^{N_e - N_{eb}} \sum_{j=1}^{N_i} [W_{ij} \int_{s_i} L_{ij} V_n^* ds + \Theta_{nij} \int_{s_i} L_{ij} M_n^* ds \\ & - M_{nij} \int_{s_i} L_{ij} \Theta_n^* ds - V_{nij} \int_{s_i} L_{ij} W^* ds] \\ & + \sum_{i=1}^{N_{eb}} \sum_{j=1}^{N_i} [W_{ij} \int_{s_i} (L_{ij} V_n^* - EI \frac{\partial^4 L_{ij}}{\partial s^4} W^*) ds \\ & + \Theta_{nij} \int_{s_i} (L_{ij} M_n^* + GJ \frac{\partial^2 L_{ij}}{\partial s^2} \Theta_n^*) ds] \\ & + \sum_{i=1}^{N_e} (WT_i^* - W^* T_i) + \int_{\Omega} qw^* d\Omega - \{ B_i^* \}^T \{ Q \} \end{aligned} \quad (6.7)$$

and

$$\begin{aligned}
C \frac{\partial w(P)}{\partial \xi} = & \sum_{i=1}^{N_e - N_{eb}} \sum_{j=1}^{N_i} [W_{ij} \int_{s_i} L_{ij} \frac{\partial V_n^*}{\partial \xi} ds + \Theta_{nij} \int_{s_i} L_{ij} \frac{\partial M_n^*}{\partial \xi} ds \\
& - M_{nij} \int_{s_i} L_{ij} \frac{\partial \Theta_n^*}{\partial \xi} ds - V_{nij} \int_{s_i} L_{ij} \frac{\partial W^*}{\partial \xi} ds] \\
& + \sum_{i=1}^{N_{eb}} \sum_{j=1}^{N_i} [W_{ij} \int_{s_i} (L_{ij} \frac{\partial V_n^*}{\partial \xi} - EI \frac{\partial^4 L_{ij}}{\partial s^4} \frac{\partial W^*}{\partial \xi}) ds \\
& + \Theta_{nij} \int_{s_i} (L_{ij} \frac{\partial M_n^*}{\partial \xi} + GJ \frac{\partial^2 L_{ij}}{\partial s^2} \frac{\partial \Theta_n^*}{\partial \xi}) ds] \\
& + \sum_{i=1}^{N_c} (W \frac{\partial T_i^*}{\partial \xi} - \frac{\partial W^*}{\partial \xi} T_i) + \int_{\Omega} q \frac{\partial w^*}{\partial \xi} d\Omega - \{ B_2^* \}^T \{ Q \}
\end{aligned} \tag{6.8}$$

There are three beam reaction forces associated with each beam end. These are shown in Fig.6.2 and are determined by the nodal deflections and rotations in the following relations:

$$R_1 = -EI \frac{\partial^3 W}{\partial s^3} = -EI \sum_{j=1}^{N_i} \frac{\partial^3 L_{ij}(s)}{\partial s^3} W_{ij} \tag{6.9}$$

$$R_2 = EI \frac{\partial^2 W}{\partial s^2} = EI \sum_{j=1}^{N_i} \frac{\partial^2 L_{ij}(s)}{\partial s^2} W_{ij} \tag{6.10}$$

and

$$R_3 = GJ \frac{\partial \Theta_n}{\partial s} = GJ \sum_{j=1}^{N_i} \frac{\partial L_{ij}(s)}{\partial s} \Theta_{nij} \tag{6.11}$$

These three reactions are resisted either by beam end supports or by the plate. For a simply supported beam along an edge with no beam on the adjacent edge, for instance, R_1 is a reaction at beam end supports, and R_2 if present is a reaction in

the plate itself. In the case of free beam end, R_1 and R_2 are resisted by the plate. In this situation, R_1 and R_2 are equal to zero theoretically and their existence as relatively very small values is due to the assumption of beam deflection shape. If the beam ends are not locked against twisting, R_3 is resisted by the plate only. Reactions, when resisted by the plate, are treated in the numerical procedure in a manner similar to loads acting on the plate, and these effects are treated accordingly in equations (6.7) and (6.8).

Fig. 6.2 Reactions at beam end

The total number of unknowns involved in equations (6.7) and (6.8) is $2N + M$ associated with the N boundary nodes and M contact element nodes. The set of $2N + M$ simultaneous equations,

$$\begin{bmatrix} A & B \\ A1 & B1 + F \end{bmatrix} \begin{Bmatrix} X \\ Q \end{Bmatrix} = \begin{Bmatrix} Y \\ Y1 - w_{se} \end{Bmatrix} \quad (3.22)$$

which are solved for the plate boundary unknowns and nodal contact pressures, are formed by applying equations (6.7) and (6.8) to N source points associated with N boundary nodes and equation (6.7) to M source points associated with M contact nodes, instead of using equations (3.17) and (3.18) as described in Section 3.3.

Using Lagrange interpolation for the plate boundary deflection function, the level of continuity of the plate edge deflection is C^0 . The tangential derivative of the plate edge deflection is discontinuous at each interelement node. For the plate edge with elastic beam boundary condition, C^0 continuity gives rise to an inconsistency with respect to elastic beam theory. For the boundary condition model, in order to model the elastic beam behaviour correctly, the continuity of C^1 for the plate boundary deflection along the beam supported edges must be implemented. A simple way to avoid numerical problems with discontinuity of the boundary functions is to model the plate edge with a single element. In more realistic engineering applications, it may be necessary to model a single straight boundary with several elements. A level of continuity of C^1 can be satisfied by applying constraint equations to equation (3.22) as follows.

At an interelement node connecting the boundary elements i and $i + 1$, the slopes left and right of the node are equal to

$$\left(\frac{\partial W}{\partial s}\right)^- = \sum_{j=1}^{N_i} \frac{\partial L_{ij}(s)}{\partial s} W_{ij} \quad (6.12)$$

and

$$\left(\frac{\partial W}{\partial s}\right)^+ = \sum_{j=1}^{N_{i+1}} \frac{\partial L_{i+1,j}(s)}{\partial s} W_{i+1,j} \quad (6.13)$$

respectively. Let $\left(\frac{\partial W}{\partial s}\right)^- = \left(\frac{\partial W}{\partial s}\right)^+$, therefore

$$\sum_{j=1}^{N_i} \frac{\partial L_{ij}(s)}{\partial s} W_{ij} = \sum_{j=1}^{N_{i+1}} \frac{\partial L_{i+1,j}(s)}{\partial s} W_{i+1,j} \quad (6.14)$$

This constraint equation prescribes a relationship between nodal deflections in the two adjacent elements. Under constraint equations, equation (3.22) is solved by using the Lagrange Multiplier method. An example (example 6.3) presented later gives results of this approach for a 6th order boundary element. Higher levels of continuity can be enforced in a similar way. For a beam with clamped ends, the

boundary conditions of zero tangential derivatives, $\frac{\partial W}{\partial s} = 0$, are implemented by constraint equations also.

6.4 "Attached Beam Element" Model

The approach here assumes that the plate and its supporting beams are connected at discrete points along the beam, as shown in Fig. 6.3(a), and these connection points are chosen as plate boundary nodes. Thus, the continuous elastic edge beam system is simplified as a set of discrete elastic point supports and the distributed shear force and normal moment (V_n and M_n) along the plate edge are given as concentrated elastic support forces and moments, acting at the connection nodes along the plate edge as indicated in Fig. 6.3(b). Adding the effects of these elastic supports reactions to the discretized integral equations (3.15) and (3.16), we have

$$\begin{aligned}
 Cw(P) = & \sum_{i=1}^{N_e - N_{eb}} \sum_{j=1}^{N_i} [W_{ij} \int_{s_i} L_{ij} V_n^* ds + \Theta_{nij} \int_{s_i} L_{ij} M_n^* ds \\
 & - M_{nij} \int_{s_i} L_{ij} \Theta_n^* ds - V_{nij} \int_{s_i} L_{ij} W^* ds] \\
 & + \sum_{i=1}^{N_{eb}} \sum_{j=1}^{N_i} [W_{ij} \int_{s_i} L_{ij} V_n^* ds + \Theta_{nij} \int_{s_i} L_{ij} M_n^* ds] - \sum_{j=1}^{N_b} (F_j W^* + M_j \Theta_n^*) \\
 & + \sum_{i=1}^{N_e} (W T_i^* - W^* T_i) + \int_{\Omega} q w^* d\Omega - \{ B_1^* \}^T \{ Q \}
 \end{aligned} \tag{6.15}$$

and

$$\begin{aligned}
C \frac{\partial w(P)}{\partial \xi} = & \sum_{i=1}^{N_e - N_b} \sum_{j=1}^{N_i} [W_{ij} \int_{s_i} L_{ij} \frac{\partial V_n^*}{\partial \xi} ds + \Theta_{nij} \int_{s_i} L_{ij} \frac{\partial M_n^*}{\partial \xi} ds \\
& - M_{nij} \int_{s_i} L_{ij} \frac{\partial \Theta_n^*}{\partial \xi} ds - V_{nij} \int_{s_i} L_{ij} \frac{\partial W^*}{\partial \xi} ds] \\
& + \sum_{i=1}^{N_{eb}} \sum_{j=1}^{N_i} [W_{ij} \int_{s_i} L_{ij} \frac{\partial V_n^*}{\partial \xi} ds + \Theta_{nij} \int_{s_i} L_{ij} \frac{\partial M_n^*}{\partial \xi} ds] - \sum_{j=1}^{N_b} (F_j \frac{\partial W^*}{\partial \xi} + M_j \frac{\partial \Theta_n^*}{\partial \xi}) \\
& + \sum_{i=1}^{N_c} (W \frac{\partial T_i^*}{\partial \xi} - \frac{\partial W^*}{\partial \xi} T_i) + \int_{\Omega} q \frac{\partial w^*}{\partial \xi} d\Omega - \{ B_2^* \}^T \{ Q \}
\end{aligned} \tag{6.16}$$

where N_b is the total number of boundary nodes connecting to beams; F_j and M_j are the concentrated force and moment at node j as shown in Fig. 6.3(b).

Unlike the attachment of a spring type elastic support where the elastic support force is proportional to only the deflection of the attached point, the elastic support force F_j at node j is dependent on all of the nodal deflections within each supporting beam. The elastic moment M_j at node j is related to the nodal normal rotations at node $j - 1$, j and $j + 1$. The relationship between the concentrated elastic reactions and the boundary nodal values are as follows:

$$\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{N_b} \end{Bmatrix} = [K_f] \{ W \} \tag{6.17}$$

and

$$\begin{Bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{N_b} \end{Bmatrix} = [K_m] \{ \Theta_n \} \tag{6.18}$$

where $[K_f]$ and $[K_m]$ of order $N_b \times N_b$ are, respectively, the beam bending and torsional stiffness matrices; and vectors $\{ W \}$ and $\{ \Theta_n \}$ contain N_b deflections and

(a) discrete connection between plate and beam

(b) concentrated elastic support forces and moments

(c) bending stiffness coefficients due to $W_j = 1$

(d) torsional stiffness coefficients due to $\theta_{nj} = 1$

Fig. 6.3 "Attached beam element" model

normal rotations at those plate boundary nodes which are connected to the attached beams and form part of the $2N$ boundary unknowns involved in the numerical solution.

The beam bending and torsional stiffness matrices are generated by using a model as shown in Figs. 6.3(c) and 6.3(d).

6.5 Numerical Results

Example 6.1

In this example the square plate has three sides simply supported, and one side supported on a beam (Fig. 6.4(a)). The boundary has one boundary element on each side. As shown in Fig. 6.4(a), boundary elements of order four are used on the simply supported (s.s.) edges and an element of order six is used on the beam supported edge. In this example the continuous edge beam "boundary condition" model is used.

The deflections and moments along the x -axis for torsional rigidity $GJ = 0$ and flexural rigidity $EI = 0$ to $20000aD$ are given in Figs. 6.5(a) and 6.5(b). The edge equivalent shear forces along the beam supported plate edge are shown in Fig. 6.5(c). Results from reference[21] for $EI = \infty$ compare favourably.

For the beam with high flexural rigidity, Figs. 6.6(a) and 6.6(b) show deflections and moments along the x axis for a range of GJ values, and again the Timoshenko's results for comparison are of a square plate with three sides simply supported and one side clamped ($GJ = \infty$). The normal plate edge moments along the beam supported plate edge are shown in Fig. 6.6(c) for the same range of GJ values.

Example 6.2

In the second example plate as shown in Fig. 6.4(b), all four edges are supported by beams, and point supports at the corners. In each case here GJ is assumed to be zero. The analytical results for this example, which are presented for comparison and validation, were given in reference[21]. The numerical results for both the continuous edge beam "boundary condition" model and the "attached beam element" model are presented.

Deflections and moments along the square plate centerline are plotted in Figs. 6.7(a) and 6.7(b) for the various flexural rigidities, for a plate model with one 6th order boundary element on each edge. The results for both models are virtually the same, but the problem solved, although quite practical, is not a particularly complicated one. The analytical results shown for comparison are, again, very close.

Fig. 6.7(c) gives the equivalent shear force along the plate edge parallel to the x axis for both models, in which the distributed shear forces for the "attached beam element" model are obtained by dividing the concentrated forces at nodes by the nodal spacing. Besides the equivalent shear forces distributed along the plate edges, there are associated corner forces at plate corners which are not plotted in this figure. Except for the case of very flexible supporting beams such as $EI = 0.5aD$, these corner forces are in tension, similar to the tensile corner forces of simply supported square plates.

Example 6.3

Stress singularity occurs at plate edge points where there are changes of the boundary conditions along plate edge. Mathematical proof has been given by Williams[38] in his paper for the combinations of free, simply supported and clamped boundary conditions. Along a straight plate edge with boundary conditions changing from clamped or simply supported to free, for example, the bending stress singularity is of order $1/\sqrt{r}$ and the shear stress singularity is of order $r^{-1.5}$ at the transition point.

In the third example, a square plate having one edge clamped, two edges half supported by cantilever beams and the others free, as shown in Fig. 6.4(c), is examined. Again, the numerical results are given for both models. The whole plate edge is divided into eight boundary elements as shown in Fig. 6.4(c) where each element is of order six. The results of the "attached beam element" model are obtained by using an identical boundary element mesh but connecting each cantilever beam to the plate edge at eight points, six points within elements 5 and 7 and two points within elements 4 and 8.

Where the boundary condition along the plate edge changes from the elastic beam support to free, the bending stress and shear force at the transition point have singular behaviour. The order of the stress singularity increases as the beam stiffness increases. The numerical results are presented for cantilever beams having a relatively small stiffness $EI = 0.5aD$ and a near rigid stiffness value $EI = 500aD$. The beams here are assumed to have $GJ = 0.0$.

Fig.6.8(a) gives the shear force distributions along the plate edge parallel to the x axis. Again, the equivalent distributed shear forces of the "attached beam element" model are calculated by dividing the discrete forces by the nodal spacing. It is seen in this figure that shear forces from both models show the singular behaviour at the transition point from elastic beam support to free.

In Fig. 6.8(b) is given the corresponding displacement distributions along the plate centerline (x axis). The differences between the results of the continuous edge beam "boundary condition" model and the "attached beam element" model increase as the beam stiffness increases. The difference between the deflections at point ($x = a/2, y = 0$), for example, calculated by using the two models is 2% when the supporting beams are very flexible ($EI = 0.5aD$), reaching 11% when the supporting beams become near rigid ($EI = 500aD$). The deflections based on the "attached beam element" model are larger, i.e. closer to "exact" than those based on the "boundary condition" model due to the fact that the polynomial

approximation of the singular shear forces in the continuous "boundary condition" model spreads the singular behaviour from one point over a certain length and adds constraints to the elastic supported plate edges. Thus, the "attached beam element" model can be considered to have this advantage when singularities are involved in the plate edge values.

The distributions of the bending moment M_{xx} , which relate to the normal surface stresses parallel to the x axis, along the y axis are given in Fig. 6.8(c). The bending stress singularity is clearly shown in this figure at two end points of the y axis where the plate boundary condition is changing from beam supported to free.

The equivalent shear forces along the beam supported plate edges are approximated as piecewise polynomials in the continuous edge beam "boundary condition" model and as discrete concentrated forces in the "attached beam element" model. The accuracy of numerical results near the singular point can be improved by element mesh refinement. As well, it is found that use of the "attached beam element" model results in more accurate results than the continuous "boundary condition" model.

Example 6.4

In the fourth example, a square plate rests on the elastic half-space foundation with all four plate edges having beams. Each plate edge is modelled as one 6th order element and the contact region between the plate and the foundation is divided uniformly by a 11×11 constant contact mesh.

The plate is subjected to a uniform load q . The results are presented for a relatively very flexible plate on the elastic half-space with relative stiffness parameter $\gamma = 3016$. The poisson's ratio value for both the plate and the half-space is taken as 0.15. The torsional stiffness of edge beam is assumed to be zero here, while the beam flexural stiffness EI varies. The continuous edge beam "boundary condition"

model is employed to analyze the plate.

The plate deflections along the plate edge for beam flexural rigidity $EI = 0.0$ to $EI = 100aD$ are given in Fig. 6.9(a). The plate deflections shown in Fig. 6.9(a) are normalized by the corresponding rigid plate deflection w_r , which is defined in equation (5.3). The associated contact pressure distribution near the plate edge (along the axis of $y = 0.455a$) is presented in Fig. 6.9(b). As the flexural stiffness of the edge beam increases, both the deflection and the contact pressure along the plate edge redistribute. The plate edge deflections become uniform (Fig. 6.9(a)) and the contact pressure beneath plate corners increases substantially (Fig. 6.9(b)). However, the increase of plate edge beam stiffness has little effect on the plate internal deflections in this particular case, as indicated in Fig. 6.9(c), which gives the plate deflections along its centerline for the same range of beam stiffnesses.

Fig. 6.4 Square plates with edge beams: (a) example 6.1,
(b) example 6.2 and (c) example 6.3

Fig. 6.5(a) Deflections along axis x of plate

Fig. 6.5(b) Moments along axis x of plate

Fig. 6.5(c) Shear forces along plate edge having beam support

Fig. 6.6(a) Deflections along axis x of plate

Fig. 6.6(b) Moments along axis x of plate

Fig. 6.6(c) Normal moments along plate edge having beam support

Fig. 6.7(a) Deflections along the plate centerline

Fig. 6.7(b) Moments the plate centerline

Fig. 6.7(c) Shear forces along plate edge

Fig. 6.8(a) Shear force distribution along plate edge

Fig. 6.8(b) Deflections along axis x of plate

Fig. 6.8(c) Moments along axis y of plate

Fig. 6.9(a) Deflections along the plate edge

Fig. 6.9(b) Contact pressure along the axis of $y=0.455a$

Fig. 6.9(c) Deflections along the plate centerline

Chapter 7

ANALYSIS OF THIN PLATES UNILATERALLY EDGE SUPPORTED

7.1 Introduction

When the corners of simply supported plates of general polygonal shape are not secured against lifting, the tendency toward corner liftoff is well-known, along with corresponding increases in deflections and bending moments in the central portion of the plate[21]. These effects have motivated many researchers to attempt to determine the actual extent of the contact and some of these are given in references[39,40,41,42,43]. The detailed examination of such deflections and moments is important in some engineering applications. Keer and Mak (1981)[39] investigated the extent of corner liftoff of a quarter infinite plate subjected to a concentrated load near the corner using the integral transform technique. Dempsey et al. (1984)[40] provided a Levy–Nadai finite integral transform solution for a square plate under a uniform pressure loading. In a subsequent study, the Levy–Nadai finite integral transform solution was extended to a rectangular plate subjected to centrally symmetric loading by Dempsey and Li (1986)[41,42]. Salamon et al. (1986)[43] also contributed to this area of study by examining a square plate unilaterally supported on discrete springs around the plate periphery in an application of the finite element method. In this latter study, the springs were connected to the external nodes of finite plate bending elements.

In the work described in the present chapter, the problems of plates on unilateral edge supports are analyzed by using the boundary element method. The support system here consists of discrete elastic springs along the plate edges, as

shown in Fig. 7.1. As unilateral supports, the elastic support system is capable of exerting compressive forces only and allows the plate to seek its natural contact. The method described here has a special versatility in that it is applicable to thin plates with arbitrary geometry and transverse loading. Examples of square, equilateral triangular, and hexagonal plates on elastic simple edge supports subject to either uniform pressure loads or concentrated loads are given for a range of elastic support stiffnesses, showing the nonlinear effects due to plate corner lift-off.

Fig. 7.1 Plate on elastic edge supports

7.2 Problem Formulation

When a plate is elastically supported only around its edges, the contact pressure $p(x, y)$ in the governing equation (2.1) is equal to zero. The corresponding integral equations for plates on elastic edge supports can be obtained from integral equations (2.13) and (2.14) by substituting zero value of $p(x, y)$ into them, and they are

$$Cw(P) = \oint_s (\Theta_n M_n^* + W V_n^* - M_n \Theta_n^* - V_n W^*) ds + \sum_{i=1}^{N_c} (W T_i^* - W^* T_i) + \int_{\Omega} q w^* d\Omega \quad (7.1)$$

and

$$C \frac{\partial w(P)}{\partial \xi} = \oint_s \left(\Theta_n \frac{\partial M_n^*}{\partial \xi} + W \frac{\partial V_n^*}{\partial \xi} - M_n \frac{\partial \Theta_n^*}{\partial \xi} - V_n \frac{\partial W^*}{\partial \xi} \right) ds + \sum_{i=1}^{N_c} \left(W \frac{\partial T_i^*}{\partial \xi} - \frac{\partial W^*}{\partial \xi} T_i \right) + \int_{\Omega} q \frac{\partial w^*}{\partial \xi} d\Omega \quad (7.2)$$

The boundary conditions of a plate on discrete elastic simple supports are defined as each having a zero normal moment condition,

$$M_n = 0 \quad (7.3)$$

and the shear boundary conditions in terms of the discrete spring stiffness can be incorporated by replacing terms in equations (7.1) and (7.2) as follows,

$$\oint_s V_n W^* ds + \sum_{i=1}^{N_c} T_i W^* = \sum_{j=1}^{N_s} k_s W_{sj} W^* \quad (7.4)$$

and

$$\oint_s V_n \frac{\partial W^*}{\partial \xi} ds + \sum_{i=1}^{N_c} T_i \frac{\partial W^*}{\partial \xi} = \sum_{j=1}^{N_s} k_s W_{sj} \frac{\partial W^*}{\partial \xi} \quad (7.5)$$

where N_s is the total number of discrete spring supports; k_s the stiffness of the springs with units of force per length; and W_{sj} is the deflection of spring support j . After substitution of these boundary conditions, integral equations (7.1) and (7.2) become

$$Cw(P) = \oint_s (\Theta_n M_n^* + W V_n^*) ds + \sum_{i=1}^{N_c} W_i T_i^* - \sum_{j=1}^{N_s} k_s W_{sj} W^* + \int_{\Omega} q w^* d\Omega \quad (7.6)$$

and

$$C \frac{\partial w(P)}{\partial \xi} = \oint_s (\Theta_n \frac{\partial M_n^*}{\partial \xi} + W \frac{\partial V_n^*}{\partial \xi}) ds + \sum_{i=1}^{N_c} W_i \frac{\partial T_i^*}{\partial \xi} - \sum_{j=1}^{N_s} k_s W_{sj} \frac{\partial W^*}{\partial \xi} + \int_{\Omega} q \frac{\partial w^*}{\partial \xi} d\Omega \quad (7.7)$$

The integral equations (7.6) and (7.7) contain the unknown boundary quantities W and Θ_n . These are the primary unknowns in the numerical procedure. In other words, a set of matrix equations is formed with these as unknown quantities which are solved in a first phase of the solution. Following the plate boundary discretization procedure described in Chapter 3, we get the discretized boundary integral equations developed from equations (7.6) and (7.7),

$$\begin{aligned} Cw(P) = & \sum_{i=1}^{N_c} \sum_{j=1}^{N_i} \Theta_{nij} \int_{s_i} M_n^* L_{ij}(s) ds + \sum_{i=1}^{N_c} \sum_{j=1}^{N_i} W_{ij} \int_{s_i} V_n^* L_{ij}(s) ds \\ & + \sum_{i=1}^{N_c} W_i T_i^* - \sum_{j=1}^{N_s} k_s W_{sj} W^* + \int_{\Omega} q w^* d\Omega \end{aligned} \quad (7.8)$$

and

$$\begin{aligned}
C \frac{\partial w(P)}{\partial n} &= \sum_{i=1}^{N_c} \sum_{j=1}^{N_i} \Theta_{n,j} \int_{s_i} \frac{\partial M_n^*}{\partial \xi} L_{ij}(s) ds + \sum_{i=1}^{N_c} \sum_{k=1}^{N_i} W_{ij} \int_s \frac{\partial V_n^*}{\partial \xi} L_{ij}(s) ds \\
&+ \sum_{i=1}^{N_c} W_i \frac{\partial T_i^*}{\partial \xi} - \sum_{j=1}^{N_s} k_s W_s \frac{\partial W^*}{\partial \xi} + \int_{\Omega} q \frac{\partial w^*}{\partial \xi} d\Omega
\end{aligned} \tag{7.9}$$

A set of $2N$ linear algebraic equations is formed by employing equations (7.8) and (7.9) respectively at each of the N boundary points as source points. In matrix form, the set of equations for solving the unknown boundary quantities W and Θ_n are,

$$[A] \begin{Bmatrix} W \\ \Theta_n \end{Bmatrix} - k_s [H] \{W_s\} = \{Y\} \tag{7.10}$$

where $\{W\}_N$ and $\{\Theta_n\}_N$ contain unknown nodal deflections and normal rotations; $\{W_s\}_{N_s}$ contains the spring support deflections; $\{Y\}_{2N}$ contains the domain integration terms associated with the applied loads; $[A]_{2N \times 2N}$ contains the boundary integral associated with the boundary unknowns, including T_i^* or $\frac{\partial T_i^*}{\partial \xi}$ where these correspond to nodal deflections at corners; and $[H]_{2N \times N_s}$ has the form of

$$[H] = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1N_s} \\ h_{21} & h_{22} & \cdots & h_{2N_s} \\ \vdots & \vdots & \vdots & \vdots \\ h_{2N1} & h_{2N2} & \cdots & h_{2NN_s} \end{bmatrix} \tag{7.11}$$

with

$$h_{ij} = W^* \quad (i = 1, 3, \dots, 2N - 1)$$

and

$$h_{ij} = \frac{\partial W^*}{\partial \xi} \quad (i = 2, 4, \dots, 2N)$$

It should be observed at this stage that the vectors $\{W\}$ and $\{W_s\}$ are not necessarily the same, in other words the springs need not be applied to node points.

A relatively simple transformation based on the shape functions, which establishes the relation between these two vectors, is given as follows:

$$\{W_s\} = [L]\{W\} \quad (7.12)$$

in which matrix $[L]_{N_s \times N}$ contains Lagrange shape functions in which polynomial coordinates have been replaced by spring support coordinates. After substitution of equation (7.12) into eqn (7.10), we have

$$[A]\left\{\begin{matrix} W \\ \Theta_n \end{matrix}\right\} - k_s[H][L]\{W\} = \{Y\} \quad (7.13)$$

The solution of equation (7.13) for the primary boundary unknowns is established once the contact region between the plate and its unilateral elastic supports is known. Again, the contact between the plate and its edge elastic supports is classified as receding type, which has been described in Section 3.4. The contact regions are independent of loading level. The contact regions do depend on relative stiffnesses, the geometry of the plate, the loading configuration and the Poisson's ratio of the plate. This is a geometrically nonlinear problem which can be solved by iteration. The initial assumption is a state of full contact. With this assumption, the equations are solved for unknown boundary quantities which includes $\{W\}$. Next the vector $\{W_s\}$ is computed, with negative values implying a state of tension between the plate and supports. In this latter case, the corresponding columns in matrix $[H]$ are replaced by zeroes, thus eliminating the tensile springs. The problem is then re-solved, and so on.

Once the boundary unknowns are found, the moments and shear forces within the plate can be obtained by taking second and third partial derivatives of the $w(P)$ expression in equation (7.8) according to their definitions in equations (2.3a-g), with $C = 1$. The compressive discrete spring reactions are equal to $k_s W_{sj}$. The equivalent distribution of shear force is given by $V = \frac{k_s W_{sj}}{d}$, where d is the space

between the discrete springs.

7.3 Numerical Examples

A dimensionless relative stiffness of the form

$$k = \frac{k_s a^3}{dD} \quad (7.14)$$

is used in the presentation of the results, with "a" as the side length of the plate. Square, equilateral triangular and hexagonal shaped plates, as shown in Fig. 7.2, are analyzed. Generally, the spring support model consists of 21 discrete springs along each of the triangular and square plates, and 17 along each edge of the hexagon. In addition some comparative results are shown for different numbers of springs for the square (as shown later for Table 7.2).

The boundary element model of the square and triangular plates consists of four 5th order boundary elements for each edge. The hexagonal plate edges each consist of four 4th order boundary elements. For the numerical examples described here, only 2 to 4 cycles of iteration are necessary to establish the contact region.

The finite element results from Salamon et al.[43] provide the comparison in Fig. 7.3(a) for the uniformly loaded square plate with different support stiffnesses. The boundary element and finite element meshes are shown in Fig. 7.3(b). The finite element model consisted of 12 d.o.f. plate bending finite elements. The negative values of deflection shown in Fig. 7.3(a) indicate plate corner lift-off.

The deflections and moments along the x axis of the uniformly loaded square plate are given in Figs. 7.4(a) and (b). The plate deflections are apparently relatively more sensitive to the elastic simple support stiffness than are the plate bending moments.

The edge deflections, and deflections and moments along the x axis of the square plate subject to a concentrated centrally applied load are given in Figs. 7.5(a) to (c), where in Fig. 7.5(c) the moment results corresponding to intermediate stiffness values fall between the two closely spaced curves shown.

A more detailed comparison between the present results of square plates on relatively very stiff supports ($k = 10^{30}$) with Dempsey and Li's[42] results of square plates on rigid supports is given in Table 7.1. These results compare very favourably except for the support shear force V near the lift-off point, where the support shear force approaches infinity for the continuous support model in Dempsey and Li's work. The discrete support system in the present method does not attempt to model this singular behaviour. The number of discrete supports influences mainly the plate edge lifting deflections and the support reaction distribution, as presented in Table 7.2. Also given in Table 7.2 are, respectively, the plate edge contact length, the plate corner deflections, the plate central point deflections and moments, and the support shear forces of the uniformly loaded square plate supported on different numbers of rigid discrete supports, along with the results of a plate on continuous rigid supports from Dempsey and Li[42].

The responses of the equilateral triangular and hexagonal plates to a uniform pressure loading are given in Figs. 7.6(a)–(c) and Fig. 7.7(a)–(c), respectively. The contact extent, which is the ratio of the contact length c to the plate edge side length a , between the plate edges and the rigid supports is shown in Table 7.3 for the three types of plates, where the values in the brackets are from Dempsey and Li's[42] work. It is seen from these results that plate geometry significantly affects the level of contact in uniformly loaded plates. The extent of contact indicated in these results for the boundary element discrete spring model is inexact but the error is essentially given within one discrete spring spacing, and given the present refinement, the influence of this error on plate internal behaviour (deflections and bending moments) is small.

Table 7.1. Square plates on rigid foundations: comparison with Dempsey and Li's results ($\nu = 0.3$)
Uniform load: $F=qa^2$

$\frac{x}{a}$	$\frac{w(x,0.5a)D}{Fa^2} \times 10^{-3}$		$\frac{w(x,0)D}{Fa^2} \times 10^{-3}$		$\frac{M_{xx}(x,0)}{F} \times 10^{-2}$		$\frac{M_{yy}(x,0)}{F} \times 10^{-2}$		$\frac{V(x,0.5a)}{F/a} \times 10$	
	D. & L.	BEM	D. & L.	BEM	D. & L.	BEM	D. & L.	BEM	D. & L.	BEM
0.0	0.000	0.0000	4.38	4.3850	5.09	5.0914	5.09	5.0914	4.47	4.488
0.1	0.000	0.0000	4.19	4.1896	4.97	4.9731	4.88	4.8811	4.48	4.590
0.2	0.000	0.0000	3.61	3.6101	4.57	4.5728	4.16	4.2521	5.28	6.780
0.3	-0.030	-0.0347	2.67	2.6709	3.75	3.7558	3.21	3.2113	0.00	0.000
0.4	-0.283	-0.2923	1.43	1.4296	2.31	2.3087	1.78	1.7786	0.00	0.000
0.5	-0.756	-0.7702	0.00	0.0000	0.00	0.0000	0.00	0.0000	0.00	0.000
Concentrated load: $F=P$										
0.0	0.00	0.000	12.9	12.876	∞	∞	∞	∞	9.00	7.809
0.1	0.00	0.000	11.8	11.749	16.3	16.266	21.6	21.587	---	18.339
0.2	-0.19	-0.202	9.45	9.453	9.40	9.403	14.0	14.026	0.00	0.000
0.3	-0.96	-0.969	6.57	6.567	5.57	5.568	9.07	9.093	0.00	0.000
0.4	-2.25	-2.269	3.36	3.361	2.81	2.810	4.69	4.722	0.00	0.000
0.5	-3.87	-3.899	0.0	0.000	0.00	0.000	0.00	0.000	0.00	0.000

Table 7.2 Influence of spring number: uniformly loaded square plate on rigid supports ($\nu = 0.3$)

No. of springs	$\frac{c}{a}$	$\frac{w(0.5a, 0.5a)}{qa^4/D}$	$\frac{w(0,0)}{qa^4/D}$	$\frac{M_{xx}(0,0)}{qa^2}$	$\frac{V(0,0.5a)}{qa}$
13	0.5	-0.000786	0.0043859	0.050926	0.4599
17	0.5	-0.000779	0.0043970	0.050938	0.4455
21	0.5	-0.000770	0.0043850	0.050914	0.4488
Continuous (D. & L.)	0.476	-0.000756	0.004380	0.0509	0.447

Table 7.3 Contact extent for plates on rigid supports

Types of plates	Uniform load		Concentrated load at the centre	
	$\nu = 0.15$	$\nu = 0.3$	$\nu = 0.15$	$\nu = 0.3$
E. Triangle	0.6	0.7	0.2	0.2
Square	0.4	0.5 (0.476)	0.2	0.2 (0.200)
Hexagon	0.25	0.375	0.125	0.125

Table 7.4 Amplification factors for plates on rigid supports

Types of plates	Uniform load				Concentrated load at the centre	
	$\nu = 0.15$		$\nu = 0.3$		$\nu = 0.15$	$\nu = 0.3$
	γ_w	γ_m	γ_w	γ_m	γ_w	γ_w
E. Triangle	1.011	1.008	1.005	1.003	1.091	1.053
Square	1.128	1.102	1.080	1.063	1.170	1.110
Hexagon	1.271	1.225	1.183	1.149	1.248	1.169

The comparison of plate central deflections and moments (in Table 7.4) between the unilateral and bilateral support cases is presented in terms of amplification factors

$$\gamma_w = \frac{w_u}{w_b} \quad \text{and} \quad \gamma_m = \frac{M_u}{M_b} \quad (7.15)$$

for central deflections and moments respectively. The subscripts 'u' and 'b' refer to the "unilateral" and "bilateral" supports. The bilateral results for the triangle and hexagonal plates are obtained by using the proposed formulation assuming that the support springs resist both the tensile and compressive forces. It is significant in these results that the amplification factors are lowest for the triangle, followed by the rectangle, and highest for the hexagon, where the corner angles for these figures range from acute to 90° to obtuse respectively. Also indicated in Table 7.4 is the effect of Poisson's ratio and loading configuration on the amplification factors.

7.4 Summary

The boundary element method has proven to be very effective in solving the class of problems of interest here. The numerical examples were intended primarily to demonstrate this, and a secondary objective was to show some of the effects of plate geometry, loading configuration and Poisson's ratio. The procedure is very practical from the point of view of computational demand including its use in the calculation of detailed internal plate values.

It was shown that for plate corners with obtuse angle, there is a stronger tendency to lift away from the support near the corner. This resulted in higher values of amplification factors than corners with smaller angles. In addition, the more concentrated loadings and smaller plate Poisson's ratio resulted in greater loss of contact.

Square

Equilateral triangle

Hexagon

c : contact length

Fig. 7.2 Unilaterally edge supported square, equilateral triangular and hexagonal plates

Fig. 7.3(a) Edge deflections of the uniformly loaded square plate on elastic edge supports

BEM

FEM

Fig. 7.3(b) Element meshes for BEM and FEM

Fig. 7.4(a) Deflections along axis x of the uniformly loaded square plate

Fig. 7.4(b) Moments along axis x of the uniformly loaded square plate

Fig. 7.5(a) Edge deflections of the concentrated loaded square plate

Fig. 7.5(b) Deflections along axis x of the concentrated loaded square plate

Fig. 7.5(c) Moments along axis x of the concentrated loaded square plate

Fig. 7.6(a) Edge deflections of the uniformly loaded equilateral triangular plate

Fig. 7.6(b) Deflections along axis y of the uniformly loaded equilateral triangular plate

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Fig. 7.6(c) Moments along axis y of the uniformly loaded equilateral triangular plate

Fig. 7.7(a) Edge deflections of the uniformly loaded hexagonal plate

Fig. 7.7(b) Deflections along axis x of the uniformly loaded hexagonal plate

Fig. 7.7(c) Moments along axis x of the uniformly loaded hexagonal plate

Chapter 8

CONCLUSIONS AND FUTURE WORK

8.1 Summary

This thesis has addressed the application of the direct boundary element method in the interaction analysis of plates on elastic foundations, specifically the Winkler foundation, the elastic half-space foundation and a number of edge support conditions. This work represents the first implementation of the direct boundary element method in the contact analysis of plates on elastic half-space, plates on unilateral edge supports, and plates with attached edge beams. The solution of such problems is not always as straightforward as might generally be expected in other typical examples of linear elastic analysis, especially in consideration of real singular conditions that may exist.

The first accomplishment of the thesis is the boundary element analysis of plates in unilateral contact with Winkler foundations. In this case, the present method is similar to the approaches presented by Bezine[14] and Sapountzakis and Katsikadelis[16] which were reviewed in Section 1.2.2. But the present method uses contact elements to model the foundation reactions and performs the domain integration over each contact element analytically. The analytical evaluation of domain integrations not only results in more accurate numerical results, but also eliminates singular problems associated with domain integration which may be introduced in quadrature[14,16].

The second contribution of the thesis is the boundary element analysis of contact problems between plates and elastic half-space foundations. Previous work related to the boundary element analysis of plates on elastic foundations are limited to the

Winkler and two-parameter foundations. A comparison of the present method to the finite element analysis of plates on elastic half-space was made in Section 3.6. In the present method, the domain discretization is applied to the foundation surface to model the contact pressure, and the plate is subjected to boundary discretization only. Therefore, most of the special advantages of the boundary element method for plate bending analysis are preserved in the present method.

The third problem solved in the thesis is the boundary element analysis of plates unilaterally edge supported. The problem was solved by other researchers by using integral transform techniques or the finite element method as described in Chapter 7. The emphasis in the present work is on the effectiveness of the proposed numerical procedure; that is, its accuracy and the minimal computational demands, both for the computer and the structural analyst; and also its potential as a general purpose approach.

Finally, the problem of plates with boundary conditions of attached elastic edge beams are solved by using the boundary element method. Previous BEM work focused on the analysis of plates with boundary conditions of free, simply supported or clamped. Two numerical models, namely a "boundary condition" type and an "attached beam element" type, were proposed in this thesis to model the plate edges with elastic beams and their applicability is validated by various numerical examples. The success of the proposed numerical models indicates the potential of the boundary element analysis of slabs with internal beams.

8.2 Conclusions

Based on the present study, the following conclusions can be drawn.

1. A general approach for the direct boundary element method combined with contact element modelling for the interaction analysis of thin elastic plates on elastic foundations has been developed.

2. Though the domain discretization is used to model the contact area between the plate and the foundation, the proposed method preserves most of the advantages of the boundary element method. For instance, the plate internal stress resultants have the same numerical accuracy as plate deflections and the solutions for plate internal actions are continuous throughout the plate domain.

3. Use of the singular solution of the plate pure bending problem, rather than the singular solution of plates on elastic foundations, as the fundamental solution in the present formulation makes it possible to evaluate all of the integrations involved in the boundary element analysis analytically. Therefore, the numerical approximations in the proposed method are performed in one phase only, involving polynomial approximations of plate boundary functions and contact pressure functions over each element. This method leads to an efficient computer program and more accurate numerical results.

4. The analytical evaluation of surface integrations eliminates the singularity problem which is encountered in the quadrature procedure when calculating plate internal moments and shear forces.

5. In terms of Winkler type foundations, the use of six node quadratic contact elements results in better numerical results than the use of constant contact elements. In the case of elastic half-space foundation, however, the constant contact elements are more suitable due to the singular behaviour of the contact pressure near plate edge areas.

6. For a plate with attached edge beams, both the continuous "boundary condition" model and the "attached beam element" model are applicable and the examples given indicate no particular difficulty with the simple configuration in which the whole plate edge has a uniform support. For a more complicated case, such as a plate with discontinuous edge beams, the "attached beam element" model becomes a more practical one due to its simplicity and versatility.

8.3 Future Work

The first suggestion for future work is that the proposed method be extended to solve contact problems of plates on elastic foundations other than Winkler or elastic half space. Using the present formulation to analyze plates resting on other types of elastic foundations, the foundation flexibility matrix $[F]$ in equations (3.22) and (3.24), which is defined in equation (3.7) for the elastic half-space model and in equation (3.23) for the Winkler model, has to be replaced by the corresponding foundation flexibility matrix.

The second recommendation for future work is boundary element analysis of plates with internal beams. The proposed method focuses on the topic of plates with edge beams. The author believes that the numerical models presented in Chapter 6 for dealing with plate edge beams, particularly the "attached beam element" model, can be used to deal with beams located within the plate domain. If that is the case, the boundary element method can be employed to analyze building frame systems consisting of plate-column-beam and bridge structures of plates on girders.

Finally, it is obvious that the present work following the initiative established by Abdel-Akher[23] will give rise sometime in the future, after further development, to very general integrated systems for analyzing a complete range of linear elastic analysis problems. It is in the nature of the approach taken that this will not only be very accurate given all of the assumptions, but also very practicable in the field.

8.4 Closing Comment

One aspect of the work which has only been touched on in the thesis, is that the numerical procedures employed are relatively general purpose, and easy to use. The plate shapes and loadings for example can be completely general. In addition all of the worked examples were done on Apollo workstations and the running time required was often less than 10 minutes.

The matrix algebra although non-sparse is of relatively very small order.

The attachment of elements and boundary conditions to thin plate models complements work done previously by Abdel-Ahkel[23]. It is the author's hope that at some future time her work might proven useful to her fellow engineers working in the field.

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Appendix A

FUNDAMENTAL SOLUTION QUANTITIES

(1) Deflection

$$W^* = \frac{r^2}{8\pi D} \ln(r) \quad (A.1a)$$

$$\Theta_n^* = \frac{e}{8\pi D} [1 + 2\ln(r)] \quad (A.1b)$$

$$M_n^* = -\frac{1}{8\pi} [(1 + 3\nu) + 2(1 + \nu)\ln(r) + 2(1 - \nu)\frac{e^2}{r^2}] \quad (A.1c)$$

$$V_n^* = \frac{1}{8\pi} [2(1 + \nu)\frac{e}{r^2} + 4(1 - \nu)\frac{e^3}{r^4}] \quad (A.1d)$$

$$M_{nt}^* = -\frac{(1 - \nu)ez}{4\pi r^2} \quad (A.1e)$$

(2) Rotation

$$\frac{\partial W^*}{\partial \xi} = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \begin{Bmatrix} e + 2e\ln(r) \\ z + 2z\ln(r) \end{Bmatrix} \quad (A.2a)$$

$$\frac{\partial \Theta_n^*}{\partial \xi} = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \begin{Bmatrix} 1 + 2\frac{e^2}{r^2} + 2\ln(r) \\ 2\frac{ez}{r^2} \end{Bmatrix} \quad (A.2b)$$

$$\frac{\partial M_n^*}{\partial \xi} = \frac{1}{4\pi} [\xi_n \quad \xi_t] \begin{Bmatrix} (1+\nu)\frac{e}{r^2} + 2(1-\nu)\frac{ez^2}{r^4} \\ (1+\nu)\frac{z}{r^2} - 2(1-\nu)\frac{e^2z}{r^4} \end{Bmatrix} \quad (A.2c)$$

$$\frac{\partial V_n^*}{\partial \xi} = \frac{1}{4\pi} [\xi_n \quad \xi_t] \begin{Bmatrix} 4(2\nu-1)\frac{e^2}{r^4} + 8(1-\nu)\frac{e^4}{r^6} - (1+\nu)\frac{1}{r^2} \\ 2(1+\nu)\frac{ez}{r^4} + 8(1-\nu)\frac{e^3z}{r^6} \end{Bmatrix} \quad (A.2d)$$

$$\frac{\partial M_{nt}^*}{\partial \xi} = -\frac{(1-\nu)}{4\pi} [\xi_n \quad \xi_t] \begin{Bmatrix} \frac{2e^2z}{r^4} - \frac{z}{r^2} \\ \frac{2ez^2}{r^4} - \frac{e}{r^2} \end{Bmatrix} \quad (A.2e)$$

(3) Second derivatives (moments)

$$\begin{Bmatrix} \frac{\partial^2 W_n^*}{\partial x^2} \\ \frac{\partial^2 W_n^*}{\partial y^2} \\ \frac{\partial^2 W_n^*}{\partial x \partial y} \end{Bmatrix} = \frac{1}{8\pi D} [T^2] \begin{Bmatrix} 1 + \frac{2e^2}{r^2} + 2\ln(r) \\ 1 + \frac{2z^2}{r^2} + 2\ln(r) \\ \frac{2ez}{r^2} \end{Bmatrix} \quad (A.3a)$$

$$\begin{Bmatrix} \frac{\partial^2 \Theta_n^*}{\partial x^2} \\ \frac{\partial^2 \Theta_n^*}{\partial y^2} \\ \frac{\partial^2 \Theta_n^*}{\partial x \partial y} \end{Bmatrix} = \frac{1}{4\pi D} [T^2] \begin{Bmatrix} \frac{3e}{r^2} - \frac{2e^3}{r^4} \\ \frac{e}{r^2} - \frac{2ez^2}{r^4} \\ \frac{z}{r^2} - \frac{2e^2z}{r^4} \end{Bmatrix} \quad (A.3b)$$

$$\begin{Bmatrix} \frac{\partial^2 M_n^*}{\partial x^2} \\ \frac{\partial^2 M_n^*}{\partial y^2} \\ \frac{\partial^2 M_n^*}{\partial x \partial y} \end{Bmatrix} = -\frac{1}{4\pi} [T^2] \begin{Bmatrix} (1+\nu)(\frac{1}{r^2} - \frac{2e^2}{r^4}) + 2(1-\nu)(\frac{1}{r^2} - \frac{5e^2}{r^4} + \frac{4e^4}{r^6}) \\ (1+\nu)(-\frac{1}{r^2} + \frac{2e^2}{r^4}) + 2(1-\nu)(\frac{3e^2}{r^4} - \frac{4e^4}{r^6}) \\ -(1+\nu)\frac{2ez}{r^4} + 2(1-\nu)(-\frac{2ez}{r^4} + \frac{4e^3z}{r^6}) \end{Bmatrix} \quad (A.3c)$$

$$\begin{Bmatrix} \frac{\partial^2 V_n^*}{\partial x^2} \\ \frac{\partial^2 V_n^*}{\partial y^2} \\ \frac{\partial^2 V_n^*}{\partial x \partial y} \end{Bmatrix} = \frac{1}{4\pi} [T^2] \begin{Bmatrix} (1+\nu)(-\frac{6\epsilon}{r^4} + \frac{8\epsilon^3}{r^6}) + (1-\nu)(\frac{12\epsilon}{r^4} - \frac{56\epsilon^3}{r^6} + \frac{48\epsilon^5}{r^8}) \\ (1+\nu)(\frac{6\epsilon}{r^4} - \frac{8\epsilon^3}{r^6}) + (1-\nu)(\frac{40\epsilon^3}{r^6} - \frac{48\epsilon^5}{r^8}) \\ (1+\nu)(-\frac{2z}{r^4} + \frac{8\epsilon^2 z}{r^6}) + (1-\nu)(-\frac{24\epsilon^2 z}{r^6} + \frac{48\epsilon^4 z}{r^8}) \end{Bmatrix} \quad (A.3d)$$

$$\begin{Bmatrix} \frac{\partial^2 M_{nt}^*}{\partial x^2} \\ \frac{\partial^2 M_{nt}^*}{\partial y^2} \\ \frac{\partial^2 M_{nt}^*}{\partial x \partial y} \end{Bmatrix} = -\frac{(1-\nu)}{4\pi} [T^2] \begin{Bmatrix} -\frac{6\epsilon z}{r^4} + \frac{8\epsilon^3 z}{r^6} \\ \frac{2\epsilon z}{r^4} - \frac{8\epsilon^3 z}{r^6} \\ -\frac{1}{r^2} + \frac{8\epsilon^2}{r^4} - \frac{8\epsilon^4}{r^6} \end{Bmatrix} \quad (A.3c)$$

in which the transformation matrix $[T^2]$ has the form of

$$[T^2] = \begin{bmatrix} x_n^2 & x_t^2 & 2x_n x_t \\ x_t^2 & x_n^2 & -2x_n x_t \\ -x_n x_t & x_n x_t & (x_n^2 - x_t^2) \end{bmatrix} \quad (A.4)$$

where x_n and x_t refer to the direction cosines for axis x in the n - t coordinate system.

(4) Third derivatives (shears)

$$\begin{Bmatrix} \frac{\partial \nabla^2 W^*}{\partial x} \\ \frac{\partial \nabla^2 W^*}{\partial y} \end{Bmatrix} = -\frac{1}{2\pi D} [T^3] \begin{Bmatrix} \frac{\epsilon}{r^2} \\ \frac{z}{r^2} \end{Bmatrix} \quad (A.5a)$$

$$\begin{Bmatrix} \frac{\partial \nabla^2 \Theta_n^*}{\partial x} \\ \frac{\partial \nabla^2 \Theta_n^*}{\partial y} \end{Bmatrix} = \frac{1}{2\pi D} [T^3] \begin{Bmatrix} -\frac{1}{r^2} + \frac{2\epsilon^2}{r^4} \\ \frac{2\epsilon z}{r^4} \end{Bmatrix} \quad (A.5b)$$

$$\begin{Bmatrix} \frac{\partial \nabla^2 M_n^*}{\partial x} \\ \frac{\partial \nabla^2 M_n^*}{\partial y} \end{Bmatrix} = -\frac{(1-\nu)}{2\pi} [T^3] \begin{Bmatrix} \frac{6\epsilon}{r^4} - \frac{8\epsilon^3}{r^6} \\ \frac{2z}{r^4} - \frac{8\epsilon^2 z}{r^6} \end{Bmatrix} \quad (A.5c)$$

$$\left\{ \begin{array}{c} \frac{\partial \nabla^2 V_n^*}{\partial x} \\ \frac{\partial \nabla^2 V_n^*}{\partial y} \end{array} \right\} = -\frac{(1-\nu)}{2\pi} [T^3] \left\{ \begin{array}{c} \frac{6}{r^4} - \frac{48e^2 z^2}{r^8} \\ -\frac{24ez}{r^6} + \frac{48e^3 z}{r^8} \end{array} \right\} \quad (A.5d)$$

$$\left\{ \begin{array}{c} \frac{\partial \nabla^2 M_{nt}^*}{\partial x} \\ \frac{\partial \nabla^2 M_{nt}^*}{\partial y} \end{array} \right\} = \frac{(1-\nu)}{\pi} [T^3] \left\{ \begin{array}{c} -\frac{z}{r^4} + \frac{4e^2 z}{r^6} \\ -\frac{e}{r^4} + \frac{4ez^2}{r^6} \end{array} \right\} \quad (A.5e)$$

in which the transformation matrix $[T^3]$ is formed as

$$[T^3] = \begin{bmatrix} x_n & x_t \\ -x_t & x_n \end{bmatrix} \quad (A.6)$$

Appendix B

BOUNDARY INTEGRATIONS

(1) Deflection

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k W^* dz = \frac{1}{8\pi D} \left[\mathbf{I}^{k+2} + e^2 \mathbf{I}^k \right] \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.1a)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \Theta_n^* dz = \frac{e}{8\pi D} \left[\frac{z^{k+1}}{k+1} + 2\mathbf{I}^k \right] \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.1b)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k M_n^* dz = -\frac{1}{8\pi} \left[(1+3\nu) \frac{z^{k+1}}{k+1} + 2(1+\nu) \mathbf{I}^k + 2(1-\nu) e^2 \mathbf{J}^k \right] \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.1c)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k V_n^* dz = \frac{1}{8\pi} \left[2(1+\nu) e \mathbf{J}^k + 4(1-\nu) e^2 \mathbf{K}^k \right] \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.1d)$$

(2) Rotation

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial W^*}{\partial \xi} dz = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{e z^{k+1}}{k+1} + 2e \mathbf{I}^k \\ \frac{z^{k+2}}{k+2} + 2\mathbf{I}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.2a)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial \Theta_n^*}{\partial \xi} dz = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{z^{k+1}}{k+1} + 2e^2 \mathbf{J}^k + 2\mathbf{I}^k \\ 2e \mathbf{J}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.2b)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial M_n^*}{\partial \xi} dz = \frac{1}{4\pi} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} (1+\nu) e \mathbf{J}^k + 2(1-\nu) \mathbf{K}^{k+2} \\ (1+\nu) \mathbf{J}^{k+1} - 2(1-\nu) e \mathbf{K}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.2c)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \frac{\partial V_n^*}{\partial \xi} dz = \frac{1}{4\pi} [\xi_n \quad \xi_l] \left\{ \begin{array}{c} 4(2\nu - 1)e\mathbf{K}^k + 8(1 - \nu)e\mathbf{L}^k - (1 + \nu)\mathbf{J}^k \\ 2(1 + \nu)\mathbf{K}^{k+1} + 8(1 - \nu)\mathbf{L}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.2d)$$

(3) Second derivatives (moments)

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \left\{ \begin{array}{c} \frac{\partial^2 W^*}{\partial x^2} \\ \frac{\partial^2 W^*}{\partial y^2} \\ \frac{\partial^2 W^*}{\partial x \partial y} \end{array} \right\} dz = \frac{1}{8\pi D} [T^2] \left\{ \begin{array}{c} \frac{z^{k+1}}{k+1} + 2e^2 \mathbf{J}^k + 2\mathbf{I}^k \\ \frac{z^{k+1}}{k+1} + 2\mathbf{J}^{k+2} + 2\mathbf{I}^k \\ 2e\mathbf{J}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.3a)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \left\{ \begin{array}{c} \frac{\partial^2 \Theta_n^*}{\partial x^2} \\ \frac{\partial^2 \Theta_n^*}{\partial y^2} \\ \frac{\partial^2 \Theta_n^*}{\partial x \partial y} \end{array} \right\} dz = \frac{1}{4\pi D} [T^2] \left\{ \begin{array}{c} 3e\mathbf{J}^k - 2e^2 \mathbf{K}^k \\ e\mathbf{J}^k - 2\mathbf{K}^{k+2} \\ \mathbf{J}^{k+1} - 2e\mathbf{K}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.3b)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \left\{ \begin{array}{c} \frac{\partial^2 M_n^*}{\partial x^2} \\ \frac{\partial^2 M_n^*}{\partial y^2} \\ \frac{\partial^2 M_n^*}{\partial x \partial y} \end{array} \right\} dz = -\frac{1}{4\pi} [T^2] \left\{ \begin{array}{c} (1 + \nu)(\mathbf{J}^k - 2e\mathbf{K}^k) + 2(1 - \nu)(\mathbf{J}^k - 5e\mathbf{K}^k + 4e\mathbf{L}^k) \\ (1 + \nu)(-\mathbf{J}^k + 2e\mathbf{K}^k) + 2(1 - \nu)(3e\mathbf{K}^k - 4e\mathbf{L}^k) \\ -2(1 + \nu)\mathbf{K}^{k+1} + 4(1 - \nu)(-\mathbf{K}^{k+1} + 2\mathbf{L}^{k+1}) \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.3c)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \left\{ \begin{array}{c} \frac{\partial^2 V_n^*}{\partial x^2} \\ \frac{\partial^2 V_n^*}{\partial y^2} \\ \frac{\partial^2 V_n^*}{\partial x \partial y} \end{array} \right\} dz = \frac{1}{4\pi} [T^2] \times \left\{ \begin{array}{c} (1 + \nu)(-6\mathbf{K}^k + 8\mathbf{L}^k) + (1 - \nu)(12\mathbf{K}^k - 56\mathbf{L}^k + 48\mathbf{M}^k) \\ (1 + \nu)(6\mathbf{K}^k - 8\mathbf{L}^k) + (1 - \nu)(40\mathbf{L}^k - 48\mathbf{M}^k) \\ \frac{1}{e}[(1 + \nu)(-2\mathbf{K}^{k+1} + 8\mathbf{L}^{k+1}) + (1 - \nu)(-24\mathbf{L}^{k+1} + 48\mathbf{M}^{k+1})] \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.3d)$$

(4) Third derivatives (shears)

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \left\{ \begin{array}{c} \frac{\partial \nabla^2 W^*}{\partial x} \\ \frac{\partial \nabla^2 W^*}{\partial y} \end{array} \right\} dz = -\frac{1}{2\pi D} [T^3] \left\{ \begin{array}{c} \epsilon \mathbf{J}^k \\ \mathbf{J}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.4a)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \left\{ \begin{array}{c} \frac{\partial \nabla^2 \Theta_n^*}{\partial x} \\ \frac{\partial \nabla^2 \Theta_n^*}{\partial y} \end{array} \right\} dz = \frac{1}{2\pi D} [T^3] \left\{ \begin{array}{c} -\mathbf{J}^k + 2\epsilon \mathbf{K}^k \\ 2\mathbf{K}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.4b)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \left\{ \begin{array}{c} \frac{\partial \nabla^2 M_n^*}{\partial x} \\ \frac{\partial \nabla^2 M_n^*}{\partial y} \end{array} \right\} dz = -\frac{(1-\nu)}{2\pi} [T^3] \left\{ \begin{array}{c} 6\mathbf{K}^k - 8\mathbf{L}^k \\ \frac{1}{\epsilon} [2\mathbf{K}^{k+1} - 8\mathbf{L}^{k+1}] \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.4c)$$

$$\int_{z_{i1}}^{z_{iN_i}} (z)^k \left\{ \begin{array}{c} \frac{\partial \nabla^2 V_n^*}{\partial x} \\ \frac{\partial \nabla^2 V_n^*}{\partial y} \end{array} \right\} dz = -\frac{(1-\nu)}{2\pi} [T^3] \left\{ \begin{array}{c} \frac{6}{\epsilon} \mathbf{K}^k - \frac{48}{\epsilon^3} \mathbf{M}^{k+2} \\ -\frac{24}{\epsilon^2} \mathbf{L}^{k+1} + \frac{48}{\epsilon^2} \mathbf{M}^{k+1} \end{array} \right\} \Big|_{z_{i1}}^{z_{iN_i}} \quad (B.4d)$$

Appendix C

SURFACE INTEGRATIONS

(1) Deflection

$$ID(w^*)_i^1 = \frac{1}{8\pi D} \left[\frac{E_i Z_i}{12} (Z_i^2 + 3E_i^2) \ln(R_i) - \frac{7}{144} E_i Z_i^3 - \frac{33}{144} E_i^3 Z_i + \frac{E_i^4}{6} \tan^{-1} \frac{Z_i}{E_i} \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.1a)$$

$$ID(w^*)_i^n = \frac{1}{8\pi D} \left[\frac{E_i^2}{15} (Z_i(Z_i^2 + 3E_i^2) \ln(R_i) - \frac{13}{5} E_i^2 Z_i - \frac{8}{15} Z_i^3 + 2E_i^3 \tan^{-1} \frac{Z_i}{E_i}) \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.1b)$$

$$ID(w^*)_i^t = \frac{1}{8\pi D} \left[\frac{E_i}{400} (20R_i^4 \ln(R_i) - 18E_i^2 Z_i^2 - 9Z_i^4) \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.1c)$$

$$ID(w^*)_i^{nt} = \frac{1}{8\pi D} \left[\frac{E_i^2}{24} R_i^4 \ln(R_i) - \frac{5}{288} E_i^2 R_i^4 \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.1d)$$

$$ID(w^*)_i^{n^2} = \frac{1}{8\pi D} \left[\frac{E_i^3}{6} \left((E_i^2 Z_i + \frac{Z_i^3}{3}) \ln(R_i) - \frac{5}{6} E_i^2 Z_i - \frac{1}{6} Z_i^3 + \frac{2}{3} E_i^3 \tan^{-1} \frac{Z_i}{E_i} \right) \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.1e)$$

$$ID(w^*)_i^{t^2} = \frac{1}{8\pi D} \left[\frac{E_i}{90} (5E_i^2 Z_i^3 + 3Z_i^5) \ln(R_i) - \frac{1}{60} E_i^3 Z_i^3 - \frac{11}{900} E_i Z_i^5 + \frac{1}{45} E_i^5 Z_i - \frac{1}{45} E_i^6 \tan^{-1} \frac{Z_i}{E_i} \right] \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.1f)$$

where

$$R_i^2 = Z_i^2 + E_i^2$$

(2) Rotation

$$ID(\frac{\partial w^*}{\partial \xi})_i^1 = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{2}{3} E_i^2 Z_i \ln(R_i) - \frac{5}{9} E_i^2 Z_i + \frac{2}{3} E_i^3 \tan^{-1} \frac{Z_i}{E_i} \\ \frac{1}{3} E_i R_i^2 \ln(R_i) - \frac{1}{9} E_i Z_i^2 \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.2a)$$

$$ID(\frac{\partial w^*}{\partial \xi})_i^n = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{1}{2} E_i^3 Z_i \ln(R_i) - \frac{3}{8} E_i^3 Z_i + \frac{1}{2} E_i^4 \tan^{-1} \frac{Z_i}{E_i} \\ \frac{1}{4} E_i^2 R_i^2 \ln(R_i) - \frac{1}{16} E_i^2 Z_i^2 \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.2b)$$

$$ID(\frac{\partial w^*}{\partial \xi})_i^t = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{1}{4} E_i^2 R_i^2 \ln(R_i) - \frac{1}{16} E_i^2 Z_i^2 \\ \frac{1}{6} E_i Z_i^3 \ln(R_i) - \frac{1}{72} E_i Z_i^3 + \frac{1}{6} E_i^3 Z_i - \frac{1}{6} E_i^4 \tan^{-1} \frac{Z_i}{E_i} \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.2c)$$

$$ID(\frac{\partial w^*}{\partial \xi})_i^{nt} = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{1}{5} E_i^3 R_i^2 \ln(R_i) - \frac{1}{25} E_i^3 Z_i^2 \\ \frac{2}{15} E_i^2 Z_i^3 \ln(R_i) - \frac{1}{225} E_i^2 Z_i^3 + \frac{2}{15} E_i^4 Z_i - \frac{2}{15} E_i^5 \tan^{-1} \frac{Z_i}{E_i} \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.2d)$$

$$ID(\frac{\partial w^*}{\partial \xi})_i^{n^2} = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{2}{5} E_i^4 Z_i \ln(R_i) - \frac{7}{25} E_i^4 Z_i + \frac{2}{5} E_i^5 \tan^{-1} \frac{Z_i}{E_i} \\ \frac{1}{5} E_i^3 R_i^2 \ln(R_i) - \frac{1}{25} E_i^3 Z_i^2 \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.2e)$$

$$ID(\frac{\partial w^*}{\partial \xi})_i^{t^2} = -\frac{1}{8\pi D} [\xi_n \quad \xi_t] \left\{ \begin{array}{c} \frac{2}{15} E_i^2 Z_i^3 \ln(R_i) - \frac{1}{225} E_i^2 Z_i^3 + \frac{2}{15} E_i^4 Z_i - \frac{2}{15} E_i^5 \tan^{-1} \frac{Z_i}{E_i} \\ \frac{E_i R_i^2}{10} (Z_i^2 - E_i^2) \ln(R_i) + \frac{E_i^3 Z_i^2}{20} + \frac{1}{200} E_i Z_i^4 \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.2f)$$

(3) Second derivatives (moments)

$$ID\left(\left\{\begin{array}{c}\frac{\partial^2 w^*}{\partial x^2} \\ \frac{\partial^2 w^*}{\partial y^2} \\ \frac{\partial^2 w^*}{\partial x \partial y}\end{array}\right\}\right)_i^1 = \frac{1}{8\pi D} [T^2] \left\{\begin{array}{c}2E_i^2 \tan^{-1}\left(\frac{Z_i}{E_i}\right) + E_i Z_i \ln(R_i) - E_i Z_i \\ E_i Z_i \ln(R_i) \\ E_i^2 \ln(R_i)\end{array}\right\} \left|_{Z_{i1}}^{Z_{i2}} \right. \quad (C.3a)$$

$$ID\left(\left\{\begin{array}{c}\frac{\partial^2 w^*}{\partial x^2} \\ \frac{\partial^2 w^*}{\partial y^2} \\ \frac{\partial^2 w^*}{\partial x \partial y}\end{array}\right\}\right)_i^n = \frac{1}{8\pi D} [T^2] \left\{\begin{array}{c}\frac{2}{3}E_i^2 Z_i \ln(R_i) - \frac{5}{9}E_i^2 Z_i + \frac{4}{3}E_i^3 \tan^{-1}\left(\frac{Z_i}{E_i}\right) \\ \frac{1}{9}E_i^2 Z_i + \frac{2}{3}E_i^2 Z_i \ln(R_i) \\ \frac{2}{3}E_i^3 \ln(R_i)\end{array}\right\} \left|_{Z_{i1}}^{Z_{i2}} \right. \quad (C.3b)$$

$$ID\left(\left\{\begin{array}{c}\frac{\partial^2 w^*}{\partial x^2} \\ \frac{\partial^2 w^*}{\partial y^2} \\ \frac{\partial^2 w^*}{\partial x \partial y}\end{array}\right\}\right)_i^t = \frac{1}{8\pi D} [T^2] \left\{\begin{array}{c}E_i^3 \ln(R_i) + \frac{1}{3}E_i Z_i^2 \ln(R_i) - \frac{1}{9}E_i Z_i^2 \\ -\frac{1}{3}E_i^3 \ln(R_i) + \frac{1}{3}E_i Z_i^2 \ln(R_i) + \frac{2}{9}E_i Z_i^2 \\ \frac{2}{3}E_i^2 Z_i - \frac{2}{3}E_i^3 \tan^{-1}\left(\frac{Z_i}{E_i}\right)\end{array}\right\} \left|_{Z_{i1}}^{Z_{i2}} \right. \quad (C.3c)$$

$$ID\left(\left\{\begin{array}{c}\frac{\partial^2 w^*}{\partial x^2} \\ \frac{\partial^2 w^*}{\partial y^2} \\ \frac{\partial^2 w^*}{\partial x \partial y}\end{array}\right\}\right)_i^{nt} = \frac{1}{8\pi D} [T^2] \left\{\begin{array}{c}(\frac{1}{4}E_i^2 R_i^2 + \frac{1}{2}E_i^4) \ln(R_i) - \frac{1}{16}E_i^2 Z_i^2 \\ (\frac{1}{4}E_i^2 R_i^2 - \frac{1}{2}E_i^4) \ln(R_i) + \frac{3}{16}E_i^2 Z_i^2 \\ \frac{1}{2}E_i^3 Z_i - \frac{1}{2}E_i^4 \tan^{-1}\left(\frac{Z_i}{E_i}\right)\end{array}\right\} \left|_{Z_{i1}}^{Z_{i2}} \right. \quad (C.3d)$$

$$ID\left(\left\{\begin{array}{c}\frac{\partial^2 w^*}{\partial x^2} \\ \frac{\partial^2 w^*}{\partial y^2} \\ \frac{\partial^2 w^*}{\partial x \partial y}\end{array}\right\}\right)_i^{n^2} = \frac{1}{8\pi D} [T^2] \left\{\begin{array}{c}\frac{1}{2}E_i^3 Z_i \ln(R_i) + E_i^4 \tan^{-1}\left(\frac{Z_i}{E_i}\right) - \frac{3}{8}E_i^3 Z_i \\ \frac{1}{2}E_i^3 Z_i \ln(R_i) + \frac{1}{8}E_i^3 Z_i \\ \frac{1}{2}E_i^4 \ln(R_i)\end{array}\right\} \left|_{Z_{i1}}^{Z_{i2}} \right. \quad (C.3e)$$

$$ID\left(\left\{\begin{array}{c}\frac{\partial^2 w^*}{\partial x^2} \\ \frac{\partial^2 w^*}{\partial y^2} \\ \frac{\partial^2 w^*}{\partial x \partial y}\end{array}\right\}\right)_i^{t^2} = \frac{1}{8\pi D} [T^2] \left\{ \begin{array}{l} \frac{1}{6} E_i Z_i^3 \ln(R_i) - \frac{1}{72} E_i Z_i^3 - \frac{2}{3} E_i^4 \tan^{-1}\left(\frac{Z_i}{E_i}\right) + \frac{2}{3} E_i^3 Z_i \\ \frac{1}{6} E_i Z_i^3 \ln(R_i) + \frac{11}{72} E_i Z_i^3 + \frac{1}{3} E_i^4 \tan^{-1}\left(\frac{Z_i}{E_i}\right) - \frac{1}{3} E_i^3 Z_i \\ \frac{1}{4} E_i^2 Z_i^2 - \frac{1}{2} E_i^4 \ln(R_i) \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.3f)$$

(4) Third derivatives (shear)

$$ID\left(\left\{\begin{array}{c}\frac{\partial \nabla^2 w^*}{\partial x} \\ \frac{\partial \nabla^2 w^*}{\partial y}\end{array}\right\}\right)_i^1 = -\frac{1}{8\pi D} [T^3] \left\{ \begin{array}{l} 4E_i \tan^{-1}\left(\frac{Z_i}{E_i}\right) \\ 4E_i \ln(R_i) \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.4a)$$

$$ID\left(\left\{\begin{array}{c}\frac{\partial \nabla^2 w^*}{\partial x} \\ \frac{\partial \nabla^2 w^*}{\partial y}\end{array}\right\}\right)_i^n = -\frac{1}{8\pi D} [T^3] \left\{ \begin{array}{l} 2E_i^2 \tan^{-1}\left(\frac{Z_i}{E_i}\right) \\ 2E_i^2 \ln(R_i) \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.4b)$$

$$ID\left(\left\{\begin{array}{c}\frac{\partial \nabla^2 w^*}{\partial x} \\ \frac{\partial \nabla^2 w^*}{\partial y}\end{array}\right\}\right)_i^t = -\frac{1}{8\pi D} [T^3] \left\{ \begin{array}{l} 2E_i^2 \ln(R_i) \\ 2E_i Z_i - 2E_i^2 \tan^{-1}\left(\frac{Z_i}{E_i}\right) \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.4c)$$

$$ID\left(\left\{\begin{array}{c}\frac{\partial \nabla^2 w^*}{\partial x} \\ \frac{\partial \nabla^2 w^*}{\partial y}\end{array}\right\}\right)_i^{nt} = -\frac{1}{8\pi D} [T^3] \left\{ \begin{array}{l} \frac{4}{3} E_i^3 \ln(R_i) \\ \frac{4}{3} E_i^2 Z_i - \frac{4}{3} E_i^3 \tan^{-1}\left(\frac{Z_i}{E_i}\right) \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.4d)$$

$$ID\left(\left\{\begin{array}{c}\frac{\partial \nabla^2 w^*}{\partial x} \\ \frac{\partial \nabla^2 w^*}{\partial y}\end{array}\right\}\right)_i^{n^2} = -\frac{1}{8\pi D} [T^3] \left\{ \begin{array}{l} \frac{4}{3} E_i^3 \tan^{-1}\left(\frac{Z_i}{E_i}\right) \\ \frac{4}{3} E_i^3 \ln(R_i) \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.4e)$$

$$ID\left(\left\{\begin{array}{c}\frac{\partial \nabla^2 w^*}{\partial x} \\ \frac{\partial \nabla^2 w^*}{\partial y}\end{array}\right\}\right)_i^{t^2} = -\frac{1}{8\pi D} [T^3] \left\{ \begin{array}{l} \frac{4}{3} E_i^2 Z_i - \frac{4}{3} E_i^3 \tan^{-1}\left(\frac{Z_i}{E_i}\right) \\ \frac{2}{3} E_i Z_i^2 - \frac{4}{3} E_i^3 \ln(R_i) \end{array} \right\} \Big|_{Z_{i1}}^{Z_{i2}} \quad (C.4f)$$

END

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