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EARTHQUAKE RESPONSE ANALYSIS OF UNANCHORED CYLINDRICAL LIQUID STORAGE TANKS

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A thesis submitted to the
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for the degree of
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**EARTHQUAKE RESPONSE ANALYSIS
OF UNANCHORED CYLINDRICAL
LIQUID STORAGE TANKS**

submitted by XIANGUANG ZENG
in partial fulfillment of the requirements for
the degree of Doctor of Philosophy in Engineering

External Examiner
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ABSTRACT

The earthquake response behaviour of unanchored cylindrical liquid storage tanks are still not fully understood. Many unanchored tank damages were observed during recent earthquakes. In this thesis, an analysis model for evaluating the nonlinear seismic uplift response of unanchored cylindrical tanks is developed using the Rayleigh-Ritz method.

An analytical formulation for the hydrodynamics of vibrating fluid in an unanchored cylindrical tank is developed by using small amplitude wave theory. The formulation given in the form of a double series is shown to be an exact and complete solution which satisfies the governing differential equation of the liquid domain and all the boundary conditions including the influence of the bottom plate uplift. The sensitivity of the solution due to numerical round-off errors is discussed and resolved properly.

The complicated nonlinear uplift mechanism of an unanchored tank under lateral load is modelled by Ritz formulation. The tank structure is discretized by several substructure elements according to the different mechanical behaviour and properties in each part of the structure. These include the bottom plate uplifted region, the bottom plate contact region, the cylindrical tank shell, and the rim wind girder. The bottom plate uplifted region is modelled by von Karman plate theory to take into consideration the coupling of the membrane action and flexural mechanism. It is shown that the inplane forces of the bottom plate have significant influence on the uplift response of an unanchored tank under lateral load. The cylindrical tank shell is modelled by the Flügge thin shell theory. The influences to the response due to the tank roof and stiffening ring girders at the base and at the top rims are also considered.

A new direct time step integration procedure is developed to analyze the dynamic response behaviour of an unanchored tank subjected to earthquake ground motion.

Two tanks, one broad and one tall, are analyzed using the ground acceleration

input of the El-Centro earthquake in this thesis. The broad tank is a 1/3 scaled model tank tested at the Earthquake Engineering Research Center, the University of California at Berkeley. The tall tank is a vinegar storage tank damaged during the 1989 Loma Prieta (San Francisco Area) earthquake.

The results show that the response of an unanchored tank can be either larger or less than that of the same tank under the anchored condition depending on the interaction between the liquid hydrodynamics and the energy dissipation mechanism due to uplift. Both the bottom uplifting mechanism and the out-of-round deformation of the tank shell have significant influences on the dynamic response behaviour of unanchored tanks.

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Chapter 1

INTRODUCTION

1.1 GENERAL

This thesis concerns the development of an efficient analysis procedure to study the seismic response behaviour of unanchored cylindrical liquid storage tanks. Several new analytical techniques have been developed to study the dynamic behaviour of both anchored and unanchored fluid-tank systems. In the present study, particular attention is focused on the nonlinear dynamic uplift mechanism of the tank bottom.

Thin shell cylindrical tanks are very efficient structures for the storage of liquid content. They are very simple to fabricate and easy to maintain. And yet, cylindrical tanks are often part of vital lifeline facilities. Some examples are fuel storage tanks in power generating stations, water reservoirs in water distribution systems, and storage tanks for hazardous liquid chemical or toxic waste found in petro-chemical industries. As in all cases of lifeline equipments, these storage tank facilities are designed to remain functional during and immediately after an earthquake to provide uninterrupted vital services to the public. Failures of these facilities usually have serious consequences. For example, the explosion or leakage due to failure of storage tanks will threaten life and pollute the environment; the collapse of water storage tanks will disrupt vital water supply. There are many examples of tank damage during earthquakes, such as those observed during the 1964 Prince William Sound earth-

quake in Alaska[1,2], the 1964 Niigata[3] and 1978 Miyagi Ken Oki [4,5] earthquakes in Japan, the 1971 San Fernando[6], 1979 Imperial Valley[7], and 1983 Coalinga[8] earthquakes in California, the 1976 Tangshan earthquake in China[9], and the 1985 Chilean earthquake[10]. During the recent 1989 Loma Prieta earthquake in California, many cylindrical tanks were also damaged[11].

Past experiences have thus shown that thin walled liquid storage tanks are very vulnerable to earthquake damage. The vulnerability of these structures to earthquake ground motions was particularly exacerbated because until recently many liquid storage tanks were designed primary to resist only hydrostatic loads. The hydrodynamic loads caused by the vibrating liquid were not properly considered in the design process.

1.2 TYPICAL LIQUID STORAGE TANK STRUCTURE

Liquid storage tanks of different structural configurations can be found in many civil engineering applications. However, ground supported cylindrical tanks are the most common type because they are simple to design, efficient to resist primary loads, and can be easily constructed.

As shown in Figure 1.1, a typical cylindrical metal tank consists essentially of a thin walled cylindrical tank shell which resists the outward liquid pressure, a thin flat bottom plate resting on the foundation, and a fixed or floating roof. The bottom plate prevents the liquid from leaking out. The roof protects the contained liquid from the atmosphere. There may also be a ring girder at the top rim and additional stiffening ring girders along the height of the tank wall to improve the stability of the tank shell. Interior columns may also be used to support the roof in the case of a large diameter tank.

In general, the tank wall is designed as a membrane to resist a purely tensile hoop stress. This provides an efficient design because metal is a very economic material especially in resisting tension. Depending on the magnitude of the tensile stress induced by the static liquid pressure, the wall of a large tank may have different wall

thicknesses along its height.

Several different types of roof may be employed to provide cover for the contained liquid: cone shaped, dome shaped, flat plate type or the more flexible type of floating roof. A common roof structure is a system of trusses supporting a thin plate. The connection between the roof and the tank shell is normally designed as a weak connection so that if the tank is over-filled, this connection will fail first before failure of the connection between the bottom plate and the cylindrical shell.

Different kinds of foundation may be used to support the tank structure: concrete ring wall, solid concrete slab, concrete base supported by piles or caissons. In most cases, particularly for medium to large capacity tanks of broad proportion with the height less than the diameter, the tanks are not anchored to the foundation, and thus may uplift or separate from the support foundation during strong ground shaking.

1.3 SEISMIC RESPONSE BEHAVIOUR OF UNANCHORED TANK

During earthquakes, the inertia effect of the vibrating fluid induces hydrodynamic pressure acting on the tank wall. This dynamic loading can be represented by a horizontal resultant force and an overturning moment acting at the tank base as shown in Figure 1.2(a). If the tank is not anchored to the foundation around its base, part of the tank bottom in the proximity of the base rim may uplift in response to the overturning moment. For an uplifting tank, the overturning of the tank system is resisted by the weight of the uplifted liquid. In the uplift mechanism, the uplifted liquid is picked up by the longitudinal tensile axial force at the tank wall. This tensile force is transferred around the cylindrical tank shell to the opposite contact side where it becomes compressive. The partial uplift of the bottom plate on the tensile side is shown in Figure 1.2(a). Different from the response behaviour of an anchored tank, of which the horizontal cross section remains circular throughout the response, the cylindrical tank shell of an unanchored tank is highly distorted during

uplift with significant out-of-round deformation along the tank height and out-of-plane warping at the base, as shown in Figure 1.2(b) and (c). As evident from the tank damage observed in past earthquakes, unanchored tanks are very vulnerable to buckling failure near the bottom of the tank shell on the contact side because of the high compressive axial stresses at those regions.

The behaviour of an unanchored tank is very complicated and highly nonlinear because of the following reasons: (1) The extent of the region that uplifts from the support foundation is a nonlinear function of the applied seismic load; (2) Nonlinear large displacement plate theory is required to model the bottom plate displacement because of the expected large amount of vertical uplift displacement; (3) Material may behave inelastically because of the large deformation. Because of the highly nonlinear uplift phenomenon, dynamic analysis of unanchored tanks is a very complicated nonlinear transient problem. At present, analysis procedures are available to predict the dynamic response of anchored tanks during earthquakes. But reliable analytical model for the prediction of the seismic uplift response of unanchored tanks are still not available.

1.4 REVIEW OF PREVIOUS RESEARCH

The study of the seismic response behaviour of cylindrical liquid storage tanks may be separated into two parts: the hydrodynamic forces induced by the ground motion, and the dynamic response of the tank system as a result of these hydrodynamic actions. In this section, reviews of the previous works done on these two subject areas are presented. A brief review on the earthquake resistant design of liquid storage tanks is also presented in this section.

1.4.1 Hydrodynamic effects

The evaluation of the hydrodynamic load induced in a vibrating container is an important topic in the investigation of the seismic response behaviour of liquid storage tanks. Many studies on this subject have been reported in the literature. In most of

these studies, the fluid is typically assumed to be incompressible and inviscid. The flow is irrotational. The governing differential equation for such an ideal fluid is given by the Laplace's equation in terms of a velocity potential function of the liquid.

Early studies on the hydrodynamic loading due to earthquakes were carried out by Housner [12], Hoskins [13] and Jacobsen [14]. In Housner's formulation, the hydrodynamic loading is derived based on the rigid tank assumption and is divided into the impulsive and convective loading components. The impulsive component is caused by the portion of the liquid accelerating as a rigid mass in unison with the rigid tank, whereas the convective component is caused by the portion of liquid which participates in the sloshing motion of the liquid surface.

In the Housner's model as shown in Figure 1.3, the convective component is modelled by an equivalent single-degree-of-freedom mechanical oscillator with f_l , m_l and ξ_l representing respectively the frequency, mass and damping ratio of the mechanical oscillator. The mass of the equivalent oscillator and its position H_l relative from the tank base are determined such that the lateral force and overturning moment exerted by the liquid and the mechanical oscillator at the base of the tank are the same. Similarly, the rigid mass m_r and its height H_r can also be determined by the impulsive lateral force and overturning moment.

In recent years, further improvement in the estimation of hydrodynamic loading have been reported by many researchers[15-23]. There have been many studies on the hydrodynamic forces of liquid content in which the interaction between the liquid motion and the flexibility of the tank shell is considered. The motivation of this advance was the observed seismic damage to liquid storage tanks, for example, during the 1964 Alaska earthquake when many liquid storage tanks designed by modern codes were damaged. In addition, the advances in digital computer and the development of powerful numerical techniques have significantly enhanced the solution capability in the modeling of complex engineering problem. Therefore, many investigations about the dynamic behaviour of flexible containers have been initiated since then. Results from experimental investigations[24-25] and analytical studies[26-27] on the dynamic

behaviour of cylindrical tanks indicate that the flexibilities of both the tank shell and the bottom plate have significant effects on the magnitude and distribution of the hydrodynamic pressure. At present, analytical solution for the hydrodynamic loading in a flexible unanchored tank which takes into account the effect of partial bottom uplift is not available. The formulation developed in this thesis and presented in later chapters is the first one reported in the literature.

Studies on the dynamic behaviour of liquid storage tanks can be categorized by the analysis assumption whether the tank bottom is considered rigid or flexible. Rigid tank bottom is assumed in the case of an anchored tank system, whereas in the case of an uplifting unanchored tank system, the flexibility of the tank bottom should be taken into consideration in the modeling of the tank system. Previous researches on the dynamic behaviour of liquid storage tanks concern mostly with anchored tanks using the rigid tank bottom assumption. The analysis of the hydrodynamic effects in unanchored tank system with flexible bottom plate is considerably more complex because of the partial separation of the tank bottom from the support foundation.

For flexible tanks partially filled with liquid, the impulsive pressure and convective pressure are coupled with the motions of the tank wall and the bottom plate. Veletsos[28] and Fischer[29] have shown that the magnitude and distribution of the impulsive pressure calculated with rigid tank assumption can be substantially different from those obtained considering the influence of tank shell flexibility.

1.4.2 Dynamic response of tank system

Dynamic analysis of anchored tanks

Several different approaches have been adopted by different researchers for determining the dynamic response of anchored tanks. These approaches include the following:

- 1) Finite element approach: In this approach, both the liquid domain and the tank wall are modelled by finite elements[30-33]. The interaction between the vibrating fluid and the flexible container wall is considered through the added mass concept. The amount of calculation required in a finite element model for the complicated

nonlinear dynamic response analysis of an unanchored tank is generally considered too high, making this approach impractical.

2) Boundary integral techniques: In this approach, the liquid response is derived from the solution of the Laplace's equation in series form, whereas the tank shell is modelled by finite elements. Since the boundary conditions at the interface between the liquid domain and the tank wall are difficult to be satisfied exactly by the liquid response shape functions, the boundary condition are thus satisfied only in an average sense. Haroun[34] used the boundary integral techniques to obtain a complete solution for flexible anchored tanks.

3) Variational method: An infinite series solution of the Laplace's equation is obtained for the liquid domain. The boundary condition of the liquid domain is satisfied by the convolution integral of the harmonic excitation response using assumed deformed shapes for the tank shell. For example, the lateral vibration of the tank wall may be assumed to behave like that of a cantilever beam in the vertical direction, with the horizontal section of the cylindrical shell maintaining a perfectly circular shape. In their studies, Veletsos[35] and Yang[36] calculate the convective hydrodynamic effect assuming the tank wall is rigid. The impulsive effect is determined in which the influence of the tank wall flexibility is considered approximately by using the ground pseudoacceleration function determined for the equivalent SDOF mechanical oscillator with the same fundamental natural frequency as that of the fluid-tank system. The results indicate that this approximation is acceptable for those anchored tanks with the H/R ratio in the range between 0.25 to 1.0, where H is the height of the liquid, and R is the tank radius.

Dynamic analysis of unanchored tanks

In practice, most of the medium to large capacity tanks in the field are unanchored. One reason is that it is difficult to properly design an adequate and safe anchorage system to prevent uplift at the tank base during severe earthquakes. Another particular concern is that an anchored tank with an improperly designed anchorage system is vulnerable to tear off and rupture failure of the tank shell or bottom plate

at the anchorage, with a possibility of losing all the liquid content and thus causing more serious damage. For this reason, Wozniak [37] suggested that it is better not to anchor the tank base in the design of some tanks for earthquake resistant purpose.

As mentioned before, the characteristics of an unanchored tank during earthquakes are inherently nonlinear and very complex. Consequently, very few analytical studies have been reported on the dynamic response behaviour of unanchored tanks.

Ishida and Kobayashi[38] investigated uplift of the bottom plate. They modelled the fluid-tank system as a mass-spring system and approximated the nonlinear rocking behaviour by a bi-linear rotational spring. Rammerstorfer *et al.*[39] analyzed cylindrical tanks under wind loads taking into consideration the nonlinear uplift behaviour of the flexible tank bottom by the finite element method with a fine mesh. Their analysis requires about 45 CPU hours on a CDC Cyber 170/730.

Natsiavas[40] derived an analytical model for unanchored tanks under seismic ground excitation by the Hamilton's principle. The fluid response was derived using small amplitude wave theory, assuming the fluid ideal and irrotational. The displacement of the tank shell was obtained using beam-type modes of which the cross section of the tank shell remains perfectly circular. The displacement of the bottom plate, as contributed by the horizontal ground excitation was modelled by nonlinear springs at the tank bottom.

Because of the complexity of the uplift mechanism, many of the approximations and assumptions adopted in the analytical procedures developed in previous studies cannot be justified. For example in some studies, the bottom plate is modelled by the linear plate bending theory[41, 42]. The serious drawback in this model is that the important membrane action of the bottom plate is ignored. For the amount of deformation experienced by an uplifting tank during earthquakes, the membrane action may surpass the bending mechanism as the dominant resistant mechanism in resisting the uplift deformation. Fischer[43] established an equivalent analytical strip model for the bottom plate based on the beam theory. Peek[16] developed an approximate scheme using solution from the axisymmetrical uplift analysis to model

the two dimensional unsymmetrical uplift problem.

Lau[19] developed an analysis procedure to study the static tilt uplift behaviour of unanchored cylindrical tanks. Under the static tilt condition produced by tilting the tank supporting platform from its initial horizontal position, the partially filled tank is subjected to an effective static lateral acceleration component which simulates the effect of the horizontal acceleration component of an earthquake. Because of the lack of an analytical model to study the dynamic uplift behaviour, the static tilt uplift analysis is carried out in order to obtain a better understanding of the uplift mechanism. The development of an efficient static tilt uplift analysis procedure is thus looked upon as an intermediate step in the development of a dynamic time history analysis model for earthquake response analysis of unanchored tanks.

In the static tilt uplift analysis model, the deformations of the bottom plate are evaluated by a Ritz-type method using substructure elements to discretize the tank system. The bottom plate attached to the base of the tank wall is subjected to both membrane action and plate bending in the uplifted region. The tank wall is analyzed using Flügge thin shell theory, and its stiffness is cast in a form compatible with that of the bottom plate for direct stiffness summation. The stiffening effects of the top rim wind girder and the bottom toe ring are also included. This procedure can be used to analyze the response of tanks under different types of lateral load including wind and earthquake loads. One particular concern in using this procedure to analyze the response due to transient loads, such as the seismic loads, is that the computation cost is too excessive for a complete dynamic time history analysis for the entire duration of an earthquake. Thus rendering the direct application of this procedure to solve the transient problem impractical and ineffective.

Experimental studies

Since recognition of the importance of the ground supported liquid storage tanks and the many uncertainties in their seismic behaviour, many experimental studies on liquid storage tanks have been carried out in recent years.

Clough[24], Manos and Clough[25], and Niwa[44] conducted a series of experi-

ments at the Earthquake Engineering Research Center in the University of California at Berkeley. These testings included both broad and tall tanks subjected to simulated earthquake loading on a shaking table and static tilt loads under the anchored and unanchored conditions. The influences of the rigid and flexible foundations on the response behaviour were also examined in the studies. In another study, Housner and Haroun[45] measured the vibration frequencies and mode shapes of several full size liquid storage tanks in the field. Zui et al[46] investigated the effects of foundation flexibility and initial irregularities of the cylindrical tanks on the seismic response. Shih and Babcock[47] studied the buckling failure of liquid storage tank under harmonic base excitation by scaled model tests. Kelly[48] studied conceptual design methods for seismic isolation of storage tanks. Sakai and Isoe[49] investigated the bottom plate uplift behaviour of liquid storage tanks by static tilt testings of a large-sized cylindrical tank.

Results from these experimental studies have shown that the seismic response behaviour of both anchored and unanchored tanks still have many uncertainties, and that of the unanchored tanks are even more complicated. One of the important seismic behaviour of liquid storage tanks, which is difficult to predicate in the analysis, is the out-of-round deformation of the tank shell and its effect on the failure mechanism. It has been observed that in the field the out-of-round deformation of the tank shell contributes significantly to the buckling failure of the tank wall due to high axial compressive stress in the tank wall.

Soil-structure interaction

From the tests carried out by Housner and Haroun[45], it was noted that the flexibility of the foundation had a noticeable influence on the vibration of the liquid storage tanks.

Fujita[50] modelled the sloshing liquid as an oscillator and the soil as an elastic foundation in the investigation of the seismic response of an anchored tank. In their study, Ishida and Kobayashi[38] modelled the supporting soil as elastic springs. Fischer and Seeber[15] analyzed the response of cylindrical liquid storage tanks subjected

to vertical support excitations by considering the soil as an elastic half space with frequency dependent stiffness and damping parameters. Shimizu[51] studied the horizontal seismic soil-tank-liquid interaction by the complex modal analysis method to take into consideration different damping characteristics of the systems. Veletsos and Tang[52] investigated the seismic soil-tank-fluid interaction behaviour in response to the horizontal and rocking ground motion input. They also studied the interaction problem under vertical earthquake excitation[53].

1.4.3 Seismic design of liquid storage tanks

The basic concept in earthquake resistant design of a structural system is to make the system capacities, which may be specified in terms of strength, stiffness or ductility, greater than the estimated demands put on the system by the earthquakes. For the seismic design of liquid storage tanks, both the capacities and demands are quite uncertain.

Surveys of seismic damage to liquid storage tank in the past decades find that typical types of tank damage can be summarized as follows[54]

1. Failure of the bottom plate
2. Buckling of the tank wall
3. Rupture of the tank wall at the anchorage
4. Failure of anchorages
5. Failure of the roof
6. Total collapse
7. Failure or partial settlement of foundation
8. Sliding or walk-off of foundations due to rocking
9. Damages to piping connections or other equipments.

Among these observed damage, the most common type of seismic damage is buckling of the tank wall.

Design Standards

There are two commonly followed design standards for the design of liquid storage tanks in North America, API Standard 650[55] (Welded steel tanks for oil storage) and AWWA D100-79[56] (AWWA standard for welded tanks for water storage). Both standards have provisions for seismic resistant design of tank. In fact, the hydrodynamic effects in both standards are determined by using the Housner's model.

In API Standard 650 and AWWA D100-79, the overturning moment M is estimated by following the Housner mechanical oscillator hydrodynamic procedure. The compressive stress in the tank shell is determined using the beam flexural formula in mechanics ($\sigma = My/I$, where I is the moment of inertia of the tank shell horizontal section at the base, y is the distance to the neutral axis) for an anchored tank. The uplift extension of an unanchored tank is determined using bending theory in modeling the bottom plate. The compressive stress in the unanchored tank shell is evaluated by the assumption that, on the tank shell horizontal cross-section at the base, the axial stress distribution of the shell is linear on the contact side, and the weight of portion of liquid resisting the overturning moment is uniformly distributed on the uplifted base rim, while the weight of the tank shell is uniformly distributed over the entire horizontal tank section. Then considering equilibrium of the base section, the maximum compressive stress of an unanchored tank can be determined. According to the estimation by the API Standard 650, for the same tank under the unanchored condition, the maximum compressive stress in the tank shell is always much larger than that under the anchored condition. This is in contradiction with some of the results obtained in the present study which shows that the maximum compressive stress in an unanchored tank shell may be larger or less than that in an anchored one.

Other design procedures

As discussed earlier, there are many accepted design procedures to evaluate the re-

sponses of anchored liquid storage tanks, such as those developed by Housner[12], Haroun-Housner[57], and Veletsos-Yang[35] procedures. Design procedures to estimate the buckling capacity of the tank wall are also available in the literature[47], [58], and [59].

In summary, the present state-of-the-art in the design of liquid storage tanks does not provide an optimal and reliable seismic design guideline to design engineers because of the many uncertainties involved in the problem. This is particularly so in the case of unanchored tanks.

1.5 OBJECTIVES AND SCOPE

The main objective of this thesis is to develop a nonlinear dynamic time history analysis model to study the seismic uplift behaviour of unanchored cylindrical liquid storage tanks, which may lead to improvement in earthquake resistant design of these structures.

As mentioned before, typical tank damages due to earthquakes can be classified into the failure of tank structure and equipment, or failure of foundations. In this thesis, the mechanism of the behaviour and damage to cylindrical liquid storage tanks as related to the structural response to earthquake ground motion will be investigated.

To analyze a nonlinear dynamic (time-dependent) problem, a step-by-step integration procedure is usually employed to solve the equation of motion. Due to the complexity of the nonlinear tank bottom uplift under lateral load, this commonly adopted approach is considered impractical in the present study. This is clearly the case as a static uplift analysis by the finite element method, with the dynamic lateral loads at a particular instant applied as pseudostatic forces, may take several hours of CPU time in a medium size computer station, as reported by Rammerstorfer and Fischer[39] and Lau[19]. Therefore, for dynamic analysis of uplifting unanchored tanks, some other more effective analysis techniques to model the fluid-tank system must be developed. Otherwise the amount of computation involved would be prohibitively

expensive, rendering seismic uplift analysis of unanchored tanks impractical.

In the present study, the formulation for dynamic uplift response analysis of unanchored cylindrical liquid storage tanks is based on the Rayleigh-Ritz method. The present approach combines advantages of the traditional Ritz formulation with the flexibility and convergence characteristics of the finite element approach. The boundary value problem, as represented by a set of partial differential equations together with the appropriate boundary conditions, is solved by the modified Rayleigh-Ritz method in which the field variables are represented by combinations of Ritz shape functions. The accuracy and effectiveness of the solution depend on the completeness and efficiency of the shape functions used to discretize the field variables.

The scope of the thesis is summarized into the following tasks:

1. Develop an analytical solution for the dynamic response of the liquid domain by boundary solution techniques.
2. Thoroughly review the theory and application of the Ritz method; and develop the theoretical bases for the efficient generation of effective Ritz shape functions.
3. Develop an efficient simplified static analysis model capable of analyzing the uplift mechanism of unanchored cylindrical tanks under static uplift conditions. Static uplift may be considered as the tank response to the dynamic earthquake load when it is applied pseudostatically. A simple simulation of the static uplift condition can be achieved by statically tilting up the rigid supporting platform of a model tank, as shown in Figure 1.4.
4. Develop a nonlinear dynamic time history analysis procedure to evaluate the seismic uplift response of unanchored tanks during earthquakes by extending the static uplift model to include the transient and inertial effects of the vibrating fluid-tank system.

1.6 OUTLINE OF THESIS

The present study concerns the development of an analytical model for dynamic uplift of unanchored cylindrical liquid storage tanks. The investigation includes studies on the liquid hydrodynamic pressure, and the seismic behaviour of liquid storage tanks including the performance of cylindrical tanks free to uplift.

In this thesis, the exact analytical formulation of the hydrodynamic effects in unanchored cylindrical tanks is presented in Chapter 2. The irregular out-of-round deformation of the tank horizontal section is included in the formulation.

Rayleigh-Ritz method is employed to model the tank structure. A liquid storage tank is analytically divided into several structural components: the tank shell, the uplifted region of the bottom plate, the contact region of the bottom plate, the tank roof, and the stiffening ring girders. Different effective Ritz shapes are employed in the modeling of each of the structural elements.

The analysis model and derivation of the Ritz shape functions for the bottom plate are discussed in Chapter 3. The Ritz shapes for the tank shell, and the stiffening effect of the roof and the rim girders are presented in Chapter 4. The assembly of the various structural components into a combined model representing the fluid-tank system is discussed in Chapter 5. The static tilt results are presented and compared with experimental measurements obtained for a broad 1/3 scale welded aluminium model tank which was tested at the Earthquake Engineering Research Center of the University of California at Berkeley[25] in Chapter 6. Chapter 7 explains the dynamic analysis model and the nonlinear time history analysis procedure of unanchored cylindrical liquid storage tanks. The dynamic analysis results are discussed in Chapters 8 and 9. Chapter 10 gives a summary and conclusion.

Figure 1.1: A typical cylindrical liquid storage tank

(a) Hydrodynamic pressure and resultants

(b) Distorted horizontal cross section

(c) Horizontal cross section warping

Figure 1.2: Uplifting of unanchored tank due to lateral forces

Figure 1.3: Housner's model for rigid tank

Figure 1.4: Unanchored tank uplift deformation by static tilt simulation

Chapter 2

HYDRODYNAMICS IN CYLINDRICAL TANKS

The behaviour of liquid in a vibrating cylindrical tank is discussed in this chapter. In the first two sections, the linear wave theory and previous research mainly on the hydrodynamic loading of anchored tanks are briefly discussed and reviewed. A new analytical formulation specifically developed for the hydrodynamic pressure in unanchored tanks is then presented. Influence of the tank wall flexibility and bottom plate uplift on the dynamic response of unanchored tanks is studied. Boundary conditions at the interface surface with the vibrating tank wall and uplifting bottom plate are established for the liquid domain. Numerical sensitivity problem and round-off error in the analytical solution are discussed. Improved formulations to overcome these difficulties are presented.

2.1 LINEAR WAVE THEORY OF HYDRODYNAMICS

2.1.1 Liquid velocity potential

The coordinate system of an unanchored cylindrical tank is shown in Figure 2.1, where r and θ are the radial and tangential coordinates, z is the longitudinal axis measuring upwards from the tank bottom, and H and H_s respectively denote the liquid depth and tank shell height. In the present study, the following assumptions on the behaviour of the liquid are made:

- The liquid is homogeneous, inviscid and incompressible.
- The flow is irrotational.
- There are no sources, sinks or cavities in the flow.
- Only small amplitude sloshing motion is considered.

For such a homogeneous, inviscid and incompressible ideal fluid initially at rest, it can be shown that the motion of the fluid induced by ground support excitation is irrotational[60]. The small amplitude sloshing assumption linearizes the hydrodynamic problem. Defining a velocity potential function by $\phi(r, \theta, z, t)$ in the cylindrical coordinate system, the velocities of a fluid particle are given by the gradients of the velocity potential function as follows[61]:

$$v_r = \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad v_z = \frac{\partial \phi}{\partial z}$$

From the continuity of the liquid flow motion, it can also be shown that the velocity potential function satisfies the Laplace's equation, or

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (2.1)$$

Any function ϕ that satisfies the Laplace's equation given above is a possible solution to the hydrodynamic problem of the liquid domain. However, the correct and unique solution among all the admissible possibilities is the one that satisfies all the boundary conditions at the domain boundaries.

To obtain the liquid pressure p , the Bernoulli-Euler equation is used which can be expressed as follows

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{V^2}{2} + g(z - H) = 0 \quad (2.2)$$

where V is the magnitude of the velocity, and ρ is the liquid mass density, g is the gravitational acceleration. This equation is obtained by integrating the equation of motion for the irrotational flow derived based on the concept of conservation of energy.

For waves of small amplitude, the flow speed is assumed to be everywhere in the domain a small quantity, and the term V^2 may then be considered negligible when compared to other terms in Equation 2.2. Hence Equation 2.2 may be simplified to

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + g(z - H) = 0 \quad (2.3)$$

Equation 2.3 represents the linear wave theory in hydrodynamics. The pressure distribution, $p(r, \theta, z, t)$, is then given as

$$p(r, \theta, z, t) = p_d + p_s = -\rho \frac{\partial \phi}{\partial t} + \rho g(H - z) \quad (2.4)$$

where

$$p_s = \rho g(H - z) \quad \text{liquid hydrostatic pressure,}$$

$$p_d = -\rho \frac{\partial \phi}{\partial t} \quad \text{hydrodynamic pressure due to liquid motion}$$

2.1.2 Boundary conditions for the liquid domain

In the formulation of the hydrodynamic effects induced in a vibrating cylindrical tank, boundary conditions are established at the top quiescent liquid free surface, the common interfaces shared by the liquid and the cylindrical tank shell and the bottom plate.

Boundary conditions on the liquid free surface

In the formulation of the surface wave theory, it is assumed that a fluid particle initially on the liquid free surface will always remain on the free surface, and the pressure on the free surface always remain zero. Substituting the height of the liquid sloshing free-surface $z = H + \xi(r, \theta, t)$ into Equation 2.3, where $\xi(r, \theta, t)$ is the free surface vertical displacement relative to the quiescent position, and neglecting higher order terms

$$\frac{\partial \phi(r, \theta, H, t)}{\partial t} + g \xi(r, \theta, t) = 0 \quad (2.5)$$

On the free surface boundary, velocity of a fluid particle normal to the free surface in the z -direction can be expressed as follows

$$\frac{\partial \phi(r, \theta, H, t)}{\partial z} = \frac{\partial \xi(r, \theta, t)}{\partial t} \quad (2.6)$$

Combining Equations 2.5 and 2.6, the following boundary condition on the liquid free surface can be obtained, in which only the velocity potential is involved

$$\frac{\partial^2 \phi(r, \theta, H, t)}{\partial t^2} + g \frac{\partial \phi(r, \theta, H, t)}{\partial z} = 0 \quad (2.7)$$

Boundary conditions on the tank wall and tank bottom

On the interface boundaries shared by the tank structure and fluid, the adjacent fluid particle and the tank structure must have the same normal velocity. Because the liquid is assumed to be inviscid, no tangential shear force can be transmitted across the interface. The boundary condition thus involves only the normal component of the motion between the adjacent liquid and the surface of the tank wall or bottom plate. Figure 2.2 shows an uplifting cylindrical tank during seismic excitation. On the surface of the cylindrical tank wall, $r = R$, the liquid must have the same vertical velocity as the wall in the r -direction, which can be expressed as follows

$$\frac{\partial \phi(R, \theta, z, t)}{\partial r} = \dot{w}(\theta, z, t) \quad (2.8)$$

where $w(\theta, z, t)$ is the displacement of the tank wall in the r -direction. The dotted notation denotes differentiation with time.

At the bottom of an uplifting tank, $z = 0$, the liquid must have the same velocity as that of the tank bottom in the z -direction

$$\frac{\partial \phi(r, \theta, 0, t)}{\partial z} = \dot{b}(r, \theta, t) \quad (2.9)$$

where $b(r, \theta, t)$ is the vertical displacement of the tank bottom in the z -direction.

2.2 HYDRODYNAMIC LOADS IN ANCHORED TANKS

In this section, hydrodynamic forces in rigid anchored tanks are first derived. Then effects of the tank shell flexibility on the hydrodynamic forces are examined. Following Housner's formulation, the total hydrodynamic force in a vibrating container is the sum of contributions from the impulsive component and the convective component. The impulsive component is the hydrodynamic force caused by the portion of liquid accelerating in unison with the tank, whereas the convective component is the hydrodynamic force caused by the portion of liquid which participates in the sloshing motion of the free surface. The velocity potential function ϕ given earlier in Equation 2.1 is derived as the sum of the impulsive velocity potential component, ϕ_i , and the convective velocity potential component, ϕ_c .

2.2.1 Rigid tank solution

For a rigid tank, there is no deformation of the tank shell ($w = 0$), and no base uplift ($b = 0$). Assuming both the impulsive and convective velocity potential components ϕ_i and ϕ_c , independently satisfy the Laplace's Equation 2.1 in the linear hydrodynamic wave theory, the boundary conditions given in Equations 2.7-9 may be rewritten in terms of the velocity potential components. In the separation of the boundary conditions it is noted that the velocity boundary condition at the cylindrical surface is satisfied by the impulsive component alone with no free surface displacement. Any discrepancy with the actual boundary condition at the liquid free surface is then corrected by the convective component. The boundary conditions then become

$$\text{at } r = R \quad \frac{\partial \phi_i}{\partial r} = \dot{x}(t) \cos \theta \quad \frac{\partial \phi_c}{\partial r} = 0 \quad (2.10)$$

$$\text{at } z = 0 \quad \frac{\partial \phi_i}{\partial z} = 0 \quad \frac{\partial \phi_c}{\partial z} = 0 \quad (2.11)$$

$$\text{at } z = H \quad \frac{\partial \phi_i}{\partial t} = 0 \quad \frac{\partial^2 \phi_c}{\partial t^2} + g \frac{\partial \phi_c}{\partial z} = -g \frac{\partial \phi_i}{\partial z} \quad (2.12)$$

where $\dot{x}(t)$ is the ground velocity. The dynamic pressure p_d is then given by the sum of p_i and p_c

$$p_d = p_i + p_c = -\rho \frac{\partial \phi}{\partial t} = -\rho \frac{\partial \phi_i}{\partial t} - \rho \frac{\partial \phi_c}{\partial t} \quad (2.13)$$

As such, the impulsive component, ϕ_i , and convective component, ϕ_c , must satisfy the Laplace's Equation 2.1 and the boundary conditions given in Equations 2.10-12, these are derived in the following.

Impulsive component

Solving the Laplace's equation by separation of variables and expressing the impulsive velocity potential, ϕ_i , as follows

$$\phi_i = \Gamma_i(r) Z_i(z) \dot{x}(t) \cos \theta \quad (2.14)$$

Only $\cos \theta$ term is involved in Equation 2.14 in the expansion of ϕ_i in the tangential direction. This is because only $\cos \theta$ type excitation is applied to the liquid as indicated in the boundary condition given in Equation 2.10. Substituting Equation 2.14 into Equation 2.1, the following can be obtained

$$\left[\Gamma_i'' Z_i + \frac{\Gamma_i'}{r} Z_i - \frac{1}{r^2} \Gamma_i Z_i + Z_i'' \Gamma_i \right] \dot{x} \cos \theta = 0 \quad (2.15)$$

where the prime symbol denotes differentiation with respect to the function variable. Since Equation 2.15 must be satisfied at all time t , this implies

$$\left[\Gamma_i'' + \frac{\Gamma_i'}{r} - \frac{1}{r^2} \Gamma_i \right] \frac{1}{\Gamma_i} = -\frac{Z_i''}{Z_i} \quad (2.16)$$

Because $Z_i(z)$ and $\Gamma_i(r)$ are independent functions, the ratio in Equation 2.16 must therefore be a constant. Assuming the ratio to be λ_0

$$\frac{Z_i''}{Z_i} = -\lambda_0 \quad (2.17)$$

and

$$\left[\Gamma_i'' + \frac{\Gamma_i'}{r} - \frac{1}{r^2} \Gamma_i \right] \frac{1}{\Gamma_i} = \lambda_0 \quad (2.18)$$

From consideration of boundary conditions, it can be proved that the ratio λ_0 above is a positive number. Thus denoting λ_0 by λ^2 , Equation 2.17 is rewritten as follows

$$Z_i''(z) + \lambda^2 Z_i(z) = 0 \quad (2.19)$$

of which the solution is given by

$$Z_i(z) = A \cos \lambda z + B \sin \lambda z \quad (2.20)$$

Consequently the boundary conditions given by Equations 2.11-12 imply

$$\begin{cases} B \lambda \cos 0 = 0 \\ A \cos \lambda H + B \sin \lambda H = 0 \end{cases} \quad (2.21)$$

For nontrivial solution of the constants A and B , the determinant of the coefficient matrix must be equal to zero, that is

$$\begin{vmatrix} 0 & \lambda \\ \cos \lambda H & 0 \end{vmatrix} = 0$$

or

$$\cos \lambda H = 0$$

For $m = 1, 2, 3, \dots$, the roots of the above equation are

$$\lambda_m = \frac{2m-1}{2} \pi \quad m = 1, 2, 3, \dots \quad (2.22)$$

Equation 2.18 is a Bessel differential equation of which the solution is given by modified Bessel function of the first kind, that is

$$\Gamma_i(r) = I_1(\lambda r)$$

Combining expressions given above, the solution for ϕ_i can be expressed as follows, which is the same as derived by Haroun [27], Veletsos[35] and Yang[36]

$$\phi_i = \sum_{m=1}^{\infty} C_m(t) I_1 \left(\frac{\lambda_m r}{H} \right) \cos \left(\frac{\lambda_m z}{H} \right) \cos \theta \quad (2.23)$$

where

$$C_m(t) = (-1)^{m+1} \frac{2H}{\lambda_m^2 I_1'(\lambda_m R/H)} \ddot{x}$$

The impulsive pressure, p_i , may be obtained by substituting the expression of ϕ_i given here into Equation 2.13

$$p_i = \rho \sum_{m=1}^{\infty} (-1)^m \frac{2H}{\lambda_m^2 I_1'(\lambda_m R/H)} \ddot{x} I_1 \left(\frac{\lambda_m r}{H} \right) \cos \left(\frac{\lambda_m z}{H} \right) \cos \theta \quad (2.24)$$

Convective component

Solving the Laplace's equation by the same procedures as above in determining the impulsive component but now by imposing the boundary conditions for the convective component, the convective velocity potential is written as

$$\phi_c = \Gamma_c(r) Z_c(z) T_c(t) \cos \theta \quad (2.25)$$

Boundary condition at the cylindrical surface given in Equation 2.10 leads to the solution

$$\Gamma_{cj}(r) = J_1 \left(\frac{\zeta_{j1} r}{R} \right) \quad (2.26)$$

where J_1 is the Bessel function of the first kind, ζ_{j1} are the zeros of its first derivative. From the boundary condition Equation (2.11), namely,

$$\frac{\partial \phi_c}{\partial z} \Big|_{z=0} = 0 ,$$

$$Z_{cj}(z) = \cosh \left(\frac{\zeta_{j1} z}{R} \right)$$

For the boundary condition on the liquid free surface given by Equation 2.12

$$\left\{ \frac{\partial^2 \phi_i}{\partial t^2} + g \frac{\partial \phi_i}{\partial z} \right\}_{z=H} = -g \left\{ \frac{\partial \phi_i}{\partial z} \right\}_{z=H}$$

Multiplying the above equation by $J_1(\zeta_{j1}r/R) r dr$ and integrating from 0 to R

$$(T_c''' + \omega_j^2 T_c) \frac{1}{2} R^2 J_1^2(\zeta_{j1}) \frac{\zeta_{j1}^2 - 1}{\zeta_{j1}^2} = g \dot{x} \sum_{m=1}^{\infty} \frac{\int_0^R 2r I_1(\lambda_m r/H) J_1(\zeta_{j1} r/R) dr}{\lambda_m I_1'(\lambda_m R/H) \cosh(\zeta_{j1} H/R)} \quad (2.27)$$

where ω_j is the sloshing frequency and is given by

$$\omega_j^2 = g \frac{\xi_{j1}}{R} \tanh\left(\frac{\xi_{j1} H}{R}\right) \quad (2.28)$$

Equation 2.27 can be further simplified to the following, based on the mathematical identities of Bessel functions [62]

$$T_c''' + \omega_j^2 T_c = \frac{2R\omega_j^2 \dot{x}}{(\zeta_{j1}^2 - 1)J_1(\zeta_{j1}) \cosh(\zeta_{j1} H/R)} \quad (2.29)$$

Solving Equation 2.29 and substituting the solution of T_c into Equation 2.25, the convective velocity potential component can be obtained

$$\phi_c = \cos \theta \sum_{j=1}^{\infty} \frac{2R \omega_j \int_0^t \ddot{x}(\tau) \sin \omega_j(t - \tau) d\tau}{(\zeta_{j1}^2 - 1)J_1(\xi_{j1}) \cosh(\xi_{j1} H/R)} J_1\left(\frac{\zeta_{j1} r}{R}\right) \cosh\left(\frac{\zeta_{j1} z}{R}\right) \quad (2.30)$$

The convective pressure p_c can then be obtained as follows

$$p_c = -\rho \frac{\partial \phi_c}{\partial t} = -\rho \cos \theta \sum_{j=1}^{\infty} \frac{2R A_j(t) J_1\left(\frac{\zeta_{j1} r}{R}\right) \cosh\left(\frac{\zeta_{j1} z}{R}\right)}{(\xi_{j1}^2 - 1)J_1(\xi_{j1}) \cosh(\xi_{j1} H/R)} \quad (2.31)$$

where

$$A_j(t) = \omega_j \int_0^t \ddot{x}(\tau) \sin \omega_j(t - \tau) d\tau \quad (2.32)$$

is the pseudoacceleration function for a single-degree-of-freedom oscillator with the same circular natural frequency ω_j as the liquid in sloshing.

2.2.2 Flexible tank solution

In the analysis of the dynamic behaviour of flexible anchored tanks, one approach to model the liquid vibrational response is to discretize the liquid domain into finite elements. This approach in modeling the fluid was adopted in the studies by Edward[30], Ma[31], Liu[32], and Balendra[33].

Another approach for analyzing the seismic response of flexible anchored tanks is adopted by Haroun[27]. In this formulation, the liquid is discretized into ring shaped boundary elements on the top free surface. An advantage of this approach as compared to the finite element approach in the modeling of the liquid domain is that the number of degrees of freedom required for the liquid part in the combined fluid-tank system is greatly reduced, minimizing the size of the problem, making this a more efficient approach. In another study, Veletsos and Yang[35] have presented an approximate analytical procedure mostly for design purpose, in which the effects of tank shell flexibility on hydrodynamic forces are included. In this approximate approach, the convective or sloshing effects are still evaluated based on the rigid tank assumption. Only the impulsive effects are modified by including the influence of the flexibility of the tank shell. The justifications for this approximation are the following:

- 1) The impulsive effects come from the lateral motions of the tank wall and the participating liquid which moves with the wall, whereas the convective effects are due to the rocking or sloshing action of the liquid. For anchored tanks, the two effects are weakly coupled, or each effect is insensitive to the influence of the other. Because the effect of the tank wall flexibility is to modify the lateral motion of the tank shell, therefore, the impulsive effects can be significantly affected by the tank wall flexibility.

- 2) The period of liquid sloshing associated with convective effects is much larger than that related to the impulsive effects. This difference in the dominant periods of vibration further reduces the coupling between the two hydrodynamic components. Consequently, the convective hydrodynamic pressure in a flexible tank may be ap

proximately given by the rigid tank solution presented in Section (2.2.1).

Veletsos-Yang procedure

With the convective effects computed based on the rigid tank assumption, the Rayleigh-Ritz method is employed by Veletsos[35] and Yang[36] to compute the impulsive component. Assuming horizontal sections of the cylindrical tank shell always remain circular throughout the response, the radial displacements of the tank wall relative to the ground support may be expressed as follows

$$w(z, \theta, t) = w(t)\phi(z) \cos \theta \quad (2.33)$$

where $\phi(z) \cos \theta$ is a Rayleigh-Ritz deformation shape function for the tank shell and $w(t)$ is the time dependent amplitude of the shape function.

Assuming $\dot{x}_0$ to be the velocity of the horizontal ground motion, the term $\dot{x}(t)$ in Equation 2.10 is then replaced by $\dot{w}(t)\phi(z) + \dot{x}_0$ in the formulation for the impulsive velocity potential component in flexible anchored tanks. Repeating the solution procedures as discussed before in obtaining Equation 2.14, the impulsive hydrodynamic pressure in a flexible tank can be determined as follows

$$p_i = \left[\sum_{k=1}^{\infty} c_{0k}(z) A_k(t) \right] \rho R \cos \theta \quad (2.34)$$

where $c_{0k}(z)$ is a dimensionless function which defines the heightwise variation of the k^{th} pressure component. $A_k(t)$ is the pseudoacceleration function corresponding to the fundamental natural frequency of the fluid-tank system similar to that given by Equation 2.32 except that the frequency ω_j is now replaced by the frequency of the fluid-tank system instead of the sloshing frequency of the fluid only. The expression for $c_{0k}(z)$ is given in reference [35]

$$c_{0k}(z) = (-1)^{k+1} \frac{2H}{\lambda_i^2 R H_1'(\lambda_i R/H)} I_1 \left(\frac{\lambda_i R}{H} \right) \cos \left(\frac{\lambda_i z}{H} \right) \quad (2.35)$$

Haroun-Housner procedure

Similar to the Veletsos-Yang procedure, the convective effects in the Haroun-Housner formulation are computed using the rigid tank solution. The impulsive effects are analyzed using the mechanical model shown in Figure 2.3.

In Figure 2.3, the mass m_l , frequency f_l , damping ratio ξ_l and the height h_l of the mass m_l are determined from the liquid fundamental sloshing mode of vibration. The mass m_r and its location h_r are evaluated from the rigid-body or uniform component of wall motion; whereas the mass m_f , frequency f_0 , damping ratio ξ_0 and location h_f simulate the deformational or flexible component of the tank vibration. The values and locations of those masses, m_l , m_f and m_r are chosen such that the mechanical system in Figure 2.3 will yield the same base shear and overturning moment as those given by the respective modes of vibration in the fluid-tank system. In the mechanical model, all the parameters are given as the functions of the ratio H/R and t/R , where H is the height of the liquid, t and R are respectively the thickness of the tank wall and the tank radius. Values of these parameters are given in design charts by Haroun[57].

The base shear contributed by m_r is $m_r\ddot{x}$, and the shear from m_f is $m_f(A_0 - \ddot{x})$, where A_0 is the pseudoacceleration function of m_f . The total impulsive base shear Q_i is following

$$Q_i = m_f A_0 + (m_r - m_f)\ddot{x} \quad (2.36)$$

Similarly, based on rigid tank solution, the convective base shear Q_c is given as

$$Q_c = m_l A_l \quad (2.37)$$

Then the total shear Q is the sum

$$Q = Q_i + Q_c = m_f A_0 + (m_r - m_f)\ddot{x} + m_l A_l \quad (2.38)$$

The maximum base shear Q_{max} and maximum overturning moment M_{max} can be approximately calculated by the SRSS (square root of the sum of the squares of all terms) rule

$$Q_{max} = \sqrt{(m_l A_l)_{max}^2 + (m_f A_0)_{max}^2 + [(m_f - m_r)\ddot{x}_{max}]^2} \quad (2.39)$$

The maximum overturning moment at the base are

$$M_{max} = \sqrt{(m_l h_l A_l)_{max}^2 + (m_f h_f A_0)_{max}^2 + [(m_f h_f - m_r h_r)\ddot{x}_{max}]^2} \quad (2.40)$$

The principle ideas in Veletsos-Yang procedure and Haroun-Housner approach are similar. Veletsos used Rayleigh-Ritz method, whereas Haroun used boundary element to model the liquid and finite elements in the discretization of the tank structure. Results obtained by the two procedures agree very well with each other.

2.3 HYDRODYNAMIC LOADS IN UNANCHORED TANKS

Unlike anchored tanks, unanchored tanks behave very differently during earthquakes which in general is very complicated. In order to understand the seismic response behaviour of unanchored tanks, particularly the uplift mechanism, a complete analytical formulation (not approximate) of the hydrodynamic loading for an uplifting tank is derived in this section.

2.3.1 General solution of Laplace equation

By the method of separation of variables, the general complete solution of the Laplace's equation given in Equation 2.1 can be expressed in series form as follows

$$\phi(r, \theta, z, t) = \sum_{n=0}^{\infty} \cos(n\theta) \left\{ T_{01n}(t) \left(\frac{r}{R}\right)^n + \dot{T}_{01n}(t) \left(\frac{r}{R}\right)^n \frac{z}{H} + \right.$$

$$\sum_{i=1}^{\infty} \left[T_{in}(t) I_n\left(\frac{\lambda_i r}{H}\right) \cos\left(\frac{\lambda_i z}{H}\right) + \check{T}_{in}(t) I_n\left(\frac{\lambda_i r}{H}\right) \sin\left(\frac{\lambda_i z}{H}\right) \right] \quad (2.41)$$

$$+ \sum_{j=1}^{\infty} \left[\bar{T}_{jn}(t) J_n\left(\frac{\zeta_{jn} r}{R}\right) \cosh\left(\frac{\zeta_{jn} z}{R}\right) + \hat{T}_{jn}(t) J_n\left(\frac{\zeta_{jn} r}{R}\right) \sinh\left(\frac{\zeta_{jn} z}{R}\right) \right] \Bigg\}$$

where $J_n(x)$ and $I_n(x)$ are respectively the n^{th} order and the modified n^{th} order Bessel functions of the first kind, which are related by

$$I_n(x) = (i)^n J_n(ix), \quad i = \sqrt{-1}$$

Due to symmetry, it is assumed in Equation 2.41 that only the cosine terms are included in the general solution as the direction of the ground motion is assumed parallel to the axis $\theta = 0^\circ$

The arbitrary time varying functions, $T_{01n}(t)$, $T_{in}(t)$, $\bar{T}_{jn}(t)$, $\check{T}_{in}(t)$ and $\hat{T}_{jn}(t)$ and the constants ζ_{jn} and λ_i in Equation 2.41, are determined by satisfying all the boundary conditions, given by Equation 2.7-9. As shown in the following, not all the arbitrary functions and constants are needed to satisfy the boundary conditions. Substituting Equation 2.41 into the boundary condition given by Equation 2.7 on the liquid free surface at $z = H$, and assuming that each of the arbitrary functions are required to satisfy the boundary condition, the following condition is required of $T_{01n}(t)$

$$T_{01n}''(t) = 0 \quad (2.42)$$

The solution of Equation 2.42 is

$$T_{01n}(t) = a_0 + b_0 t$$

Substituting this back into the velocity potential expression given in Equation 2.41 and calculating the hydrodynamic pressure, it is noted that the constant term a_0 does not contribute to the hydrodynamic effect according to Equation 2.4. Furthermore, the term associated with b_0 gives rise to a constant hydrodynamic pressure, which

is non periodic and thus incompatible with the expected oscillatory behaviour of the fluid-tank system during earthquakes. Therefore without loss of generality, it is taken in the derivation that

$$a_0 = b_0 = 0 \quad \text{or} \quad T_{01n}(t) = 0$$

Upon examining the velocity potential term associated with $\ddot{T}_{01n}(t)$ in Equation 2.41, it is noted that the velocity component in the z -direction given by gradient of the velocity potential in the z -direction is

$$\dot{T}_{01n}(t) \left(\frac{r}{R} \right)^n \frac{1}{H}$$

This represents a constant vertical velocity for the liquid throughout the height of the liquid, which is unlikely excited by the horizontal component of the ground motions. Therefore it is taken in the derivation here that

$$\dot{T}_{01n} = 0$$

The general expression for the velocity potential function ϕ is then given as follows

$$\begin{aligned} \phi(r, \theta, z, t) = & \sum_{n=0}^{\infty} \cos(n\theta) \\ & \left\{ \sum_{i=1}^{\infty} \left[T_{in}(t) I_n \left(\frac{\lambda_i r}{H} \right) \cos \left(\frac{\lambda_i z}{H} \right) + \ddot{T}_{in}(t) I_n \left(\frac{\lambda_i r}{H} \right) \sin \left(\frac{\lambda_i z}{H} \right) \right] \right. \\ & \left. + \sum_{j=1}^{\infty} \left[\dot{T}_{jn}(t) J_n \left(\frac{\zeta_{jn} r}{R} \right) \cosh \left(\frac{\zeta_{jn} z}{R} \right) + \ddot{T}_{jn}(t) J_n \left(\frac{\zeta_{jn} r}{R} \right) \sinh \left(\frac{\zeta_{jn} z}{R} \right) \right] \right\} \end{aligned} \quad (2.43)$$

Orthogonal expansion of velocity potential

In order to ensure convergence of all the infinite series in the velocity potential function given in Equation 2.41 for tanks of different size, the following conditions on the completeness of the expansion functions in the cylindrical coordinates (r, θ, z) are necessary

1. The series related to θ must be complete in the tangential direction over the interval $0 < \theta < \pi$
2. The series related to z must be complete in the vertical direction over the interval $0 < z < H$
3. The series related to r must be complete in the radial direction over the interval $0 < r < R$

For expansion of the velocity potential function ϕ in the tangential direction θ , it is shown in Appendix A that the infinite series $\{\cos(n\theta)\}$ is complete and orthogonal over the interval $(0 < \theta < \pi)$ for $n = 0, 1, 2, 3, \dots$

The remaining time function and constants λ_i and ζ_{jn} given in Equation 2.43 are determined by the three boundary conditions of an uplifting cylindrical tank. The boundary condition at the top quiescent free surface and the tank bottom involve the vertical velocity of the liquid, whereas the boundary condition at the cylindrical interface between the liquid and the tank wall involves the radial velocity of the fluid. Recalling from earlier discussion, the liquid velocity is given by the gradient of the velocity potential function in the direction considered. In the formulation of the hydrodynamic effects, it is convenient to uncouple the determination of the arbitrary time functions if possible through the separation of the boundary conditions. In other words, the arbitrary time functions determined from the boundary condition on the cylindrical tank wall are made not to contribute to the other two boundary conditions at the liquid top free surface and the tank bottom. The uncoupling of the boundary conditions can be accomplished by proper choices of the arbitrary constants. Upon choosing

$$\lambda_i = \frac{2i-1}{2}\pi$$

and ζ_{jn} as the zeros of the first derivative of the n^{th} order of Bessel function J_n , such that

$$\cos\left(\frac{\lambda_i z}{H}\right) = 0, \quad \text{for } z = H$$

$$\frac{\partial}{\partial r} J_n \left(\frac{\zeta_{jn} r}{R} \right) = 0 \quad \text{for } r = R$$

then the terms $T_{in}(t)$ and $\check{T}_{in}(t)$ are determined from the boundary condition on the cylindrical tank wall given by Equation 2.8, whereas $\bar{T}_{jn}(t)$ and $\check{T}_{jn}(t)$ are determined by the remaining boundary conditions at the top free surface and tank bottom given by Equations 2.7 and 2.9. As shown in Appendix A, the series $\{\cos(\lambda_i z/H)\}$ is complete and orthogonal for the expansion of the velocity potential along the tank height ($0 < z < H$), with the advantage that this series do not contribute to the hydrodynamic pressure on the liquid free surface. Consequently, $\check{T}_{in}(t)$ may be conveniently chosen to be zero because the cos function has already proven complete for expansion of the velocity potential in the z -direction and satisfying the boundary condition and thus the sin function is not necessary. This also has the additional advantage that the remaining infinite series associated with $I_n(\lambda_i r/H)$ does not contribute to the vertical velocity at the tank bottom $z = 0$.

In order to eliminate the terms associated with $\bar{T}_{jn}(t)$ and $\check{T}_{jn}(t)$ in the boundary condition on the tank wall at $r = R$ given in Equation 2.8, the constants ζ_{jn} are chosen as the values for which the first derivative of J_n is zero. As shown in Appendix A, the series of n^{th} order Bessel functions $J_n(k_j^n r)$ is complete and orthogonal with respect to the weighting function r in the interval $(0, R)$, that is

$$\int_0^R r J_n(k_j^n r) J_n(k_i^n r) dr = 0 \quad \text{for } i \neq j$$

and

$$\int_0^R r J_n(k_j^n r) J_n(k_j^n r) dr = \frac{1}{2} \left\{ R^2 J_n'^2(k_j^n R) + \left[R^2 - \left(\frac{n}{k_j^n} \right)^2 \right] J_n^2(k_j^n R) \right\} \quad (2.44)$$

where $k_j^n R$ are the zeros of the first derivative of the Bessel function J_n .

For an uplifting tank, the boundary condition at the tank bottom given by Equation 2.9 states that the liquid flow has the same vertical velocity as the bottom plate

$$\frac{\partial}{\partial z} \phi(r, \theta, 0, t) = \sum_{n=0}^{\infty} \cos(n\theta) \sum_{j=1}^{\infty} \check{T}_{jn}(t) \frac{\zeta_{jn}}{R} J_n \left(\frac{\zeta_{jn} r}{R} \right) = \dot{b}(r, \theta, t) \quad (2.45)$$

with ζ_{jn} chosen as the zeros of the first derivative of the n^{th} order Bessel function J_n . Using the orthogonal property of the Bessel function given in Equation 2.44, $\tilde{T}_{jn}(t)$ can be determined as follows

$$\tilde{T}_{jn}(t) = \frac{\kappa_n \int_0^{2\pi} \cos n\theta d\theta \int_0^R r \dot{b}(r, \theta, \tau) J_n(\frac{\zeta_{jn} r}{R}) dr}{\pi \zeta_{jn} R (1 - n^2/\zeta_{jn}^2) J_n^2(\zeta_{jn})} \quad (2.46)$$

where

$$\kappa_n = \begin{cases} 2 & \text{for } n = 1, 2, 3, \dots \\ 1 & \text{for } n = 0 \end{cases}$$

Next, applying the boundary condition at the interface between the liquid domain and the tank wall as given in Equation 2.8, the following can be obtained

$$\sum_{n=0}^{\infty} \cos(n\theta) \sum_{i=1}^{\infty} T_{in}(t) \frac{\lambda_i}{H} I'_n(\frac{\lambda_i r}{H}) \cos(\frac{\lambda_i z}{H}) = \dot{w}(\theta, z, t) \quad (2.47)$$

where $T_{in}(t)$ can be determined by making use of the orthogonal property of the cosine functions

$$T_{in}(t) = \frac{\kappa_n \int_0^{2\pi} \int_0^H \dot{w}(\theta, z, t) \cos n\theta \cos \frac{\lambda_i z}{H} d\theta dz}{\pi H R_{in}} \quad (2.48)$$

where

$$R_{in} = \frac{\lambda_i}{H} I'_n(\frac{\lambda_i R}{H})$$

On the liquid free surface, the velocity potential must satisfy the free surface boundary condition given by Equation 2.7. Substituting the expression of ϕ into the boundary condition

$$\begin{aligned} & \sum_{j=1}^{\infty} \left[\tilde{T}_{jn}''(t) \cosh(\frac{\zeta_{jn} H}{R}) + \tilde{T}_{jn}''(t) \sinh(\frac{\zeta_{jn} H}{R}) \right] J_n(\frac{\zeta_{jn} r}{R}) + \\ & g \left\{ - \sum_{i=1}^{\infty} \frac{\lambda_i}{H} T_{in}(t) I_n(\frac{\lambda_i r}{H}) \sin(\lambda_i) + \right. \\ & \left. \sum_{j=1}^{\infty} \frac{\zeta_{jn}}{R} \left[T_{jn}(t) \sinh(\frac{\zeta_{jn} H}{R}) + \tilde{T}_{jn}(t) \cosh(\frac{\zeta_{jn} H}{R}) \right] J_n(\frac{\zeta_{jn} r}{R}) \right\} = 0 \end{aligned} \quad (2.49)$$

Making use of the orthogonal property of the Bessel functions, Equation 2.49 can be simplified as

$$T_{jn}'' + \omega_{jn}^2 T_{jn} = -\dot{T}_{jn}(t) \frac{g\zeta_{jn}}{R} - \dot{T}_{jn}''(t) \tanh\left(\frac{\zeta_{jn}H}{R}\right) + \frac{2g \int_0^R \sum_{i=1}^{\infty} \lambda_i I_n\left(\frac{\lambda_i r}{H}\right) \sin \lambda_i T_{in}(t) r J_n\left(\frac{\zeta_{jn}r}{R}\right) dr}{R^2 H (1 - n^2/\zeta_{jn}^2) J_n^2(\zeta_{jn}) \cosh(\zeta_{jn}H/R)} \quad (2.50)$$

where

$$\omega_{jn}^2 = g \frac{\zeta_{jn}}{R} \tanh\left(\frac{\zeta_{jn}H}{R}\right) \quad (2.51)$$

In Equation 2.50, the functions $\dot{T}_{jn}(t)$ and $T_{in}(t)$ are given by Equation 2.46 and 2.48, only the function T_{jn} is unknown. Solving Equation 2.50 by the familiar Duhamel integral approach, the following can be obtained

$$\begin{aligned} T_{jn} = & a_{jn} \sin \omega_{jn} t + b_{jn} \cos \omega_{jn} t \\ & - \frac{g\zeta_{jn}}{R} \frac{1}{\omega_{jn}} \int_0^t \dot{T}_{jn}(\tau) \sin \omega_{jn}(t - \tau) d\tau \\ & - \tanh\left(\frac{\zeta_{jn}H}{R}\right) \frac{1}{\omega_{jn}} \int_0^t \dot{T}_{jn}''(\tau) \sin \omega_{jn}(t - \tau) d\tau \\ & + \frac{2g \int_0^R \sum_{i=1}^{\infty} \lambda_i I_n\left(\frac{\lambda_i r}{H}\right) \sin(\lambda_i r) J_n\left(\frac{\zeta_{jn}r}{R}\right) dr \int_0^t T_{in}(\tau) \sin \omega_{jn}(t - \tau) d\tau}{\omega_{jn} R^2 H (1 - n^2/\zeta_{jn}^2) J_n^2(\zeta_{jn}) \cosh(\zeta_{jn}H/R)} \end{aligned} \quad (2.52)$$

With all the time varying functions determined, the combined hydrodynamic pressure is then given by the following

$$\begin{aligned} p_d = & -\rho \sum_{n=0}^{\infty} \cos n\theta \left\{ \sum_{i=1}^{\infty} \dot{T}_{in}(t) I_n\left(\frac{\lambda_i r}{H}\right) \cos\left(\frac{\lambda_i z}{H}\right) \right. \\ & \left. + \sum_{j=1}^{\infty} \left[\dot{T}_{jn}(t) \cosh\left(\frac{\zeta_{jn}z}{R}\right) + \dot{T}_{jn}''(t) \sinh\left(\frac{\zeta_{jn}z}{R}\right) \right] J_n\left(\frac{\zeta_{jn}r}{R}\right) \right\} \end{aligned} \quad (2.53)$$

Natsiavas[40] also derived a similar general solution as Equation 2.53.

2.3.2 Convergence and round-off errors

Using Equation 2.53 to calculate the hydrodynamic pressure, numerical results obtained in seismic uplift response analyses of unanchored cylindrical tanks indicate that the response behaviour is sensitive to numerical error, especially for higher sloshing modes. This numerical problem is investigated here by examining the relative order of magnitude of the terms in the hydrodynamic pressure expression.

Upon examination of each term in Equation 2.53, it is noted that the cosh and sinh functions associated with $\dot{T}_{jn} \cosh(\zeta_{jn}z/R)$ and $\dot{T}_{jn} \sinh(\zeta_{jn}z/R)$ respectively are the sources of the numerical problem. In Equation 2.53, the infinite series of index i is stable and therefore does not contribute to the numerical convergence problem because the magnitude of the term $\dot{T}_{in} I_n(\lambda_i r/H)$ is in the order of

$$\frac{I_n(\lambda_i r/H)}{\lambda_i I'_n(\lambda_i R/H)}$$

as $\dot{T}_{in}$ from Equation 2.48 has the order of magnitude of $1/[\lambda_i I'_n(\lambda_i R/H)]$. Since the Bessel function I_n and I'_n above are of the same order of magnitude, increase in i causes increase in λ_i and thus the magnitude of $\dot{T}_{in} I_n(\lambda_i r/H)$ is well behaved and should diminish for increasing i . However, this is not the case for the cosh and sinh terms in the summation of the infinite series for j . When j increases in the infinite series, it is noted that the term $\dot{T}_{jn} \sinh(\zeta_{jn}z/R)$ approximately has the order of magnitude of $\sinh(\zeta_{jn}z/R)/\zeta_{jn}$. This is because from Equation 2.46, $\dot{T}_{jn}$ has the order of $1/\zeta_{jn}$. In determining the order of the corresponding term $\dot{T}_{jn} \cosh(\zeta_{jn}z/R)$, it is noted that the order of magnitude of $\dot{T}_{jn}$ is controlled by the third term in that equation, that is

$$\left| \frac{g\zeta_{jn}}{R} \frac{1}{\omega_{jn}} \int_0^t \dot{T}_{jn}(\tau) \sin \omega_{jn}(t - \tau) d\tau \right|,$$

which is approximately $1/\omega_{jn}$, since $\dot{T}_{jn}$ has the order of $1/\zeta_{jn}$. In the estimation here, the first two terms in Equation 2.52, which represent free vibration response, are neglected because their contribution will eventually be diminished due to damping

effect. Therefore, as j increases

$$\lim_{j \rightarrow \infty} \dot{T}_{jn} \cosh \left(\frac{\zeta_{jn} z}{R} \right) = \lim_{j \rightarrow \infty} \cosh \left(\frac{\zeta_{jn} z}{R} \right) / \omega_{jn} = \infty$$

$$\lim_{j \rightarrow \infty} \dot{T}_{jn} \sinh \left(\frac{\zeta_{jn} z}{R} \right) = \lim_{j \rightarrow \infty} \sinh \left(\frac{\zeta_{jn} z}{R} \right) / \omega_{jn} = \infty$$

Both of these terms increase rapidly without bound as the orthogonal parameter ζ_{jn} increases, resulting in the loss of accuracy in the calculation of the convective hydrodynamic pressure component, which is expected to become smaller for higher sloshing modes as

$$\lim_{j \rightarrow \infty} \left\{ T_{jn} \cosh \left(\frac{\zeta_{jn} z}{R} \right) + \bar{T}_{jn} \sinh \left(\frac{\zeta_{jn} z}{R} \right) \right\} = 0 \quad (2.54)$$

As such, convergence of the series is based on the subtraction of two large numbers, which is difficult to maintain high level of accuracy. For instance, if $\bar{T}_{jn} \cosh (\zeta_{jn} z/R)$ and $\bar{T}_{jn} \sinh (\zeta_{jn} z/R)$ in Equation 2.54 both have the orders of magnitude of 10^{20} , while the summation of them has the order of magnitude of 10^{-3} , then it requires at least 24 significant figures in the calculation to obtain the correct result. When double precision of approximately 14 digit accuracy in computation is used to evaluate the higher sloshing modes, the accuracy of the result is clearly lost.

On the other hand, it is also noted that the impulsive pressure given by the term $T_m I_n (\lambda_i r/H) \cos (\lambda_i z/H)$ in Equation 2.53 is stable. It has an order of magnitude of $I_n (\lambda_i r/H) / [I'_n (\lambda_i R/H) \lambda_i]$ because T_{in} in Equation 2.48 has the order of $1 / (I'_n (\lambda_i R/H) \lambda_i)$ when i increases in the infinite sense.

Example of round-off errors

As an example to demonstrate the round-off errors on convergence of the solution, the rocking effects of two rigid tanks are considered. The first one is a rigid tall tank of 10 m in diameter and 20 m water depth, and the other is a rigid broad tank of 20 m in diameter and 20 m water depth. Both are excited by the rocking motions

which are the same as the El-Centro N-S earthquake component with a peak rocking acceleration of 3.4 rad./s^2 .

As discussed before, bottom plate displacement affects the accuracy in the determination of the convective (sloshing) component. For demonstration purpose, the results obtained for the two rigid tanks are the sloshing hydrodynamic pressure components at the water top surface on the exciting axis adjacent to the tank wall due to rocking base motion input only.

For a rigid tank, the dynamic pressure response in Equation 2.53 is simplified to the following as $T_{i1} = 0$ with $w = 0$

$$p_d = -\rho \cos \theta \sum_{j=1}^{\infty} \left[\dot{T}_{j1}(t) \cosh\left(\frac{\zeta_{j1} z}{R}\right) + \dot{T}_{j1}(t) \sinh\left(\frac{\zeta_{j1} z}{R}\right) \right] J_1\left(\frac{\zeta_{j1} r}{R}\right) \quad (2.55)$$

In the following discussion in this section, the j -mode of the response of convective component contributed from the base rocking is denoted as the corresponding terms in Equation 2.55 with the subscript j .

Time history of the convective pressures induced in the rigid tall tank is plotted in Figure 2.4 (a). Responses given by fundamental sloshing mode and the two modes solution have no noticeable difference. However, it is noted that the response of the three modes solution becomes divergent due to round-off error as discussed.

For the case of a rocking broad rigid tank, the results for which are shown in Figure 2.4 (b), the fundamental mode response is noticeably different from that of the first-three modes in the analysis. This suggests that the second and third modes contribute significantly to the overall response of the system. However, contrary to the trend of convergence, the response using the first-seven modes clearly shows divergence due to numerical round-off error.

The numerical results of the two rigid tanks have shown that the round-off errors involved in the subtraction of two large number may have a significant adverse effect on the dynamic response of unanchored tanks which allows bottom plate to uplift.

2.3.3 Improved formulation of hydrodynamic forces in flexible uplifting tanks

Because of the numerical problem discussed in the previous section, an improved formulation for the hydrodynamic effect is discussed here. Improvement in the formulation is obtained by rewriting the cosh and sinh functions associated with the terms T_{jn} and $\hat{T}_{jn}$ in their exponential forms

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

Separating the velocity potential $\phi(r, \theta, z, t)$ into three components, $\phi_1(r, \theta, z, t)$, $\phi_2(r, \theta, z, t)$ and $\phi_3(r, \theta, z, t)$

$$\phi(r, \theta, z, t) = \phi_1(r, \theta, z, t) + \phi_2(r, \theta, z, t) + \phi_3(r, \theta, z, t) \quad (2.56)$$

where $\phi_1(r, \theta, z, t)$, $\phi_2(r, \theta, z, t)$ and $\phi_3(r, \theta, z, t)$ are independent solutions for the Laplace's equation given in Equation 2.1

$$\phi_1(r, \theta, z, t) = \sum_{n=0}^{\infty} \cos(n\theta) \sum_{i=1}^{\infty} T_{in}(t) I_n\left(\frac{\lambda_i r}{H}\right) \cos\left(\frac{\lambda_i z}{H}\right) \quad (2.57)$$

$$\phi_2(r, \theta, z, t) = \sum_{n=0}^{\infty} \cos(n\theta) \sum_{j=1}^{\infty} T_{jn}(t) J_n\left(\frac{\zeta_{jn} r}{R}\right) e^{-\frac{\zeta_{jn} z}{R}} \quad (2.58)$$

$$\phi_3(r, \theta, z, t) = \sum_{n=0}^{\infty} \cos(n\theta) \sum_{j=1}^{\infty} \hat{T}_{jn}(t) J_n\left(\frac{\zeta_{jn} r}{R}\right) e^{-\frac{\zeta_{jn} z}{R}} \quad (2.59)$$

It is noted that the time varying function $\bar{T}_{jn}(t)$ and $\hat{T}_{jn}(t)$ are not the same ones as previously defined in Equation 2.43

The boundary conditions given by Equations 2.7-9 can also be rewritten as follows:

For ϕ_1 :

$$\text{for } z=0 \quad \frac{\partial \phi_1}{\partial z} = 0 \quad (2.60)$$

$$\text{for } r=R \quad \frac{\partial \phi_1}{\partial r} = \dot{w}(\theta, z, t) \quad (2.61)$$

$$\text{for } z=H \quad \frac{\partial \phi_1}{\partial t} = 0 \quad (2.62)$$

For ϕ_2 and ϕ_3 :

$$\text{for } z=0 \quad \frac{\partial \phi_2}{\partial z} + \frac{\partial \phi_3}{\partial z} = \dot{b}(r, \theta, t) \quad (2.63)$$

$$\text{for } r=R \quad \frac{\partial \phi_2}{\partial r} = \frac{\partial \phi_3}{\partial r} = 0 \quad (2.64)$$

$$\text{for } z=H \quad \frac{\partial^2(\phi_2 + \phi_3)}{\partial t^2} + g \frac{\partial(\phi_2 + \phi_3)}{\partial z} = -g \frac{\partial \phi_1}{\partial z} \quad (2.65)$$

It is noted here that the boundary conditions at $z = 0$ and $z = H$, given respectively in Equations 2.60 and 2.62, are automatically satisfied by ϕ_1 . It is also observed here that more restrictive boundary conditions concerning ϕ_2 and ϕ_3 are imposed at the cylindrical surface $r = R$, requiring each term independently satisfies the prescribed condition, rather than just their sum as implied in Equation 2.11.

Substituting the expression of ϕ_1 given by Equation 2.57 into the boundary condition at $r = R$,

$$\sum_{n=0}^{\infty} \cos(n\theta) \sum_{i=1}^{\infty} T_{in}(t) \frac{\lambda_i}{H} I'_n\left(\frac{\lambda_i r}{H}\right) \cos\left(\frac{\lambda_i z}{H}\right) = \dot{w}(\theta, z, t) \quad (2.66)$$

of which $T_{in}(t)$ is determined to be the same as before

$$T_{in} = \frac{\kappa_n \int_0^{2\pi} \int_0^H \dot{w}(\theta, z, t) \cos n\theta \cos \frac{\lambda_i z}{H} d\theta dz}{\pi H R_{in}} \quad (2.67)$$

where

$$R_{in} = \frac{\lambda_i}{H} I'_n\left(\frac{\lambda_i R}{H}\right)$$

$$\kappa_n = \begin{cases} 2 & \text{for } n = 1, 2, 3, \dots \\ 1 & \text{for } n = 0 \end{cases}$$

Since ζ_{jn} is chosen as the zeros of the first derivative of the Bessel function J_n , it is clear that the boundary condition at $r = R$ given in Equation 2.64 is satisfied

automatically by ϕ_2 and ϕ_3 . Substituting the expressions for ϕ_1 and ϕ_2 given in Equations 2.31 and 2.32 into the boundary condition at $z = 0$ given in Equation 2.63, the following is obtained

$$\sum_{n=0} \cos(n\theta) \sum_{j=1} (T_{jn} - \dot{T}_{jn}) J_n\left(\frac{\zeta_{jn}r}{R}\right) \frac{\zeta_{jn}}{R} = \dot{b}(r, \theta, t) \quad (2.68)$$

For convenience, $\dot{T}_{jn}$ may be expressed in terms of T_{jn} as follows

$$\dot{T}_{jn} = T_{jn} + S_{jn} \quad (2.69)$$

Substituting $-S_{jn} = (T_{jn} - \dot{T}_{jn})$ into Equation 2.68 and using the orthogonal property of Bessel function given in Equation 2.44, S_{jn} can be determined as

$$S_{jn} = -\frac{\kappa_n \int_0^{2\pi} \cos n\theta d\theta \int_0^R r \dot{b}(r, \theta, t) J_n\left(\frac{\zeta_{jn}r}{R}\right) dr}{\pi \zeta_{jn} R (1 - n^2/\zeta_{jn}^2) J_n^2(\zeta_{jn})} \quad (2.70)$$

It is evident from Equation 2.70 that S_{jn} has the order of magnitude of $1/\zeta_{jn}$. Now, substituting ϕ_1 , ϕ_2 and ϕ_3 into Equation 2.65, the following is obtained

$$\begin{aligned} & \sum_{j=1} \left[T_{jn}'' \epsilon^{\zeta_{jn}H/R} + \dot{T}_{jn}'' \epsilon^{-\zeta_{jn}H/R} + g \left(T_{jn} \epsilon^{\zeta_{jn}H/R} - \dot{T}_{jn} \epsilon^{-\zeta_{jn}H/R} \right) \frac{\zeta_{jn}}{R} \right] J_n\left(\frac{\zeta_{jn}r}{R}\right) \\ &= g \sum_{i=1} T_{in} I_n\left(\frac{\lambda_i r}{H}\right) \frac{\lambda_i}{H} \sin \lambda_i \end{aligned} \quad (2.71)$$

Using the orthogonal properties of the functions $\cos n\theta$ and $J_n(\frac{\zeta_{jn}r}{R})$, and the relationship between $\dot{T}_{jn}$ and T_{jn} (Equation 2.69), Equation 2.71 can be simplified as follows

$$\begin{aligned} & (\epsilon^{\zeta_{jn}H/R} + \epsilon^{-\zeta_{jn}H/R}) T_{jn}'' + g \frac{\zeta_{jn}}{R} (\epsilon^{\zeta_{jn}H/R} - \epsilon^{-\zeta_{jn}H/R}) T_{jn} \\ &= \frac{2g \int_0^R \sum_{i=1}^{\infty} \lambda_i I_n\left(\frac{\lambda_i r}{H}\right) \sin \lambda_i T_{in}(t) r J_n\left(\frac{\zeta_{jn}r}{R}\right) dr}{R^2 H (1 - n^2/\zeta_{jn}^2) J_n^2(\zeta_{jn})} \\ & \quad - \epsilon^{-\zeta_{jn}H/R} S_{jn}''(t) + g \frac{\zeta_{jn}}{R} \epsilon^{-\zeta_{jn}H/R} S_{jn}(t) \end{aligned} \quad (2.72)$$

Taking

$$\omega_{jn}^2 = g \frac{\zeta_{jn}}{R} \frac{e^{\zeta_{jn}H/R} - e^{-\zeta_{jn}H/R}}{e^{\zeta_{jn}H/R} + e^{-\zeta_{jn}H/R}} = g \frac{\zeta_{jn}}{R} \tanh\left(\frac{\zeta_{jn}H}{R}\right) \quad (2.73)$$

Equation 2.72 becomes

$$\begin{aligned} \bar{T}_{jn}'' + \omega_{jn}^2 T_{jn} = & -\frac{e^{-2\zeta_{jn}H/R}}{1 + e^{-2\zeta_{jn}H/R}} S_{jn}''(t) + g \frac{\zeta_{jn}}{R} \frac{e^{-2\zeta_{jn}H/R}}{1 + e^{-2\zeta_{jn}H/R}} S_{jn}(t) \\ & + \frac{2g \int_0^R \sum_{i=1}^{\infty} \lambda_i I_n\left(\frac{\lambda_i r}{H}\right) \sin \lambda_i T_{in}(t) r J_n\left(\frac{\zeta_{jn} r}{R}\right) dr}{R^2 H (1 - n^2 / \zeta_{jn}^2) J_n^2(\zeta_{jn}) (1 + e^{-2\zeta_{jn}H/R})} e^{-\zeta_{jn}H/R} \end{aligned} \quad (2.74)$$

According to the theory of ordinary differential equation, a linear differential equation of the form

$$y'' + \omega^2 y = f(t), \text{ where } y'' = \frac{d^2 y}{dt^2} \quad (2.75)$$

has a general solution

$$y = a \sin \omega t + b \cos \omega t + \frac{1}{\omega} \int_0^t f(\tau) \sin \omega(t - \tau) d\tau \quad (2.76)$$

where constants a and b are determined from the initial conditions at $t = 0$.

Following this standard procedure, the general solution of Equation 2.74 is given by the familiar Duhamel integration as follows

$$\begin{aligned} \bar{T}_{jn} = & a_{jn} \sin \omega_{jn} t + b_{jn} \cos \omega_{jn} t \\ & - \frac{e^{-2\zeta_{jn}H/R}}{1 + e^{-2\zeta_{jn}H/R}} \frac{1}{\omega_{jn}} \int_0^t S_{jn}''(\tau) \sin \omega_{jn}(t - \tau) d\tau \\ & + g \frac{\zeta_{jn}}{R} \frac{e^{-2\zeta_{jn}H/R}}{1 + e^{-2\zeta_{jn}H/R}} \frac{1}{\omega_{jn}} \int_0^t S_{jn}(\tau) \sin \omega_{jn}(t - \tau) d\tau \\ & + \frac{2g \int_0^R \sum_{i=1}^{\infty} \lambda_i I_n\left(\frac{\lambda_i r}{H}\right) \sin \lambda_i r J_n\left(\frac{\zeta_{jn} r}{R}\right) dr \int_0^t T_{in}(\tau) \sin \omega_{jn}(t - \tau) d\tau}{\omega_{jn} R^2 H (1 - n^2 / \zeta_{jn}^2) J_n^2(\zeta_{jn}) (1 + e^{-2\zeta_{jn}H/R})} e^{-\zeta_{jn}H/R} \end{aligned} \quad (2.77)$$

where

$$\omega_{jn} = \sqrt{g \frac{\zeta_{jn}}{R} \tanh\left(\frac{\zeta_{jn}H}{R}\right)}$$

Neglecting the first two free vibration terms given in Equation 2.77, the order of magnitude for the respective remain terms are

$$\frac{1}{\zeta_{jn}} e^{-2\zeta_{jn}H/R}, \quad e^{-2\zeta_{jn}H/R} \text{ and } e^{-\zeta_{jn}H/R}$$

With $\bar{T}_{jn}(t)$ and $S_{jn}(t)$ determined from Equations 2.70 and 2.77, $\hat{T}_{jn}(t)$ can be easily obtained from Equation 2.69.

The dynamic pressure for an uplifting tank is then given as follows

$$p_d = -\rho \frac{\partial \phi}{\partial t} = -\rho \left[\frac{\partial \phi_1}{\partial t} + \frac{\partial \phi_2}{\partial t} + \frac{\partial \phi_3}{\partial t} \right] \quad (2.78)$$

or

$$p_d = -\rho \sum_{n=0}^{\infty} \cos n\theta \left\{ \sum_{j=1}^{\infty} \left\{ \frac{dT_{jn}}{dt} I_n \left(\frac{\lambda_j r}{H} \right) \cos \left(\frac{\lambda_j z}{H} \right) + \left[\frac{dT_{jn}}{dt} e^{\frac{\zeta_{jn}z}{R}} + \frac{d\hat{T}_{jn}}{dt} e^{-\frac{\zeta_{jn}z}{R}} \right] \cdot J_n \left(\frac{\zeta_{jn}r}{R} \right) \right\} \right\} \quad (2.79)$$

The expression given in Equation 2.79 is for the undamped case. For damped sloshing motions, the corresponding expression can be obtained by including the exponential decay term inside each of the Duhamel integrals in Equation 2.79, and replacing the undamped natural frequencies of the sloshing modes, ω_{jn} , by the damped frequencies, $\omega_{Djn} = \omega_{jn} \sqrt{1 - \xi_{jn}^2}$, where ξ_{jn} is the damping ratio of the j^{th} sloshing mode of circumferential wave number n .

Upon examination of each term in Equation 2.79, it is noted that the term associated with T_{jn} has the order of magnitude of approximately $e^{-\zeta_{jn}(H-z)/R}$, as obtained from Equation 2.77 of which the dominant term in $\hat{T}_{jn} e^{\zeta_{jn}z/R}$ (when j increases) is

$$\left| \frac{\int_0^R \sum_{i=1}^{\infty} \lambda_i I_n \left(\frac{\lambda_i r}{H} \right) \sin \lambda_i r J_n \left(\frac{\zeta_{jn} r}{R} \right) dr \int_0^t T_{in}(\tau) \sin \omega_{jn}(t - \tau) d\tau}{R^2 H (1 - \nu^2 / \zeta_{jn}^2) J_n^2(\zeta_{jn}) (1 + e^{-2\zeta_{jn}H/R})} e^{-\zeta_{jn}(H-z)/R} \right|$$

which has the order of $e^{-\zeta_n(H-z)/R}$ and diminishes with increasing j . The improved formulation for the hydrodynamic pressure is therefore numerically stable.

2.4 EXAMPLES

Numerical results of two tanks, a broad tank and a tall tank are presented in this section. The difference between the case of fixed bottom plate (anchored tank) and the case of uplifting bottom plate (unanchored tank) is examined. In the study here, the influence of the uplifting bottom plate on the shell motion (rocking) is considered approximately as a function of the uplifted bottom rim.

The uplift motion of the bottom plate is modeled as a given motion directly related to the resultant overturning moment due to the hydrodynamic pressure induced in the vibrating fluid-tank system. The complete formulation of the interactions between the liquid and the flexible tank shell and the uplifting bottom plate in an unanchored tank is presented in later chapters.

2.4.1 Bottom plate uplift model

During earthquakes, the hydrodynamic pressure acting on the tank structure including the bottom plate, can be represented by a horizontal resultant force and an overturning moment at the base of the support. To balance the overturning moment, axial compressive stress exists on one side of the tank shell, and axial tensile stress on the other. The axial compressive force is resisted by the support reaction between the bottom plate and the foundation distributed over the contact area. Because the tank is not anchored to the foundation, the tensile force separates part of the bottom plate from the foundation until the weight of the liquid on the uplifting part and the vertical component of the internal forces from deformation of the bottom plate are in equilibrium to balance the tensile uplifting force. From analytical calculations and experimental observations, it is observed that the uplifting mechanism has the

following characteristics:

1. The magnitude of vertical uplift is strongly related to the magnitude of the overturning moment.
2. The uplift of the bottom plate occurs mostly on the uplifted side of the tank
3. Uplift of the bottom plate occurs only when the overturning moment reaches some critical level.

As a preliminary study to assess the effect of bottom plate uplift on the dynamic response behaviour of unanchored tanks, several assumptions on the uplift mechanism are adopted here without giving full justification in order to simplify the analysis. These simplifications are later removed in a more complete and exact formulation of the fluid-structure interaction problem presented in later chapters. The assumptions are:

1. The bottom plate uplift is induced by the horizontal ground acceleration only
2. As shown in Figure 2.5(a), during uplifting, it is assumed that only the half of the bottom plate, which is in the same direction of the horizontal ground acceleration, separates from the foundation. The uplifting bottom plate deforms into two plane surfaces about the axis perpendicular to the excitation.
3. The bottom plate does not uplift unless the horizontal ground acceleration exceeds a critical level which can be determined by considering the equilibrium around the base of the tank shell. The excitation overturning moment is equal to the product of the total fluid-tank mass multiplied with the ground acceleration and the height of the center of mass from the foundation. The resisting moment is contributed by the weight of the uplifted portion of the tank shell and the transverse shear force of the bottom plate at the edge of the uplifted base rim due to the liquid pressure.

4. As shown in Figure 2.5(b), bottom plate uplift occurs during interval when the ground acceleration is above the critical acceleration level. Within this level, the uplift motion of the bottom plate is assumed to be related to the rate of change of the ground acceleration. The bottom plate is assumed to accelerate upwards when the slope of the ground acceleration time history is positive, and vice versa. In other words, the vertical acceleration of the bottom plate is assumed to have the same sign as the derivative of horizontal ground acceleration. The critical acceleration level can be determined by static tilt analysis[63]. For the purpose of demonstration, the critical level is assumed as 0.05 g in the analysis of this section.
5. The magnitude of the bottom plate vertical uplift acceleration is obtained by scaling the horizontal ground acceleration in proportion to the ratio of the expected maximum base rim vertical uplift to the maximum ground displacement. Based on assumption 3, the edge uplift can be determined within the time interval defined between time t_s and t_r as shown in Figure 2.5(b).

2.4.2 Tank shell rocking model

For a flexible unanchored tank subjected to earthquake excitation, the shell rocks with the uplifting base rim when the bottom plate uplifts. Since the base rim on one side remains in contact with the foundation, the rocking displacements of the tank shell on the contact side is significantly smaller than that on the uplifted side because of the restraining action from the vertical foundation support. On the other side, the tank shell is relatively free to rock with the base rim. In the analysis, this rocking behaviour of the tank shell is approximated by a model shown in Figure 2.6(a). Furthermore, because the inplane stiffness of the thin walled tank shell is so large it is unlikely that the rocking motion of the cylindrical shell can cause any significant change in the circumferential length of tank shell perimeter. Assuming that the perimeter length remains constant during uplift for the distorted tank shell,

the cross section of the tank shell deforms into elliptical out-of-round shape as shown in Figure 2.6(b).

In analysis here, the following assumptions are applied to take into consideration the influence of the tank shell out-of-round deformation on the dynamic response of unanchored tanks during bottom plate uplift

1. All generatrices on the tank shell remain straight throughout response.
2. The tank shell remains perpendicular with the uplifted bottom plate at the excitation axis as shown in Figure 2.6(a).
3. The circumferential length of any horizontal cross section remains constant (inplane stiffness of the tank shell is infinite).
4. The circular shape of the tank shell horizontal cross section becomes an ellipse.

2.4.3 Numerical results

The hydrodynamic pressures and overturning moments at the base of two tanks, one a broad tank of 20 m diameter with 5 m depth of water, and another a tall tank of 20 m diameter with 20 m depth of water, are presented here.

The acceleration input is the north-south acceleration component record of the 1940 El Centro earthquake, with a peak ground acceleration amplitude of 0.35 g as shown in Figure 2.7. The maximum vertical base uplift displacement and the critical ground acceleration of uplift for both tanks are assumed to be 12 cm and 0.05 g respectively. The maximum uplift and critical ground acceleration are determined by static tilt analysis. Using the analysis model of bottom plate uplift, the base rim uplift time history of the unanchored tank can be calculated as shown in Figure 2.8. For comparison, the hydrodynamic pressures induced in corresponding rigid tanks, in which the bottom plate is fixed and the flexibility of the tank wall is neglected, are also presented. In the numerical results presented in this section, the unanchored

tank results with uplifting bottom plate are denoted by the label 'uplifting bottom plate', whereas the results of rigid anchored tank are denoted by the label 'rigid'.

Broad tank results

Figures 2.9-15 are the broad tank results. Time histories of the normalized overturning moments are given in fig 2.9. The overturning moments are normalized by the total weight of the liquid. The maximum overturning moment in the case of considering bottom plate uplifting is about 5% larger than that of the rigid tank. But the overall overturning moment responses for the two cases are similar. This is because the uplift motion of the bottom plate influences mostly the convective component only, and the rocking effect of the tank shell is not very significant in a broad tank. Uplifting takes place only occasionally during intervals when the ground acceleration exceeds the critical value. As the impulsive component depends mostly on the motion of the tank shell, it contributes more to the overturning moment than the convective component.

The time histories of the normalized impulsive components near the bottom plate and at the mid-height on the north side of the tank shell are shown in Figures 2.10 and 2.11 respectively. The impulsive pressures are normalized by the liquid hydrostatic pressure at the bottom of the tank, and the maximum values of the impulsive pressure on the tank wall are about 14 kPa at the base, and about 12 kPa at the mid-height. There are no noticeable differences between the impulsive components of the unanchored and the rigid tanks.

The time histories of the normalized convective components near the bottom plate at the north side of the tank shell are plotted in Figure 2.12. The convective pressures are normalized by the same way as the impulsive pressure, and the maximum value of the convective pressure on the tank wall at the base is about 7.96 kPa. As shown in Figure 2.12, the bottom plate uplift influences significantly the response of the convective pressure.

The time histories of normalized convective components at mid-height on the north side of tank shell are given in Figure 2.13. Comparing with Figure 2.12, the uplift has less influence at the mid-height than near the bottom.

The time histories of the normalized wave height at the liquid surface of the north shell are plotted in Figure 2.14. The surface waves are normalized by the diameter of the tank, and the maximum value of the wave height is about 0.22 m. The difference between the case of bottom plate uplifting and the case of rigid tank is noticeably small.

At the instant of maximum dynamic pressure, the normalized convective pressure profiles along the north tank wall are shown in Figure 2.15. The influence of the bottom plate uplift is significant at the bottom of the liquid, and vanishes gradually from the bottom to the top of the liquid free surface.

Tall tank results

Figures 2.16-22 are the tall tank results. Time histories of the normalized overturning moment at the base level are given in Figure 2.16. Because the rocking motion has greater influence on the tank shell motion in a tall tank and the overturning moment is mostly contributed by the impulsive component which depends on the shell motion, the time history of the overturning moment response in the unanchored tall tank is very different from that of the rigid tank.

The impulsive pressures on the tank wall normalized by the hydrostatic pressure at the base are plotted in Figure 2.17 for response at the base, and in Figure 2.18 for response at the mid-height. As shown in Figures 2.17 and 2.18, due to rocking motion of tank shell, the influence of the bottom plate uplift is greater at the mid-height of the tank shell than that at the base, and both cases show that the bottom plate uplift has significant influence in the impulsive pressure of a tall tank during the dynamic response.

The convective pressures at the bottom and at the mid height are given in Figure 2.19 and Figure 2.20 respectively. The time histories of the normalized wave height are shown in Figure 2.21. The wave height with a maximum of 0.32 m is normalized by the diameter of the tank. There is no noticeable difference between the unanchored and rigid tanks in the surface wave. The convective pressure profiles along the north side at the instant of maximum pressure are given in Figure 2.22.

Comparing the tall tank results with that of the broad tank, the influence of bottom plate uplift is much more significant in the case of the tall tank than in the broad tank since the broad tank shell has less rocking response associated with the base rim uplifting.

2.4.4 Summary and conclusions

This chapter presents a computational procedure to evaluate the hydrodynamic forces in vibrating flexible cylindrical liquid storage tanks. Flexibility of the cylindrical tank system is considered in the analysis model. In addition to the boundary conditions at the cylindrical tank-liquid interface and the liquid free surface at the top, boundary condition of partial uplift at the tank bottom is also satisfied. Thus, the present hydrodynamic formulation can be used in the modeling of the dynamic behaviour of both anchored and unanchored tanks. Numerical accuracy in the computation of hydrodynamic pressure is also investigated. It is shown that accuracy in the convective hydrodynamic pressure can be significantly affected due to loss of significant figures in finite floating point computation. An improved formulation for the vibrating liquid which eliminates this numerical difficulty is presented. Numerical results on the study of the effect of bottom plate uplift on the magnitude and distribution of hydrodynamic pressure are also presented. The results show that the bottom plate uplift has a significant effect on hydrodynamic response.

Figure 2.1: Coordinate system for unanchored cylindrical tank

Figure 2.2: Bottom plate uplifting

Figure 2.3: Haroun-Housner model for a flexible tank

(a) Broad tank of 20m dia., 20m dep. water

(b) Tall tank of 10m dia., 20m dep. water

Figure 2.4: Divergence of round-off errors

(a) bottom plate uplifting model

(b) a phase of ground acceleration

Figure 2.5: Bottom plate uplifting model

(a) elevation

(b) plan

Figure 2.6: Tank shell rocking model

Figure 2.7: Ground acceleration excitation

Figure 2.8: Base rim uplift

Figure 2.9: Normalized overturning moment of broad tank

Figure 2.10: Normalized impulsive pressure at bottom of broad tank

Figure 2.11: Normalized impulsive pressure at mid-height of broad tank

Figure 2.12: Normalized convective pressure at bottom of broad tank

Figure 2.13: Normalized convective pressure at mid-height of broad tank

Figure 2.14: Normalized surface wave of broad tank

Figure 2.15: Normalized convective pressure profile of broad tank

Figure 2.16: Normalized overturning moment of tall tank

Figure 2.17: Normalized impulsive pressure at bottom of tall tank

Figure 2.18: Normalized impulsive pressure at mid-height of tall tank

Figure 2.19: Normalized convective pressure at bottom of tall tank

Figure 2.20: Normalized convective pressure at mid-height of tall tank

Figure 2.21: Normalized surface wave of tall tank

Figure 2.22: Normalized convective pressure profile of tall tank

Chapter 3

RITZ FORMULATION OF BOTTOM PLATE

In this chapter, the mathematical model for the flexible bottom plate of an unanchored cylindrical liquid storage tank, partially uplifted under the static tilt condition, is presented. In a previous study on static tilt uplift behavior, Lau and Clough[63] employed derived displacement Ritz shape functions in the discretization of the bottom plate. The derived Ritz shape functions are obtained analytically by solving the bottom plate governing differential equations. Using these derived Ritz shape functions, details of the unanchored tank uplift resistant mechanism are studied. However, because of the complexity and the amount of computation required to generate these derived Ritz shape functions, it is considered not practical to implement this bottom plate model for earthquake time history response analysis of uplifting unanchored tanks. Therefore an initial important task in this thesis research is to develop a simplified analysis model of the bottom plate, suitable for nonlinear time history analysis based on the results obtained previously and understanding of the uplift mechanism using the earlier vigorous Ritz formulation of the bottom plate.

In the following sections, the derivation of the bottom plate Ritz shape function is briefly discussed. A new simplified analysis model for the bottom plate is developed based on the results obtained using the derived Ritz shape functions. This new analysis model for the bottom plate is then later extended to include the inertia effect in the dynamic formulation.

3.1 ANALYSIS MODEL UNDER LATERAL LOADS

When an unanchored tank is subjected to lateral loads, such as earthquake loads induced by the hydrodynamic pressure in vibrating liquid, or the wind load acting on the tank shell, a lateral resultant force and an overturning moment are developed at the base of the tank shell. The tank shell will rock in response to the overturning moment. The tank shell tends to uplift on one side, pulling part of the bottom plate on the proximity of the uplifted tank wall from the foundation, as shown in Figure 3.1(a-b). On the contact side of the tank shell, a narrow strip of the bottom plate also separates from the foundation due to the rotation of the bottom plate-tank wall connection. In Figure 3.1(a), the separation of the bottom plate narrow strip on the contact side (left side) is exaggerated to illustrate the uplift phenomenon. During dynamic response with partial uplift under lateral load, the lateral load is resisted by friction distributed over the contact bottom. The overturning moment is equilibrated by the couple due to W as shown in Figure 3.1(a), where W is the resultant of the vertical reaction from the support foundation.

Nonlinearity of the uplift mechanism mainly results from two factors:

1. The deflection of the bottom plate is so large that the nonlinear Von Karman plate theory has to be used to describe the bottom plate deformation.
2. The extent of the bottom plate separation from the foundation, that is the inward extent of bottom uplift, is not a linear function of the amount of uplift at the base rim edge.

Results from experiment [24] [25] and analytical studies[63] indicate that major portion of the bottom plate will remain in contact with the foundation during uplift. A large portion of the vertical force W shown in Figure 3.1 comes from the bottom plate contact area support reaction. The remaining contribution to W comes from the weight of the uplifted liquid, the uplifted part of the bottom plate and the cylindrical tank structure. It is assumed in the present study that the lateral fric-

tion is distributed uniformly over the contact area between the bottom plate and the foundation.

Observations from experimental investigations [24-25] and results from previous analytical studies[63] on the uplift behavior of unanchored tanks have indicated that the bottom plate contact pattern closely resembles that of a circular shape, which is adopted in the present investigation.

3.2 BASIC ASSUMPTIONS

In the formulation of the bottom plate uplift model, the following assumptions are made:

1. The Von Karman plate theory for moderately large deflection is employed to model the bottom plate uplift. Although the expected vertical uplift deflection is large when compared to the bottom plate thickness, it is still expected to be small when compared to the dimension of the uplifting area of the bottom plate. Also, the radii of curvature of the vertical displacements are much larger than the thickness of the plate, such that the approximations $\tan \theta \approx \theta$ and $\sin \theta \approx \theta$ are acceptable.
2. The material is assumed to remain linear elastic throughout the response. This assumption may not be applicable under very large uplift. But in the present study, this assumption is adopted in order to simplify the complex problem.
3. The support foundation is assumed to be rigid, which implies that the bending moment at the bottom plate contact-uplift boundary is exactly zero.
4. An uniform friction is assumed to be distributed over the contact area between the foundation and the bottom plate to resist the lateral resultant force due to the hydrodynamic effects.

5. The shape of the portion of the bottom plate that remains in contact with the foundation is assumed to be circular. Its location and dimension are unknown transient variables, to be determined in the analysis.
6. The lateral load is assumed to be symmetric about a plane through the center of the tank and parallel to the excitation axis. Consequently, the load does not cause any rotation about the longitudinal axis of the tank.

3.3 RITZ FORMULATION OF UPLIFTED REGION SUBSTRUCTURE ELEMENT

In the formulation, the bottom plate is divided analytically into the interior contact region and the outer uplifted part. Each of these two regions of an uplifted bottom plate is modeled by substructure element. In this section, the formulations of the substructure element modeling the uplifted region are discussed.

A Rayleigh-Ritz solution begins with the approximation of the displacement field variable by Ritz shape functions. Each of the discretizing Ritz function has an amplitude, a_i , the magnitude of which is yet to be determined. The amplitudes of the Ritz shape function, a_i , are generally referred to as the generalized coordinates in Rayleigh-Ritz formulation.

Each of the shape functions must satisfy compatibility conditions, or essential boundary conditions. For the uplifted region substructure element, essential boundary conditions concern continuity of all the displacements and their first derivatives. It is not required in the discretization process that the shape functions satisfy nonessential boundary conditions (force boundary conditions). But in doing so, more accurate results can be obtained for number of degrees of freedom in the discretization process.

Using Ritz displacement shape functions, the total potential energy of the system, Π_p , can be evaluated in terms of the Ritz generalized coordinates. According to the principle of stationary potential energy, the equilibrium configuration as defined by the numerical values of the Ritz coefficients, a_i , are determined by solving the n

algebraic equations as follows

$$\frac{\partial \Pi_p}{\partial a_i} = 0 \quad \text{for } i = 1, 2, \dots, n \quad (3.1)$$

3.4 DERIVED DISPLACEMENT RITZ SHAPE FUNCTIONS

In earlier studies on the static tilt uplift behavior of unanchored tanks, Lau and Clough[63] employed derived displacement Ritz shape functions in the discretization of the bottom plate.

In the uplifted region, the bottom plate displacements in polar coordinates are given as a linear combination of the Ritz shape functions derived from the von Karman plate theory. The von Karman plate theory assumes that the deflections are no longer small in comparison with the plate thickness but are still small when compared to the remaining plate dimensions. In the small deflection theory, the inplane stresses depend only upon the external load acting in the inplane direction. But in the von Karman plate theory, the inplane stresses are also affected by stretching of the midplane of the plate produced by large vertical deflections due to bending. Thus, the inplane stresses are coupled to the flexural mechanism in the von Karman plate theory. The midplane strain-displacement relations in polar coordinates are then given as follows

$$\begin{aligned} \varepsilon_r &= \frac{\partial u_r}{\partial r} + \frac{1}{2} \left(\frac{\partial w}{\partial r} \right)^2, & \varepsilon_\theta &= \frac{u_r}{r} + \frac{\partial u_\theta}{r \partial \theta} + \frac{1}{2r^2} \left(\frac{\partial w}{\partial \theta} \right)^2 \\ \gamma_{r\theta} &= \frac{\partial u_\theta}{\partial r} + \frac{1}{r} \left(\frac{\partial u_r}{\partial \theta} - u_\theta \right) + \frac{1}{r} \frac{\partial w}{\partial r} \frac{\partial w}{\partial \theta} \end{aligned} \quad (3.2)$$

where w , u_r and u_θ are respectively the components of the displacement at the mid-plane in the z , r and θ directions.

The governing differential equations of the von Karman plate theory in polar coordinate system are

$$\frac{\partial N_r}{\partial r} + \frac{1}{r} \frac{\partial N_{r\theta}}{\partial \theta} + \frac{N_r - N_\theta}{r} = 0 \quad (3.3)$$

$$\frac{1}{r} \frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_{r\theta}}{\partial r} + \frac{2}{r} N_{r\theta} = 0 \quad (3.4)$$

$$\begin{aligned} & \frac{\partial^2 M_r}{\partial r^2} + \frac{2}{r} \frac{\partial^2 M_{r\theta}}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial^2 M_\theta}{\partial \theta^2} + \frac{2}{r} \frac{\partial M_r}{\partial r} - \frac{1}{r} \frac{\partial M_\theta}{\partial r} + \frac{2}{r} \frac{\partial M_{r\theta}}{\partial \theta} \\ & + \frac{\partial^2 w}{\partial r^2} N_r + \left[\frac{\partial w}{r \partial r} + \frac{\partial^2 w}{r^2 \partial \theta^2} \right] N_\theta + \left[\frac{2}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{2}{r^2} \frac{\partial w}{\partial \theta} \right] N_{r\theta} + p_z = 0 \end{aligned} \quad (3.5)$$

where N_r , N_θ and $N_{r\theta}$ are the midplane stress resultants, M_r , M_θ and $M_{r\theta}$ are the corresponding midplane moment resultants, and p_z is the normal pressure loading.

In the study by Lau and Clough[63], each derived Ritz shape function is a coupled set of displacement patterns consisting of three components: radial inplane u_r , tangential inplane u_θ , and vertical w displacement components. The Ritz shapes for the vertical displacement w is obtained by solving the linear plate bending differential equation. The governing differential equation for plate bending in polar coordinates is given as follows

$$\begin{aligned} \nabla^4 w &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) = \frac{p_z}{D} \\ D &= \frac{Et^3}{12(1 - \nu^2)} \end{aligned} \quad (3.6)$$

where D is the plate flexural rigidity. The general solution for w is given in the form of an infinite series as follows

$$\begin{aligned} w &= \bar{w} + (c_{10} + c_{20}r^2 + c_{30} \ln r + c_{40}r^2 \ln r) + \\ & (c_{11}r + c_{21}r^3 + c_{31}r^{-1} + c_{41}r \ln r) \cos \theta + \\ & \sum_{m=2}^{\infty} (c_{1m}r^m + c_{2m}r^{m+2} + c_{3m}r^{-m} + c_{4m}r^{-m+2}) \cos m\theta \end{aligned} \quad (3.7)$$

where $\bar{w}$ is the interior w Ritz shape functions obtained by solving the following differential equation with fixed boundary conditions

$$\nabla^4 \bar{w} = \frac{p_z}{D} \quad (3.8)$$

The corresponding inplane components (u_r and u_θ) for the vertical Ritz shape w are determined by solving the two inplane equilibrium Equations 3.3, 3.4 of the von Karman plate theory. The amplitudes of the Ritz shapes are determined by minimizing the potential energy of the discretized system. In this formulation, the equilibrium of the bottom plate in the inplane direction is always satisfied. Details of the formulations can be found in references [63] and [64].

3.5 SIMPLIFIED DISPLACEMENT RITZ SHAPE FUNCTIONS

A simplified analysis model for the bottom plate uplifted region is presented here with the objective of extending this model to dynamic formulation for seismic response analysis in later chapters.

3.5.1 Coordinate transformation

Due to symmetry of the problem, it is sufficient to consider only half of the bottom plate in the analysis. In the simplified formulation, the Ritz shape functions are defined in a normalized natural coordinate systems, as shown in Figure 3.2(a). The uplifted region of the bottom plate is mapped into a curvilinear natural coordinate system, ξ and η , as shown in Figure 3.2(b) by linear mapping functions. The natural coordinate axis ξ varies along the radius of the bottom plate, and the axis η essentially varies in the tangential direction.

The linear mapping functions which transform the crescent shape uplifted region as shown in Figure 3.2(a) to the normalized square region shown in Figure 3.2(b) are given as follows

$$\begin{aligned}\vartheta &= \frac{\pi}{2}(\eta + 1) \\ r &= \frac{1}{2} [(R - r_c) \xi + R + r_c] \\ r_c &= b_c \sin \frac{\pi \eta}{2} + \sqrt{b^2 - b_c^2 \cos^2 \frac{\eta \pi}{2}}\end{aligned}\tag{3.9}$$

where $r_c = b_c$ is the radial distance of the contact-uplift interface from the center of the bottom plate.

The radial lines with $\theta = \text{constant}$ in the geometric coordinate system are mapped to the straight lines $\eta = \text{constant}$ in the $\xi - \eta$ coordinate system, whereas the uniformly spaced curves between the contact-uplift interface DC and the outer base rim boundary AB are mapped to the lines $\xi = \text{const.}$

The displacement shape functions are defined, for convenience, in the natural (ξ, η) coordinate system by separation of the two variables having the general form of

$$N(\xi, \eta) = N_\xi(\xi)N_\eta(\eta) \quad (3.10)$$

where N_ξ is a function of ξ , and N_η a function of η only. Typically in harmonic analysis, the variation in the tangential direction is taken to be the linear combination of simple $\cos(m\theta)$ or $\sin(m\theta)$ harmonic terms. For the complex bottom plate uplift problem, this approach is inefficient.

3.5.2 $N_\eta(\eta)$ shape functions

For an uplifting bottom plate, there are no concentrated load applied within the uplifted region. Therefore, the first, second and third derivatives of the bottom plate vertical displacement should be smooth within the uplifted domain. The stresses should therefore also vary smoothly along the constant lines of ξ , and η . Considering the symmetric deflections along a curve $\xi = \text{constant}$ about the axis $\theta = 0$ or $\theta = \pi$ (that is, $\eta = -1$ or $\eta = 1$ respectively), the function $N_\eta(\eta)$ must satisfy the symmetry condition $dN_\eta(\pm 1)/d\eta = 0$. Because the radial dimension of the bottom plate uplifted region is typically much smaller than that in the tangential direction, which is one half circumference of the cylindrical tank, the behavior of the uplifted region may be considered as closely resembling that of an one way plate. For such an one way plate, the variation of the vertical displacement along the tangential direction is very similar to that of a beam deformed only by imposed displacement at its end. Under

such loading condition, the curvature of the deflection will only change once along the length of the beam, with only one inflection point.

The function $N_\eta = \cos[(\pi(\eta + 1)/2]$ satisfies $dN_\eta(\pm 1)/d\eta = 0$, and its curvature $d^2N_\eta/d\eta^2$ has only one zero point within the interval -1 to 1. A typical uplift displacement variation along a normalized coordinate grid line, that is variation in the η or $\xi = \text{constant}$ direction, from the uplifted side at $\theta = 0^\circ$ to the contact side at $\theta = 180^\circ$, is shown in Figure 3.3(a). To simulate this distribution, an infinite series of cosine functions is typically utilized in harmonic or Fourier analysis. However, this approach generally needs to include many cosine terms in the infinite series in order to have satisfactory accuracy, particularly for stress calculation. Therefore, the infinite cosine series representation of the bottom plate uplift is considered inefficient for the modeling purpose in the present study.

Instead, the variation of the Ritz shape function in the η -direction is constructed by modifying the simple $\cos\theta$ or $\cos[\pi(\eta + 1)/2]$ function. The function $N_\eta(\eta)$ is assumed to have the form

$$N_\eta = \cos \left[\frac{\pi}{2}(g(\eta) + 1) \right]$$

where $g(\eta)$ is a function of η instead of just simply η . The function $g(\eta)$ is chosen such that $g(\pm 1) = \pm 1$ or ∓ 1 , in order to satisfy the symmetry condition $dN(\pm 1)/d\eta = 0$. Instead of having a zero point at $\eta = 0$ as the original function $\cos[\pi(\eta + 1)/2]$ does, as indicated by the dotted line in Figure 3.3(b), the function $g(\eta)$ has a zero point at the offset position, $\eta = x_0$, such that $g(x_0) = 0$. The accuracy of the approximation function depends on the accuracy of $g(\eta)$. In the present formulation, the function $g(\eta)$ is assumed to be quadratic and is determined from the following conditions:

$$g(-1) = 1 \quad ; \quad g(x_0) = 0 \quad ; \quad g(1) = -1$$

The quadratic function $g(\eta)$ is then obtained as follows

$$g(\eta) = \frac{x_0}{1 - x_0^2}(\eta^2 - 1) - \eta \quad (3.11)$$

The variation of $N_\eta(\eta)$ is plotted in Figure 3.3(b), which is a good approximation to the bottom plate uplift shown in Figure 3.3(a). As compared to the infinite cosine series approach, the modified cosine function approach is much more efficient as it needs only two terms in the representation, a constant term and the modified cosine function term.

Taking derivative of $N_\eta(\eta)$ once,

$$\frac{dN_\eta}{d\eta} = \frac{\pi}{2} \cos(\pi g/2) \left(\frac{2x_0}{1-x_0^2} \eta - 1 \right)$$

For $|x_0| \leq 0.5$, there is no zero first derivative located within the interval (-1,1), which implies that the shape N_η is monotonous within the interval. Therefore, in the derivation of the Ritz shape function, x_0 should be taken $0 \leq x_0 \leq 0.5$.

Here in the proposed simplified formulation, the N_η function is chosen either as a constant term or a modified $\cos\theta$ term

$$N_\eta = 1, \quad \cos \frac{\pi}{2} (g(x_0, \eta) + 1) \quad (3.12)$$

The offset parameter x_0 is considered a variable in the formulation. Its value is determined in the analysis.

3.5.3 $N_\xi(\xi)$ shape functions

The form of the shape function N_ξ in the radial or ξ direction depends on the displacement component under consideration. For the vertical displacement component w , N_ξ^w is given as follows

$$N_\xi^w = \psi_1(\xi), \psi_2(\xi), \psi_3(\xi), \psi_4(\xi), \dots \quad (3.13)$$

where ψ_1 and ψ_2 are the boundary shapes of unit displacement or rotation respectively at the base rim ($\xi = 1$). The remaining terms in N_ξ^w are the interior shapes with zero displacement and zero rotation at the other boundary. Note that all the shape

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functions have zero displacement and rotation at the contact-uplift interface boundary ($\xi = -1$). The boundary shapes are chosen as the simple cubic beam shape functions

$$\begin{aligned}\psi_1(\xi) &= (2 - \xi)(\xi + 1)^2/4 \\ \psi_2(\xi) &= (\xi - 1)(\xi + 1)^2/4\end{aligned}\quad (3.14)$$

The interior shapes $\psi_3, \psi_4, \dots$ are given by a complete series of polynomial functions as follow

$$(1 - \xi^2)^2, (1 - \xi^2)^2\xi, (1 - \xi^2)^2\xi^2, (1 - \xi^2)^2\xi^3, \dots \quad (3.15)$$

Physically, these are the deflected shapes of a fixed-end beam subjected to polynomial distributed loadings of increasing order. The variations of the Ritz shapes N_ξ^r are shown in Figure 3.4.

Following Equation 3.11, the vertical displacement field is then given by a linear combination of all the chosen vertical shape functions

$$\begin{aligned}w(\xi, \eta) &= a_{11}\psi_1 + a_{12}\psi_2 + a_{13}\psi_3 + \dots + \\ &\quad \cos \left[\frac{\pi}{2} (\alpha(\eta^2 - 1) - \eta + 1) \right] (a_{21}\psi_1 + a_{22}\psi_2 + a_{23}\psi_3 + \dots)\end{aligned}\quad (3.16)$$

where the a 's are the magnitudes of the vertical shape functions, $\alpha = x_0/(1 - x_0^2)$ is the parameter for the modified cosine function.

For the inplane displacement components, the radial variations are also described by a complete series of shape functions. The boundary shapes are the linear polynomial functions with unit displacement at the two ends.

$$\begin{aligned}\phi_1(\xi) &= (1 - \xi)/2 \\ \phi_2(\xi) &= (1 + \xi)/2\end{aligned}\quad (3.17)$$

The interior shapes are given as follows

$$(1 - \xi^2), (1 - \xi^2)\xi, (1 - \xi^2)\xi^2, (1 - \xi^2)\xi^3, \dots \quad (3.18)$$

These shapes are shown in Figure 3.5. Physically, these are the displacement shapes of an axially loaded bar. Again, the axial load is given as polynomials of increasing order. Similar to the vertical displacement defined in Equation 3.16, the inplane displacements are given as combinations of the simplified inplane displacement shape functions. The Ritz shapes for the inplane displacements u_r and u_θ are therefore expressed as follows

$$u_r(\xi, \eta) = a_{11}^r \phi_1 + a_{12}^r \phi_2 + a_{13}^r \phi_3 + \dots + \cos \left[\frac{\pi}{2} \left(\alpha_1 \left(\frac{1+\xi}{2} \right)^{\alpha_2} (\eta^2 - 1) - \eta + 1 \right) \right] (a_{21}^r \phi_1 + a_{22}^r \phi_2 + a_{23}^r \phi_3 + \dots) \quad (3.19)$$

and

$$u_\theta = \sin \left[\frac{\pi}{2} \left(\alpha_3 \left(\frac{1+\xi}{2} \right)^{\alpha_4} (\eta^2 - 1) - \eta + 1 \right) \right] (a_1^\theta \phi_1 + a_2^\theta \phi_2 + a_3^\theta \phi_3 + \dots) \quad (3.20)$$

where α_1 and α_3 have the same form of the parameter α which is appeared in Equation 3.16. In Equations 3.20 and 3.21, $[(1+\xi)/2]^{\alpha_2}$ and $[(1+\xi)/2]^{\alpha_4}$ are the offset coefficients which further modify the parameters α_1 and α_3 respectively. All these offset parameters are unknown variables in the formulation. Their magnitudes are determined in the analysis process.

3.6 MATRIX FORMULATION OF RITZ SHAPES

The Ritz shapes for the vertical displacement, $w(\xi, \eta)$, given in Equation 3.16 can be expressed in matrix form as follows

$$w(\xi, \eta) = \mathbf{N}^w(\xi, \eta) \mathbf{a}^w \quad (3.21)$$

$\mathbf{N}^w$ – the Ritz shape for vertical displacement

$\mathbf{a}^w$ – the magnitudes of Ritz shape for vertical displacement

Similarly, the Ritz shapes for the inplane displacements, u_r and u_θ , given in Equations 3.19 and 3.20 can also be expressed in matrix form

$$u_r(\xi, \eta) = \mathbf{N}^r(\xi, \eta) \mathbf{a}^r \quad (3.22)$$

$$u_\theta(\xi, \eta) = \mathbf{N}^\theta(\xi, \eta) \mathbf{a}^\theta \quad (3.23)$$

$\mathbf{N}^r$ – the Ritz shapes for radial displacement

$\mathbf{a}^r$ – the magnitudes of Ritz shapes for radial displacement

$\mathbf{N}^\theta$ – the Ritz shapes for tangential displacement

$\mathbf{a}^\theta$ – the magnitudes of Ritz shapes for tangential displacement

Using the simplified Ritz shape functions presented herein, the potential energy of the deformed bottom plate uplifted region can be calculated in terms of the unknown Ritz coefficients, magnitudes of the Ritz shape functions in static tilt uplift analysis are then determined by the minimum potential energy theorem.

The offset parameter α 's($\alpha_1 \dots$) in Equations 3.16, 3.19, 3.20 are also considered as independent Ritz coefficients which are determined by minimization of the potential energy.

3.7 RITZ FORMULATION OF CONTACT REGION SUBSTRUCTURE ELEMENT

The contact region of the bottom plate is modeled as a two dimensional plane stress element. The Ritz shape of the contact element is obtained from solution of the two dimensional plane stress problem. The solution of the contact region for an uplifted bottom plate is obtained by the following steps: First the deformation of a disc subjected to an uniformly distributed body force parallel to the direction of the exciting axis is obtained as the first part of the solution. The discrepancies or violations of continuity at the contact-uplift interface are then corrected by the homogeneous solution obtained from the plane stress problem with zero applied loading.

3.7.1 Plane disc solution with uniform load

A disc of diameter $2b$ with uniform load q acting in the plane is shown in Figure 3.6(a). The governing differential equations are given as follows

$$\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y &= 0\end{aligned}\tag{3.24}$$

where considering a disc under the given loading condition, $X = q/t$ and $Y = 0$, where t is the thickness of the bottom plate. Because the first step in the derivation of the Ritz shape functions for the contact region involves a particular displacement solution of the governing differential equation, the boundary conditions are not considered at this step. Any suitable boundary condition assumption may be adopted. Violations of the actual boundary conditions can be corrected later. If there is no hole or concentrated load applied within the disc, there are two independent boundary conditions available to identify the solution, which are the distributions of the inplane displacement along the x and y axes. Accordingly, two assumptions can be made to obtain the particular solution. First, assuming $\tau_{xy} = 0$, from Equation 3.24

$$\begin{aligned}\frac{\partial \sigma_y}{\partial y} &= 0 \\ \frac{\partial \sigma_x}{\partial x} &= -\frac{q}{t}\end{aligned}\tag{3.25}$$

Solving Equation 3.25 gives

$$\sigma_x = -\frac{q}{t}x + f(y)$$

where $f(y)$ is a function of y . Considering the contact bottom plate satisfies the plane stress assumption in elasticity, the following stress-strain relations are employed

$$\begin{aligned}\sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \\ \sigma_y &= \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \\ \tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy}\end{aligned}\tag{3.26}$$

where

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\tag{3.27}$$

Secondly, assuming $f(y) = 0$, then from Equation 3.27, a particular displacement solution is obtained as follows

$$\begin{aligned}u &= -\frac{q}{2Et}x^2(1-\nu^2) + u_0 \\ v &= v_0\end{aligned}$$

where u_0 and v_0 are constants representing the rigid body motions. As the rigid body motion of the bottom plate can be taken into account by the boundary conditions, therefore, without loss of generality, it is convenient to choose $u_0 = 0$, and $v_0 = 0$

By the coordinate transformation of

$$\begin{aligned}x &= r \cos \theta \\ y &= r \sin \theta \\ u_r &= u \cos \theta + v \sin \theta \\ u_\theta &= -u \sin \theta + v \cos \theta\end{aligned}\tag{3.28}$$

The displacement in the bottom plate contact region can be expressed in terms of u_r and u_θ as follows

$$\begin{aligned} u_r &= -\frac{q(1-\nu^2)r^2}{8Et}(3\cos\theta + \cos 3\theta) \\ u_\theta &= \frac{q(1-\nu^2)r^2}{8Et}(\sin\theta + \sin 3\theta) \end{aligned} \quad (3.29)$$

At the boundary between the uplifted and contact regions, the displacements u_r and u_θ are obtained by substituting $r = b$ into Equation 3.38 as follows

$$\begin{aligned} u_r(b, \theta) &= -\frac{q(1-\nu^2)b^2}{8Et}(3\cos\theta + \cos 3\theta) \\ u_\theta(b, \theta) &= \frac{q(1-\nu^2)b^2}{8Et}(\sin\theta + \sin 3\theta) \end{aligned} \quad (3.30)$$

Any discrepancies in $u_r(b, \theta)$ and $u_\theta(b, \theta)$ with the actual correct boundary conditions can be corrected by superposing a proper homogeneous solution under imposed boundary displacements equal to the negative of the discrepancies.

3.7.2 Homogeneous solution

In Figure 3.6(b), the boundary displacements, $u_r(b, \theta)$ together with the opposite of $u_r(b, \theta)$, and $u_\theta(b, \theta)$ together with the opposite of $u_\theta(b, \theta)$, are imposed at the boundary of the contact region circular disc. In polar coordinates, the governing differential equations with zero body force are given as follows

$$\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} &= 0 \\ \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} &= 0 \end{aligned} \quad (3.31)$$

Introducing here the stress function ϕ , such that

$$\begin{aligned} \sigma_r &= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \\ \sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} \quad \tau_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) \end{aligned} \quad (3.32)$$

Since the stress functions automatically satisfy equilibrium, the stress function ϕ need only to satisfy the compatibility condition expressed as follows

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 \phi = 0 \quad (3.33)$$

A general expression for the stress function ϕ which satisfies Equation 3.33 is given as follows[65]

$$\begin{aligned} \phi = & a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0 r^2 \theta + a'_0 \theta \\ & + \frac{a_1}{2} r \theta \sin \theta + (b_1 r^3 + a'_1 r^{-1} + b'_1 r \ln r) \cos \theta \\ & - \frac{c_1}{2} r \theta \cos \theta + (d_1 r^3 + c'_1 r^{-1} + d'_1 r \ln r) \sin \theta \\ & + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a'_n r^{-n} + b'_n r^{-n+2}) \cos n\theta \\ & + \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c'_n r^{-n} + d'_n r^{-n+2}) \sin n\theta \end{aligned} \quad (3.34)$$

From symmetry, the following condition must be satisfied at the symmetry axis

$$\tau_{r\theta} = \left[\frac{1}{r} \frac{\partial \phi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} \right]_{\theta=0, \theta=\pi} = 0$$

In polar coordinates, the strain-displacement relationships are

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{u_r}{r} + \frac{\partial u_\theta}{r \partial \theta} \quad (3.35)$$

For a plane stress problem, the Hooke's Law reduces to

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu \sigma_\theta) \quad \varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu \sigma_r) \quad (3.36)$$

It is note here that there should be no singularity for u_r , u_θ , ε_r , ε_θ and $\gamma_{r\theta}$ at $r = 0$.
The stress function ϕ can be expressed as follows

$$\begin{aligned}\phi = & a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r \\ & + \frac{a_1}{2} r \theta \sin \theta + (b_1 r^3 + a'_1 r^{-1} + b'_1 r \ln r) \cos \theta \\ & + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a'_n r^{-n} + b'_n r^{-n+2}) \cos n\theta \\ & + \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c'_n r^{-n} + d'_n r^{-n+2}) \sin n\theta\end{aligned}\quad (3.37)$$

Substituting Equation 3.37 into Equation 3.32, and then using Equations 3.35 and 3.36, the following can be obtained by integrating displacements u_r and u_θ

$$\begin{aligned}u_r = & b_{01} r + [b_{11}(1 - 3\nu)r^2 + b_{12}] \cos \theta \\ & + \sum_{n=2}^{\infty} \{b_{n1} r^{n-1} + [-n + 2 - \nu(n + 2)]b_{n2} r^{n+1}\} \cos n\theta\end{aligned}\quad (3.38)$$

$$u_\theta = [b_{11}(5 + \nu)r^2 - b_{12}] \sin \theta + \sum_{n=2}^{\infty} \{-b_{n1} r^{n-1} + (n + 4 + \nu n)b_{n2} r^{n+1}\} \sin n\theta$$

or putting in matrix form

$$\begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \mathbf{N}_b \mathbf{a}_b \quad (3.39)$$

where $\mathbf{N}_b$ — the Ritz shape of contact region

$\mathbf{a}_b$ — the unknown Ritz parameter vector

The constants $b_{01}, b_{11}, \dots$ in Equation 3.38 will be determined by the values u_r, u_θ on the boundary and the correction to the particular solution given in Equation 3.30.

Figure 3.1: Phenomenon of uplift under lateral loads

(a) uplift area in normal coordinate

(b) mapped uplift area

Figure 3.2: Coordinate mapping

(a) typical uplift of bottom

(b) cosine and modified cosine function

Figure 3.3: Modeling of displacement variation in tangential direction

Figure 3.4: Function N_ξ for vertical Ritz shapes (uplifted region)

Figure 3.5: Function N_ξ for inplane Ritz shapes (uplifted region)

Figure 3.6: Models of contact bottom plate

Chapter 4

RITZ FORMULATION OF TANK SHELL

In this chapter, the contribution of the cylindrical tank shell to the uplift mechanism is determined using the linear thin shell theory presented by Flügge[66]. Influences of the tank shell flexibility, particularly the out-of-round deformation and out-of-plane warping of the cylindrical shell are considered in the formulation. In order to model realistically the behaviour of an uplifting tank, the semi-infinite assumption commonly adopted in analysis of thin walled cylindrical shell is not utilized in present formulation.

Membrane theory may give a good approximation for the stress distribution far away from the two ends of the shell. But the displacement and stress distributions near the ends of the shell are dominated by flexural behaviour of the thin shell. Since resistance to bottom plate uplift is greatly influenced by the flexibility of tank shell at the base, it is therefore necessary to use a shell theory which includes both flexural and membrane mechanisms in the modelling of uplifting fluid-tank system, the bending theory of shell will be used in this thesis.

Analytical solutions of cylindrical shell with uniform thickness are used as Ritz shapes to discretize the cylindrical tank structure. The Ritz shapes are separated into boundary and interior shapes. Boundary shape functions have imposed specified displacement at the shell ends, whereas the interior shape functions have fixed boundary conditions. The response of tank shell due to the liquid pressure can be separated into two parts. The first part of the response is described by a linear combination of boundary Ritz shape functions which are solutions to the homogeneous

governing differential equations. The second part is the response produced by the liquid pressure acting on a fixed-end cylindrical shell. Common to many metal tanks in the field, the thickness of the cylindrical shell plate may vary along the height of the tank. Thinner plate may be used in the fabrication of the top part of the tank. The variation of the tank wall thickness along the height of the tank is considered approximately in the formulation.

4.1 FLÜGGE THIN SHELL THEORY

In Flügge thin shell theory for cylindrical shell, the stress components are represented by the resultants at the middle surface of the shell. These stress resultances can be expressed in terms of the displacement components at the middle surface. Consider a cylindrical shell of radius R , height H and uniform thickness t in the cylindrical coordinate system as shown in Figure 4.1, with z and θ denoting the axial and circumferential directions respectively, and u , v and w denoting the displacements at the middle surface of the tank shell in the axial, circumferential and radial directions respectively. If the tank shell has several thickness along the height of the tank, an average thickness is used in the derivation of the displacement shape.

By integrating the stresses over the thickness t , the stress resultants of forces and moments per unit length of the middle surface are given as follows

$$\begin{aligned}
 N_z &= \int_{-t/2}^{t/2} \sigma_z(1 + x/R)dx & M_z &= - \int_{-t/2}^{t/2} \sigma_z(1 + x/R)x dx \\
 N_\theta &= \int_{-t/2}^{t/2} \sigma_\theta dx & M_\theta &= - \int_{-t/2}^{t/2} \sigma_\theta x dx \\
 N_{z\theta} &= \int_{-t/2}^{t/2} \tau_{z\theta}(1 + x/R)dx & M_{z\theta} &= - \int_{-t/2}^{t/2} \tau_{z\theta}(1 + x/R)x dx \\
 N_{\theta z} &= \int_{-t/2}^{t/2} \tau_{\theta z} dx & M_{\theta z} &= - \int_{-t/2}^{t/2} \tau_{\theta z} x dx \\
 Q_z &= \int_{-t/2}^{t/2} \tau_{zx}(1 + x/R)dx & Q_\theta &= - \int_{-t/2}^{t/2} \tau_{\theta x} dx
 \end{aligned} \tag{4.1}$$

For a thin shell, $(1 + x/R) \approx 1$. Equation 4.1 can then be simplified as follows

$$\begin{aligned}
 N_z &= \frac{Et}{1 - \nu^2} \left[\frac{\partial u}{\partial z} + \nu \left(\frac{\partial v}{R \partial \theta} + \frac{w}{R} \right) \right] \\
 N_\theta &= \frac{Et}{1 - \nu^2} \left[\left(\frac{\partial v}{R \partial \theta} + \frac{w}{R} \right) + \nu \frac{\partial u}{\partial z} \right] \\
 N_{z\theta} &= N_{\theta z} = \frac{Et}{1 - \nu^2} \left(\frac{\partial u}{R \partial \theta} + \frac{\partial v}{\partial z} \right) \\
 M_z &= -\frac{Et^3}{12(1 - \nu^2)} \left(\frac{\partial^2 w}{\partial z^2} + \nu \frac{\partial^2 w}{R^2 \partial \theta^2} \right) \\
 M_\theta &= -\frac{Et^3}{12(1 - \nu^2)} \left(\frac{\partial^2 w}{R^2 \partial \theta^2} + \nu \frac{\partial^2 w}{\partial z^2} \right) \\
 M_{z\theta} &= M_{\theta z} = -\frac{Et^3}{12(1 + \nu)} \frac{\partial^2 w}{R \partial \theta \partial z}
 \end{aligned} \tag{4.2}$$

where ν is the Poisson's ratio.

The cylindrical shell equilibrium equations are obtained by considering equilibrium of the shell element shown in Figure 4.1[66]

$$\begin{aligned}
 R^2 \frac{\partial^2 u}{\partial z^2} + \frac{1 - \nu}{2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1 + \nu}{2} R \frac{\partial^2 v}{\partial z \partial \theta} + \nu \frac{\partial w}{\partial z} R + p_z &= 0 \\
 \frac{1 + \nu}{2} R \frac{\partial^2 u}{\partial z \partial \theta} + \frac{\partial^2 v}{\partial \theta^2} + \frac{1 - \nu}{2} R^2 \frac{\partial^2 v}{\partial z^2} + \frac{\partial w}{\partial \theta} + p_\theta &= 0 \\
 R \nu \frac{\partial u}{\partial z} + \frac{\partial v}{\partial \theta} + w + k \left(R^4 \frac{\partial^4 w}{\partial z^4} + 2R^2 \frac{\partial^4 w}{\partial z^2 \partial \theta^2} + \frac{\partial^4 w}{\partial \theta^4} \right) + p_r &= 0
 \end{aligned} \tag{4.3}$$

$$k = \frac{t^2}{12R^2}$$

where p_r , p_θ and p_z are the components of the liquid pressure in the radial, circumferential and axial direction respectively.

4.2 BOUNDARY RITZ SHAPE FUNCTIONS

The boundary Ritz shape functions of the tank shell are defined from the homogeneous solution of the equilibrium equations given in Equation 4.3. Assuming the cylindrical shell displacements can be written in the following form

$$\begin{aligned} u(z, \theta) &= \sum_{m=0} A e^{\lambda \frac{z}{R}} \cos(m\theta) \\ v(z, \theta) &= \sum_{m=1} B e^{\lambda \frac{z}{R}} \sin(m\theta) \\ w(z, \theta) &= \sum_{m=0} C e^{\lambda \frac{z}{R}} \cos(m\theta) \end{aligned} \quad (4.4)$$

The governing differential equation in Equation 4.3 are solved by the standard separation of variable procedure. Taking $p_z = p_\theta = p_r = 0$, and substituting Equation 4.4 into 4.3, the determinant of the coefficient matrix of the variables A , B and C must be equal to zero for nontrivial solution

$$\begin{vmatrix} \lambda^2 - \frac{1-\nu}{2}m^2 & \frac{1+\nu}{2}\lambda m & \nu\lambda \\ -\frac{1+\nu}{2}\lambda m & \lambda^2 \frac{1-\nu}{2} - m^2 & -m \\ \nu\lambda & m & k(\lambda^2 - m^2)^2 + 1 \end{vmatrix} = 0 \quad (4.5)$$

or

$$\lambda^4 + \frac{k}{1-\nu^2}(\lambda^2 - m^2)^4 = 0 \quad (4.6)$$

For a particular harmonic wave number m , (where $m = 0, 1, 2, 3, \dots$), Equation 4.6 has 8 complex roots, which may be expressed as follows

$$\begin{aligned} \lambda_1 &= -\alpha_1 + \beta_1 i & \lambda_2 &= -\alpha_1 - \beta_1 i & \lambda_3 &= -\alpha_2 + \beta_2 i & \lambda_4 &= -\alpha_2 - \beta_2 i \\ \lambda_5 &= \alpha_1 + \beta_1 i & \lambda_6 &= \alpha_1 - \beta_1 i & \lambda_7 &= \alpha_2 + \beta_2 i & \lambda_8 &= \alpha_2 - \beta_2 i \end{aligned} \quad (4.7)$$

The parameters ξ_l and η_l where $l = 1, \dots, 8$ are then defined in terms of the complex roots given by Equation 4.7

$$\xi_1 + \eta_1 i = -\frac{\lambda_1}{(\lambda_1^2 - m^2)^2}(\nu\lambda_1^2 + m^2) \quad (4.8)$$

$$\xi_3 + \eta_3 i = -\frac{\lambda_3}{(\lambda_3^2 - m^2)^2}(\nu \lambda_3^2 + m^2) \quad (4.9)$$

$$\xi_5 + \eta_5 i = -\frac{\lambda_5}{(\lambda_5^2 - m^2)^2}(\nu \lambda_5^2 + m^2) \quad (4.10)$$

$$\xi_7 + \eta_7 i = -\frac{\lambda_7}{(\lambda_7^2 - m^2)^2}(\nu \lambda_7^2 + m^2) \quad (4.11)$$

$$\xi_2 + \eta_2 i = \frac{m}{(\lambda_1^2 - m^2)^2}[(2 + \nu)\lambda_1^2 - m^2] \quad (4.12)$$

$$\xi_4 + \eta_4 i = \frac{m}{(\lambda_3^2 - m^2)^2}[(2 + \nu)\lambda_3^2 - m^2] \quad (4.13)$$

$$\xi_6 + \eta_6 i = \frac{m}{(\lambda_5^2 - m^2)^2}[(2 + \nu)\lambda_5^2 - m^2] \quad (4.14)$$

$$\xi_8 + \eta_8 i = \frac{m}{(\lambda_7^2 - m^2)^2}[(2 + \nu)\lambda_7^2 - m^2] \quad (4.15)$$

Using Equation 4.7 - 4.15, the homogeneous solution of Equation 4.3 can be obtained by separation of variables and expressed in terms of $\bar{z}$, where $z = z/R$, as follows

$$\begin{aligned} w(\bar{z}, \theta) = & e^{-\alpha_{01}\bar{z}}(c_{01} \cos \beta_{01}\bar{z} + c_{02} \sin \beta_{01}\bar{z}) \\ & + e^{\alpha_{01}\bar{z}}(c_{03} \cos \beta_{01}\bar{z} + c_{04} \sin \beta_{01}\bar{z}) - \nu c_{06} \\ & \sum_{m=1} \left\{ e^{-\alpha_{m1}\bar{z}}(c_{m1} \cos \beta_{m1}\bar{z} + c_{m2} \sin \beta_{m1}\bar{z}) + \right. \\ & e^{-\alpha_{m2}\bar{z}}(c_{m3} \cos \beta_{m2}\bar{z} + c_{m4} \sin \beta_{m2}\bar{z}) + \\ & e^{\alpha_{m1}\bar{z}}(c_{m5} \cos \beta_{m1}\bar{z} + c_{m6} \sin \beta_{m1}\bar{z}) + \\ & \left. e^{\alpha_{m2}\bar{z}}(c_{m7} \cos \beta_{m2}\bar{z} + c_{m8} \sin \beta_{m2}\bar{z}) \right\} \cos(m\theta) \end{aligned} \quad (4.16)$$

$$\begin{aligned} u(\bar{z}, \theta) = & e^{-\alpha_{01}\bar{z}}[(c_{01}\xi_{01} + \eta_{01}c_{02}) \cos \beta_{01}\bar{z} + (c_{02}\xi_{01} - c_{01}\eta_{01}) \sin \beta_{01}\bar{z}] + \\ & e^{\alpha_{01}\bar{z}}[(c_{03}\xi_{05} + \eta_{05}c_{04}) \cos \beta_{01}\bar{z} + (c_{04}\xi_{05} - c_{03}\eta_{05}) \sin \beta_{01}\bar{z}] + c_{05} + c_{06}z + \\ & \sum_{m=1} \left\{ e^{-\alpha_{m1}\bar{z}}[(c_{m1}\xi_{m1} + \eta_{m1}c_{m2}) \cos \beta_{m1}\bar{z} + (c_{m2}\xi_{m1} - c_{m1}\eta_{m1}) \sin \beta_{m1}\bar{z}] + \right. \\ & e^{-\alpha_{m2}\bar{z}}[(c_{m3}\xi_{m3} + \eta_{m3}c_{m4}) \cos \beta_{m2}\bar{z} + (c_{m4}\xi_{m3} - c_{m3}\eta_{m3}) \sin \beta_{m2}\bar{z}] + \\ & e^{\alpha_{m1}\bar{z}}[(c_{m5}\xi_{m5} + \eta_{m5}c_{m6}) \cos \beta_{m1}\bar{z} + (c_{m6}\xi_{m5} - c_{m5}\eta_{m5}) \sin \beta_{m1}\bar{z}] + \\ & \left. e^{\alpha_{m2}\bar{z}}[(c_{m7}\xi_{m7} + \eta_{m7}c_{m8}) \cos \beta_{m2}\bar{z} + (c_{m8}\xi_{m7} - c_{m7}\eta_{m7}) \sin \beta_{m2}\bar{z}] \right\} \cos(m\theta) \end{aligned} \quad (4.17)$$

$$\begin{aligned}
v(z, \theta) = & \sum_{m=1} \left\{ e^{-\alpha_{m1}z} [(c_{m1}\xi_{m2} + \eta_{m2}c_{m2}) \cos \beta_{m1}\bar{z} + (c_{m2}\xi_{m2} - c_{m1}\eta_{m2}) \sin \beta_{m1}\bar{z}] + \right. \\
& e^{-\alpha_{m2}z} [(c_{m3}\xi_{m4} + \eta_{m4}c_{m4}) \cos \beta_{m2}\bar{z} + (c_{m4}\xi_{m4} - c_{m3}\eta_{m4}) \sin \beta_{m2}\bar{z}] + \\
& e^{\alpha_{m1}z} [(c_{m5}\xi_{m6} + \eta_{m6}c_{m6}) \cos \beta_{m1}\bar{z} + (c_{m6}\xi_{m6} - c_{m5}\eta_{m6}) \sin \beta_{m1}\bar{z}] + \\
& \left. e^{\alpha_{m2}z} [(c_{m7}\xi_{m8} + \eta_{m8}c_{m8}) \cos \beta_{m2}\bar{z} + (c_{m8}\xi_{m8} - c_{m7}\eta_{m8}) \sin \beta_{m2}\bar{z}] \right\} \sin(m\theta)
\end{aligned} \tag{4.18}$$

The constants $c_{01}, \dots, c_{m1}, \dots, c_{m8}$ in Equations 4.16 - 4.18 are determined from the boundary conditions at the two ends of the tank shell. Corresponding to $m = 0$, there are six constants $c_{01}, c_{02}, \dots, c_{06}$ that need to be determined, whereas for $m = 1, 2, \dots$, there are eight constants $c_{m1}, c_{m2}, \dots, c_{m8}$ for each wave number $m > 0$.

At the base of the cylindrical tank shell, the radial, rotational, axial (vertical) and tangential displacements, denoted respectively $w_b, \partial w_b / \partial z, u_b$, and v_b can be expressed in terms of Fourier series as follows

$$\begin{aligned}
w_b &= w_0 + w_1 \cos \theta + w_2 \cos 2\theta + \dots + w_m \cos m\theta \\
\frac{\partial w_b}{\partial z} &= w'_0 + w'_1 \cos \theta + w'_2 \cos 2\theta + \dots + w'_m \cos m\theta \\
u_b &= u_0 + u_1 \cos \theta + u_2 \cos 2\theta + \dots + u_m \cos m\theta \\
v_b &= v_1 \sin \theta + v_2 \sin 2\theta + \dots + v_m \sin m\theta
\end{aligned}$$

For the harmonic case of $m = 0$, three displacement boundary conditions can be specified at the bottom of the tank shell. These are related to w_0, w'_0 and u_0 , which are used to determine the coefficients $c_{01}, c_{02}, \dots, c_{06}$ given in Equation 4.16 and 4.17. For the case of higher harmonic, $m > 0$, four boundary conditions at the tank bottom edge are specified involving w_m, w'_m, u_m and v_m . Subsequently, these are used to determine the coefficient $c_{m1}, c_{m2}, \dots, c_{m8}$ given in Equations 4.16-18. Using similar methods at the top of the shell, another three conditions are specified for $c_{01}-c_{06}$, and four conditions for $c_{m1}-c_{m8}$. Thus, six equations can be obtained to solve for the six unknowns $c_{01}, c_{02}, \dots, c_{06}$ and other eight equations for the eight unknowns $c_{m1}, c_{m2}, \dots, c_{m8}$ for each wave number $m > 0$.

4.3 INTERIOR RITZ SHAPE FUNCTION

The interior Ritz shape functions for the tank shell is derived in this section. The tank shell interior Ritz shape is given by the tank shell deformed shape due to the applied fluid pressure with fixed boundary conditions at both ends of the cylindrical shell. In the derivation, the fluid pressure is represented by a double series. A solution which satisfies equilibrium with the fluid pressure loading but not the boundary conditions at the two ends is obtained. The violations of the boundary conditions are then corrected by superposing properly scaled boundary Ritz shape functions derived in the previous section.

The fluid pressure loading is specified by the pressure term p_r normal to the tank wall with

$$p_z = p_\theta = 0$$

In the governing differential equations of the cylindrical shell, Equation 4.3, there are three equations for three unknowns, u , v and w . A displacement function F is introduced for convenience to represent u , v and w such that the three governing equations with three unknowns can be reduced to one equation with one unknown. The interior Ritz shape function is derived by assuming a displacement function F as follows[67]

$$u = \frac{\partial}{\partial z} \left[\frac{\partial^2}{R^2 \partial \theta^2} - \nu \frac{\partial^2}{\partial z^2} \right] F \quad (4.19)$$

$$v = -\frac{\partial}{R \partial \theta} \left[\frac{\partial^2}{R^2 \partial \theta^2} + (2 + \nu) \frac{\partial^2}{\partial z^2} \right] F \quad (4.20)$$

$$w = R \nabla^4 F \quad (4.21)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{R^2 \partial \theta^2}$$

Substituting Equations 4.19-21 into Equation 4.3, the first two equations in Equation 4.3 are automatically satisfied, the third one is reduced to

$$\nabla^4 F + \frac{Et}{R^2 D} \frac{\partial^4 F}{\partial z^4} = \frac{p_r}{RD} \quad (4.22)$$

where t is the thickness of the tank shell, E is the elastic modulus, and D is the flexural rigidity given as follows

$$D = \frac{Et^3}{12(1 - \nu^2)}$$

Since the two series

$$\cos m\theta \quad \sin \frac{n\pi z}{H}$$

are complete for $0 < \theta < 2\pi$ and $0 < z < H$, any specified fluid pressure distribution, p_r , can be expressed in terms of the double series

$$p_r = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} p_{mn} \cos m\theta \sin \frac{n\pi z}{H} \quad (4.23)$$

In the formulation of the interior Ritz shape functions, it is assumed that the displacement function F can also be expressed by the harmonic series $\cos m\theta$ and $\sin n\pi z/H$ as follows

$$F = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} F_{mn} \cos m\theta \sin \frac{n\pi z}{H} \quad (4.24)$$

Defining

$$\frac{p_{mn}}{RD} = p_{mn}, \quad \frac{n\pi}{H} = \gamma_n$$

The following equation can be obtained by substituting Equations 4.23 and 4.24 into Equation 4.22

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[\gamma_n^8 + \frac{4m^2}{R^2} \gamma_n^6 + \left(\frac{6m^4}{R^4} + \frac{Et}{R^2 D} \right) \gamma_n^4 + \frac{4m^6}{R^6} \gamma_n^2 + \frac{m^8}{R^8} \right] F_{mn} \cos m\theta \sin \gamma_n z \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} p_{mn} \cos m\theta \sin \gamma_n z \end{aligned} \quad (4.25)$$

from which the coefficient F_{mn} can be easily determined

$$F_{mn} = \frac{P_{mn}}{\gamma_n^8 + \frac{4m^2}{R^2}\gamma_n^6 + \left(\frac{6m^4}{R^4} + \frac{Et}{R^2D}\right)\gamma_n^4 + \frac{4m^6}{R^6}\gamma_n^2 + \frac{m^8}{R^8}} \quad (4.26)$$

The displacements can therefore be obtained by substituting Equation 4.26 into Equations 4.19 - 4.21 as follows

$$\begin{aligned} u &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left(\nu \gamma_n^2 - \frac{m^2}{R^2} \right) \gamma_n F_{mn} \cos m\theta \cos \gamma_n z \\ v &= -\frac{1}{R} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left((2 + \nu) \gamma_n^2 + \frac{m^2}{R^2} \right) m F_{mn} \sin m\theta \sin \gamma_n z \\ w &= R \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left(\gamma_n^2 + \frac{m^2}{R^2} \right)^2 F_{mn} \cos m\theta \sin \gamma_n z \end{aligned} \quad (4.27)$$

Using Equation 4.2, the internal forces are then determined to be the following

$$\begin{aligned} N_z &= \frac{Et}{R^2} \frac{\partial^4 F}{\partial z^2 \partial \theta^2} = \frac{Et}{R^2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} m^2 \gamma_n^2 F_{mn} \cos m\theta \sin \gamma_n z \\ N_\theta &= Et \frac{\partial^4 F}{\partial z^4} = Et \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \gamma_n^4 F_{mn} \cos m\theta \sin \gamma_n z \\ N_{z\theta} &= -\frac{Et}{R} \frac{\partial^4 F}{\partial z^3 \partial \theta} = -\frac{Et}{R} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} m \gamma_n^3 F_{mn} \sin m\theta \cos \gamma_n z \end{aligned} \quad (4.28)$$

The displacements at the two ends of the tank shell can be calculated by substituting $z = 0$ into Equation 4.27 for the displacements at the base and $z = H$ for the displacements at the top rim. Any discrepancy in the boundary conditions (at $z = 0$ and $z = H$) can be corrected by the homogeneous solution presented in section 4.2.

Figure 4.1: Tank shell displacements stress results

Chapter 5

ASSEMBLED MODEL OF TANK SYSTEM

The derivations of the Ritz shapes for the tank shell, uplifted region and contact region of the bottom plate, have been presented in earlier chapters. The Ritz method is basically a variational method which employs a stationary principle. In elasticity, the well known principle is the minimum potential energy theorem. In the analysis, the tank system is divided into substructure elements including the bottom plate contact region, the bottom plate uplift region, the tank shell, the roof and the ring girder. Continuity of the displacements must be maintained at the boundaries between different substructures.

The stiffness matrix and loading vector of each of the substructure elements are evaluated numerically using Gaussian quadrature. The stiffness matrix of the combined fluid-tank system including contributions from the tank shell, the uplifted bottom plate and the contact bottom plate are assembled by the direct stiffness procedure.

Nonlinearity of the uplift mechanism comes from the von Karman plate theory of the uplifted region and the variable geometry of the bottom plate contact region. For every iteration on the bottom plate contact-uplift geometry, the deformations of the nonlinear fluid-tank structural system is determined by the Newton-Raphson method. The determination of the contact-uplift boundary of the bottom plate is determined in another level of iteration using the bisection method.

5.1 ASSEMBLED MODEL OF BOTTOM PLATE UPLIFTED REGION

As discussed in Chapter 3, each component of the bottom plate displacement in the uplifted region is approximated by the linear combination of series of Ritz shape functions which can be expressed as follows

$$\mathbf{u} = \begin{bmatrix} u_r \\ u_\theta \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{N}^r(\xi, \eta) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}^\theta(\xi, \eta) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}^w(\xi, \eta) \end{bmatrix} \begin{bmatrix} \mathbf{a}^r \\ \mathbf{a}^\theta \\ \mathbf{a}^w \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} \mathbf{a}^r \\ \mathbf{a}^\theta \\ \mathbf{a}^w \end{bmatrix} \quad (5.1)$$

$\mathbf{N}^r$ – the Ritz shape function matrix for the radial displacement

$\mathbf{N}^\theta$ – the Ritz shape function matrix for the tangential displacement

$\mathbf{N}^w$ – the Ritz shape function matrix for the vertical displacement

where $\mathbf{a}^r$, $\mathbf{a}^\theta$ and $\mathbf{a}^w$ are respectively the unknown Ritz parameter amplitude vector.

The potential energy corresponding to the uplifted region of the bottom plate is given as follows

$$\begin{aligned} \Pi_p = & \frac{D}{2} \int \int \left\{ (\nabla^2 w)^2 - 2(1 - \nu) \left[\frac{\partial^2 w}{\partial r^2} \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right. \right. \\ & \left. \left. - \left(\frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right)^2 \right] \right\} r dr d\theta \\ & + \frac{Et}{2(1 - \nu^2)} \int \int \left[\epsilon_r^2 + 2\nu \epsilon_r \epsilon_\theta + \epsilon_\theta^2 + \frac{1 - \nu}{2} \gamma_{r\theta}^2 \right] r dr d\theta \\ & - \int \int p_z w r dr d\theta \end{aligned} \quad (5.2)$$

From the stationary process (Equation 3.1), and assuming M is the total number of unknown Ritz coefficients for the bottom plate uplifted region, stationarity of the potential energy function implies[68]

$$\frac{\partial \Pi_p}{\partial a_i} = 0 \quad \text{for } i = 1, 2, \dots, M$$

or

$$\begin{aligned}
\frac{\partial \Pi_p}{\partial a_i} = & \int \int D \left\{ \left[\frac{\partial^2 w}{\partial r^2} + \nu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right] \frac{\partial}{\partial a_i} \frac{\partial^2 w}{\partial r^2} \right. \\
& + \left(\frac{1}{r} \frac{\partial w}{\partial r} + \nu \frac{\partial^2 w}{r^2} \right) \frac{\partial}{\partial a_i} \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \\
& + 2(1 - \nu) \left(\frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right) \frac{\partial}{\partial a_i} \left(\frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right) \left. \right\} r \, dr \, d\theta \quad (5.3) \\
& + \int \int \frac{Et}{1 - \nu^2} \left\{ (\epsilon_r + \nu \epsilon_\theta) \frac{\partial \epsilon_r}{\partial a_i} + (\epsilon_\theta + \nu \epsilon_r) \frac{\partial \epsilon_\theta}{\partial a_i} + \frac{1 - \nu}{2} \gamma_{r\theta} \frac{\partial \gamma_{r\theta}}{\partial a_i} \right\} r \, dr \, d\theta \\
& - \int \int p_z \frac{\partial w}{\partial a_i} r \, dr \, d\theta
\end{aligned}$$

Using the kinematic displacements and inplane strains relationships of the von Karman plate theory given in Equation 3.2, Equation 5.3 produces a set of M simultaneous algebraic equations which are then solved to determine the unknown Ritz parameters $\mathbf{a}$

$$\mathbf{K}'_u \mathbf{a} = \mathbf{R}'_u \quad (5.4)$$

$\mathbf{K}'_u$ – stiffness matrix of uplifted region

$\mathbf{R}'_u$ – loading vector of uplifted region

The matrix $\mathbf{K}'_u$ is nonlinear due to the coupling of the membrane and flexural mechanisms.

Tangent Stiffness Matrix $\mathbf{K}_u$

Using Newton-Raphson method[69], the tangent stiffness matrix $\mathbf{K}_u$ can be obtained as follow

$$\mathbf{K}_u = \begin{bmatrix} \frac{\partial \mathbf{K}'_1}{\partial a_1} & \cdots & \frac{\partial \mathbf{K}'_1}{\partial a_i} & \cdots & \frac{\partial \mathbf{K}'_1}{\partial a_M} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{K}'_M}{\partial a_1} & \cdots & \frac{\partial \mathbf{K}'_M}{\partial a_i} & \cdots & \frac{\partial \mathbf{K}'_M}{\partial a_M} \end{bmatrix} \quad (5.5)$$

where $\mathbf{K}'_i$ is i -th column vector in $\mathbf{K}'_u \mathbf{a}$.

5.2 ASSEMBLED MODEL OF BOTTOM PLATE CONTACT REGION

In Figure 5.1, u_r and u_θ are the displacements in the polar coordinates with the origin located at the center of the tank bottom, whereas u'_r and u'_θ are the same displacements expressed in the polar coordinates with the origin located at the center of the circular contact area.

In the Ritz formulation, the Ritz displacement shape functions for the contact region are obtained from the analytical solutions presented in Chapter 3. Discretizing the bottom plate displacement using these Ritz shape functions, displacement in the contact region can be expressed as follows

$$\mathbf{u}' = \begin{bmatrix} u'_r \\ u'_\theta \end{bmatrix} = \mathbf{N}_b \mathbf{a}_b \quad (5.6)$$

where

$\mathbf{N}_b$ – the Ritz shape function matrix for the contact region

$\mathbf{a}_b$ – the unknown Ritz parameter vector for the contact region

The potential energy of the bottom plate contact region can be expressed as follows

$$\begin{aligned} \Pi_p = & \frac{Et}{2(1-\nu^2)} \int_0^b \int_0^{2\pi} \left[\epsilon_{r'}^2 + 2\nu\epsilon_{r'}\epsilon_{\theta'} + \epsilon_{\theta'}^2 + \frac{1-\nu}{2}\gamma_{r'\theta'}^2 \right] r' dr' d\theta' \\ & - \int \int q(\cos \theta' u'_r - \sin \theta' u'_\theta) r' dr' d\theta' \end{aligned}$$

where q is the uniform friction between the contact bottom plate and the support foundation. From the stationary process (Equation 3.1), and assuming M_b is the number of unknown Ritz coefficients for the bottom plate contact region, stationarity of the potential energy function

$$\frac{\partial \Pi_b}{\partial a_{b,i}} = 0 \quad \text{for } i = 1, 2, \dots, M_b$$

leads to

$$\begin{aligned} \frac{\partial \Pi_b}{\partial a_{b,i}} = & \int \int \frac{Et}{1-\nu^2} \left\{ (\epsilon_{r'} + \nu \epsilon_{\theta'}) \frac{\partial \epsilon_{r'}}{\partial a_{b,i}} + (\epsilon_{\theta'} + \nu \epsilon_{r'}) \frac{\partial \epsilon_{\theta'}}{\partial a_{b,i}} + \right. \\ & \left. \frac{1-\nu}{2} \gamma_{r'\theta'} \frac{\partial \gamma_{r'\theta'}}{\partial a_{b,i}} \right\} r' dr' d\theta' - \int \int q \frac{\partial (\cos \theta' u'_r - \sin \theta' u'_\theta)}{\partial a_{b,i}} r' dr' d\theta' = 0 \end{aligned} \quad (5.7)$$

Considering the plane stress problem as a particular case of the von Karman plate, thus using the kinematic relationships of the inplane strains and displacements given in Equation 3.2 by neglecting the nonlinear terms, Equation 5.7 gives a set of M_b simultaneous algebraic equations which can then be solved to determine the unknown Ritz parameters $\mathbf{a}_b$

$$\mathbf{K}_b \mathbf{a}_b = \mathbf{R}_b \quad (5.8)$$

$\mathbf{K}_b$ – stiffness matrix of contact bottom plate

$\mathbf{R}_b$ – loading vector of contact bottom plate

As the Ritz shape functions in the uplifted region are defined in normalized coordinates (ξ, η) given in Equations 3.23 and 3.24, the inplane displacements at the contact boundary in terms of u_r and u_θ are given as follows (from Equation 5.1)

$$\begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \begin{bmatrix} \mathbf{N}^r(-1, \eta) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}^\theta(-1, \eta) & \mathbf{0} \end{bmatrix} \mathbf{a} = \bar{\mathbf{N}}^b \mathbf{a} \quad (5.9)$$

Correspondingly, the displacements at the contact boundary given in terms of u'_r and u'_θ can be expressed as follows (from Equation 5.8)

$$\begin{bmatrix} u'_r \\ u'_\theta \end{bmatrix} = \bar{\mathbf{N}}_b \mathbf{a}_b \quad (5.10)$$

Considering the displacement continuity at the interface between the uplifted and contact regions, the following relationship along the interface can be obtained

$$\begin{bmatrix} u'_r \\ u'_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta - \theta') & -\sin(\theta - \theta') \\ \sin(\theta - \theta') & \cos(\theta - \theta') \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \mathbf{N}'_c \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \quad (5.11)$$

From Equations 5.9-5.11, the following relation between the Ritz coefficients $\mathbf{a}$ and $\mathbf{a}_b$ can be derived

$$\bar{\mathbf{N}}_b \mathbf{a}_b = \bar{\mathbf{N}}'_c \bar{\mathbf{N}}^b \mathbf{a} \quad (5.12)$$

which implies that

$$\mathbf{a}_b = \mathbf{N}_c \mathbf{a} \quad \mathbf{N}_c = (\bar{\mathbf{N}}_b^t \bar{\mathbf{N}}_b)^{-1} \bar{\mathbf{N}}_b^t \bar{\mathbf{N}}'_c \bar{\mathbf{N}}^b \quad (5.13)$$

where $\mathbf{N}_c$ is the transformation matrix from $\mathbf{a}$ to $\mathbf{a}_b$. The notation t in $\bar{\mathbf{N}}_b^t$ denotes transpose of the matrix.

Substituting Equation 5.13 into Equation 5.8 and premultiplying both sides by $\mathbf{N}_c^t$, the equilibrium equation for the contact region expressed in terms of the uplifted region Ritz coefficients $\mathbf{a}$ can be written as follows

$$\bar{\mathbf{K}}_b \mathbf{a} = \bar{\mathbf{R}}_b \quad (5.14)$$

where

$$\bar{\mathbf{K}}_b = \mathbf{N}_c^t \mathbf{K}_b \mathbf{N}_c \quad \bar{\mathbf{R}}_b = \mathbf{N}_c^t \mathbf{R}_b$$

5.3 ASSEMBLED MODEL OF TANK SHELL

The derivations of the Ritz shapes functions for the tank shell have been presented in section 4.2. The displacements of the tank shell can be expressed as follows in matrix form using the derived Ritz shape functions

$$\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{N}'_s \mathbf{a}'_s \quad (5.15)$$

where

$\mathbf{N}'_s$ — the Ritz shape function matrix for the tank shell

$\mathbf{a}'_s$ — the unknown Ritz parameter vector corresponding to the degree of

freedom at the bottom and top of the tank shell

The potential energy of the shell can be expressed as follows

$$\begin{aligned} \Pi_s = & \int \int \frac{D}{2} \left\{ (\nabla^2 w)^2 - 2(1 - \nu) \left[\frac{\partial^2 w}{\partial z^2} \frac{\partial^2 w}{R^2 \partial \theta^2} - \left(\frac{\partial^2 w}{R \partial z \partial \theta} \right)^2 \right] \right\} R dz d\theta \\ & + \int \int \frac{Et}{2(1 - \nu^2)} \left\{ \left(\frac{\partial u}{\partial z} \right)^2 + 2\nu \frac{\partial u}{\partial z} \frac{1}{R} \left(w + \frac{\partial v}{\partial \theta} \right) \right. \\ & \left. + \left(\frac{w}{R} + \frac{\partial v}{R \partial \theta} \right)^2 + \frac{1 - \nu}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial u}{R \partial \theta} \right)^2 \right\} R dz d\theta - \int \int p_r w R dz d\theta \end{aligned} \quad (5.16)$$

Most thin walled cylindrical steel tanks have ring girders on the top rim and along the tank height. These ring girders have stiffening effects on the flexibility of the tank shell. In this formulation, it is assumed that they offer no resistance to any out-of-plane deformations, including rotation, because of their small width compared to the radius of the cylindrical shell. Derivation of the stiffness contribution from the ring girders is based on shape functions which are compatible with the displacement of the tank shell.

The potential energy of the ring girder Π_g can be expressed as follows

$$\Pi_g = \int_0^{2\pi} \left\{ EA \left(\frac{\partial v}{R \partial \theta} + \frac{w}{R} \right)^2 + EI_x \left(\frac{\partial^2 w}{R^2 \partial \theta^2} \right)^2 + EI_y \left(\frac{\partial^2 u}{R^2 \partial \theta^2} \right)^2 \right\} \frac{R}{2} d\theta \quad (5.17)$$

The tank roof, which protects the inside liquid from the atmosphere, is modeled as a plane stress disc. The Ritz shape functions of the roof are the same as the shapes for the bottom plate contact region derived in Chapter 3. Denoting the displacements over the roof by u' and v' , the roof displacements can be expressed by the Ritz coefficients $\mathbf{a}'$, as the following matrix form

$$\mathbf{u}' = \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \mathbf{N}_b \\ \mathbf{0} \end{bmatrix} \mathbf{a}', \quad (5.18)$$

where

$\mathbf{N}_b^r$ — the Ritz shape function matrix as the same formulation as in Equation 5.3.

$\mathbf{0}$ — zero matrix corresponding to the degree of freedoms at the bottom and u , $\partial w/\partial z$ at the top.

Applying the minimization procedure of the potential energy theorem (Equation 3.1) and summing the stiffness contribution from the ring girders and roof by the direct stiffness procedure, the influence of the roof and ring girders is included in the tank shell formulation in the assembled matrix equilibrium equations

$$\mathbf{K}'_s \mathbf{a}'_s = \mathbf{R}'_s \quad (5.19)$$

$\mathbf{K}'_s$ — stiffness matrix of shell

$\mathbf{R}'_s$ — loading vector

By static condensation, all the degrees of freedom at the top rim of the cylindrical shell can be condensed out, leaving only those at the tank base. Equation 5.19 is then rewritten as follows

$$\mathbf{K}_s \mathbf{a}_s = \mathbf{R}_s \quad (5.20)$$

$\mathbf{K}_s$ — condensed stiffness matrix of shell

$\mathbf{R}_s$ — condensed loading vector

$\mathbf{a}_s$ — condensed degrees of freedom at base rim

Then displacements over the tank shell can be expressed by the condensed Ritz coefficients $\mathbf{a}_s$

$$\mathbf{u} = \mathbf{N}_{sb} \mathbf{a}_s \quad (5.21)$$

Using the Ritz shape functions defined in Chapter 3 for the bottom plate uplifted region, the displacements at the base rim expressed in terms of the base plate uplifted region displacement components u_r , u_θ and w are given in the matrix form as follows

$$\begin{bmatrix} u_r \\ u_\theta \\ -w \\ -\frac{\partial w}{\partial r} \end{bmatrix} = \begin{bmatrix} \mathbf{N}^r(1, \eta) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}^\theta(1, \eta) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}^w(1, \eta) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \partial \mathbf{N}^w(1, \eta)/\partial r \end{bmatrix} \mathbf{a} = \bar{\mathbf{N}}_{bs} \mathbf{a} \quad (5.22)$$

The displacements at the base rim expressed in terms of the condensed degrees of freedom $\mathbf{a}_s$ of the tank shell are given as follows

$$\begin{bmatrix} w \\ v \\ u \\ \frac{\partial w}{\partial z} \end{bmatrix} = \bar{\mathbf{N}}_{sb} \mathbf{a}_s \quad (5.23)$$

where $\bar{\mathbf{N}}_{sb}$ is $\mathbf{N}_{sb}$ in Equation 5.21 at the base rim.

Because the displacements from the tank shell and the bottom plate must be continuous at the tank base, the following relationship must be satisfied

$$\bar{\mathbf{N}}_{sb} \mathbf{a}_s = \bar{\mathbf{N}}_{bs} \mathbf{a} \quad (5.24)$$

which implies

$$\mathbf{a}_s = (\bar{\mathbf{N}}_{sb}^t \bar{\mathbf{N}}_{sb})^{-1} \bar{\mathbf{N}}_{sb}^t \bar{\mathbf{N}}_{bs} \mathbf{a} = \mathbf{N}_s \mathbf{a} \quad (5.25)$$

where $\mathbf{N}_s$ is the transformation matrix from $\mathbf{a}$ to $\mathbf{a}_s$

Substituting Equation 5.24 into Equation 5.20 and premultiply both sides by the transposed matrix $\mathbf{N}_s^t$, the following stiffness equation for the tank shell response can be obtained

$$\bar{\mathbf{K}}_s \mathbf{a} = \bar{\mathbf{R}}_s \quad (5.26)$$

where

$$\bar{\mathbf{K}}_s = \mathbf{N}_s^t \mathbf{K}_s \mathbf{N}_s \quad \bar{\mathbf{R}}_s = \mathbf{N}_s^t \mathbf{R}_s$$

5.4 ASSEMBLED MODEL OF THE COMBINED TANK SYSTEM

The tangent stiffness Equations 5.5, 5.14 and 5.26 are combined together following the direct stiffness procedure. The combined equilibrium equation becomes

$$\bar{\mathbf{K}}\mathbf{a} = \bar{\mathbf{R}} \quad (5.27)$$

where $\bar{\mathbf{K}}$ is the tangent stiffness matrix for the combined tank system.

In formulating the analysis model of the tank system, there are no displacement restriction boundary support conditions imposed in the horizontal direction. Resistance to lateral tank base displacement in the horizontal direction is provided by the frictional force between the contact bottom plate and support foundation. Therefore, rigid body motion in horizontal direction of the tank system is included in Equation 5.27. The stiffness matrix $\mathbf{K}$ is singular. In order to eliminate the rigid body motion, without loss of generality, the horizontal motion at the center of the tank is assumed to be zero. As discussed in Chapter 3, the displacements in the contact region of the bottom plate is obtained by superposition of a fixed boundary solution under uniform load and a homogeneous solution with specific boundary displacements. Substituting Equation 5.13 into Equations 5.6, and evaluating the displacement at the bottom plate center

$$\mathbf{N}_b(b_c, 0)\mathbf{N}_c\mathbf{a} = \begin{bmatrix} d_0 \\ 0 \end{bmatrix} \quad (5.28)$$

where b_c is the distance between the bottom plate center and center of the circular contact region on the excitation axis, as shown in Figure 5.1, then $\mathbf{N}_b(b_c, 0)$ is the Ritz shape matrix for the contact region evaluated at the center of the tank. In Equation 5.28, d_0 is the lateral displacement at the bottom center. Due to symmetry, displacement perpendicular to the excitation direction is zero, the second equation in Equation 5.28 is automatically satisfied. Expanding the first equation in Equation 5.28 as follows

$$N_1a_1 + N_2a_2 + \dots + N_{M-1}a_{M-1} + N_Ma_M = d_0 \quad (5.29)$$

which can be interpreted as a condition relating the Ritz coefficients a_1 to a_M . A new reduced Ritz vector $\bar{\mathbf{a}}$ with only $(M - 1)$ coefficients can be defined

$$\bar{\mathbf{a}} = [a_1 \ a_2 \ \dots \ a_{M-1}]^t \quad (5.30)$$

From Equations 5.29 and 5.30, the relation between $\mathbf{a}$ and $\bar{\mathbf{a}}$ can be established

$$\mathbf{a} = \mathbf{N}_a \bar{\mathbf{a}} + \mathbf{d}_0 \quad (5.31)$$

where

$$\mathbf{N}_a = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots \\ \dots & & & \\ 0 & \dots & 0 & 1 \\ -N_1/N_M & -N_2/N_M & \dots & -N_{M-1}/N_M \end{bmatrix}_{M \times (M-1)} \quad \mathbf{d}_0 = \begin{bmatrix} 0 \\ 0 \\ \dots \\ d_0/N_M \end{bmatrix}_{M \times 1} \quad (5.32)$$

Using the reduced Ritz coefficient vector, and substituting into Equation 5.27, the new reduced matrix equilibrium equation can be written as

$$\mathbf{K} \bar{\mathbf{a}} = \mathbf{R} \quad (5.33)$$

where

$$\mathbf{K} = \mathbf{N}_a^t \bar{\mathbf{K}} \mathbf{N}_a$$

$$\mathbf{R} = \mathbf{N}_a^t \bar{\mathbf{R}} - \mathbf{N}_a^t \bar{\mathbf{K}} \mathbf{d}_0 \quad (5.34)$$

The stiffness matrix $\mathbf{K}$ is positive definite and nonsingular. Solution of the nonlinear equilibrium equation given in Equation 5.34 is discussed in the next section.

5.5 NONLINEAR ITERATIONS

The nonlinearity of the unanchored tank uplift problem results from two factors, the von Karman plate theory of the uplifted region and the determination of the extent of the bottom plate separation from the foundation. A two-level iteration scheme is formulated to deal with the nonlinearity. For every iteration of contact bottom plate geometry, the nonlinear structural response is calculated by Newton-Raphson method under the given contact bottom plate geometry, while the contact bottom plate geometry is determined by the bisection method.

5.5.1 Iteration on nonlinear structural deformation

Given a contact bottom plate geometry, the nonlinear structural deformation is determined by Newton-Raphson iteration. The Newton-Raphson is summarized as follows (Figure 5.3)

$$\begin{aligned} \mathbf{K}' \Delta \bar{\mathbf{a}}^{i+1} &= \mathbf{R} - \mathbf{R}_c^i \\ \bar{\mathbf{a}}^{i+1} &= \bar{\mathbf{a}}^i + \Delta \bar{\mathbf{a}}^{i+1} \end{aligned} \quad (5.35)$$

where $\mathbf{R}_c^i$ is the structural internal resistant force vector corresponding to $\bar{\mathbf{a}}^i$.

The convergence of the iteration is defined by norm measure of the maximum error in the all coefficients of the displacement and rotation. In order to have the same unit in the evaluation of the error norm for displacement and rotation, the errors in rotations are multiplied by the radius of the tank, R . An error norm can then be defined and used for convergence check:

$$\|\Delta \bar{\mathbf{a}}^{i+1}\| \leq \epsilon \quad (5.36)$$

where ϵ is the tolerance.

5.5.2 Iteration on contact region

As one of the basic assumptions adopted in the present study as discussed in Chapter 3, the shape of the contact portion of the bottom plate is assumed to be circular. As the bottom plate is supported by rigid foundation, the bending moment at the contact-uplift boundary of the bottom plate is always zero. The objective of the iteration on the contact region is then to identify two points on the contact-uplift interface which satisfy the iteration condition of zero bending moment. Bisection method which guarantees convergence is used[70] (Figure 5.4) to iterate for the correct bottom plate uplift geometry.

At the iteration step i , assuming $r_c^{i,1}$ and $r_c^{i,2}$ are two points on the contact side of the bottom plate, and the corresponding bending moments are $M_c^{i,1}$ and $M_c^{i,2}$ respectively, such that $M_c^{i,1}M_c^{i,2} \leq 0$, and $r_u^{i,1}$, $r_u^{i,2}$, $M_u^{i,1}$ and $M_u^{i,2}$ are similar coefficients on the uplifted side of the bottom plate, the new estimations of r_c^{i+1} and r_u^{i+1} are obtain by

$$\begin{aligned} r_c^{i+1} &= \frac{r_c^{i,2} + r_c^{i,1}}{2} \\ r_u^{i+1} &= \frac{r_u^{i,2} + r_u^{i,1}}{2} \end{aligned} \tag{5.37}$$

Using the new r_c^{i+1} and r_u^{i+1} as the new contact geometry, the response M_c^{i+1} and M_u^{i+1} can be calculated by Newton-Raphson method. The new $r_c^{i+1,1}$ and $r_c^{i+1,2}$ are selected from $M_c^{i,1}$, $M_c^{i,2}$ and M_c^{i+1} so that $M_c^{i+1,1}M_c^{i+1,2} \leq 0$. Thus the new $r_u^{i+1,1}$ and $r_u^{i+1,2}$ are selected by the same way as the $r_c^{i+1,1}$ and $r_c^{i+1,2}$.

Convergence is defined by comparing the errors with a preset tolerance as follows

$$\begin{aligned} |r_c^{i+1} - r_c^i| &\leq \epsilon \\ |r_u^{i+1} - r_u^i| &\leq \epsilon \end{aligned} \tag{5.38}$$

where ϵ is a given tolerance.

Figure 5.1: Displacement continuity at the boundary

Figure 5.2: Ring Girder of Tank Shell

Figure 5.3: Newton-Raphson iteration

Figure 5.4: Bisection method

Chapter 6

STATIC TILT ANALYSIS

In this chapter, static tilt analysis results are presented and compared with experimental measurements obtained for a broad 1/3 scale welded aluminum model tank tested in previous studies[25] at Earthquake Engineering Research Center of the University of California at Berkeley. The open top aluminum model tank of 12 ft.(3.66 m) in diameter, 6 ft.(1.83 m) high, with 5 ft.(1.52 m) depth of water, is tilted to an angle with the horizontal plane which varies from 0 to 16 degrees at increments of 4 on a rigid support platform. The construction detail of the tank is shown in Figure 6.1.

The tank system is analyzed by modelling the tank using the Ritz energy approach presented in earlier chapters. The number of degrees of freedom, that is the number of Ritz shape functions considered in the analysis model, is selected based on the requirement in achieving convergence in the analysis results and maintaining displacement continuity at the boundaries between different modelling substructure elements.

For the uplifted region of the bottom plate, the displacements are discretized by two types of boundary Ritz shape functions, the displacement boundary shapes and rotational boundary shapes. The displacement boundary shapes include shape functions of either vertical or inplane displacement at the base rim, whereas the rotational shapes involve rotational displacement at the boundary. Two kinds of boundary shapes and one kind of interior shape are chosen as Ritz shape functions

in the uplift region of bottom plate.

The Ritz shape functions for contact region of bottom plate are approximated by up to $\cos 3\theta$ or $\sin 3\theta$, which can correct any displacement discrepancy from the solution of a disc with uniform friction in Equation 3.29, Chapter 3. They can also give smooth continuity of displacements and stresses at the boundary between contact and uplifted regions of the bottom plate.

The Ritz shape functions chosen in the modelling of the shell allow up to $\cos 4\theta$ or $\sin 4\theta$ type of variation in the displacements of the cylindrical shell, which is shown from the results in this chapter that the displacements and stresses are smooth at the base rim of tank shell.

Analytical results for the 12 degrees tilt angle case are presented. These are correlated with recorded test data obtained in previous studies. The analysis results are shown in two groups: the results of the bottom plate, and those of the tank shell. The orientation and definitions of terminology commonly used throughout the discussions are described in Figure 6.2. The location of the contact-uplift interface is plotted by the dotted line within the bottom plate. The model tank is tilted to the north, which is shown as the left side in Figure 6.2. Because of symmetry, only the eastern half of the results of tank are presented. These are shown as the upper half in the configuration of the model tank in Figure 6.2. The uplifted side of the model tank is shown as the right hand side in Figure 6.2, while the contact side is on the left. The north-south tilting axis is also referred to as the uplift axis in the following discussion. The angle θ is measured counterclockwise from the uplift axis on the uplifted side. The uplift extent along the uplift axis is denoted by the symbol e .

6.1 BOTTOM PLATE RESULTS

For a static tilt angle of 12 degrees with the horizontal support platform, the displacements, membrane stresses, bending stresses, and geometric stiffness index of the bottom plate which represents the effect of the membrane action in the uplifted region

are presented in the following sections.

6.1.1 Bottom plate vertical displacements along radial direction

The bottom plate vertical displacements in the uplifted region along the radial directions at $\theta = 0^\circ$ (uplift-axis), $\theta = 45^\circ$, $\theta = 90^\circ$, and $\theta = 180^\circ$ (contact-axis) are shown in Figure 6.3. The scale of the vertical displacement along the contact-axis is much larger than that on the uplift side in Figure 6.3(a), because of the much smaller magnitude of separation on the contact side.

As shown in Figure 6.3(a), the edge rotates upwards because of the uplifting of the tank shell, and the maximum bottom plate vertical uplift occurs at the rim. The curvature at the edge rotates downwards because of the liquid pressure acting downwards on the uplifted portion of the bottom plate and the liquid pressure pushing against the tank shell. The deformed shape of the bottom plate on contact-axis shown in Figure 6.3(b) is different from that obtained in the uplifted side shown in Figure 6.3(a). Separation of the bottom plate on the contact side is caused by the liquid pressure acting outwards on the tank shell, causing the contact edge to rotate outwards. To maintain continuity at the edge near the tank base rim on the contact side, a narrow region of the bottom plate has to separate from the foundation due to the outward rotation of the edge. The extent of the bottom plate separation on the contact side is much smaller than that on the uplifted side because a significantly greater amount of uplift is obtained on the uplifted side under the given tilt angle (12°), and the downward liquid pressure on the contact side is larger than that on the uplifted side. The magnitude of the vertical separation on the uplifted side, with a maximum of about 0.12 in, is much larger than that on the contact side, with a maximum of about 0.0005 in. These differences are because the rotation of the edge on the contact side is much less significant than the effect of uplift tendency from the tank shell on the uplifted side.

The experimental measurements of the vertical uplift are also plotted in Figure 6.3(a). The analytical result on the extent of separation on the uplifted side agrees well with the measured data. The analytical results on the vertical displacement are less than those observed in the experiment. The errors are from about 1% to 30%. As discussed in the next section, these discrepancies may be caused by yielding in some of rivets reducing the stiffness of the bottom plate.

6.1.2 Displacements around base rim

The vertical, radial and tangential displacements around the tank bottom rim are presented in Figure 6.4.

The vertical displacement around the bottom rim plotted in Figure 6.4(a) shows that the tank shell has warped significantly out-of-plane at its base during uplift. This is observed by comparing the vertical displacement distribution shown in Figure 6.4(a) with a $\cos\theta$ type of variation. The deformed base rim does not lie on in the same plane as before uplift. The uplifted side has a greater uplift separation gradient than that on the contact side.

Resisted by friction in the center portion of the bottom plate (contact region) which prevents the tank from slipping, there is significant translational displacement at the base rim towards the tilting direction. On the contact side, the base rim moves away from the tank center with a positive radial displacement, whereas that on the uplifted side moves towards the center with a negative radial displacement as shown in Figure 6.4(b). The flexibility of the base rim in the inplane direction has significant effect on the stress distribution in bottom plate.

The tangential displacement of the bottom rim is plotted in Figure 6.4(c). Due to symmetry, zero tangential displacement is observed at the uplift axis. The maximum tangential displacement occurs at about the position $\theta = 90^\circ$. This tangential displacement around the base rim generally has a $\sin\theta$ type of distribution.

6.1.3 Bottom plate membrane stresses along uplift axis

The bottom plate membrane stresses on the uplifted side and on the contact side along the uplift axis are respectively shown in Figures 6.5(a) and 6.5(b).

Significantly different stress distributions are observed between the contact region and separated region of the bottom plate. Figure 6.5 shows that both radial and tangential membrane stresses reverse their signs from positive on contact side to negative on uplift side. Resisted by uniform friction acting in the contact region of the bottom plate which prevents the tilted tank from slipping towards the tilt direction, the lateral translational displacement of the base rim pulls at the bottom plate on the contact side, while it pushes against the tank bottom on the uplifted side. This causes tensile stresses on the contact side as shown in Figure 6.5(b), and the reversal of membrane stresses on the uplifted side which become compressive.

The stress distribution in the separated region on the uplifted side is also observed to be much different from that on the contact side. As mentioned before, the vertical displacement on the contact side is essentially very small. The membrane stresses in the separated region on the contact side are of the same magnitudes as those in the adjacent contact region. Because of the influence of large uplift, the membrane stresses in the separated region on the uplifted side have large jumps from the stresses in the adjacent contact region. The location of the peak membrane stresses shown in Figure 6.5(a) correlates with the location with the largest vertical displacement gradient in the radial direction as observed in Figure 6.3(a).

Comparing the analytical results with the experimental observations which are also plotted in Figure 6.5, the overall variations (distribution tendency) are similar, but the magnitudes are different. This discrepancy may be explained by the difference in lateral support as provided by the uniform friction assumption in the analysis model here, and the actual center point bolted fixity in the experiment.

Discussion of discrepancies between analysis and experiment observations

From the construction details of the model tank shown in Figure 6.1, the bottom plate is constructed by rivetting an outer annular plate to an inner circular piece. The rivets

are spaced at 3" c-c staggered. However, in the analysis, the bottom plate is assumed to be a completely smooth and continuous plate in one piece. It is found from analysis that if the connection between the girder and the bottom plate were strong enough, the analytical results should agree better with the experimental observations. The construction details, the location of gages and estimates of the stresses in the rivets are shown in Figure 6.6. The strain gages are located on the uplift axis. Assuming that the stresses are linearly distributed between the measuring gages, the stresses at the mid-point radial location between two rivets located on the excitation axis are here taken as an estimate of the stresses in both rivets. On the symmetric excitation axis with zero inplane shear stress and maximum principal stresses, the estimated radial stress σ_r is 2.302 ksi, and the estimated tangential stress σ_θ is 1.638 ksi, as shown in the Figure 6.6. The stress τ_v which is transmitted to the rivets can be determined as follows

$$\tau_v = \sqrt{\sigma_r^2 + \sigma_\theta^2} = 2.83 \text{ ksi} \quad (6.1)$$

The shear force $\bar{F}$ resisted by each rivet is then estimated to be

$$\bar{F} = \tau_v a_v t / 2$$

where a_v is the distance between the rivets (3 in.), t is the thickness of the bottom plate (0.08 in.). The shear force is divided by two because there are two lines of rivets.

$$\bar{F} = (2830)(3)(0.08)/2 = 339 \text{ (lb)}$$

The shear stress $\bar{\tau}$ estimated in each rivet is therefore

$$\bar{\tau} = \frac{\bar{F}}{A_v} = \frac{339}{\pi(3/32)^2/4} = 49.1 \text{ ksi}$$

where A_v is the area of the cross section of one rivet.

Comparing this with the shear yield stress for aluminium which is about 7 – 20 ksi[71], the material of the rivets must have yielded. After yielding of the rivets, the stresses in bottom plate are redistributed, which may cause the discrepancies between the analysis results and the experiment recorded data. The yielding of the rivets may also lead to reduction of the stiffness of the bottom plate, resulting in higher uplift displacements observed in the experiments than that obtained in analysis.

6.1.4 Membrane principal stresses

Contours of the bottom plate principal membrane stresses and the principal direction are plotted in Figure 6.7. The larger principal stress is denoted by σ_1 . The principal direction is defined as the angle between the direction of σ_1 and the positive uplift axis measured in the counterclockwise direction.

As shown in Figure 6.7, the contour lines of the principal stress are heavily concentrated within in the uplifted area between $\theta = 0^\circ$ and $\theta = 90^\circ$. The stress levels in this concentrated area are much higher than that in other parts of the bottom plate. Due to the partially uplifting of the bottom plate, most of the variations of the membrane stresses in the bottom plate are concentrated in the uplifted area on the uplifted side. The significantly higher membrane stresses in the uplifted area is an important parameter with a significant influence on the uplift mechanism of the bottom plate. This is discussed further in a later section using a defined geometric stiffness index.

6.1.5 Bending stresses along radial directions

The bottom plate bending stresses in the uplifted region along the radial direction at $\theta = 0^\circ$ (uplift-axis), $\theta = 45^\circ$, $\theta = 90^\circ$, and $\theta = 180^\circ$ (contact-axis) are shown in Figure 6.8.

The experimental measurements on the uplifted axis ($\theta = 0^\circ$) are also plotted in this figure. With the exception of the experimental data recorded by the gage near the rivet connection, the analytical results agree well with the experimental observations.

6.1.6 Importance of membrane action

In order to evaluate the relative importance of membrane action compared to flexural mechanism in the deformation of the bottom plate, a geometric stiffness index function $G(r, \theta)$ is defined here.

The governing differential equations of von Karman plate theory for the nonlinear

uplifted region are given in Equations 3.3 - 3.5. The first two equations, Equations 3.3 and 3.4, represent the inplane equilibrium, whereas Equation 3.5 represents the vertical equilibrium. For the purpose of derivation, the vertical equilibrium equation is rewritten as

$$\begin{aligned} & \frac{\partial^2 M_r}{\partial r^2} + \frac{2}{r} \frac{\partial^2 M_{r\theta}}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial^2 M_\theta}{\partial \theta^2} + \frac{2}{r} \frac{\partial M_r}{\partial r} - \frac{1}{r} \frac{\partial M_\theta}{\partial r} + \frac{2}{r} \frac{\partial M_{r\theta}}{\partial \theta} \\ & + \frac{\partial^2 w}{\partial r^2} N_r + \left[\frac{\partial w}{r \partial r} + \frac{\partial^2 w}{r^2 \partial \theta^2} \right] N_\theta + \left[\frac{2}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{2}{r^2} \frac{\partial w}{\partial \theta} \right] N_{r\theta} = p_z \end{aligned} \quad (6.3)$$

where N_r , N_θ and $N_{r\theta}$ are the midplane force resultants, M_r , M_θ and $M_{r\theta}$ are the corresponding midplane moment resultants, and p_z is the liquid pressure loading.

The first line in Equation 6.3 denotes the contribution from the bending theory (transversal shear), while the second line gives the effect of membrane action. Two variables, $\bar{B}$ and $\bar{M}$, are denoted as follows

$$\begin{aligned} \bar{B} &= -\frac{1}{p_z} \left[\frac{\partial^2 M_r}{\partial r^2} + \frac{2}{r} \frac{\partial^2 M_{r\theta}}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial^2 M_\theta}{\partial \theta^2} + \frac{2}{r} \frac{\partial M_r}{\partial r} - \frac{1}{r} \frac{\partial M_\theta}{\partial r} + \frac{2}{r} \frac{\partial M_{r\theta}}{\partial \theta} \right] \\ \bar{M} &= -\frac{1}{p_z} \left[\frac{\partial^2 w}{\partial r^2} N_r + \left[\frac{\partial w}{r \partial r} + \frac{\partial^2 w}{r^2 \partial \theta^2} \right] N_\theta + \left[\frac{2}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{2}{r^2} \frac{\partial w}{\partial \theta} \right] N_{r\theta} \right] \end{aligned}$$

and

$$\bar{B} + \bar{M} = 1 \quad (6.4)$$

The $\bar{B}$ presents the effect of bending mechanism, and the $\bar{M}$ denotes the contribution from membrane action. Then the relative importance of membrane action in the overall vertical resistance mechanism can be obtained by defining a geometric stiffness index G as follows

$$G(r, \theta) = \frac{\bar{M}}{|\bar{B}| + |\bar{M}|} \quad (6.5)$$

The possible range of the geometric stiffness index G is from -0.5 to 1.0 . The possibility of a negative values for G is explained in Figure 6.9. For the purpose of demonstration, the one dimension case is shown here. When a compressive axial force

combined with a positive curvature or a tensile axial force combined with a negative curvature, the value of the geometric stiffness index G is negative.

The variations of the bottom plate geometric stiffness index G along the radial directions at $\theta = 0^\circ$ (uplift-axis), $\theta = 45^\circ$, $\theta = 90^\circ$, and $\theta = 180^\circ$ (contact-axis) are shown in Figure 6.10. The influence of the membrane stresses to overall vertical equilibrium is contributed by two factors, the magnitudes of the membrane stresses and the slope variations of the vertical displacements of the bottom plate. With large membrane stresses and significant changes in slopes on uplift-axis ($\theta = 0^\circ$), the geometric stiffness index G has the largest distribution. With more gentle slopes and smaller membrane stresses, the index G along the radial direction $\theta = 90^\circ$ has the smallest distribution among the four curves plotted. Around the rim of the bottom plate, the geometric stiffness index G is negative because tensile stresses are combined with negative curvatures around the edge. As shown in Figure 6.10, the maximum index G is about 0.6 and with significant changes in magnitude over the uplifted area. This indicates that the membrane action has a significant effect on the uplift behaviour of unanchored tank and can not be neglected.

6.2 TANK SHELL RESULTS

The analytical results obtained for the cylindrical tank shell during static tilt uplift and the comparison with experimental data are presented in this section. The results are arranged in the order of displacements, membrane stresses and bending stresses.

6.2.1 Tank shell displacements

The tank shell radial, tangential and axial displacements at several cross sections are shown in Figures 6.11 and 6.12.

The displacements at 0 in. height (base rim), 1.44 in. (1/50 of tank height), 2.88 in. (2/50 of tank height), and 5.76 in. (4/50 of tank height) are plotted in Figure 6.11. As shown in Figure 6.11(a), due to the strong resistance of the bottom plate,

the radial displacement at the bottom rim is very small, but it increases in magnitude significantly at the higher horizontal sections. The tank shell on the uplifted side (i.e. $\theta = 0^\circ$) moves inwards (negative radial displacement), whereas on the contact side (i.e. $\theta = 180^\circ$), it moves further outwards (positive radial displacement) for those cross sections near the tank bottom. However, the outward radial displacement on the contact side is larger in magnitude than the inward displacement on the uplifted side. This is because the liquid pressure acting on the tank wall pushes the shell outwards, cancelling some of the rocking effect of the cylindrical tank on the uplifted side due to the bottom uplift.

The corresponding tangential displacements of the tank shell are plotted in Figure 6.11(b). From the figure, it is noted that the peaks of the tangential displacements gradually shift their locations away from the section perpendicular to the exciting or uplift axis ($\theta = 90^\circ$) to the uplifted side ($\theta < 90^\circ$) at the higher cross sections in the shell. This indicates that the tank uplift mechanism induces the tank shell to displace more in the tangential direction on the uplifted side than on the contact side. This behavior together with the uneven distribution of radial displacement observed in Figure 6.11(a) indicate that the cylindrical tank shell undergoes significant out-of-round deformation during uplift. The results also indicate that the out-of-round deformation is more pronounced on the uplifted side than on the contact side.

There is not noticeable difference in the longitudinal or axial displacement of the tank shell at those cross sections near the bottom as shown in Figure 6.11(c). This is because for a thin shell, it is much stiffer in the axial direction than in the radial and tangential directions.

The displacements of the tank shell at 18 in. (quarter height), 36 in. (mid-height) and 72 in. (top) are given in Figure 6.12. As shown in Figure 6.12(a) and contrary to the behavior at sections close to the base shown in Figure 6.11(a), the radial displacements on the uplifted side are greater than that on the contact side at horizontal section, and even greater for the sections further away from the base. This change in the behavior of the tank shell is due to the uplifting of the base rim.

The tangential and axial displacement distributions shown in Figures 6.12(b) and 6.12(c) have similar characteristics as those near the base which are discussed earlier in Figures 6.11(b) and 6.11(c).

The radial displacements along the tank wall on the uplifted side ($\theta = 0^\circ$) is shown in Figure 6.13(a), whereas that on the contact side ($\theta = 180^\circ$) is given in Figure 6.13(b).

The deformed shapes of cross sections at 18 in (quarter height) 36 in (mid-height) and 72 in (top) are given in Figure 6.14.

Upon examination of Figures 6.12, 6.13 and 6.14, the tank wall on the uplifted side ($\theta = 0$) seems rotating with the uplifting base rim, but the tank wall on the contact side ($\theta = \pi$) seems still straight, the cross sections of the tank wall are out-of-round. That behavior may be explained as follows.

When the tank bottom partially uplifts from the foundation, if the tank base rim would remain a plane, response of the tank shell would look as a rigid body rotating about a contact point at $\theta = 180^\circ$ on the base rim. In such case, the magnitude of the radial displacement will be the same on the contact side as that on the uplifted side. However, as shown in Figure 6.11(c), a significant portion of the tank base rim remains in contact with the foundation, which prevents the shell on the contact side rotation while the shell on the uplifted side rotates with the uplifting base rim. If the perimeter of the tank section would reduce without restricting, the cross section of the tank shell would remain circular. But the perimeter of the cross section is hardly to change its length due to the strong inplane stiffness, the tank shells between the uplift side and the contact side are pushed outwards when the tank shell on the uplift side is moving inwards. The horizontal cross sections become more distorted for the higher cross sections as shown as Figure 6.14.

6.2.2 Tank shell membrane stresses

The tank shell membrane stresses at horizontal sections 0, 2 and 5.76 in. from the tank bottom are shown in Figure 6.15. The axial and hoop membrane stresses along

the tank height are presented in Figures 6.16 and 6.17, respectively. The experimental measurements are also plotted in Figures 6.16 and 6.17. Considering the uncertainty in the experiment and the analysis, the errors are within the acceptable ranges in most results. It seems that the errors for axial stresses are larger than that for hoop stresses. This is because the axial stresses are more sensitive to the base rim uplifting than the hoop stresses. The axial stress in the tank shell is highly influenced by axial deformation imposed at the base rim of the thin walled structure. Because of the difference in the analysis model and the actual test tank, the difference in the analysis results and the recorded test data are not unexpected. On other hand, the analytical hoop stresses, which are mainly influenced by the liquid pressure and the inplane displacements (radial displacement and tangential displacement) of the shell, agree better with the experimental measurements.

6.2.3 Tank shell bending stresses

The shell bending stresses on the several cross sections near the bottom of the tank, 0 in., 2 in. and 5.76 in., are given in Figure 6.18. It is shown that the bending mechanism is a local effect for the thin shell. For the distance of 5.76 in. about the bottom, the bending stresses almost vanish. In Figure 6.19, the bending stresses on the uplifted side ($\theta = 0^\circ$) and the contact side ($\theta = 180^\circ$) are presented. Once again, it is proved that the bending effect becomes neglectable for a small distance about the bottom.

Because the extremely thin tank structure is a very flexible, local deformation of the shell is greatly affected by the presence of the residual stresses induced during the fabrication process, particularly near the welded joint at the base rim. As shown in Figure 6.18, the errors of the bending stresses of the shell are expected near the bottom. The better agreement is obtained for the bending stresses far away from the base rim as shown in Figure 6.19.

DETAIL A

DETAIL B

Figure 6.1: Dimension and construction detail of model tank

Figure 6.2: Orientation and definitions of terminology

(a) vertical displacements on uplifted side

(b) vertical displacements on contact axis

Figure 6.3: Vertical displacements along radial directions

Figure 6.4: Displacements around bottom rim

Figure 6.5: Bottom plate membrane stresses along uplift axis

Figure 6.6: Estimation of stresses in rivets

Figure 6.7: Bottom plate principal membrane stresses and directions

(a) radial bending stress along radius

(b) tangential bending stress along radius

Figure 6.8: Bending stresses along radius directions

Figure 6.9: Negative geometric stiffness index G

Figure 6.10: Bottom plate geometric stiffness index G

(a) radial displacement

(b) tangential displacement

(c) axial displacement

Figure 6.11: Tank shell displacements at horizontal sections near the bottom

Figure 6.12: Tank shell displacements at horizontal sections at quarter-height, mid-height and top

(a) displacement (w) on uplift side

(b) displacement (w) on contact side

Figure 6.13: Tank shell radial displacements at vertical section on uplift axis

maximum is 0.19 in.

Figure 6.14: Tank shell out-of-round deformation at quarter-height, mid-height and top

(a) membrane axial stress

(b) membrane hoop stress

(c) membrane shear stress

Figure 6.15: Tank shell membrane stresses at horizontal sections near the bottom

(a) axial stress on uplift-axis

(b) axial stress on contact-axis

Figure 6.16: Tank shell membrane axial stresses along uplift axis

(a) hoop stress on uplift-axis

(b) hoop stress on contact-axis

Figure 6.17: Tank shell membrane hoop stresses along uplift axis

Figure 6.18: Tank shell bending stresses at horizontal sections near the bottom

(a) bending stress on uplift-axis

(b) bending stress on contact-axis

Figure 6.19: Bending stresses along uplift axis

Chapter 7

DYNAMIC MODEL OF UNANCHORED CYLINDRICAL TANKS

The response behaviour of liquid storage tanks subjected to strong ground motion is still not fully understood, especially in the case of flexible unanchored ones. During an earthquake, an unanchored tank uplifts in response to the overturning moment induced by the liquid inertia effect. The uplift mechanism of unanchored tanks is a very complex nonlinear problem. In this chapter, a direct time step integration procedure is developed to analyze the dynamic response behaviour of an uplifting unanchored tank. The analysis models presented in earlier chapters for static tilt analysis are extended to include the inertial effect for dynamic response analysis in this chapter.

7.1 DYNAMIC MODEL OF FLUID-TANK SYSTEM

An unanchored cylindrical liquid storage tank system is shown in Figure 7.1. Response of the tank structure to external loads is described by the motion of the middle surface of the tank system, which includes the cylindrical shell, the bottom plate, and also the roof plate in the case of a covered tank. As shown in Figure 7.1, u denotes the middle surface displacement in the z direction, while w and v_θ respectively denote those in the radial and tangential directions.

During an earthquake, vibration of the liquid generates hydrodynamic pressure

loading on the tank shell and the bottom plate. The system considered in this study is a flexible cylindrical tank, which may have several different thicknesses along the height, and may have a roof cover at the top and ring girders around the tank wall. The bottom of the tank is free to uplift from the foundation. The liquid is assumed to be incompressible, inviscid and has a free surface at the top. The damping effect of a floating roof on the sloshing motion of the liquid is not considered in this thesis.

From observation, the fluid-tank system has two distinct resonant frequency components:

1. Low frequency vibrations associated with the sloshing of the liquid
2. High frequency vibrations associated with the coupled vibration modes in which tank deformation takes place.

In order to capture the responses of the two distinctly different frequency vibration components, a new nonlinear time history analysis procedure is developed.

As discussed in chapter 2, the hydrodynamic pressure in a vibrating tank can be expressed in terms of the motion of the tank structure. The remaining problem in the formulation of the analysis model for the dynamic response analysis of an unanchored tank is then to formulate the equation of motion for the system considered. This equation of motion may be established using the Lagrange principle. It is highly nonlinear because of possible partial uplift of the tank bottom from the foundation during excitation.

Lagrange equation of motion

For a tank structure consisting of a tank shell, a bottom plate, a tank roof and ring girders at the top and bottom, it is assumed that the kinetic energy T , potential energy V , and virtual work δW can be expressed in terms of a series of generalized

coordinate $q_1, q_2, \dots, q_N$, and their derivatives with time $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_N$

$$\begin{aligned} T &= T(q_1, q_2, \dots, q_N, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_N) \\ V &= V(q_1, q_2, \dots, q_N) \\ \delta W &= Q_1 \delta q_1 + Q_2 \delta q_2 + \dots + Q_N \delta q_N \end{aligned} \quad (7.1)$$

where Q_i is the generalized loading corresponding to the generalized coordinate q_i . Equivalent to the Hamilton's principle, the Lagrange equation of motion is obtained as follows

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad (i = 1, 2, \dots, N) \quad (7.2)$$

Equation 7.2 represents a set of N simultaneous equations of motion for $i = 1, 2, \dots, N$

7.2 DYNAMIC ANALYSIS APPROACH

The dynamic equation of motion given in Equation 7.2 may be solved by direct time-step integration. However, because the uplift behaviour of the system is highly nonlinear, it is therefore not efficient to apply time step integration on the equation of motion of the multi-degree-of-freedom system represented by Equation 7.2. The numerical time step integration process may have convergence, stability and other numerical problems.

The exact response of a mathematical model is usually obtained by analyzing the complete system including all the degrees of freedom in the analysis process. However, the dynamic response of many systems can be approximately obtained with acceptable accuracy in terms of only a few generalized coordinates, the number of which is much smaller than the total number of degrees of freedom in the original system. In developing a more efficient approach in the present study, the Rayleigh-Ritz reduction of degrees of freedom is applied to the dynamic system represented by Equation 7.2. In this regard, the Rayleigh-Ritz method uses a set of assumed displacement shapes as the generalized coordinates. The selected displacement shapes

should satisfy all the geometric or essential boundary conditions of the system. Better accuracy can be obtained if the selected shapes satisfy some or all of the natural or force boundary conditions.

Possible sources of error in the numerical time integration of a nonlinear system may include the following [72]

1. Use of tangent stiffness instead of secant stiffness in the modeling of the response mechanism.
2. Delay in detecting transitions in the force-displacement relationship.
3. Insufficient number or inaccurate Rayleigh-Ritz shapes are employed in the analysis.

The selection of a suitably small time step Δt in the numerical time integration scheme can usually minimize some of these errors. But this is an inefficient and costly approach to improve the accuracy of the solution when some other alternatives are available.

The errors due to the use of tangent stiffness and the delay in detecting the occurrence of a transition in the force-displacement relationship in the uplift process may be eliminated by a process of iteration. Newton Raphson method is an useful procedure to eliminate the error due to use of the tangent stiffness. In addition, an effective iterative scheme is developed here to minimize the error due to changes in the support during uplift.

In order to obtain better accuracy on the result and to reduce the computing time, an integration scheme with variable time step, Δt , is developed in this study. Throughout the response at each time step, different Rayleigh-Ritz shapes are derived and used in the reduction of the multi-degree-of-freedom system to a generalized single-degree-of-freedom system. This is necessary because of the changing base support configuration for an uplifting tank.

In the Rayleigh-Ritz formulation, it is assumed that the response is given by $\psi v(t)$, where ψ is the generalized Rayleigh-Ritz shape function, and $v(t)$ is the generalized

coordinate. The generalized Rayleigh-Ritz shape ψ may be looked upon as a transformation matrix, or a displacement or strain distribution field over the tank structure, which reduces the multi-degree-of-freedom fluid-tank system into the generalized single-degree-of-freedom system.

The Lagrange equation of motion for the generalized coordinate v can then be rewritten as

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{v}} \right) - \frac{\partial T}{\partial v} + \frac{\partial V}{\partial v} = Q \quad (7.3)$$

For the tank structure, the kinetic energy T can be obtained by

$$T = \frac{1}{2} \int \bar{m} [(\dot{w} + \dot{x} \cos \theta)^2 + (\dot{v}_\theta - \dot{x} \sin \theta)^2 + \dot{u}^2] ds \quad (7.4)$$

where u denotes the displacement in the z direction, w and v_θ are respectively the displacements in the radial and tangential directions as shown in Figure 7.1, x is the horizontal earthquake ground displacement, and $\bar{m}$ is the mass distribution function of the tank structure. The integration in Equation 7.4 is evaluated over the entire tank structure.

In Ritz formulation, the following relations can be established for the displacements and velocities of the tank structure

$$\begin{aligned} u &= \bar{u} v & \dot{u} &= \bar{u} \dot{v} \\ v_\theta &= \bar{v}_\theta v & \dot{v}_\theta &= \bar{v}_\theta \dot{v} \\ w &= \bar{w} v & \dot{w} &= \bar{w} \dot{v} \end{aligned} \quad (7.5)$$

where $\bar{u}$, $\bar{v}_\theta$ and $\bar{w}$ are respectively the displacement components of the Ritz shape in the longitudinal, tangential and radial directions as shown in Figure 7.1. Using Equation 7.5, Equation 7.4 can be rewritten as follow

$$\begin{aligned}
T = & \frac{\dot{v}^2}{2} \int m[\dot{u}^2 + \dot{v}_\theta^2 + \dot{w}^2] ds \\
& + \dot{v} \int \dot{m}[\dot{x}\dot{w} \cos \theta - \dot{x}\dot{v}_\theta \sin \theta] ds \\
& + \frac{1}{2} \int m\dot{x}^2 ds
\end{aligned} \tag{7.6}$$

Neglecting the potential energy contributed by the body forces, the potential energy of the tank structure is given by

$$V = \frac{1}{2} \int \int \int (\sigma_z \epsilon_z + \sigma_r \epsilon_r + \sigma_\theta \epsilon_\theta + \tau_{z\theta} \gamma_{z\theta} + \tau_{zr} \gamma_{zr} + \tau_{r\theta} \gamma_{r\theta}) r \, dr \, d\theta \, dz \tag{7.7}$$

In Ritz formulation, the stresses σ_z , σ_r , σ_θ , τ_{zr} , $\tau_{z\theta}$, $\tau_{r\theta}$, and the strains ϵ_z , ϵ_r , ϵ_θ , γ_{zr} , $\gamma_{z\theta}$, $\gamma_{r\theta}$ can be expressed as follows

$$\begin{aligned}
\sigma_z &= \bar{\sigma}_z v & \epsilon_z &= \bar{\epsilon}_z v \\
\sigma_r &= \bar{\sigma}_r v & \epsilon_r &= \bar{\epsilon}_r v \\
\sigma_\theta &= \bar{\sigma}_\theta v & \epsilon_\theta &= \bar{\epsilon}_\theta v \\
\tau_{zr} &= \bar{\tau}_{zr} v & \gamma_{zr} &= \bar{\gamma}_{zr} v \\
\tau_{z\theta} &= \bar{\tau}_{z\theta} v & \gamma_{z\theta} &= \bar{\gamma}_{z\theta} v \\
\tau_{r\theta} &= \bar{\tau}_{r\theta} v & \gamma_{r\theta} &= \bar{\gamma}_{r\theta} v
\end{aligned} \tag{7.8}$$

where $\bar{\sigma}_z$, $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, $\bar{\tau}_{zr}$, $\bar{\tau}_{z\theta}$, $\bar{\tau}_{r\theta}$ and $\bar{\epsilon}_z$, $\bar{\epsilon}_r$, $\bar{\epsilon}_\theta$, $\bar{\gamma}_{zr}$, $\bar{\gamma}_{z\theta}$, $\bar{\gamma}_{r\theta}$ are the stresses and strains of the tank structure corresponding to the generalized Ritz shape function ψ . Substituting Equation 7.8 into Equation 7.7, the potential energy V can be rewritten as

$$\begin{aligned}
V &= \frac{v^2}{2} \int \int \int (\bar{\sigma}_z \bar{\epsilon}_z + \bar{\sigma}_r \bar{\epsilon}_r + \bar{\sigma}_\theta \bar{\epsilon}_\theta + \bar{\tau}_{z\theta} \bar{\gamma}_{z\theta} + \bar{\tau}_{zr} \bar{\gamma}_{zr} + \bar{\tau}_{r\theta} \bar{\gamma}_{r\theta}) r \, dr \, d\theta \, dz \\
&= v^2 \bar{V}
\end{aligned} \tag{7.9}$$

where $\bar{V}$ is the potential energy of the tank structure corresponding to the generalized Ritz shape ψ .

From Equations 7.6 and 7.9, it is easy to obtain the following

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{v}} \right) &= \ddot{v} \int \bar{m} [\bar{u}^2 + \bar{v}_\theta^2 + \bar{w}^2] ds \\ &\quad + \int \bar{m} \ddot{x} (\bar{w} \cos \theta - v_\theta \sin \theta) ds \\ \frac{\partial T}{\partial v} &= 0 \end{aligned} \quad (7.10)$$

$$\frac{\partial V}{\partial v} = 2 \bar{V} v$$

For convenience, the hydrodynamic pressure p_d derived in Equation 2.79 is restated here as follows

$$\begin{aligned} p_d = -\rho \sum_{m=0}^{\infty} \cos m\theta \left\{ \sum_{j=1}^{\infty} \left\{ \frac{dT_{jm}}{dt} I_m \left(\frac{\lambda_j r}{H} \right) \cos \left(\frac{\lambda_j z}{H} \right) \right. \right. \\ \left. \left. + \left[\frac{d\bar{T}_{jm}}{dt} e^{\frac{\zeta_{jm} z}{R}} + \frac{d\bar{T}'_{jm}}{dt} e^{-\frac{\zeta_{jm} z}{R}} \right] J_m \left(\frac{\zeta_{jm} r}{R} \right) \right\} \right\} \end{aligned}$$

Using the Ritz formulation, the above expression for p_d can be rewritten as follows

$$p_d = \bar{P}_d \ddot{v} + P'_d \quad (7.11)$$

where $\bar{P}_d \ddot{v}$ is the contribution to hydrodynamic pressure from the terms related to $\ddot{v}$, P'_d is the remaining part of the dynamic pressure due to the rigid body motion and free vibrational response of the vibrating liquid.

The virtual work done by δv is $Q\delta v$, where the generalized load Q is given as follows

$$\begin{aligned} Q &= \int_{shell} p_d \bar{w} ds + \int_{bottom} p_d \bar{u} ds \\ &= \ddot{v} \left[\int_{shell} \bar{P}_d \bar{w} ds + \int_{bottom} \bar{P}_d \bar{u} ds \right] \\ &\quad + \int_{shell} P'_d \bar{w} ds + \int_{bottom} P'_d \bar{u} ds \end{aligned} \quad (7.12)$$

where the subscripts, shell and bottom, denote that the integrals are evaluated accordingly over the tank shell and bottom plate.

Substituting Equation 7.10 and 7.12 into the Lagrange equation of motion given in Equation 7.3, the equation of motion in terms of the Ritz generalized coordinate can be obtained as follows

$$M^* \ddot{v} + K^* v = P^* \quad (7.13)$$

where

$$M^* = \int \bar{m}[\bar{u}^2 + \bar{v}_\theta^2 + \bar{w}^2] ds - \left[\int_{shell} \bar{P}_d \bar{w} ds + \int_{bottom} \bar{P}_d \bar{u} ds \right] \quad (7.14)$$

$$K^* = 2 V \quad (7.15)$$

where $\bar{V}$ is defined in Equation 7.9, and

$$P^* = \int_{shell} P'_d \bar{w} ds + \int_{bottom} P'_d \bar{u} ds - \int \bar{m}[\bar{x} \bar{w} \cos \theta - \bar{x} \bar{v}_\theta \sin \theta] ds \quad (7.16)$$

To include energy dissipation mechanism in the response or damping of the fluid-tank system, Equation 7.13 can be modified as follows

$$M^* \ddot{v} + C^* \dot{v} + K^* v = P^* \quad (7.17)$$

where C^* is the generalized damping for the Ritz generalized single-degree-of-freedom system. Equation 7.17 is a nonlinear equation of motion for an unanchored tank structure. It can be rewritten in incremental form for the time interval from $t = t_n$ to $t = t_{n+1}$

$$M_{n+1}^* \Delta \ddot{v} + C_{n+1}^* \Delta \dot{v} + K_{n+1}^* \Delta v = \Delta P^* \quad (7.18)$$

where

$$\Delta \ddot{v} = \ddot{v}_{n+1} - \ddot{v}_n; \quad \Delta \dot{v} = \dot{v}_{n+1} - \dot{v}_n; \quad \Delta v = v_{n+1} - v_n; \quad \Delta P^* = P_{n+1}^* - P_n^*$$

or

$$\Delta \ddot{v} + 2\omega_{n+1}^* \xi_{n+1} \Delta \dot{v} + \omega_{n+1}^{*2} \Delta v = \frac{\Delta P^*}{M^*} \quad (7.19)$$

where

$$\omega_{n+1}^{*2} = \frac{K_{n+1}^*}{M_{n+1}^*} \quad \xi_{n+1} = \frac{C_{n+1}^*}{2M_{n+1}^*} \quad (7.20)$$

Applying direct integration techniques, the unknown variables v_{n+1} , $\dot{v}_{n+1}$ and $\ddot{v}_{n+1}$ can then be obtained.

7.2.1 Generalized Ritz shape function

For typical thin walled metal liquid storage tanks, the weight of the tank structure is much smaller than that of the liquid content. Consequently, the static deflections of the tank shells with the sum of the hydrostatic liquid load and hydrodynamic loading p_d applied pseudostatically gives a good approximation to the actual dynamic deflections of the tank structure. The generalized Ritz shape ψ_n at time step t_n can thus be obtained by solving the response of the tank structure subjected to the following pseudostatic loading

$$p_n = p_{d,n} + p_{m,n} \quad (7.21)$$

where $p_{d,n}$ is the hydrodynamic load on the tank structure, and $p_{m,n}$ is the inertial force contributed by the mass of the tank structure.

Using the Ritz method presented in earlier chapters for the static tilt analysis of unanchored tank system, the displacement field of the generalized Ritz shape ψ_n can be determined.

7.2.2 Direct time step integration procedures

Several alternative numerical integration methods, such as constant acceleration method, average acceleration method, linear acceleration method and Wilson- θ method, may be used to carry out the time step integration of the equation of motion of the generalized single degree of freedom system to determine the response. It is well known

that most numerical methods are stable when $\omega\Delta t$ (where ω is the frequency of the system, Δt is the interval of time) is smaller than a certain value. An exception is the average acceleration method, which is unconditionally stable. Therefore to ensure adequate accuracy in the numerical results, $\omega\Delta t$ must be selected to be considerably smaller than 1. After comparison of the numerical results determined in a convergence study[72], it has been shown that for the integration of a single-degree-of-freedom system, the linear acceleration method, which gives no amplitude decay and the minimum effect on period elongation, gives accurate results. The linear acceleration method is not adopted in the present study. This is because, for the nonlinear tank system, the exact generalized period is not known before obtaining the parameters in Equation 7.19. These parameters and the time step interval $\Delta t = t_{n+1} - t_n$ are therefore dependent on the estimation of the generalized Ritz shape in the previous time step. In order to guarantee a stable solution, the average acceleration method which is unconditionally stable and gives good accuracy in the calculation is selected in the dynamic formulation presented in this thesis.

Time step Δt

The time step Δt for the integration of the incremental equation of motion is taken as $T/10$ ($T = 2\pi/\omega_{n+1}^*$) in Equation 7.19.

For the liquid response, because the sloshing modes have different frequencies which are usually significantly lower than that associated with the coupled vibration modes represented in Equation 7.19, the time step selected in the calculation of the hydrodynamic pressure by the Duhamel integral in Equation 2.77 is taken as

$$(\Delta t)_{liquid} = \frac{2\pi}{10\omega_{jn}}$$

where ω_{jn} is the frequency of the j^{th} liquid sloshing mode.

Average acceleration method

If it is assumed that the acceleration remains constant over the small interval of time $\Delta t = t_{n+1} - t_n$ and its value is equal to the average of the values of the acceleration at the beginning and at the end of the interval:

$$\ddot{v} = \frac{1}{2} (\ddot{v}_n + \ddot{v}_{n+1})$$

then

$$\begin{aligned}\dot{v}_{n+1} &= \dot{v}_n + \frac{1}{2} \Delta t (\ddot{v}_n + \ddot{v}_{n+1}) \\ v_{n+1} &= v_n + \Delta t \dot{v}_n + \frac{1}{4} (\Delta t)^2 (\ddot{v}_n + \ddot{v}_{n+1})\end{aligned}\tag{7.22}$$

From Equation 7.18

$$\Delta v = \frac{M_{n+1}^* \left[\frac{4}{\Delta t} \dot{v}_n + 2\ddot{v}_n \right] + C_{n+1}^* \left[2\dot{v}_n + \frac{\Delta t}{2} \ddot{v}_n \right] + \Delta P^*}{\frac{4}{(\Delta t)^2} M_{n+1}^* + \frac{2}{\Delta t} C_{n+1}^* + K_{n+1}^*}\tag{7.23}$$

7.2.3 Iteration strategy

As discussed in Section 6.3.1, the generalized Ritz shape is selected as the deflection obtained due to the loading given by Equation 7.21. In other words, the loading given in Equation 7.21 is applied as static loads to the tank system, the resulting static deformation is then chosen as the generalized Ritz shape ψ . Assuming the response at time $t = t_n$ is known, this response is used as the initial generalized Ritz shape to determine the response at the next time step $t = t_{n+1}$. Following the analysis procedure discussed in Section 6.3, the displacement, velocity and acceleration responses, v_{n+1} , $\dot{v}_{n+1}$ and $\ddot{v}_{n+1}$, at time t_{n+1} can be obtained. However, the dynamic loading used to calculate the response v_{n+1} , $\dot{v}_{n+1}$ and $\ddot{v}_{n+1}$ has changed in accordance to the response of the uplifting tank system during the duration of this time step, which must therefore also has an effect on the derivation of the generalized Ritz shape ψ . The generalized Ritz shape ψ defined at the beginning of the time step in general is not the suitable shape for the Ritz reduction of the system at the end of the time

step. To alleviate this problem, in the analysis procedure, an iteration strategy is developed to generate an optimum converged generalized Ritz shape ψ for the next time step t_{n+1} .

Iteration procedure

The iteration procedures for the nonlinear dynamic response analysis of an unanchored liquid-tank system are stated as follows

1. Start with the converged response, v_n , $\dot{v}_n$ and $\ddot{v}_n$ with the generalized Ritz shape ψ_n , at the end of time step t_n .

Take $\psi_{n+1}^{(1)} = \psi_n$.

2. Obtain the incremental equation of motion of the generalized single degree of freedom system (Equation 7.18) using $\psi_{n+1}^{(i)}$ for iteration i .
3. Solve Equation 7.18 using average acceleration or other time step integration method.

Get estimated response at t_{n+1} : $v_{n+1}^{(i)}$, $\dot{v}_{n+1}^{(i)}$ and $\ddot{v}_{n+1}^{(i)}$

4. Calculate the updated hydrodynamic loading based on $\ddot{v}_{n+1}^{(i)}$
5. Apply the updated hydrodynamic loading together with the hydrostatic pressure as static loads, and solve for $\psi_{n+1}^{(i+1)}$
6. Check for convergence

$$\left| 1 - \frac{\langle \psi_{n+1}^{(i)}, \psi_{n+1}^{(i+1)} \rangle}{\langle \psi_{n+1}^{(i+1)}, \psi_{n+1}^{(i+1)} \rangle} \right| < \varepsilon \quad \text{where } \varepsilon \text{ is the tolerance}$$

$\langle \psi_{n+1}^{(i)}, \psi_{n+1}^{(i+1)} \rangle$ is defined as follows

$$\langle \psi_{n+1}^{(i)}, \psi_{n+1}^{(i+1)} \rangle = \int m [u_{n+1}^{(i)} \bar{u}_{n+1}^{(i+1)} + \bar{v}_{\theta, n+1}^{(i)} v_{\theta, n+1}^{(i+1)} + w_{n+1}^{(i)} \bar{w}_{n+1}^{(i+1)}] ds \quad (7.24)$$

where m is the mass distribution of the tank structure, $\bar{u}_{n+1}^{(i)}$, $\bar{v}_{\theta, n+1}^{(i)}$, $\bar{w}_{n+1}^{(i)}$ and $u_{n+1}^{(i+1)}$, $v_{\theta, n+1}^{(i+1)}$, $w_{n+1}^{(i+1)}$ are respectively the displacement components of the Ritz shapes $\psi_{n+1}^{(i)}$ and $\psi_{n+1}^{(i+1)}$.

If converged : go to next time step

If not converged : go to step 7

7. If the iteration process does not converge within a prescribed number of cycles, the time step size is reduced, and then go to step 3

8. Express the initial conditions in terms of the new Ritz shape $\psi_{n+1}^{(i+1)}$:

$$v_n^{(i+1)} = v_n \frac{\langle \psi_n, \psi_{n+1}^{(i+1)} \rangle}{\langle \psi_{n+1}^{(i+1)}, \psi_{n+1}^{(i+1)} \rangle}$$

9. Go back to step 2

The above iteration procedures are repeated for all the time steps until the end of the ground excitation record.

7.3 NUMERICAL EXAMPLES

In order to demonstrate the variety of the developed dynamic analysis procedure, two numerical examples are given in this section. A 10-story shear building structure with the mass 100,000 kg a story, and a total lateral stiffness 3,600 MN/cm at each story, is shown in Figure 7.2(a). The second example is the displacement response of a steel tank with 120 ft diameter, 40 ft height, 1 in thickness of the wall and full of water. In this section, the earthquake acceleration input for both of these two examples is the north-south acceleration component recorded in the 1940 El Centro earthquake, as shown in Figure 2.8, with a peak ground acceleration amplitude of 0.35g.

The dynamic procedure discussed in the previous sections is applied in this section to calculate the 10-story structure and the anchored tank which are multi-degree-of-freedom systems. The concept of dynamic analysis in this thesis is to obtain the dynamic response of a multi-degree-of-freedom by a single Rayleigh-Ritz shape. To cover the contributions from all modes, the iteration and changing the time step are used. Although the two examples are linear elastic, the dynamic procedure can be extended to the nonlinear multi-degree-of-freedom system.

The time history of the displacement on the roof (top of the building) is plotted in Figure 7.2(b). The exact solution in Figure 7.2 is the response which is calculated

from the mode superposition of the all 10 modes. The exact solution are almost the same as the Ritz solution from the Figure 7.2(b).

The earthquake response of an open top, fixed base broad tank is given in Figure 7.3. The damping ratio is assumed to be 2%. The comparison with the results by Haroun[27], which was calculated using the finite element method under the same earthquake input and damping ratio, is also plotted in Figure 7.3(b). The displacement is normalized by the maximum displacement which is equal to 0.236 in or 5.99 mm. The result from the Ritz method introduced in this thesis agrees well with the previous research by Haroun.

The both examples indicate that there is not inherent difference between the exact method and Ritz method.

Figure 7.1: Liquid storage tank excited by earthquake

(a) Shear structure elevation

(b) Time history of displacement at top of the structure

Figure 7.2: Dynamic analysis of a shear building structure

(a) configurations of an anchored tank (steel)

(b) Time history of displacement at mid height of tank

Figure 7.3: Dynamic analysis of an anchored tank

Chapter 8

SEISMIC RESPONSE OF A BROAD TANK

The seismic performance of a broad tank is discussed in this chapter. The broad model tank analyzed here is the same model tank described in Chapter 6, 12 feet in diameter and 6 feet high, previously tested on the shaking table at the Earthquake Engineering Research Center of the University of California, Berkeley[25]. The differences between the response behaviour of anchored and unanchored tanks are discussed in this chapter.

The aluminum model tank is a 1/3 scale model of a steel prototype, 10.97 m (36 feet) in diameter by 5.49 m (18 feet) high. The scaling requirements to maintain similitude for the prototype can be found in reference[25].

In the dynamic test, the model tank was attached on the 20x20 feet square shaking table. Only horizontal excitations were applied in the test. Because of the similitude requirements, the time scale of the earthquake input, the N-S component of the El Centro 1940 earthquake acceleration, was multiplied by a factor of $1/\sqrt{3}$. The peak of the acceleration was scaled to 0.5 g. The same table acceleration used in the experiment as shown in Figure 8.2, is used as input motion in the analysis of the broad tank.

The analysis model of the broad model tank is shown in Figure 8.1. The analytical model shown in Figure 8.1 is slightly different from the test model shown in Figure 6.1. The difference is in the modeling of the bottom plate. The analytical model assumes that the bottom plate is a uniform 2 mm thick plate, whereas the bottom plate of the actual tested model is constructed by riveting a 2 mm thick annular ring

piece to the center plate. Possible sources of error in the experiment are the following: 1. residual stresses from fabrication of the models; 2. initial imperfection; 3. extreme flexibility of the tank structure. It is very difficult to obtain consistent and error-free measurements. Furthermore, there are additional errors due to differences between the analytical model and tested model in the weakened rivet connections and overlap of the annular plate with the centre piece. Using the static tilt analysis, the effects of the possibly yielded rivets together with the other discrepancies have been found to be equivalent to reduce the elastical modulus of the bottom plate to 1/3 of the original value. With this approximation, the uplift of the analytical model is about the same magnitude as the experimental measurement for the static tilt analysis under the 12° tilt angle.

In this chapter, analytical results are presented for both unanchored and the anchored tank cases. Comparison of these results and studies on the difference the seismic response behaviour of an anchored broad tank and that of the same tank under the unanchored case are reported. Correlations of the analytical results with the experimental measurements are also discussed.

8.1 DISPLACEMENTS

The displacement analytical results of the unanchored and anchored tanks are discussed in this section.

The vertical uplift of the bottom plate at the base rim on the north and south side of the excitation axis are shown in Figure 8.3. The maximum uplift on the north side is 4.1 mm at 1.226 second, whereas that on the south side is 2.1 mm at 1.309 second. It is noted that occurrence of the maximum uplift coincides with that of the peak ground acceleration.

The radial displacements of the tank shell at the mid height (911 mm above the bottom) on the south side ($\theta = 0^\circ$, as shown in Figure 6.2) for both the unanchored and anchored tanks are shown in Figure 8.4. The maximum displacement is about 2.9 mm at 1.309 second for the unanchored tank, and 0.23 mm at 1.272 second for

the anchored tank. As noted, the displacement of the unanchored tank is much larger than that of the anchored tank. The dynamic displacement behaviour of the unanchored tank is inherently different from that of the anchored tank.

During the motion, the anchored tank behaves like an oscillator, as the anchored tank displaces are about the same amount in the positive and negative radial directions from the static position. This response behaviour justifies the adoption of the assumed Ritz shape of a cantilever beam by Veletsos and Young [28], [36] in their studies of the dynamic behaviour of anchored tanks.

Because of the bottom uplift, the vibration behaviour of the unanchored tank is not like an oscillator any more. The radial displacement in the negative direction is much larger than that in the positive direction. The reason of the larger negative displacement has been discussed in Chapter 6 for the static tilt case. The base rim on the uplifted side separates from the foundation, allowing the tank shell on the uplifted side to move radially inwards, whereas that on the contact side, the portion of the arc of the base rim remaining in contact with the foundation, restrains the radial outward displacement of the tank shell. In the modelling of an unanchored tank, the Ritz shape functions adopted in the discretization of the system must have the capacity to capture this behaviour in order to obtain accurate results.

The radial displacements of the two tank shells at the top rim on the south side are given in Figure 8.5. The maximum displacements are respectively 5.1 mm at the 1.309 second for the unanchored tank, and 0.12 mm at 1.212 second for the anchored tank. The displacement at the top rim has similar characteristics as those at the mid-height.

Time histories of the top rim out-of-round deformation are shown in Figure 8.6. For the unanchored tank, the deformed shapes at 1.226, 1.262 and 1.309 seconds with a maximum radial displacement of 7.1 mm are shown in Figure 8.6(a). The maximum displacement of 7.1 mm is located at the north side of the tank shell at the time of the maximum overturning moment which occurs at 1.226 second. In Figure 8.6(b), the deformed shapes of the anchored tank at 1.212, 1.253 and 1.272 seconds with a

maximum of 0.12 mm are shown. As can be noted in the figures, the horizontal section of the anchored tank remains circular with $\cos\theta$ type of translational displacement during the vibration. In comparison, the deformed shape of the unanchored tank has significant out-of-round components. The higher mode components, such as $\cos 2\theta$, $\cos 3\theta$, and perhaps even higher, have significant contributions to the vibration of the unanchored tank.

The deformed shapes of the unanchored tank at the top rim and mid height at 1.309 second are plotted in Figure 8.7. As observed from the figures, the deformed shape of the cross-section has relatively higher contributions from the higher order out-of-round modes, giving it a more pronounced pinching appearance as the height of the cross section increases. Also supported by results presented in the later section, it is shown that the higher out-of-round modes are very important in the dynamic response behaviour of unanchored tanks and should not be ignored.

8.2 LATERAL RESULTANT FORCE AND OVERTURNING MOMENT

The lateral resultant force acting on the tank is calculated from the lateral resultant force of the liquid pressure acting on the tank shell. The overturning moment is the total moment from the liquid pressure acting on the tank shell at the base horizontal section. In this thesis, both the lateral resultant force and the overturning moment are normalized by the total weight of the tank.

In Figure 8.8, the maximum normalized lateral resultant force is about 0.28 g at 1.226 second for the case of the unanchored tank, and 0.32 g at 1.272 second for the anchored tank. Correspondingly, the maximum normalized overturning moment is 0.172 m at 1.226 second for the case of the unanchored tank and 0.196 m at 1.272 second for the anchored tank, as shown in Figure 8.9. As noted from the results, the occurrence of the maximum lateral resultant force coincides with that of the maximum overturning moment. Furthermore, the vibration phases of the lateral resultant force

and overturning moment are similar for both the unanchored tank and the anchored tank. This is because the lateral resultant force and the overturning moments are mainly affected by the $\cos\theta$ type mode only. The hydrodynamic pressure given in Equation 2.79 is dominated by the first $\cos\theta$ component. As such, the relationship of the lateral resultant force and the overturning moment for the unanchored and anchored tanks is similar to that of a single degree-of-freedom system under the earthquake excitation, of which the moment is equal to the product of the lateral force and a constant height of the oscillating mass from the ground.

8.3 HYDRODYNAMIC PRESSURE AND WAVE HEIGHT

The wave height on the south side ($\theta = 0^\circ$), and the impulsive and convective pressures near the bottom (2.54 cm) and at the mid-height of the liquid (762 mm above the bottom) are presented in this section.

The time history of the impulsive pressure for the unanchored tank at 2.54 cm above the bottom is shown in Figure 8.10(a). The maximum pressure is 6.82 kPa occurring at 2.226 second. The pressure for the anchored tank is given in Figure 8.10(b), of which the maximum is 8.10 kPa occurring at 1.267 second.

The time histories of the impulsive pressure at the mid-height of the liquid are plotted in Figure 8.11. Unlike the impulsive pressure near the bottom, which has the maximum response occurs at 1.226 second, the maximum impulsive pressure for the unanchored tank is 6.22 kPa at 1.309 second. Like the impulsive pressure near the bottom, which has the maximum response at 1.267 second, the maximum impulsive pressure at the mid-height is 7.26 kPa for the anchored tank occurring at the same time 1.267 second as the peak near the bottom.

The impulsive pressure profiles along the tank height on the south side at the time corresponding to the maximum overturning moment (1.226 second for the unanchored tank, 1.272 second for the anchored tank) are shown in Figure 8.12. The shapes of the profile for the unanchored and anchored tanks are similar. For the anchored tank, the fundamental vibration mode of the tank system represents the dominant response

of the system and consequently also that of the impulsive pressure.

Comparing the results shown in Figures 8.10 and 8.11, it is noted that the variations in the time histories of the impulsive pressure at the bottom and at the mid-height are similar except in the magnitude of the response. This indicates that contribution from the fundamental vibrating mode essentially dominates the total response of the tank system for the broad tank analyzed.

In contrast, for the unanchored tank, the higher modes have greater significance in the response of the impulsive pressure. There is a sharp peak at 1.309 second as shown in Figure 8.11(a). Recalling the out-of-round effects shown in the Figure 8.7 at that instant, this peak must be contributed significantly by the higher modes. Further discussion on this is presented in the discussion of the membrane stress.

Similar plots for the time histories of the convective pressure are shown in Figures 8.13 to 8.16. The maximum of convective pressures at 25.4 cm above the bottom on the south side of the tank shell ($\theta = 0^\circ$) in the unanchored tank, as shown in Figure 8.13(a), is 0.12 kPa which occurs at 2.872 second. Similarly for the anchored tank, as shown in Figure 8.13(b), the maximum is 0.75 kPa at 2.974 second. The convective pressure at the mid-height of the liquid (762 mm above the bottom) and on the quiescent surface of the liquid on the south side of the tank shell ($\theta = 0^\circ$) are shown in Figures 8.14(a) and 8.15(a) for the unanchored tank, and in Figures 8.14(b) and 8.15(b) for the anchored tank. In general, the convective pressure response of the anchored tank is smoother than that of the unanchored tank. This is because of the effect of bottom plate uplifting. As shown in Figure 8.13(a), the response is the least smooth at the location (25.4 mm above the bottom) nearest to the bottom. In Figure 8.14(a), at a location farther from the bottom (762 mm above the bottom), the response become smoother. In Figure 8.15(a), where the location is farthest from the bottom, the response is the smoothest of the three plots.

The impulsive pressure in an anchored tank is mostly dominated by the higher frequency tank shell-liquid vibration mode, whereas the convective pressure is contributed by the low frequency liquid sloshing modes.

On the other hand, the impulsive pressure in an unanchored tank is dominated by both the shell and bottom plate vibration modes with the higher frequency modes interacting with that of the liquid. The convective pressure of an unanchored tank is contributed by the liquid sloshing as well as by the bottom plate uplift motion.

Time histories of the surface wave height are shown in Figure 8.16. The maximum wave height is 0.136 m at 5.443 second for the unanchored tank, and 0.160 m at 2.921 second for the anchored tank. The frequency of the wave motion in the unanchored tank seems slightly higher than that for the anchored tank. This is probably because the bottom plate uplifting from the foundation increases the higher frequency components of the wave. It is observed that the surface wave height is related to the convective pressure by a coefficient of $1/(\rho g)$.

8.4 MEMBRANE STRESSES OF TANK SHELL

Investigations of earthquake damage of liquid storage tanks indicates that there are two typical kinds of observed buckling pattern in the tank shell. One is the bulging out of the shell surface commonly known as "elephant foot" buckling, usually located near the base. The other is the failure of the shell surface commonly known as "diamond shaped" buckling which may occur at some distance above the bottom plate. The causes of these two kinds of failures are discussed in this section.

8.4.1 MEMBRANE AXIAL STRESS

Time histories of the membrane axial stress of the shell at the location 25.4 mm above the bottom, on the south side ($\theta = 0^\circ$), are shown in Figure 8.17. The maximum axial stress is 1.62 MPa at 1.226 second for the unanchored tank, and 1.23 MPa at 1.272 second for the anchored tank. In both cases, the occurrence of the maximum axial stress coincides with the time of maximum overturning moments. The maximum compressive stress in the unanchored tank is about 21% higher than that in the anchored tank. Similar time histories of the membrane axial stress at the mid-height are shown in Figure 8.18. The stress of the unanchored tank is much higher than

that in the anchored tank.

Comparing the axial stress near the bottom with that at the mid height, it is noted that the axial stress is highest at the bottom and decreases rapidly with the height for the anchored tank analyzed. Two major reasons can be attributed to this. First, the impulsive pressure, which is zero at the top, and maximum at the bottom with a distribution of a parabola, seems to dominate the response behaviour of the broad tank. Secondly, for a cantilever beam kind of structure, like the anchored tank, the overturning moment vanishes quickly from the bottom to the top under distributed parabolic loading.

The profiles of the membrane axial stress on the contact side are shown in Figure 8.19(a) with the maximum of 1.7 MPa for the unanchored tank corresponding to the maximum overturning moment which occurs at 1.226 second, and in Figure 8.19(b) for the anchored tank with the maximum of 1.27 MPa during the maximum overturning moment occurring at 1.272 second. As shown in the figures, the axial stress distribution in the anchored tank is slightly more concentrated near the bottom than that in the unanchored tank.

The circumferential distribution of the membrane axial stress corresponding to the maximum overturning moments at a horizontal section near the bottom (25.4 mm) are shown in Figure 8.20. For the anchored tank, the stress distribution is $\cos\theta$ type, which agrees with the plane assumption of the cross section. The axial stress of the anchored tank can be estimated by the beam formula $\sigma = My/J$, where M is the overturning moment, y is the distance to the neutral axis of the tank horizontal section, J is the inertia moment of the section to the neutral axis. The stress is symmetrically distributed about the neutral axis. The maximum compressive (negative) stress is equal to the maximum tensile (positive) stress for the anchored tank. On the other hand, the stress distribution in the unanchored tank is very different from a $\cos\theta$ distribution. The compressive stress is more concentrated on the contact side. As a result, the compressive stress on the contact side is significantly higher than the tensile stress on the uplifted side.

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

PRECISIONSM RESOLUTION TARGETS

Because of partially uplifting of the bottom rim, the cross-section of the tank shell no longer remains a plane. The formula $\sigma = M y/J$, is therefore not applicable to unanchored tanks.

At the time of the maximum compressive stresses at the mid-height, the stress distributions around the tank wall are given in Figure 8.21(a) for the unanchored tank at 1.309 second, and in the Figure 8.21(b) for the anchored tank at 1.272 second. Again, for the anchored tank, as shown in Figure 8.21, the stress distribution of the anchored tank is of $\cos\theta$ type. Occurrence of the maximum stress coincides with that of the maximum overturning moment, like the stress near the bottom.

However, similar to the stress distribution near the bottom as shown in Figure 8.20(a), the stress distribution of the unanchored tank at mid-height is different from that of the $\cos\theta$ type. The maximum stress occurs at a different time from the time of maximum overturning moment. It is believed that significant contributions from higher modes (including $\cos 2\theta$, ..., $\cos n\theta$) result in this kind of stress distribution. As shown in Figures 8.4(a) and 8.5(a), the radial displacements at the mid height and at the top rim have sharp peak values at 1.309 second. The impulsive pressure shown in Figure 8.11(a) also shows similar sharp peak value at 1.309 second. From vibration analysis theory, the sharp peaks are due to contributions from higher modes in the dynamic response. The deformed shapes of the unanchored tank at the mid-height and at the top rim presented in Figure 8.7 show that the out-of-round effect is very significant at 1.309 second. The dynamic out-of-round effects, contributed by the vibration of the higher modes, play an important role in the seismic response behaviour of unanchored tanks.

8.4.2 MEMBRANE HOOP STRESS

The profiles of the membrane hoop stress on the contact side corresponding to the maximum overturning moment for the unanchored and anchored tanks are shown in Figure 8.22. The overall shapes of the two tanks are similar, except that the compressive hoop stress of the unanchored tank near the bottom rim is much higher

than that of the anchored tank. This is because of the different edge boundary conditions at the tank base between the anchored and unanchored tanks. At the bottom edge of the shell, it is assumed that there is no radial displacement and the edge is fixed to the foundation in the case of the anchored tank. As such, the hoop stress at the bottom edge for the anchored tank is zero. For the unanchored tank, the bottom edge of the shell has significant displacements due to the bottom plate uplift. Thus the hoop stress near the bottom is very large.

The membrane hoop stress along the horizontal section near the bottom (25.4 mm) and at the mid-height of the tank (762 mm above the bottom) are respectively shown in Figures 8.23 and 8.24 corresponding to the maximum overturning moments, which occurs at 1.226 second for the unanchored tank, and at 1.272 second for the anchored tank. As noted from the figures, the hoop stress in the unanchored tank (± 15 MPa) is much larger than that in the anchored tank (± 0.4 MPa) at a section near the bottom. However, the hoop stress distributions in the unanchored and anchored tanks are similar in magnitude at the mid-height of the tank. The distributions in both cases are smooth, and are of $\cos\theta$ type. This is because the hoop stress are essentially related to the distribution of the liquid pressure, and during the maximum overturning moment, the $\cos\theta$ pressure component is the dominant component in the dynamic pressure.

8.4.3 BUCKLING OF TANK SHELL

Upon examination of seismic damages to tanks due to buckling of the shells, it is observed that bucklings may be located near the bottom, or at the middle of the shell, or even near the top of the tank depending on the aspect ratios of the tanks. The buckled shell may bulge out, such as in the case of "elephant-foot" buckling near the tank bottom as shown in Figure 8.25(a), or squeeze in, in the case of the "diamond shape" buckling shown in Figure 8.25(b). As discussed earlier, the maximum overturning moment, which is contributed mainly by the $\cos\theta$ type liquid hydrodynamic pressure, results in the high compressive axial membrane stress near the bottom on

the contact side. The shell near the bottom bulges out on the contact side because the liquid pressure is pushing outward the tank near the base and at the same time the base rim rotates outwards as discussed in Chapter 6. If the compressive axial stress exceeds the buckling stress, the shell near the bottom on the contact will lose the axial capacity, and the tank shell will fail by "elephant-foot" buckling.

On the other hand, as discussed in the previous sections of 8.1 and 8.1.1, the high compressive axial stress may occur far away from the bottom, such as at the mid height, due to the contribution from higher vibration modes. The high compressive axial stress is related to the out-of-round deformation of the tank wall, in which some portion of the deformed shape bulges out and some of that squeezes in. If the axial stress exceeds the stability limit at the location where the deformed shape bulges out, the shell will buckle outward. However, because the squeezing in portion corresponds to compressive hoop strain in the shell which may affect the buckling strength of the shell, therefore the squeezing in portion of the shell may be weaker. When the axial stress exceeds the buckling limit, the tank shell will lose the capacity. The failure will then be in the form of "diamond shape" buckling.

8.5 GENERALIZED VIBRATION PERIOD

The generalized frequency of a tank system is defined in chapter 7. The generalized period of a tank system is then given as follows

$$T = \frac{2\pi}{\omega}$$

where ω is the generalized frequency. Time histories of both the generalized periods of the unanchored and the anchored tanks are shown in Figure 8.26.

According to previous studies by Veletsos[73] and Haroun[57], the fundamental frequency, ω_0 , of an anchored tank may be estimated as follows

$$\omega_0 = \frac{C_1}{H} \sqrt{\frac{E}{\rho}} \quad (8.1)$$

where E is the Young's modulus of elasticity for the material of the tank shell; ρ is its mass density; H is the depth or height of the liquid; and C_1 is a dimensionless

coefficient which depends on the tank proportions, such as the ratios of H/R , t/R , the Poisson's ratio ν , and the relative mass densities of the liquid and the tank shell, ρ_l/ρ .

Table of the dimensionless coefficient C_w can be found in reference[73] and is reproduced here as shown in Table 8.1. In the table, C_w is a dimensionless coefficient for water content. For liquid content other than water, a parameter C_l can be determined from C_w as follows

$$C_l = C_w \sqrt{\frac{\rho_w}{\rho}} \quad (8.2)$$

where ρ_w is the mass density of water, and ρ_l is the mass density of the liquid considered.

Table 8.1 Dimensionless coefficient C_w

$\frac{H}{R}$	C_w		$\frac{H}{R}$	C_w	
	$h/R = 0.0005$	$h/R = 0.001$		$h/R = 0.0005$	$h/R = 0.001$
0.3	0.0121	0.0601	1.1	0.0650	0.0915
0.4	0.0168	0.0667	1.6	0.0651	0.0917
0.5	0.0506	0.0719	1.8	0.0646	0.0909
0.6	0.0538	0.0763	2.0	0.0637	0.0896
0.7	0.0561	0.0799	2.2	0.0625	0.0879
0.8	0.0587	0.0829	2.4	0.0611	0.0859
0.9	0.0605	0.0851	2.6	0.0596	0.0838
1.0	0.0620	0.0875	2.8	0.0580	0.0815
1.2	0.0641	0.0903	3.0	0.0563	0.0792

For the model tank discussed in this chapter, using the above formula and Table 8.1, the fundamental period of the anchored tank is estimated to be

$$T_0 = 0.03 \text{ s}$$

This estimated period is approximately equal to the lower limit of the analytical results obtained for the unanchored tank as shown in the Figure 8.26(a), and about the same as the average of that obtained for the anchored tank in the Figure 8.26(b).

For the anchored tank, the response of the system is linear elastic. The vibration modes and natural frequencies can be determined by an eigenvalue analysis. The dynamic response of the system may theoretically involve all the modes of the system. In this thesis, a generalized period is defined to describe the dynamic behaviour of the tank system at a particular instant of time. When the generalized period is about the fundamental vibration period indicated by the upper limit of the time history shown in Figure 8.26(b), it denotes that the fundamental mode dominates the dynamic response of the tank system. When the generalized period is lower than the estimated fundamental period, it denotes that the higher modes are important in the dynamic response. Because the response is contributed not only by the fundamental mode in a multi-degree-of-freedom system, the time history of the generalized period for the anchored tank is always below the fundamental period as shown in Figure 8.26(b).

For the nonlinear unanchored tank system, the vibration modal concept does not apply. Therefore, the generalized period for the unanchored tank in this thesis is defined as a response parameter to explain the transient dynamic behaviour of the nonlinear system. As shown in Figure 8.26(a), when the bottom plate uplifts from the foundation, the structure becomes less stiff due to the loss of some restraint at the bottom, the generalized period then becomes higher. When there is no bottom plate uplift, the structure behaves like an anchored tank, the generalized period should then be similar to that of the anchored tank shown in Figure 8.26(b).

Upon examination of the overall performances of the unanchored and anchored tanks, the generalized period of the unanchored tank is much longer than that of the anchored tank.

8.6 CORRELATIONS WITH EXPERIMENT DATA

Comparisons of the analytical results with the experimental measurement are discussed in this section. Because of significant differences in the details between the tested model and the analysis model, the influence of the residual stresses due to the welding process in the construction of the tested models, and also the measurement are highly affected by the test environment because of the slenderness of the tested models, it is sometimes very difficult to match the analytical results with those of the experiment.

In the experiment, many response quantities are recorded, such as the Fourier coefficients of the displacements and stress time histories. Because some uncertainties are involved in the indirect measurements such as stresses, the direct measurement results, such as dynamic pressure and wave height, are selected as the basic test data for comparison with the analytical calculations.

The locations of two measurement gages, and gage (1) for the wave height, gage (9) for the dynamic pressure, are shown in Figure 8.27.

For comparison, the time histories of both the experimental and the analytical results are plotted in the same figures. The experimental measurements are presented by the solid lines, while the analytical calculations are presented by the dot lines.

As shown in Figure 8.28, the phase and magnitude of the dynamic pressure from the analysis agree well with the experiment results, except that some discrepancy occurs around 1.309 second. When the experimental results suddenly jumps to about +12 kPa and -9.3 kPa.

In the report of the experiment study[25], the authors indicate that these dynamic peaks do not coincide with the maximum uplift of the bottom plate, which is associated with the maximum overturning moment that is mainly contributed by the lower modes (or the fundamental mode). One reasonable explanation for this kind of behaviour is that the effects of the higher modes, like the "whip effect", result in the sudden jumps. Actually, the analytical calculation has shown the effects of the

higher modes, as discussed in the last section. The reason that the analytical results do not match at the sudden jumps may also be due to the difficulty in the modeling of the bottom plate.

As shown in Figure 8.29, the analytical prediction of the wave height agrees well with the experimental measurement at gage (1). The better correlation is due to the fact that the wave height is essentially influenced by the convective pressure at the liquid surface, which is less sensitive to the bottom uplifting and the displacement of the tank shell than the impulsive pressure. For other dynamic response quantities of the unanchored tanks, the analytical results correlate reasonably well with the experimental results.

Figure 8.1: Structural detail of unanchored and anchored broad tanks

Figure 8.2: Input ground acceleration for analysis of broad tanks

Figure 8.3: Bottom rim uplifting time history of unanchored broad tank

(a) unanchored tank

(b) anchored tank

Figure 8.4: Radial displacements at mid-height (914 mm above bottom) of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.5: Radial displacements at top rim of broad tanks

maximum radial displacement is 7.1 mm

1.226 seconds

1.262 seconds

1.309 seconds

(a) unanchored tank

maximum radial displacement is 0.12 mm

1.212 seconds

1.253 seconds

1.272 seconds

(b) anchored tank

Figure 8.6: Time histories of top rim deformed shapes of broad tank

maximum radial displacement is 5.1 mm

(a) mid-height

(b) top rim

Figure 8.7: Deformed shapes of top rim and mid-height (at 1.309 s) of broad tank

(a) unanchored tank

(b) anchored tank

Figure 8.8: Lateral resultants of unanchored and anchored broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.9: Overturning moments of unanchored and anchored broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.10: Impulsive pressures near (25.4 mm above) bottom of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.11: Impulsive pressures at mid-height of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.12: Impulsive pressure profiles of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.13: Convective pressures near (25.4 mm above) bottom of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.14: Convective pressures at mid-height of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.15: Convective pressures on surface of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.16: Wave height of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.17: Membrane axial stresses near (25.4 mm above) bottom of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.18: Membrane axial stresses at mid-height of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.19: Membrane axial stress profiles on contact side of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.20: Membrane axial stresses along circumference near (25.4 mm above) bottom of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.21: Membrane axial stresses along circumference at mid-height of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.22: Membrane hoop stress profiles on contact side of broad tanks

(a) unanchored tank

(b) anchored tank

Figure 8.23: Membrane hoop stresses along circumference near (25.4 mm above) bottom of broad tanks

Figure 8.24: Membrane hoop stresses along circumference at mid-height (914 mm above bottom) of broad tanks

(a) elephant foot buckling

(b) diamond shaped buckling

Figure 8.25: Buckling of tank shell

(a) unanchored tank

(b) anchored tank

Figure 8.26: Generalized periods of broad tanks

Figure 8.27: Instrumentation of experimental broad tank

Figure 8.28: Dynamic pressures of experimental and analytical results at gage (9) of broad tank

Figure 8.29: Wave heights of experimental and analytical results at gage (1) of broad tank

Chapter 9

SEISMIC RESPONSE OF A TALL TANK

In order to better understand the seismic behaviour of unanchored tanks, an additional tank, a tall vinegar storage tank[74], which was damaged during the 1989 Loma Prieta (San Francisco Area) earthquake, is analyzed in this chapter.

Figure 9.1 shows the configuration of the tall tank[74]. This is a stainless steel tank, with 8.534 m (28 ft) in diameter and 8.534 m (28 ft) high. The actual tank is anchored to a continuous concrete foundation pad by six welded anchored straps. During the earthquake, the anchorages were torn off the tank wall, rupturing the tank shell and losing the vinegar contents. This damaged tank underwent serious "elephant-foot" buckling near the bottom all around the tank shell. The buckled shell had completely collapsed into 15 cm wide flap for about one third of the circumference. A second elephant foot buckling occurred above the closed flap at these locations. The damaged tank was full at the time of the earthquake.

The analytical model shown in the Figure 9.1 is assumed to be full with vinegar (density of 1.05 relative to water), open top, and is excited by the 1940 EL-Centro earthquake NS acceleration component which is shown in Figure 2.2 but scaled to the estimated peak observed at the site of 0.2 g.

9.1 DISPLACEMENTS

The time histories of the base rim uplift at the exciting axis are given in Figure 9.2. The uplift on the south side, with a maximum vertical separation of 15.2 mm occurring at 2.123 second, is plotted by the solid line, while the uplift on the north side, with a maximum of 12.9 mm at 2.200 second, is plotted by the dotted line.

The radial displacements at the mid-height (4.267 m above the bottom) and at the top rim on the south side ($\theta = 0^\circ$) are shown in Figures 9.3 and 9.4. It is obvious that the displacement of the unanchored tank is much larger than that of the anchored tank. For the unanchored tank, the maximum radial displacements are 28.8 mm at the mid-height, 54.8 mm at the top rim, both occurring at 2.200 second. For the anchored tank, the maximum radial displacements are 1.8 mm at the mid-height occurring at 2.246 second, and 3.3 mm at the top rim occurring at 4.840 second.

The time histories of the deformed shapes at the top rim are given in Figure 9.5. The deformed shapes at 2.123, 2.180, and 2.220 seconds with a maximum of 57.1 mm radial displacement are shown in Figure 9.5(a) for the unanchored tank. The cross section of the tank rim are highly distorted and the radial inward displacements (negative) are much larger than the radial outward displacements (positive).

The deformed shapes of the top rim for the anchored tank at 2.152, 2.215 and 2.246 seconds with a maximum of 1.5 mm radial displacement are shown in Figure 9.5(b). It is noted here again that the cross section for the anchored tank remains circular.

The out-of-round deformed shapes of the unanchored tank at the top rim and at the mid-height at 2.313 second are shown in Figure 9.6. It is obvious from the results shown that the higher mode effects are significant. The deformed shape at the mid-height shows significant $\cos 4\theta$ component. The magnitudes of the displacements are relatively small with maximum radial displacement only about 2 mm.

The vibration behaviour in terms of displacements in the tall tank shows similar characteristics as the broad tank:

1. The unanchored tank has much larger radial displacement than that in the anchored tank.
2. Unlike the anchored tank which behaves like an oscillator, the negative radial displacement of the unanchored tank, which is related to the uplifting of the bottom rim, are much larger than the positive displacement on the opposite of the tank wall.
3. Out-of-round deformation is significant in the unanchored tank.

9.2 OVERTURNING MOMENTS

The overturning moments induced by the hydrodynamic pressure in the tall tank are shown in Figure 9.7. The maximum normalized overturning moment of the anchored tank is larger than that in the unanchored tank. The maximum normalized moment in the anchored tank with 0.9235 m occurring at 2.246 second is about 2.1 times that in the unanchored tank with 0.4394 m occurring at 2.226 second.

9.3 HYDRODYNAMIC PRESSURE AND WAVE HEIGHT

The hydrodynamic pressures near the bottom (25.4 mm) and at the mid-height (4267 mm above the bottom) on the south side of the tank shell ($\theta = 0^\circ$) are shown in Figures 9.8 and 9.9 respectively. The dynamic pressure of the anchored tank is relatively higher than that of the unanchored tank, comparing the maximums of 10.11 kPa (near the bottom) and 12.85 kPa (mid-height) at 2.246 second for the anchored tank with the maximums of 8.02 kPa (near the bottom) and 6.98 kPa (mid-height) at 2.123 second for the unanchored tank.

The contours corresponding to the occurrence of the maximum dynamic pressure, which coincide with the maximum overturning moment, are shown in Figure 9.10 for the unanchored tank, and in Figure 9.11 for the anchored tank. The contour lines for the unanchored tank range from -3 kPa to 11 kPa, whereas that for the anchored

tank, the contour lines range from -10 kPa to 14 kPa.

From the contour plots, it is clear that the hydrodynamic pressure in the anchored tank is higher than that in the unanchored tank. Because contours for a $\cos\theta$ distribution is symmetric with the axis $\theta = 90^\circ$, it is observed that the pressure distributions of both cases are not of the $\cos\theta$ type in the tall tank, although the anchored tank has a higher $\cos\theta$ component in its response than that in the unanchored tank.

The time histories of the wave heights are shown in Figure 9.12. The maximum wave height of the unanchored tank is 0.330 m occurring at 2.829 second, whereas the maximum for the anchored tank is 0.308 m at 9.248 second. It is also observed that the sloshing in the anchored tank is smoother than that in the unanchored tank.

For the anchored tank, the wave height is primarily related to the convective pressure. The wave height is considered not as much influenced by the impulsive component which is related to the displacement of the tank shell[73].

But for the unanchored tank, the sloshing waves are also related to the impulsive vibration components because both the broad tank and the tall tank show significant differences between the behaviour of the unanchored and the anchored tanks in the wave responses.

9.4 MEMBRANE STRESSES OF TANK SHELL

The membrane stresses in the tall tank shell reported in this section show the similar behaviour as the broad tank discussed in the last chapter.

The time histories of the axial stresses near the bottom (25.4 mm) on the south side ($\theta = 0^\circ$) are given in Figure 9.13. Comparing Figure 9.13 with 9.9, it is observed that there is a strong correlation between the overturning moment and the axial stress near the bottom.

In Figure 9.14(a), the axial stress for the unanchored tank at the horizontal section near the bottom corresponding to the time of the maximum axial stresses (at 2.123 second) shows that the stress distribution in the unanchored tank is concentrated to

the compressive (negative) part, and that the maximum compressive stress is larger than the maximum tensile (positive) stress. But for the anchored tank, the stress distribution is $\cos\theta$ type as shown in Figure 9.14(b), in which the compressive stress is about the same as the tensile stresses. Because the maximum overturning moment of the anchored tank is much larger than that of the unanchored tank, the maximum compressive axial stress of the anchored tank, with 25.9 MPa occurring at 2.246 second, is about 1.98 times that of the unanchored tank with 13.10 MPa at 2.123 second.

The time histories of the axial stresses at the mid-height (4.267 m above the bottom) are given in Figure 9.15. There are obvious contributions from the higher modes in the unanchored tank same as in the case of the broad tank described in the last chapter.

The axial stress around the circumference at the mid-height is presented in Figure 9.16 at the time of the maximum axial stress. For the unanchored tank, the stress is 10.09 MPa occurring at 2.313 second, and for the anchored tank, it is 3.165 MPa at 2.246 second.

As shown in Figure 9.16, the stress distribution of the unanchored tank is dominated by higher mode components, whereas the stress distribution in the anchored tank is again simply of the $\cos\theta$ type.

The membrane stress contours occurring at the time of maximum overturning moment, are shown in Figures 9.17 to 9.20. The axial stress contours of the unanchored tank (occurring at 2.123 second) in Figure 9.17 shows that the stress distribution is very complicated over the tank shell. The axial stress contour of the anchored tank (occurring at 2.246 second) in Figure 9.18 is again shown to be of the $\cos\theta$ type. The stress is more concentrated within the lower half part of the tank shell, which is the same as observed in the broad tank presented in the last chapter.

The hoop stresses in the unanchored tank (at 2.123 second) and the anchored tank (at 2.246 second) are given in Figures 9.19 and 9.20 respectively. Because the dynamic pressure in the anchored tank is higher and the hoop stress depends on the

liquid pressure, the hoop stresses in the anchored tank are larger than those in the unanchored tank. On the other hand, there is significant compressive hoop stress near the bottom of the unanchored tank similar to that observed in the case of the broad tank.

9.5 GENERALIZED VIBRATION PERIOD

The time histories of the generalized periods defined in chapter 7 are shown in Figure 9.21 for both the unanchored and anchored tall tanks.

Using the method recommended by Veletsos[73] and Haroun[57], the fundamental period of the anchored tank is estimated to be

$$T_0 = 0.16 \text{ s}$$

T_0 is about the same as the upper limit of the data plotted in Figure 9.21(b) for the anchored tank, and the lower limit in Figure 9.21(a) for the unanchored tank.

The characteristics of the generalized periods of the unanchored and the anchored tall tanks are similar to those observed in the broad tank.

Figure 9.1: Configuration of tall tank

Figure 9.2: Bottom rim uplifting time history of unanchored tall tank

(a) unanchored tank

(b) anchored tank

Figure 9.3: Radial displacements at mid-height of tall tanks

(a) unanchored tank

(b) anchored tank

Figure 9.4: Radial displacements at top rim of tall tanks

maximum radial displacement is 57.1 mm

2.123 seconds

2.180 seconds

2.220 seconds

(a) unanchored tank

maximum radial displacement is 1.5 mm

2.152 seconds

2.215 seconds

2.246 seconds

(b) anchored tank

Figure 9.5: Time histories of top rim deformed shapes of tall tank

maximum radial displacement is 2.0 mm

(a) mid-height

(b) top rim

Figure 9.6: Deformed shapes of top rim and mid-height (at 2.313 s) of tall tank

(a) unanchored tank

(b) anchored tank

Figure 9.7: Overturning moments of unanchored and anchored tall tanks

(a) unanchored tank

(b) anchored tank

Figure 9.8: Dynamic pressures near (25.4 mm above) bottom of tall tanks

(a) unanchored tank

(b) anchored tank

Figure 9.9: Dynamic pressures at mid-height (4267 mm above bottom) of tall tanks

Figure 9.10: Contour of dynamic pressure for unanchored tall tank (at 2.123 s)

Figure 9.11: Contour of dynamic pressure for anchored tall tank (at 2.246 s)

(a) unanchored tank

(b) anchored tank

Figure 9.12: Wave height of tall tanks

Figure 9.13: Membrane axial stresses near (25.4 mm above) bottom of tall tanks

(a) unanchored tank

(b) anchored tank

Figure 9.14: Membrane axial stresses along circumference near bottom of tall tanks

(a) unanchored tank

(b) anchored tank

Figure 9.15: Membrane axial stresses at mid-height (914 mm above bottom) of tall tanks

Figure 9.16: Membrane axial stresses along circumference at mid-height of tall tanks

Figure 9.17: Contour of membrane axial stress for unanchored tall tank (at 2.123 s)

Figure 9.18: Contour of membrane axial stress for anchored tall tank (at 2.246 s)

Figure 9.19: Contour of membrane hoop stress for unanchored tall tank (at 2.123 s)

Figure 9.20: Contour of membrane hoop stress for anchored tall tank (at 2.246 s)

(a) unanchored tank

(b) anchored tank

Figure 9.21: Generalized periods of tall tanks

Chapter 10

SUMMARY AND CONCLUSIONS

In this chapter, a brief summary of the thesis and conclusions on the seismic response behaviour of both anchored and unanchored liquid storage tanks are presented.

10.1 SUMMARY

Based on the linear wave theory of hydrodynamics, an analytical formulation for the hydrodynamic loading in a cylindrical tank which takes into consideration the bottom plate uplift, has been developed. The effects of the tank shell flexibility and bottom plate uplift are considered through the boundary conditions of the liquid domain in the dynamic response analysis of the unanchored tanks. The numerical problems of sensitivity and round-off errors in the analytical solution are discussed and resolved. Numerical examples are presented to demonstrate the validity of the new formulation.

The Ritz formulation of the bottom plate of an unanchored cylindrical liquid storage tank partially uplifted under static tilt lateral loading is presented to study the uplift resistant mechanism and the interactions between the cylindrical thin shell and the flexible bottom plate. This static tilt uplift model is extended to dynamic analysis in later chapters. Because of the large bottom plate uplifting, the nonlinear von Karman plate theory for moderately large deflection is adopted in modeling the bottom plate uplift from the foundation. Simplified Ritz shape functions, which are defined in the transformed normalized coordinates, are given as polynomials of increasing order

to determine the behaviour of the bottom plate uplifted region. The contact region is modeled as a two dimensional plane stress substructure element. The Ritz shape functions for the contact region are obtained by the analytical displacement solutions which satisfy the displacement and stress continuities with the uplifted region.

The Ritz formulation of the tank shell are based on boundary shapes and interior shapes. The shape functions are obtained from the linear thin shell theory presented by Flügge. The boundary shapes are defined as the homogeneous displacement solutions with imposed Fourier displacement conditions at the ends of the shell, while the interior shapes are the deformation of the tank shell due to the applied liquid pressure with fixed boundary condition at both ends of the tank shell.

The assembled model of the tank system is then presented. There are several different mechanisms involved in the response of the tank system. These include the deformation resistant mechanisms in the contact bottom plate, the uplifted bottom plate, the cylindrical shell, the ring girders and the roof. Using the derived Ritz shapes in a variational method which employs the stationary principle, the stiffness matrix and loading vector for each substructure element is obtained. The equation for the whole system is then obtained following the direct stiffness procedure. The static solution of tank system under the lateral load is solved by a two level iteration strategy. For every given bottom plate contact-uplift geometry, the nonlinear structural response is calculated by Newton-Raphson method. The contact-uplift geometry is then determined by the bisection method.

Analysis of a statically tilt model tank on rigid support platform with a 12 degree tilt angle is presented. The analysis shows that the method in this thesis is efficient and accurate. The comparison with the experiment indicates that the analytical results agree well with the experimental measurements.

The dynamic analysis model and the iterative procedures in a nonlinear time step integration scheme for the earthquake response behaviour of unanchored tanks are developed based on the static tilt uplift model. The accuracy and effectiveness of the new dynamic procedures are demonstrated by analyzing a 10-storey shear building

and an anchored tank. The results obtained agree well with the solution obtained by other well known established analysis procedures. The seismic performances of two tanks, a broad tank and a tall tank, are presented next. There are three significant contributions which lead toward a better understanding on the dynamic analysis and behaviour of liquid storage tanks:

1. A new, exact formulation for the vibrating fluid dynamics has been developed, which includes the effect of the bottom plate uplift on the hydrodynamic loading.
2. A reliable, effective and efficient technique to analyze nonlinear tank system due to uplifting of the bottom plate under lateral forces (wind, earthquake) has been developed.
3. A practical and numerically stable dynamic iterative procedure has been developed, which shows that both the bottom uplifting mechanism and the out-of-round deformation of the tank shell have significant effects on the dynamic response behaviour of unanchored tanks.

10.2 CONCLUSIONS

Using the dynamic analysis procedure developed in this thesis, a broad 1/3 scaled model aluminium tank (12 feet diameter, 6 feet high) and a tall steel tank (28 feet diameter, 28 feet high) are analyzed for both unanchored and anchored conditions under the earthquake ground motion input. From the analysis results of the broad and tall tanks analyzed in this thesis, and comparison with the experimental observations, it is noted that some of the observations may be extrapolated to explain the response behaviour of other unanchored liquid storage tanks in general cases. The seismic response behaviour of unanchored cylindrical tanks may be summarized as follows:

1. The displacement response of an unanchored tank is much larger than that of the same under the anchored condition.

2. A simple assumed Ritz shape mode of the cantilever beam type, is not adequate to represent the deformation in the tank shell of an unanchored tank. The time histories of the radial displacements show that the negative (inwards) displacement is always much larger than that in the positive (outwards) direction.
3. For an initially perfect circular unanchored tank, the out-of-round deformation of the tank shell, combined with the bottom plate uplift, are significant mechanisms in the dynamic response of the tank system. On the other hand, an initially perfectly circular anchored tank remains circular throughout the dynamic response.
4. The dynamic overturning moment in an unanchored tank can be less than that in an anchored tank, as demonstrated in the two examples presented in this thesis. For the tall tank example, the overturning moment of anchored tank is more than twice that in the unanchored case. This is because the dissipation of the seismic energy by the rocking motion of the tank system more than offsets the increase in hydrodynamic loading due to increased displacement response in the unanchored case.
5. The timing of maximum bottom plate uplift coincides with that of the maximum overturning moment and maximum lateral resultant force, which indicates that the overturning moment is the most important factor directly relating to the uplift mechanism.
6. The convective hydrodynamic pressure component is strongly coupled with the impulsive component in determining the dynamic response of an unanchored tank, because the analytical results of the convective pressures show that there is a significant difference between that in the unanchored tank and anchored tank. On the other hand, the convective pressure in an anchored tank is weakly coupled with the impulsive pressure.

7. The maximum axial membrane stress near the bottom of the tank is observed to be related to the maximum overturning moment in both the unanchored and anchored tanks.
8. On the tank wall near the bottom, the axial stress distribution around the circumference of an unanchored tank is concentrated on the contact side of the tank shell. As a consequence, the compressive axial stress on the contact side of the tank shell is significantly higher than the tensile stress on the uplifted side at the time of maximum overturning moment.
9. There is relatively larger compressive hoop stress near the bottom on the contact side in an unanchored tank.
10. The axial stress distribution over the cylindrical thin shell in an unanchored tank is much more complicated than that in an anchored tank. The axial stress distribution in an anchored tank is a $\cos\theta$ type of distribution throughout the height of the tank.
11. Large compressive axial stress may occur at the middle or upper part of an unanchored tank shell, which is not directly related to the maximum overturning moment, but attributed to the higher modes in the dynamic response.
12. The frequency range of an unanchored tank is much wider than that of an anchored tank. Frequencies of an unanchored tank may be lower than that of an anchored tank because the additional flexibility from the uplifting bottom plate. On the other hand, the frequencies of an unanchored tank may be higher because of the more pronounced participations of the higher modes in the dynamic response. It is also noted that an unanchored tank may be more sensitive to the frequency content of the earthquake excitation than an anchored tank.

10.3 RECOMMENDATIONS FOR FURTHER STUDY

The research of seismic response behaviour of cylindrical liquid storage tanks has been carried out for decades. Optimal design procedures, particularly those for unanchored tank, are still not available. Based on the studies and research presented in this thesis, a better understanding of the seismic behaviour of cylindrical liquid storage tank has been achieved. Additional research effort to develop a practical design procedure for unanchored tank is still needed. Recommendations for future analytical and experimental studies on seismic behaviour of liquid storage tank are the following:

1. A simplified analytical procedure based on the analysis model presented in this thesis.
2. Better modelling of the structural characteristics of the tank, such as dimensions, material properties and plate thickness of the tank.
3. Better correlations between uplift and earthquake input.
4. Dynamic buckling of cylindrical liquid storage tanks.
5. Effect of out-of-round deformation on failure mechanisms.
6. Influence of vertical excitation on the uplift behaviour.
7. Interaction between the tank system and support foundation.
8. Design chart or table for earthquake resistant design.

Appendix A

In functional analysis theory [74], considering a differential equation of function $u(x)$ given in the form

$$\frac{d}{dx} \left(A(x) \frac{du}{dx} \right) + B(x)u(x) = -\lambda^2 p(x)u(x) \quad (A.1)$$

where λ is a constant parameter and $A(x)$, $B(x)$ and $p(x)$ are given real functions which satisfy the following conditions:

$A(x) > 0$, continuous in $[a, b]$, and continuously differentiable in (a, b)

$B(x)$ is continuous in $[a, b]$

$p(x) > 0$, continuous and integrable in (a, b)

and the boundary condition at $x = a$ and $x = b$ are given by the following:

$$\begin{aligned} \alpha u(a) + \beta u'(a) &= 0 & \alpha^2 + \beta^2 &\neq 0 \\ \gamma u(b) + \delta u'(b) &= 0 & \gamma^2 + \delta^2 &\neq 0 \end{aligned} \quad (A.2)$$

Definition: The parameter λ , for which a non-trivial solution $u(x)$ of the boundary problem Equations A.1-2 exists, is called an eigenvalue of this boundary value problem and the non-trivial solution $u(x)$ is the eigenfunction corresponding to the eigenvalue λ .

If $u(x)$ is an eigenfunction, so is all functions of the form $c u(x)$, c being an arbitrary constant. Two propositions are stated here without proof:

1. With the above assumptions on the boundary value problem, there are infinitely eigenvalues. All these eigenvalues are real and can be arranged in the order

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots$$

such that $\lim_{n \rightarrow \infty} \lambda_n = +\infty$

2. For each eigenvalue λ_i , there is exactly one corresponding eigenfunction $u_i(x)$.

For a series of eigenfunctions $u_1(x)$, $u_2(x)$, ..., $u_n(x)$, ..., in L_2 space, $L_2(a, b)$, the following theorem can be stated

Theorem: The series of eigenfunctions which satisfies Equation A.1, and boundary conditions Equation A.2 is complete and orthogonal in $L_2(a, b)$.

A L_2 space denoted by $L_2(a, b)$ is defined as the set of all measurable complex-valued functions such that the Lebesgue integral

$$\int_a^b |f(x)| dx \text{ is finite.}$$

The definition and properties of the Lebesgue integral can be found in any standard mathematics handbooks, for example, in reference[75].

Complete and orthogonal properties of $\cos n\theta$

The completeness and orthogonality of the series $\cos(n\theta)$ in $(0, \pi)$, can be proved with reference to the boundary value problem given by Equations A.1-2

$$\frac{d^2 u}{dx^2} = -n^2 u \tag{A.3}$$

$$u'(0) = 0 \quad u'(\pi) = 0$$

which corresponds to

$$A(x) = 1, \quad B(x) = 0, \quad p(x) = 1, \quad \lambda_n = n$$

$$\alpha = 0, \quad \beta = 1, \quad \gamma = 0, \quad \delta = 1$$

Complete and orthogonal properties of $\cos(\frac{\lambda}{H}z)$

Similarly, the series $\{\cos(\lambda_n x/H)\}$ is proven complete and orthogonal in $(0 < z < H)$ with reference to the boundary value problem

$$\begin{aligned}\frac{d^2 u}{dx^2} &= -\left(\frac{2n-1}{2}\pi\right)^2 u \\ u'(0) &= 0 \quad u(H) = 0\end{aligned}\tag{A.4}$$

which corresponds to

$$\begin{aligned}A(x) &= 1, & B(x) &= 0, & p(x) &= 1, & \lambda_n &= \frac{2n-1}{2}\pi \\ \alpha &= 0, & \beta &= 1, & \gamma &= 1, & \delta &= 0\end{aligned}$$

Complete and orthogonal properties of Bessel function

By definition, J_n , the $J_n(k_j^n x)$ is the solution of the Bessel differential equation

$$\frac{d}{dx}\left(x\frac{dJ_n}{dx}\right) + \left((k_j^n)^2 x - \frac{n^2}{x}\right)J_n = 0\tag{A.5}$$

Comparing Equation A.5 with A.1, the following can be identified

$$A(x) = x, \quad B(x) = -\frac{n^2}{x}, \quad p(x) = x, \quad \lambda_n = k_j^n$$

For Equation A.5, four possible boundary value problems can be defined according to the boundary conditions prescribed for the problem

$$J_n(0) = 0, \quad J_n(R) = 0 \quad (n \neq 0) \quad (A.6)$$

$$J'_0(0) = 0, \quad J_0(R) = 0 \quad (n = 0) \quad (A.7)$$

$$J_n(0) = 0, \quad J'_n(R) = 0 \quad (n \neq 0) \quad (A.8)$$

$$J'_0(0) = 0, \quad J'_0(R) = 0 \quad (n = 0) \quad (A.9)$$

Equations A.6 and A.7 correspond to the cases of which the parameters $\lambda_n = k_j^n$ are the zeros of Bessel function, whereas Equations A.8 and A.9 correspond the cases where the parameters λ_n are the zeros of the first derivative of the Bessel function.

For Equation A.6:

$$\alpha = 1, \quad \beta = 0, \quad \gamma = 1, \quad \delta = 0$$

For Equation A.7:

$$\alpha = 0, \quad \beta = 1, \quad \gamma = 1, \quad \delta = 0$$

For Equation A.8:

$$\alpha = 1, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 1$$

For Equation A.9:

$$\alpha = 0, \quad \beta = 1, \quad \gamma = 0, \quad \delta = 1$$

Therefore, it is proved that, for zeros of Bessel function and zeros of its first derivative, the n^{th} order Bessel functions $J_n(k_j^n x)$ are complete and orthogonal with the weighting function x in $(0, R)$, or

$$\int_0^R x J_n(k_j^n x) J_n(k_i^n x) dx = 0 \quad i \neq j$$

and

$$\int_0^R x J_n(k_j^n x) J_n(k_j^n x) dx = \frac{1}{2} \left\{ R^2 J_n'^2(k_j^n R) + \left[R^2 - \left(\frac{n}{k_j^n} \right)^2 \right] J_n^2(k_j^n R) \right\} \quad (A.10)$$

where $k_j^n R$ are the zeroes of Bessel function or its first derivative.

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