Asymptotic analysis of generating functions of alg-log type

by

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Abstract

A *decomposable combinatorial structure* is one that is made up of simpler objects which cannot be themselves further decomposed. These simpler structures are called *components*. For example, Dyck paths are made up of excursions, and graphs can be decomposed into their connected components. The size of an object is the number of elements it has. For example, a graph on n vertices is of size n , and a Dyck path with $2n$ steps has size $2n$.

Techniques from complex analysis have proven to be very valuable tools for asymptotically enumerating decomposable structures. The application of complex analysis techniques to combinatorics is known as *analytic combinatorics*. In this work we apply methods from analytic combinatorics to determine asymptotic enumeration results for several classes of decomposable structures. We also determine asymptotic expansions for some functions that are not combinatorial in nature but are of interest from a historical point of view.

We study two common classes of decomposable structures: the exp-log class, and the class of decomposable structures with alg-log components. There are many combinatorial objects that fall into one of these classes, including function graphs, 2-regular graphs, and sets of bicycles. In addition, we consider these structures with *restricted patterns*. A restricted pattern specifies that components of certain sizes must appear exactly a given number of times. For example, one might wish to know how many permutations of size n have exactly 2 cycles of size 8.

This work is divided into five parts: the first provides the necessary background on enumerative combinatorics and analytic combinatorics; the second part improves some previous results for decomposable structures with alg-log components; the third and fourth prove several new theorems for both labeled and unlabeled classes of structures with alg-log components that have not been previously considered; the final part studies exp-log structures with restricted patterns and provides several combinatorial examples.

KEYWORDS:

Analytic combinatorics; asymptotic enumeration; generating functions; restricted patterns

For my wife, Erin. Without her unconditional love and support, this work would not have been possible.

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List of Notations

$[z^n]f(z)$	the coefficient of z^n in the formal power series $f(z)$	10
$\arg(z)$	the principal argument of the complex number z	20
$\binom{a}{b}$	the binomial coefficient	10
$\mathbb{E}(X)$	the expected value of a random variable X	109
$\Gamma(z)$	the Gamma function	14
$\lceil a \rceil$	the least integer greater than or equal to a	34
$\lfloor a \rfloor$	the greatest integer less than or equal to a	21
$\mathbb{C}$	the set of all complex numbers	14
$\mathbb{N}$	the set of nonnegative integers	7
$\mathbb{R}$	the set of real numbers	32
$\mathbb{Z}$	the set of all integers	14
$\mathbb{Z}^+$	the set of positive integers	32
$\mathbb{Z}_{\leq 0}$	the set of nonpositive integers	14
$\Re(z)$	the real part of the complex number z	67
$\text{Var}(X)$	the variance of a random variable X	109

Chapter 1

Introduction

In the mid-1900s, Edward Maitland Wright wrote two papers [44, 45] examining the coefficients of certain power series with exponential singularities. These papers generalized results of Faber [13], Häusler [27], and Perron [33–35]. Wright also wrote a paper with Barbara Yates [46] that determined the asymptotic expansion of a general class of integrals.

The functions and integrals Wright studied arose in several areas of mathematics, including number theory (weighted integer partitions) and linear difference-differential equations. The techniques he used were primarily from complex analysis. Specifically he used the saddle point method and method of steepest descents to approximate several contour integrals. Though his research was not directly combinatorial in nature, many problems in combinatorics were later studied using functions and techniques similar to those used by Wright.

The study of decomposable combinatorial structures lends itself well to examination using techniques from complex analysis. A decomposable structure is one that is made up of simpler combinatorial objects called components, which cannot themselves be further decomposed. There are many examples of such structures in combinatorics: graphs decompose into connected components, permutations into cycles, Dyck paths into excursions, *etc.* These structures arise in many scientific

fields, including computer science, cryptography, biostatistics, and the analysis of algorithms.

In the enumeration of decomposable structures, the generating function is a powerful tool since the coefficients of a generating function can be used to enumerate many properties of these structures. However, it is often the case that these coefficients cannot be determined exactly. Instead, one often determines asymptotic expressions for these coefficients.

The main tools for finding such asymptotic expressions come from complex analysis. The application of these tools to combinatorics is commonly known as *analytic combinatorics*, and it developed much more recently than Wright's research. In 1990, Flajolet and Odlyzko [18] introduced a technique which they called *singularity analysis*, which applied to several classes of generating functions. Then, in 1994, Flajolet published papers with Grabner, Kirschenhofer, Prodinger, and Tichy [15] and with Golin [14] which developed *mellin transforms* as a tool for combinatorial enumeration.

As analytic combinatorics continued to be studied, some standard classes were defined. The *exp-log* (exponential-logarithmic) and *alg-log* (algebraic-logarithmic) classes were introduced by Flajolet and Soria in 1990 [20] and 1993 [21], respectively. In particular, they studied the probability distribution of the number of components in random decomposable structures in these classes (see [3, 5, 6, 36] for related research). Exp-log functions are so called because they are exponential functions with logarithmic components, whereas an alg-log function is the product of an algebraic factor with a logarithmic factor. The relevant coefficients in the latter are the algebraic exponent and the logarithmic exponent.

The decomposable structures studied by Flajolet and Soria are some of a large number of structures that can be constructed in a standardized method using the sequence (SEQ), cycle (CYC), and set (SET) constructions. A full description of these methods is given in [19] with a wide variety of examples.

Gourdon, in his doctoral thesis [24] and a conference proceedings [25], further extended the understanding of the exp-log and alg-log classes of combinatorial structures. He determined the distributions of the r th largest components of structures in both classes. Once again, the main methods used in that thesis were contour integration techniques from complex analysis. In 2001, Panario and Richmond [32] found the distribution of the r th smallest component of structures in the exp-log class. Omar, Panario, Richmond, and Whitely [31] then used elementary techniques (as opposed to analytic techniques) to examine the smallest components in exp-log structures.

Most recently, Dong, Gao, and Panario [11] formalized the notion of decomposable structures with *restricted patterns*. These are decomposable structures where some components of given sizes must appear a precise number of times. First, Dong *et al.* determined the generating function for decomposable structures constructed via the SET construction, and they gave some asymptotic probability results for the exp-log class. Then, with Richmond [12], they extended their previous results to the r th smallest components of decomposable structures with alg-log components and restricted patterns.

In this thesis we use techniques from complex analysis to find asymptotic expressions for the coefficients of several classes of power series. There are two main purposes of the work in this thesis. First, we examine many power series that lend themselves well to combinatorial examples, such as sets of bicycles, and functional graphs. In doing so we prove several new theorems, and we also extend some existing theorems. Second, we examine classes of power series that are not necessarily combinatorial but are natural extensions of the work initiated by Wright.

The work in this thesis falls naturally into five parts.

In Chapter 2, we briefly review the mathematical theory required for the following chapters. First, in Section 2.1, we give a formal definition of a combinatorial class. Next, we describe generating functions and their use in enumerative combinatorics,

and we then define the decomposable combinatorial classes that appear throughout this thesis. In Section 2.3, we describe the complex analysis technique of singularity analysis as it applies to asymptotic enumeration. We conclude the chapter with Section 2.4, which gives a description of the saddle point method — another technique from complex analysis that has applications in enumerative combinatorics.

We revisit and extend some previously known theorems for structures with certain alg-log components in Chapter 3. This chapter is an extension of work by Dong *et al.* [11, 12]. In Section 3.3, we use singularity analysis to generalize theorems by considering a wider range of generating function and by relaxing the assumptions required for the proof. Specifically, we greatly extend the range of the algebraic exponent considered, and we relax the conditions required of the restricted patterns. In the following section, we use the saddle point method to improve the accuracy of the asymptotic expansion of the coefficients of a generating functions with alg-log components whose algebraic exponent is negative and whose logarithmic exponent is zero.

In Chapter 4, we prove new asymptotic enumeration results for both labeled and unlabeled decomposable structures with alg-log components and restricted patterns. We consider alg-log components with positive algebraic exponent and non-zero logarithmic exponent. The generating functions examined in this chapter have previously not been considered in any literature. The main analytic technique used in Chapter 4 is the saddle point method.

In Chapter 5, we consider decomposable structures whose components are of alg-log type with only a positive logarithmic exponent, *i.e.*, the algebraic exponent is zero. This chapter is divided into three parts. In Section 5.1, we consider alg-log components with logarithmic exponent no less than two, and we use the saddle point method to determine asymptotic enumeration results. These generating functions represent combinatorial objects such as sets of bicycles, so we consider them with

restricted patterns. Next, in Section 5.2, we consider logarithmic exponents between zero and one. These generating functions do not correspond to any combinatorial objects, but they fill an interesting gap in the literature, and we determine asymptotic results for their coefficients using singularity analysis. Finally, we offer a conjecture for the asymptotic expression of the coefficients of generating functions in the remaining range of logarithmic exponent—between one and two.

Finally, in Chapter 6, we extend the results of Gourdon [24] by examining the largest component of an exp-log structure with a restricted pattern. We conclude the thesis with some questions that we came across in the course of this study that are left for further investigation.

Chapter 2

Mathematical Background

The purpose of this chapter is to briefly introduce enumerative combinatorics and to describe the main mathematical tools we use in this thesis. In Section 2.1, we define combinatorial structures, generating functions, and decomposable structures. Sections 2.3 and 2.4 describe singularity analysis and the saddle point method, respectively. Those two tools are the main techniques of asymptotic enumeration used in later chapters. For more comprehensive descriptions of the concepts in this chapter, refer to [19, 39, 43].

2.1 Enumerative Combinatorics

Stanley [39] writes in *Enumerative Combinatorics Volume 1*,

The basic problem of enumerative combinatorics is that of counting the number of elements of a finite set. Usually we are given an infinite class of finite sets S_i where i ranges over some index set $I \dots$, and we wish to count the number $f(i)$ of elements of each S_i “simultaneously.”

For example, consider the class of integer permutations. For a given integer n , there are precisely $n!$ permutations. In this case, we can consider permutations to be

the infinite class of finite sets S_i , where S_i is the set of all permutations of size i , and the index set is simply $\mathbb{N}$.

The main purpose of this thesis is to enumerate various combinatorial objects according to their size. These objects can be quite complicated, and directly counting them may not be feasible. We must therefore use more powerful tools of enumeration to obtain the desired results.

2.1.1 Combinatorial Structures

As in Flajolet and Sedgewick's *Analytic Combinatorics* [19], in this thesis we consider a combinatorial class to be a set of discrete objects, *i.e.*, objects that can be described finitely by some rules of construction. There are many examples of such classes including words, trees, graphs, permutations, polynomials over finite fields, functions from a finite set into itself, and so on.

Definition 1 (Flajolet and Sedgewick [19], p. 16). A *combinatorial class* is a finite or denumerable set of elements on which a size function is defined, satisfying the following conditions:

1. the size of an element is a nonnegative integer;
2. the number of elements of any given size is finite.

An element of a combinatorial class is also called a *combinatorial structure*.

One measure of a combinatorial structure is its size. For example, a permutation on n elements, a polynomial over a finite field of degree n , and a binary tree on n vertices are all considered to be of size n . For a given combinatorial class $\mathcal{A}$, the size of an element $\alpha \in \mathcal{A}$ is denoted by $|\alpha|$. Furthermore, we let a_n denote the number of objects in $\mathcal{A}$ of size n .

We focus mainly on the enumeration of objects according to their size, but in combinatorics there are many other characteristic parameters by which one might enumerate objects. For example, one might wish to know the number of Dyck paths with a given number of peaks. Here Dyck paths are defined to be paths in the (x, y) plane from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$ that never pass below the x -axis. Figure 1 is an example of a Dyck path of length 22 with 5 peaks.

Figure 1: A Dyck path of length 22 with 5 peaks.

Definition 2 (Flajolet and Sedgewick [19], p. 16). The *counting sequence* of a combinatorial class $\mathcal{A}$ is the sequence of integers $(a_n)_{n \geq 0}$ where a_n is the number of objects in a class $\mathcal{A}$ that have size n .

2.1.2 Generating Functions

Generating functions are formal power series whose coefficients represent information about the counting sequences of combinatorial classes. This is useful because it often allows to represent counting sequences in a compact form. There are two categories of generating functions.

Definition 3. The *ordinary generating function* (OGF) of a sequence (a_n) is the power series

$$\sum_{n \geq 0} a_n z^n,$$

and the *exponential generating function* (EGF) of (a_n) is

$$\sum_{n \geq 0} \frac{a_n}{n!} z^n.$$

Though the OGF and EGF of (a_n) differ only by a factor $1/n!$, they typically count different types of combinatorial structures. Specifically, OGFs usually count *unlabeled structures*, whereas EGFs usually count *labeled structures*.

Labeled structures are those in which the atoms of an object (for example, nodes in a graph or tree) are distinguished from one another by the fact that they have distinct labels. Without loss of generality, we assume that objects of size n are labeled by the set of integers $\{1, 2, \dots, n\}$. The classic example of a labeled structure is integer permutations, which may be viewed as a linear arrangement of distinct integers, as shown Figure 2.

1	2	3	4	5	6	7	8	9	10	11	12
7	11	12	5	3	2	8	1	4	9	10	6

Figure 2: The permutation $\rho = (1\ 7\ 8)(2\ 11\ 10\ 9\ 4\ 5\ 3\ 12\ 6)$.

The EGF for the class $\mathcal{P}$ of permutations is therefore

$$P(z) = \sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

Notice that we have not mentioned the convergence of generating functions. That is because analyticity does not have any bearing on the coefficients of the generating function for a given combinatorial class. However, as we will see throughout this thesis, we can indeed use the analytic properties of a generating function to determine information about its coefficients.

Conversely, the atoms of unlabeled structures bear no such labels. For example,

consider the class $\mathcal{C}$ of expressions made up of pairs correctly matched parentheses, *i.e.*, no initial string of parentheses has more right parentheses than left. Figure 3 gives the elements of $\mathcal{C}$ for sizes 0 through 3.

•, (), ()() (()), ()()() ()()() ((()()) ((()() ((()))), ...

Figure 3: Elements of $\mathcal{C}$ for size 0, 1, 2, 3.

Therefore, $\mathcal{C}$ has the counting sequence 1, 1, 2, 5, ... In fact, these objects are counted by the Catalan numbers (see [38] for dozens of examples of structures counted by these numbers), whose OGF is

$$C(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} z^n = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

Finally, we let $[z^n]f(z)$ denote the coefficient of z^n in the formal power series $f(z) = \sum f_n z^n$, *i.e.*,

$$[z^n] \sum_{n \geq 0} f_n z^n = f_n.$$

From this choice of notation, it is plain that for a power series $f(z)$ and constant ρ , $[z^n]f(\rho z) = \rho^n [z^n]f(z)$ —a fact we use frequently in this thesis.

2.1.3 Decomposable Structures

Often the structures in a combinatorial class can be uniquely decomposed into simpler objects. In fact, many structures can be described as being constructed from simpler elements. For example, a permutation may be thought of as a set of disjoint cycles. Another example is polynomials over a finite field, which are constructed as products of irreducible factors.

The size of such an object is equal to the sum of the sizes of its components. For

example, the permutation

$$\rho = (134)(27)(58)(6)$$

has size $3 + 2 + 2 + 1 = 8$.

In this thesis, we focus on two related constructions: the *set* construction in the labeled case and the *multiset* construction, which we describe below. Here multisets are unordered collections of elements with arbitrary repetitions allowed. For example, $\{1, 1, 4, 3, 1\}$ is a multiset of integers.

Flajolet and Sedgewick [19] give a comprehensive description of several such constructions. We state without proof two required results of theirs.

Lemma 1 (p. 26, [19]). *Let $\mathcal{B}$ be an unlabeled combinatorial class with OGF $B(z) = \sum_{n \geq 1} b_n z^n$, and let $\mathcal{A}$ be the collection of multisets of elements of $\mathcal{B}$. Then we write $\mathcal{A} = \text{MSET}(\mathcal{B})$, and the OGF for $\mathcal{A}$ is given by*

$$A(z) = \prod_{n \geq 1} (1 - z^n)^{-b_n} = \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} B(z^k) \right).$$

Lemma 2 (pp. 103–104, [19]). *Let $\mathcal{B}$ be a labeled combinatorial class with EGF $B(z) = \sum_{n \geq 1} \frac{b_n}{n!} z^n$, and let $\mathcal{A}$ be the collection of sets of elements of $\mathcal{B}$. Then we write $\mathcal{A} = \text{SET}(\mathcal{B})$, and the EGF for $\mathcal{A}$ is given by $A(z) = \exp(B(z))$.*

2.2 Asymptotic Notation

Throughout this thesis we use *big-oh notation* to illustrate asymptotic growth. Specifically, we use $o(f(z))$, $O(f(z))$, and $\Theta(f(z))$ as defined below.

Definition 4. Let $f(z)$ and $g(z)$ be two functions defined on the non-negative real numbers. Then we write

1. $f(z) = O(g(z))$ if and only if there exist positive constants k and N such that

$$|f(z)| \leq k|g(z)| \text{ for all } z > N;$$

2. $f(z) = o(g(z))$ if and only if

$$\lim_{z \rightarrow \infty} \frac{f(z)}{g(z)} = 0;$$

3. $f(z) = \Theta(g(z))$ if and only if there exists a positive constant m such that

$$\lim_{z \rightarrow \infty} \frac{|f(z)|}{|g(z)|} = m.$$

Note that the definition of $\Theta(f(z))$ given in Definition 4 differs slightly from the definition commonly used in computer science. However, the definition we use is more appropriate for our work.

2.3 Singularity Analysis

So far the examples given in this chapter have been of combinatorial structures where the number of elements of size n has a closed formula in terms of n . However, there are many structures for which such a closed formula is not known even if the generating function is. Fortunately, there has been much work done applying techniques from complex analysis to extract coefficients of generating functions, which is practical for combinatorics (for example, [16, 17, 23]). Typically these techniques do not provide exact formulae for the desired quantities: instead they give asymptotic information. There are several very good surveys of combinatorial asymptotics, including [2–4, 30].

In 1990, Flajolet and Odlyzko published *Singularity analysis of generating functions* [18], which introduced a standardized *singularity analysis* by which one can

Figure 4: The region $\Delta(\eta, \phi)$.

analyze the coefficients of certain generating functions. They restricted their attention to functions with a unique dominant singularity, although their methods may be extended very easily to a function with a finite number of distinct singularities lying on the circle of convergence [19]. Their approach is based on contour integration using the Cauchy integral formula and Hankel-like contours [42]. We use similar contours in Sections 3.3 and 5.2. What follows is a brief description of their results.

Let $\Delta \equiv \Delta(\phi, \eta)$ denote the closed domain

$$\Delta(\phi, \eta) = \{z : |z| \leq 1 + \eta, z \neq 1, |\arg(z - 1)| \geq \phi\}, \quad (1)$$

where $\eta > 0$ and $0 < \phi < \pi/2$. Such a region is known as a *delta domain*, and it has the shape of an indented disk as shown in Figure 4 [18]. This region is also colloquially known as the “Pac-Man” domain because of its resemblance to the popular video game character. A function that is analytic in such a delta domain is said to be Δ -*analytic*.

2.3.1 Transfer Theorems

We now state without proof three theorems and one corollary that were proven by Flajolet and Odlyzko [18] that apply to a large class of functions that arise in enumerative combinatorics. Theorem 1 is an asymptotic version of the binomial theorem.

Theorem 1 (p. 381, [19]). *Let α be an arbitrary complex number in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. The coefficient of z^n in*

$$f(z) = (1 - z)^{-\alpha}$$

admits for large n a complete asymptotic expansion in descending powers of n ,

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{k=1}^{\infty} \frac{e_k}{n^k} \right),$$

where e_k is a polynomial in α of degree $2k$, and $\Gamma(z)$ is the Gamma function [1]. In particular,

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \frac{\alpha(\alpha-1)}{2n} + \frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24n^2} + \frac{\alpha^2(\alpha-1)^2(\alpha-2)(\alpha-3)}{48n^3} + O\left(\frac{1}{n^4}\right) \right).$$

Flajolet and Odlyzko [20] showed that the proof of Theorem 1 is easily extended to a wider class of singular functions, most notably ones that involve logarithmic terms.

Theorem 2 (p. 385, [19]). *Let α be an arbitrary complex number in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and let β be an arbitrary complex number. The coefficient of z^n in the function*

$$f(z) = (1 - z)^{-\alpha} \left(\frac{1}{z} \log \frac{1}{1 - z} \right)^{\beta}$$

admits for large n a full asymptotic expression in descending powers of $\log n$,

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^\beta \left[1 + \frac{C_1}{\log n} + \frac{C_2}{(\log n)^2} + \cdots \right],$$

where

$$C_k = \binom{\beta}{k} \Gamma(\alpha) \frac{d^k}{ds^k} \frac{1}{\Gamma(s)} \Big|_{s=\alpha}.$$

Theorem 3 (p. 390, [19]). *Let α be an arbitrary real number, let β be an arbitrary complex number, and let $f(z)$ be a function that is Δ -analytic.*

(i) *Assume that $f(z)$ satisfies in the intersection of a neighbourhood of 1 with its Δ -domain the condition*

$$f(z) = O \left((1-z)^{-\alpha} \left(\log \frac{1}{1-z} \right)^\beta \right).$$

Then

$$[z^n]f(z) = O \left(n^{\alpha-1} (\log n)^\beta \right).$$

(ii) *Assume that $f(z)$ satisfies in the intersection of a neighbourhood of 1 with its Δ -domain the condition*

$$f(z) = o \left((1-z)^{-\alpha} \left(\log \frac{1}{1-z} \right)^\beta \right).$$

Then

$$[z^n]f(z) = o \left(n^{\alpha-1} (\log n)^\beta \right).$$

Theorem 2 immediately implies the following corollary.

Corollary 1 (p. 392, [19]). *Assume that $f(z)$ is Δ -analytic and*

$$f(z) \sim (1-z)^{-\alpha} \quad \text{as } z \rightarrow 1, \quad z \in \Delta,$$

with $\alpha \notin \{0, -1, -2, \dots\}$. Then,

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}.$$

Theorem 3 and Corollary 1 are known as the *transfer theorems* because they transfer the error terms from singular functions to their coefficient expansions. Throughout this thesis, we will make frequent use of the theorems in this section.

2.4 Saddle Point Method

Singularity analysis is a very powerful tool, but it is limited in its application to generating functions of a certain form. Fortunately, there are other methods we may employ as well, including the *saddle point method*. Flajolet and Sedgewick [19] give the following characterization of this method:

Precisely, the principle of the *saddle point method* for the estimation of contour integrals is to choose a path crossing a saddle point, then estimate the integrand locally near this saddle point (where the modulus of the integrand achieves its maximum on the contour), and deduce, by local approximations and termwise integration, an asymptotic expansion of the integral itself. Some sort of “localization” or “concentration” property is required to ensure that the contribution near the saddle point captures the essential part of the integral.

The scope of singularity analysis is limited to functions that have moderate (e.g. polynomial) growth at their singularities [19]. Conversely, the saddle point method applies to functions that grow exponentially near their singularities. We only give a

brief overview of this method. For a more comprehensive description, we refer the reader to Flajolet and Sedgewick's *Analytic Combinatorics* [19].

Consider a generating function $G(z)$, which we assume to be analytic at the origin and to have nonnegative coefficients. Our goal is to seek an asymptotic form of the coefficients of $G(z)$, which are given in integral form by the Cauchy integral formula as

$$[z^n]G(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} G(z) \frac{dz}{z^{n+1}}, \quad (2)$$

where $\mathcal{C}$ is a positively oriented contour that encircles the origin and is within the domain of analyticity of $G(z)$.

To utilize the saddle point method, we specifically choose $\mathcal{C}$ to be a circle centred at the origin and passing through (or very near to) a saddle point on the positive real line. We make use of polar coordinates and set $z = re^{i\theta}$, where r will be chosen to be equal to (or very near to) the positive saddle point. Thus, (2) becomes

$$[z^n]G(z) = \frac{r^{-n}}{2\pi} \int_{-\pi}^{+\pi} G(re^{i\theta}) e^{-in\theta} d\theta. \quad (3)$$

For the saddle point method to work, the main contribution of the integral (3) must come from an arc of the circle of radius r determined by $|\theta| \leq \theta_0$ for some suitably chosen θ_0 . Therefore we divide $\mathcal{C}$ into two contours:

$$\mathcal{C}^{(0)} = \{z = re^{i\theta}, |\theta| \leq \theta_0\},$$

$$\mathcal{C}^{(1)} = \{z = re^{i\theta}, |\theta| > \theta_0\}.$$

Letting $f(z) = \log G(z) - n \log z$, we also require that on $\mathcal{C}^{(0)}$, a quadratic approximation for $f(z)$ holds. That is,

$$f(re^{i\theta}) = f(r) + rf'(r)(e^{i\theta} - 1) + \frac{r^2}{2}f''(r)(e^{i\theta} - 1)^2 + O(r^3h'''(r)\theta^3),$$

must be accurately approximated by $f(r) - \frac{b(r)}{2}\theta^2$ for $|\theta| \leq \theta_0$, where

$$b(r) = rf'(r) + r^2 f''(r).$$

Therefore it is natural to choose r such that $f'(r) = 0$, *i.e.*,

$$r \frac{G'(r)}{G(r)} = n$$

which is a version of the *saddle point equation* in polar coordinates. This choice of r gives locally a quadratic approximation of $f(re^{i\theta})$ without linear terms, *i.e.*,

$$f(re^{i\theta}) - f(r) = -\frac{1}{2}b(r)\theta^2 + o(\theta^3),$$

which is valid for fixed r as $\theta \rightarrow 0$.

The angle θ_0 is chosen as a function of n (or r) such that

$$r^2 f''(r)\theta_0^2 \rightarrow \infty, \quad r^3 f'''(r)\theta_0^3 \rightarrow 0, \quad (4)$$

as $n \rightarrow \infty$. The condition of (4) is known as the *saddle point dimensioning heuristic*.

Then

$$\begin{aligned} [z^n]G(z) &= \frac{r^{-n}}{2\pi} \int_{-\theta_0}^{+\theta_0} G(re^{i\theta}) e^{-in\theta} d\theta + O\left(\frac{r^{-n}}{2\pi} \int_{\theta_0}^{+\pi} G(re^{i\theta}) e^{-in\theta} d\theta\right) \\ &= \frac{r^{-n}}{2\pi} \int_{-\theta_0}^{+\theta_0} \exp\left(f(r) - \frac{1}{2}b(r)\theta^2 + o(1)\right) d\theta + O\left(\frac{r^{-n}}{2\pi} \int_{\theta_0}^{+\pi} G(re^{i\theta}) e^{-in\theta} d\theta\right). \end{aligned} \quad (5)$$

The final steps of the saddle point method are to prove that the first integral in (5) can be approximated by a Gaussian integral and that the second integral in (5) is negligible compared to that Gaussian integral.

Chapter 3

Improved Results for Known Cases with Alg-Log Components

Figure 5: An illustration of the ranges of α and β examined in Chapter 3: $\alpha > 0$, $\beta \in \mathbb{R}$ and $\alpha < 0$, $\beta = 0$.

3.1 Alg-Log Components

In this chapter we extend some of the known results from Dong *et al.* [12] for decomposable objects with alg-log components. We begin by defining the *alg-log* class of

component generating functions.

Definition 5. Let

$$\Delta(\nu, \theta) = \{z : |z| \leq \rho(1 + \nu), z \neq \rho, |\arg(z - \rho)| \geq \theta\}$$

for some constants $\rho > 0, \nu > 0$, and $0 < \theta < \pi/2$. Then $C(z)$ is of *alg-log type* at singularity ρ with algebraic exponent α and logarithmic exponent β , if $C(z)$ is analytic in $\Delta(\nu, \theta)$, and for some constants c and d ,

$$C(z) = c + d(1 - z/\rho)^\alpha \left(\log \frac{1}{1 - z/\rho} \right)^\beta (1 + o(1)), \quad (6)$$

as $z \rightarrow \rho$ in $\Delta(\nu, \theta)$.

3.2 Restricted Patterns

In [12], Dong *et al.* consider decomposable objects with two ranges of exponents. First, they assume $0 < \alpha < 1$ with $\beta \in \mathbb{R}$. We improve this range to $\alpha > 0, \beta \in \mathbb{R}$ in Section 3.3. Next, they consider $\alpha < 0, \beta = 0$. In Section 3.4 we consider the same range. In both sections we consider structures with *restricted patterns*, which we define as in [11].

Definition 6. The *restricted pattern* of an object of size n is a mapping $S : J \mapsto \mathbb{N}$, where J is a set of components' sizes, $\mathbb{N}$ is the set of nonnegative integers, and $S(j)$ is the number of components of size j . If we do not have a restricted pattern, J is the empty set. For convenience we also use the notation

$$S = \left[\prod_{j \in J} j^{S(j)} \right].$$

For example, let $S = \{(1, 5), (2, \lfloor \log n \rfloor), (\lfloor \sqrt{n} \rfloor, 3)\}$. Then the pattern S consists of 5 elements of size 1, $\lfloor \log n \rfloor$ of size 2, and 3 of size $\lfloor \sqrt{n} \rfloor$, and we denote S by $[1^5 2^{\lfloor \log n \rfloor} \lfloor \sqrt{n} \rfloor^3]$.

Our goal in this chapter is to enumerate objects that contain precisely $S(j)$ components of size j for each $j \in J$; in these objects, components whose sizes are not in J may appear any number of times. Hence a restricted pattern can be thought of as a compulsory set of component sizes for a given object.

As in [11], $C(z)$ will always denote a component generating function. For an integer k , let C_k be the number of components of size k . Then we have

$$C(z) = \begin{cases} \sum_{k=1}^{\infty} \frac{C_k}{k!} z^k & \text{in the labeled case,} \\ \sum_{k=1}^{\infty} C_k z^k & \text{in the unlabeled case,} \end{cases}$$

i.e., $C(z)$ is an exponential generating function in the labeled case and an ordinary generating function in the unlabeled case.

Furthermore, denote by $C(z; J)$ the generating function for the components whose sizes are in the given set J , *i.e.*,

$$C(z; J) = \begin{cases} \sum_{j \in J} \frac{C_j}{j!} z^j & \text{in the labeled case,} \\ \sum_{j \in J} C_j z^j & \text{in the unlabeled case.} \end{cases}$$

In order to save space, we will also use the following notation:

$$\mathcal{S} = \begin{cases} \prod_{j \in J} \left(\frac{C_j}{j!} \right)^{S(j)} \frac{1}{S(j)!} & \text{in the labeled case,} \\ \prod_{j \in J} \binom{C_j + S(j) - 1}{S(j)} & \text{in the unlabeled case.} \end{cases}$$

Let $m' = \sum_{j \in J} jS(j)$ denote the size of the restricted pattern, and $m = n - m'$ denote the total size of the unrestricted components in an object of size n with restricted pattern S of size m' . In this thesis we will also frequently use the following lemma, which was proven by Dong *et al.* [12].

Lemma 3. *Let $S : J \mapsto \mathbb{N}$ be a given restricted pattern and $F(z; S)$ be the generating function of structures with the restricted pattern S . For the labeled case,*

$$F(z; S) = \mathcal{S} z^{m'} \exp(C(z) - C(z; J)).$$

For the unlabeled case,

$$F(z; S) = \mathcal{S} z^{m'} \exp \left(\sum_{k \geq 1} \frac{C(z^k) - C(z^k; J)}{k} \right).$$

3.3 Alg-Log Components with Positive Algebraic Exponent

This section is based on Section 3 of [12] by Dong *et al.* However, rather than assume $0 < \alpha < 1$ and $\beta \in \mathbb{R}$ as they did, we consider instead all $\alpha > 0$ and $\beta \in \mathbb{R}$. The main analytic tool in this section is singularity analysis.

In this section we only consider labeled objects. However, the method used to prove Theorem 4 can be easily modified to prove an analogous theorem for unlabeled objects. We choose not to include this proof because it is essentially identical to the proof of Theorem 3.2 in [12].

The statement of Theorem 4 is virtually identical to that of Theorem 3.1 in [12]. Specifically, both theorems give the same expression for $[z^n]F(z; S)$. However, in that paper the following restrictions on the pattern S were required:

$$\begin{aligned} \hat{j} &= O(m/\log m) \\ \sum_{j \in J} j^{-\alpha} (\log j)^\beta &= o(m^{1-\alpha} (\log m)^{\beta-1}), \end{aligned} \tag{7}$$

where $\hat{j}$ is defined to be the maximum element in J . In Theorem 4, the only restriction required is $\hat{j} = o(m/\log m)$, and there is no explicit restriction on the sum (7). Strictly speaking, the restriction $\hat{j} = o(m/\log m)$ is stronger than $\hat{j} = O(m/\log m)$. However, in practice this is not a substantial difference.

Theorem 4. *Assume $\alpha > 0$, $\alpha \notin \mathbb{Z}$, and*

$$\hat{j} = o(m/\log m). \tag{8}$$

Then, as $m \rightarrow \infty$,

$$[z^n]F(z; S) \sim de^c \frac{\mathcal{S}}{\Gamma(-\alpha)} m^{-\alpha-1} (\log m)^\beta \rho^{-m} \exp(-C(\rho; J)),$$

uniformly for any S satisfying (8).

Proof. Since $\alpha > 0$,

$$\begin{aligned}
F(z; \emptyset) &= \exp(C(z)) = e^c \exp \left(d(1 - z/\rho)^\alpha \left(\log \frac{1}{1 - z/\rho} \right)^\beta (1 + o(1)) \right) \\
&= e^c \left(1 + d(1 - z/\rho)^\alpha \left(\log \frac{1}{1 - z/\rho} \right)^\beta (1 + o(1)) \right. \\
&\quad \left. + o \left(d(1 - z/\rho)^\alpha \left(\log \frac{1}{1 - z/\rho} \right)^\beta (1 + o(1)) \right) \right) \\
&= e^c + de^c(1 - z/\rho)^\alpha \left(\log \frac{1}{1 - z/\rho} \right)^\beta (1 + o(1)),
\end{aligned}$$

as $z \rightarrow \rho$ in $\Delta(\rho, \theta)$. It follows from Theorems 2 and 3 that

$$[z^n]F(z; \emptyset) \sim e^c \frac{d}{\Gamma(-\alpha)} \rho^{-n} n^{-\alpha-1} (\log n)^\beta,$$

and

$$[z^n]C(z) \sim \frac{d}{\Gamma(-\alpha)} \rho^{-n} n^{-\alpha-1} (\log n)^\beta. \quad (9)$$

The goal now is to estimate $[z^n]F(z; S)$. Let $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ be positively oriented with

$$\begin{aligned}
\gamma_1 &= \left\{ z : |z - 1| = \frac{1}{m}, \theta < \arg(z - 1) < 2\pi - \theta \right\}, \\
\gamma_2 &= \left\{ z : \frac{1}{m} \leq |z - 1| \leq E \frac{\log m}{m}, \arg(z - 1) < \theta \right\}, \\
\gamma_3 &= \left\{ z : |z| = 1 + A \frac{\log m}{m}, \theta < \arg(z - 1) < 2\pi - \theta \right\}, \\
\gamma_4 &= \left\{ z : \frac{1}{m} \leq |z - 1| \leq E \frac{\log m}{m}, \arg(z - 1) < 2\pi - \theta \right\},
\end{aligned}$$

where $A = \alpha + 2$, and $E \in \mathbb{R}$ satisfies

$$\left| 1 + E \frac{\log m}{m} e^{i\theta} \right| = 1 + A \frac{\log m}{m}.$$

Figure 6 illustrates the contour γ .

Figure 6: The contour γ .

Then by Lemma 3,

$$[z^n]F(z; S) = \rho^{-n}[z^n]F(\rho z; S) = \rho^{-m} \mathcal{S} \frac{1}{2\pi i} \int_{\gamma} f(z) dz, \quad (10)$$

where

$$f(z) = z^{-m-1} \exp(C(\rho z) - C(\rho z; J)).$$

On γ , $|z| \leq 1 + A \frac{\log m}{m}$, and since $\hat{j} = o\left(\frac{m}{\log m}\right)$, we have

$$|z^j| \leq \left(1 + A \frac{\log m}{m}\right)^j = 1 + O\left(j \frac{\log m}{m}\right) = O(1).$$

From (9), we have

$$\sum_{j \in J} C_j \frac{\rho^j}{j!} = O(1), \quad (11)$$

which implies that on γ ,

$$\sum_{j \in J} C_j \frac{(\rho z)^j}{j!} = O(1). \quad (12)$$

Note that (11) is also implied by the fact that as $z \rightarrow \rho$ in $\Delta(\nu, \theta)$, $C(z) \rightarrow c$.

From (6) we obtain the following expression for $f(z)$, which is valid on γ :

$$f(z) = z^{-m-1} \exp \left(c + d(1-z)^\alpha \left(\log \frac{1}{1-z} \right)^\beta (1 + o(1)) - \sum_{j \in J} C_j \frac{(\rho z)^j}{j!} \right).$$

Because of (12), we obtain the following expression for $f(z)$, which depends on the Taylor expansion of the exponential function:

$$\begin{aligned} f(z) &= e^c \exp \left(- \sum_{j \in J} C_j \frac{\rho^j}{j!} \right) z^{-m-1} \\ &\quad \times \exp \left(d(1-z)^\alpha \left(\log \frac{1}{1-z} \right)^\beta (1 + o(1)) + \sum_{j \in J} C_j \frac{\rho^j}{j!} (1 - z^j) \right). \end{aligned}$$

Next, let

$$\begin{aligned} H(z) &= d(1-z)^\alpha \left(\log \frac{1}{1-z} \right)^\beta (1 + o(1)), \\ R(z; J) &= \sum_{j \in J} C_j \frac{\rho^j}{j!} (1 - z^j), \end{aligned}$$

and

$$g(z; J) = H(z) + R(z; J).$$

Then

$$f(z) = e^c \exp(-C(\rho; J)) z^{-m-1} \left(1 + \sum_{k=1}^{A-1} \frac{g(z; J)^k}{k!} + O(g(z; J)^A) \right). \quad (13)$$

We substitute (13) for $f(z)$ in (10) to obtain

$$\begin{aligned} [z^n]F(z; S) &= \rho^{-m} \mathcal{S} e^c \exp(-C(\rho; J)) \frac{1}{2\pi i} \int_{\gamma} z^{-m-1} \left(1 + \sum_{k=1}^{A-1} \frac{g(z; J)^k}{k!} + O(g(z; J)^A) \right) dz \\ &= \rho^{-m} \mathcal{S} e^c \exp(-C(\rho; J)) \\ &\quad \times \frac{1}{2\pi i} \left(\int_{\gamma} z^{-m-1} dz + \int_{\gamma} z^{-m-1} \sum_{k=1}^{A-1} \frac{g(z; J)^k}{k!} dz + \int_{\gamma} z^{-m-1} O(g(z; J)^A) dz \right). \end{aligned} \quad (14)$$

We now consider each integral in (14) separately. By the Cauchy integral theorem,

$$\frac{1}{2\pi i} \int_{\gamma} z^{-m-1} dz = 0.$$

Next, we estimate the integral

$$\frac{1}{2\pi i} \int_{\gamma} z^{-m-1} \sum_{k=1}^{A-1} \frac{g(z; J)^k}{k!} dz \quad (15)$$

using the binomial theorem:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} z^{-m-1} \sum_{k=1}^{A-1} \frac{g(z; J)^k}{k!} dz &= \sum_{k=1}^{A-1} \frac{1}{k!} \frac{1}{2\pi i} \int_{\gamma} z^{-m-1} (H(z) + R(z; J))^k dz \\ &= \sum_{k=1}^{A-1} \frac{1}{k!} \frac{1}{2\pi i} \int_{\gamma} \sum_{\ell=0}^k \binom{k}{\ell} H(z)^{\ell} R(z; J)^{k-\ell} z^{-m-1} dz \\ &= \sum_{k=1}^{A-1} \frac{1}{k!} \sum_{\ell=0}^k \binom{k}{\ell} \frac{1}{2\pi i} \int_{\gamma} H(z)^{\ell} R(z; J)^{k-\ell} z^{-m-1} dz, \end{aligned}$$

which we examine in three cases.

Case I: $\ell = 0$.

In the case $\ell = 0$,

$$\frac{1}{2\pi i} \int_{\gamma} R(z; J)^k z^{-m-1} dz = [z^m] \left(\sum_{j \in J} C_j \frac{\rho^j}{j!} (1 - z^j) \right)^k = 0,$$

since $\hat{j} = O\left(\frac{m}{\log m}\right)$.

Case II: $\ell = 1$.

In this case, we estimate

$$\sum_{k=1}^{A-1} \frac{1}{(k-1)!} \frac{1}{2\pi i} \int_{\gamma} H(z) R(z; J)^{k-1} z^{-m-1} dz.$$

First, let $k = 1$. Then

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} H(z) z^{-m-1} dz &= \frac{1}{2\pi i} \int_{\gamma} d(1-z)^{\alpha} \left(\log \frac{1}{1-z} \right)^{\beta} (1+o(1)) z^{-m-1} dz \\ &= [z^m] d(1-z)^{\alpha} \left(\log \frac{1}{1-z} \right)^{\beta} (1+o(1)) \\ &= \frac{d}{\Gamma(-\alpha)} m^{-\alpha-1} (\log m)^{\beta} + o(m^{-\alpha-1} (\log m)^{\beta}) \\ &\sim \frac{d}{\Gamma(-\alpha)} m^{-\alpha-1} (\log m)^{\beta} \end{aligned} \tag{16}$$

by Theorems 2 and 3. In fact, we will prove that (16) is the main contribution to (15).

To do so, we next assume $k \geq 2$ and examine the integrals of the form

$$\frac{1}{2\pi i} \int_{\gamma} H(z) R(z; J)^{k-1} z^{-m-1} dz,$$

by breaking γ into two sections: γ_3 and $\gamma' = \gamma_1 \cup \gamma_2 \cup \gamma_4$.

On γ_1 ,

$$\begin{aligned}
 z &= 1 + \frac{1}{m}e^{i\psi}, \quad \theta < \psi < 2\pi - \theta \\
 z^j &= \left(1 + \frac{1}{m}e^{i\psi}\right)^j \\
 &= 1 + O\left(\frac{j}{m}\right) \\
 1 - z^j &= O\left(\frac{1}{\log m}\right) = o(1),
 \end{aligned}$$

since $\hat{j} = O\left(\frac{m}{\log m}\right)$.

On γ_2 (and analogously on γ_4),

$$\begin{aligned}
 z &= 1 + re^{i\theta}, \quad \frac{1}{m} \leq r \leq E \frac{\log m}{m} \\
 z^j &= (1 + re^{i\theta})^j \\
 &= 1 + O(jr) \\
 1 - z^j &= O(jr) = o\left(\frac{m}{\log m} \times \frac{\log m}{m}\right) = o(1).
 \end{aligned}$$

Therefore, on γ' , $R(z; J)^k = o(1)$, which implies that

$$H(z)R(z; J)^{k-1} = o\left((1-z)^\alpha \left(\log \frac{1}{1-z}\right)^\beta\right),$$

so

$$\begin{aligned}
 \frac{1}{2\pi i} \int_{\gamma'} H(z)R(z; J)^{k-1} z^{-m-1} dz &= \frac{1}{2\pi i} \int_{\gamma'} o\left((1-z)^\alpha \left(\log \frac{1}{1-z}\right)^\beta\right) dz \\
 &\leq \frac{1}{2\pi i} \int_{\gamma'} o\left((1-z)^\alpha \left(\log \frac{1}{1-z}\right)^\beta\right) dz \\
 &= o\left(m^{-\alpha-1}(\log m)^\beta\right).
 \end{aligned}$$

Next, we estimate

$$\frac{1}{2\pi i} \int_{\gamma_3} H(z) R(z; J)^{k-1} z^{-m-1} dz.$$

On γ_3 ,

$$\begin{aligned} |z|^{-m-1} &= \left| 1 + A \frac{\log m}{m} \right|^{-m-1} \\ &= \exp \left((-m-1) \log \left(1 + A \frac{\log m}{m} \right) \right) \\ &= \exp \left((-m-1) \left(A \frac{\log m}{m} + O \left(\frac{(\log m)^2}{m^2} \right) \right) \right) \\ &= \exp (-A \log m + o(1)) \\ &= O(m^{-A}). \end{aligned}$$

Furthermore, on γ_3 we have $H(z) = O(1)$ and $R(z; J) = O(1)$. Therefore,

$$\frac{1}{2\pi i} \int_{\gamma_3} H(z) R(z; J)^{k-1} z^{-m-1} dz = O(m^{-A}).$$

Case III: $2 \leq \ell \leq k$.

In this case,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} H(z)^\ell R(z; J)^{k-\ell} z^{-m-1} dz &= \frac{1}{2\pi i} \int_{\gamma} \left(d(1-z)^\alpha \left(\log \frac{1}{1-z} \right)^\beta (1+o(1)) \right)^\ell O(1) \frac{dz}{z^{m+1}} \\ &= \frac{1}{2\pi i} \int_{\gamma} O \left(d^2(1-z)^{2\alpha} \left(\log \frac{1}{1-z} \right)^{2\beta} \right) \frac{dz}{z^{m+1}} \\ &= [z^m] O \left(d^2(1-z)^{2\alpha} \left(\log \frac{1}{1-z} \right)^{2\beta} \right) \\ &= O \left(m^{-1-2\alpha} (\log m)^{2\beta} \right), \end{aligned}$$

where the last line follows by Theorem 3.

Therefore, we may conclude that

$$\frac{1}{2\pi i} \int_{\gamma} z^{-m-1} \sum_{k=1}^{A-1} \frac{g(z; J)^k}{k!} dz \sim \frac{d}{\Gamma(-\alpha)} m^{-\alpha-1} (\log m)^{\beta}.$$

Finally, we are left to show that the term

$$\frac{1}{2\pi i} \int_{\gamma} z^{-m-1} O(g(z; J)^A) dz = \sum_{\ell=0}^A \frac{1}{2\pi i} \int_{\gamma} O(H(z)^{\ell} R(z; J)^{A-\ell}) \frac{dz}{z^{m+1}} \quad (17)$$

is negligible compared to (16). However, by the arguments in the above cases, the only term in (17) that we must consider is

$$\frac{1}{2\pi i} \int_{\gamma} O(R(z; J)^A) \frac{dz}{z^{m+1}},$$

which we examine in two cases.

Case I: Estimation of $\frac{1}{2\pi i} \int_{\gamma} O(R(z; J)^A) \frac{dz}{z^{m+1}}$ on $\gamma' = \gamma_1 \cup \gamma_2 \cup \gamma_4$.

From (11) and the fact that $\hat{j} = o\left(\frac{m}{\log m}\right)$,

$$R(z; J) = \sum_{j \in J} C_j \frac{\rho^j}{j!} (1 - z^j) = \sum_{j \in J} C_j \frac{\rho^j}{j!} O\left(\frac{j \log m}{m}\right) = O(1) \cdot o(1) = o(1).$$

Furthermore, on γ ,

$$|z^{-m-1}| = O\left(\left(1 - \frac{1}{m}\right)^{-m-1}\right) = O(1).$$

Therefore, since $\alpha > 0$ and $A = \alpha + 2$,

$$\frac{1}{2\pi i} \int_{\gamma'} O(R(z; J)^A) \frac{dz}{z^{m+1}} = o(1) |\gamma'| = o\left(\frac{(\log m)^{A+1}}{m^{A+1}}\right) = o\left(\frac{(\log m)^{\beta}}{m^{\alpha+1}}\right),$$

which is negligible compared to (21).

Case II: Estimation of $\frac{1}{2\pi i} \int_{\gamma} O(R(z; J)^A) \frac{dz}{z^{m+1}}$ on γ_3 .

In this case, $|z^{-m-1}| = O(m^{-A})$, and $R(z; J) = O(1)$. Therefore,

$$\frac{1}{2\pi i} \int_{\gamma_3} O(R(z; J)^A) z^{-m-1} dz = O(m^{-A}),$$

which is clearly negligible compared to (16). $\square$

The conditions required of J in Theorem 4 are different from those in [12], but we cannot say that they are a universal improvement. This is because in Theorem 4, we require $\hat{j} = o(\log m/m)$, whereas in [12], $\hat{j} = O(\log m/m)$. However, in practical terms this is a minor difference, and we completely eliminate the need for a restriction on the sum

$$\sum_{j \in J} j^{-\alpha} (\log j)^{\beta}.$$

The method used to prove Theorem 4 can be used to prove an analogous theorem for unlabeled objects.

We now consider the case where $\alpha \in \mathbb{Z}^+$. Flajolet and Odlyzko (see [18], pp. 231–232) prove that the transfer theorems may be applied in a modified form in this case. Indeed, if $\alpha \in \mathbb{Z}^+$ and $\beta \in \mathbb{R}$, then

$$[z^n](1-z)^{\alpha} \left(\log \frac{1}{1-z} \right)^{\beta} = n^{-\alpha-1} (\log n)^{\beta} \left[-\binom{\beta}{1} \frac{C_1}{\log n} + \binom{\beta}{2} \frac{C_2}{(\log n)^2} + \cdots \right],$$

where $C_j = \left. \frac{d^k}{ds^k} \frac{1}{\Gamma(-s)} \right|_{s=\alpha}$ i.e.,

$$[z^n](1-z)^{\alpha} \left(\log \frac{1}{1-z} \right)^{\beta} \sim C n^{-\alpha-1} (\log n)^{\beta-1}, \quad (18)$$

where $C = (-1)^{\alpha} \alpha! \beta$. With slight modification of the proof, a change in the definition of $C(z)$, and a stronger condition on $\hat{j}$, we may prove an analogue to Theorem 4 for

the case when α is a positive integer, which we state below with a sketch of the proof.

Suppose that

$$C(z) = c + d(1 - z/\rho)^\alpha \left(\log \frac{1}{1 - z/\rho} \right)^\beta (1 + O((1 - z/\rho)^K))$$

as $z \rightarrow \rho$ in $\Delta(\nu, \theta)$ for some positive constant $K < 1$. Note that this $C(z)$ is still alg-log in the sense of Definition 5. Using this definition for $C(z)$, we prove the following theorem.

Theorem 5. *Let $\alpha \in \mathbb{Z}^+$, and assume that*

$$\hat{j} = o(m/(\log m)^2). \quad (19)$$

Then, as $m \rightarrow \infty$,

$$[z^n]F(z; S) \sim (-1)^\alpha d e^c \alpha! \beta \mathcal{S} m^{-\alpha-1} (\log m)^{\beta-1} \rho^{-m} \exp(-C(\rho; J)),$$

uniformly for any S satisfying (19).

Proof. Since $\alpha > 0$,

$$F(z; \emptyset) = e^c + d e^c (1 - z/\rho)^\alpha \left(\log \frac{1}{1 - z/\rho} \right)^\beta (1 + O((1 - z/\rho)^K)),$$

as $z \rightarrow \rho$ in $\Delta(\rho, \theta)$. From the above discussion and the transfer theorems, it follows that

$$[z^n]F(z; \emptyset) \sim (-1)^\alpha d e^c \alpha! \beta \rho^{-n} n^{-\alpha-1} (\log n)^{\beta-1},$$

and

$$[z^n]C(z) \sim (-1)^\alpha d\alpha! \beta \rho^{-n} n^{-\alpha-1} (\log n)^{\beta-1}.$$

We proceed to estimate $[z^n]F(z; S)$ using the same contour γ as in the proof of Theorem 4, *i.e.*, $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, with

$$\begin{aligned} \gamma_1 &= \left\{ z : |z-1| = \frac{1}{m}, \theta < \arg(z-1) < 2\pi - \theta \right\}, \\ \gamma_2 &= \left\{ z : \frac{1}{m} \leq |z-1| \leq E \frac{\log m}{m}, \arg(z-1) < \theta \right\}, \\ \gamma_3 &= \left\{ z : |z| = 1 + A \frac{\log m}{m}, \theta < \arg(z-1) < 2\pi - \theta \right\}, \\ \gamma_4 &= \left\{ z : \frac{1}{m} \leq |z-1| \leq E \frac{\log m}{m}, \arg(z-1) < 2\pi - \theta \right\}, \end{aligned}$$

where $A = \lceil \alpha \rceil + 2$, and $E \in \mathbb{R}$ satisfies

$$\left| 1 + E \frac{\log m}{m} e^{i\theta} \right| = 1 + A \frac{\log m}{m}.$$

Then by Lemma 3,

$$[z^n]F(z; S) = \rho^{-n} [z^n]F(\rho z; S) = \rho^{-m} \mathcal{S} \frac{1}{2\pi i} \int_{\gamma} f(z) dz,$$

where

$$f(z) = z^{-m-1} \exp(C(\rho z) - C(\rho z; J)).$$

As before, on γ we obtain the following expression for $[z^n]F(z; S)$:

$$\begin{aligned}
[z^n]F(z; S) &= \rho^{-m} \mathcal{S} e^C \exp(-C(\rho; J)) \\
&\times \frac{1}{2\pi i} \left(\int_{\gamma} z^{-m-1} dz + \int_{\gamma} z^{-m-1} \sum_{k=1}^{A-1} \frac{g(z; J)^k}{k!} dz + \int_{\gamma} z^{-m-1} O(g(z, J)^A) dz \right),
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
H(z) &= d(1-z)^\alpha \left(\log \frac{1}{1-z} \right)^\beta (1 + o((1-z)^K)), \\
R(z; J) &= \sum_{j \in J} C_j \frac{\rho^j}{j!} (1 - z^j), \\
g(z; J) &= H(z) + R(z; J).
\end{aligned}$$

We consider each integral in (20) separately. By the Cauchy integral formula,

$$\frac{1}{2\pi i} \int_{\gamma} z^{-m-1} dz = 0.$$

By the binomial theorem,

$$\frac{1}{2\pi i} \int_{\gamma} z^{-m-1} \sum_{k=1}^{A-1} \frac{g(z; J)^k}{k!} dz = \sum_{k=1}^{A-1} \frac{1}{k!} \sum_{\ell=0}^k \binom{k}{\ell} \frac{1}{2\pi i} \int_{\gamma} H(z)^\ell R(z; J)^{k-\ell} z^{-m-1} dz,$$

which we examine in three cases.

Case I: $\ell = 0$. As in the proof of Theorem 4, when $\ell = 0$,

$$\frac{1}{2\pi i} \int_{\gamma} R(z; J)^k z^{-m-1} dz = 0.$$

Case II: $\ell = 1$. For $k = 1$,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\gamma} H(z) z^{-m-1} dz &= \frac{1}{2\pi i} \int_{\gamma} d(1-z)^{\alpha} \left(\log \frac{1}{1-z} \right)^{\beta} (1 + O((1-z)^K)) dz \\
&= (-1)^{\alpha} \alpha! \beta d m^{-\alpha-1} (\log m)^{\beta-1} + O(m^{-\alpha-1} (\log m)^{\beta-2}) \\
&\quad + O(m^{-\alpha-K-1} (\log m)^{\beta-1}) \\
&\sim (-1)^{\alpha} \alpha! \beta d m^{-\alpha-1} (\log m)^{\beta-1},
\end{aligned} \tag{21}$$

and we will prove that (21) is the asymptotically dominant term in (20). Next, assume $k \geq 2$ and consider integrals of the form

$$\frac{1}{2\pi i} \int_{\gamma} H(z) R(z; J)^{k-1} z^{-m-1} dz,$$

breaking γ into two sections: γ_3 and $\gamma' = \gamma_1 \cup \gamma_2 \cup \gamma_4$.

On γ_1 , $\hat{j} = o(m/(\log m)^2)$ implies that

$$1 - z^j = O\left(\frac{j}{m}\right) = o\left(\frac{1}{(\log m)^2}\right). \tag{22}$$

On γ_2 (and analogously on γ_4),

$$1 - z^j = O(jr) = o\left(\frac{m}{(\log m)^2} \times \frac{\log m}{m}\right) = o\left(\frac{1}{\log m}\right).$$

Therefore, on γ' , $R(z; J) = o(1/\log m)$, which implies that

$$H(z) R(z; J)^{k-1} = o\left(\frac{1}{\log m} (1-z)^{\alpha} \left(\log \frac{1}{1-z}\right)^{\beta}\right),$$

so by the transfer theorems,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma'} H(z) R(z; J)^{k-1} z^{-m-1} dz &= \frac{1}{2\pi i} \int_{\gamma'} o \left(\frac{1}{\log m} (1-z)^\alpha \left(\log \frac{1}{1-z} \right)^\beta \right) dz \\ &= o \left(m^{-\alpha-1} (\log m)^{\beta-1} \right). \end{aligned}$$

By the same method as in the proof of Theorem 4,

$$\frac{1}{2\pi i} \int_{\gamma_3} H(z) R(z; J)^{k-1} z^{-m-1} dz = O \left(m^{-A} \right).$$

Case III: $2 \leq \ell \leq k$.

In this case, the transfer theorems imply that

$$\frac{1}{2\pi i} \int_{\gamma} H(z)^\ell R(z; J)^{k-\ell} z^{-m-1} dz = O \left(m^{-1-2\alpha} (\log m)^{2\beta-2} \right),$$

and we conclude that

$$\frac{1}{2\pi i} \int_{\gamma} z^{-m-1} \sum_{k=1}^{A-1} \frac{g(z; J)^k}{k!} dz \sim (-1)^\alpha \alpha! \beta m^{-\alpha-1} (\log m)^{\beta-1}.$$

All that remains is to show that the term

$$\frac{1}{2\pi i} \int_{\gamma} z^{-m-1} O \left(g(z; J)^A \right) dz = \sum_{\ell=0}^A \frac{1}{2\pi i} \int_{\gamma} O \left(H(z)^\ell R(z; J)^{A-\ell} \right) \frac{dz}{z^{m+1}}$$

is negligible compared to (21). This again reduces to considering only

$$\frac{1}{2\pi i} \int_{\gamma} O \left(R(z; J)^A \right) \frac{dz}{z^{m+1}}.$$

Fortunately, the same argument used in the proof of Theorem 4 applies here, and this term is indeed negligible. $\square$

3.4 Algebraic Components with Negative Algebraic Exponent

This section is an improvement of the results in [12], and uses a very similar method to the one used by Dong *et al.*. In this section, assume that the component generating function has the form

$$C(z) = d(1 - z/\rho)^{-\sigma} + c + o(1) \text{ as } z \rightarrow \rho \text{ in the open disk } |z| < \rho,$$

where c , d , and σ are real constants with $\sigma, d > 0$. Further, assume that $C(z)$ is analytic in the open disc $|z| < \rho$. We use $-\sigma$ with $\sigma > 0$ instead of α to avoid confusion and emphasize that the algebraic exponent is negative. In contrast with Section 3.3, the main tool used in this section is the saddle point method.

As in the previous section, we once again only consider labeled objects. The method used to prove Theorem 6 can also be used to prove an analogous theorem for unlabeled objects. We choose not to include this proof because it is essentially identical to the proof of Theorem 4.2 in [12].

Let $\hat{j} = \max\{j : j \in J\}$, let $h(z) = d(1 - z)^{-\sigma}$, and let $r = r(m)$ be defined by the saddle point equation

$$rh'(r) = m, \quad 0 < r < 1. \quad (23)$$

Throughout this section assume that the following conditions hold:

$$\hat{j} = o\left(m^{\frac{1}{1+\sigma}}\right), \quad \sum_{j \in J} j^{\sigma+1} = o\left(m^{\frac{2}{1+\sigma}}\right), \quad \text{and} \quad \sum_{j \in J} j^{\sigma-1} = o\left(m^{\frac{\sigma}{6(1+\sigma)}}\right). \quad (24)$$

As in Section 3.3, the statement of Theorem 6 is very similar to the statement of Theorem 4.1 in [12]. However, in the case of Theorem 6, we give a statement for

$[z^n]F(z; S)$ which is asymptotically equivalent to that given in [12], but is slightly more accurate. In [12] there is also the required restriction that

$$\sum_{j \in J} j^\sigma = o\left(m^{\frac{1}{1+\sigma}}\right), \quad (25)$$

but the first and last restrictions in (24) imply that

$$\sum_{j \in J} j^\sigma = o\left(m^{\frac{6+\sigma}{6(1+\sigma)}}\right),$$

which, while a weaker restriction than (25), is sufficient to prove Theorem 6.

Theorem 6 is an improvement on its analogue in [12] in two ways: the multiplicative exponential term in (35) is more precise with the inclusion of the $C'(\rho; J)$ term, and the condition on $\sum_{j \in J} j^\sigma$ is relaxed. This appears to come at a cost, however, as there is now a restriction on $\sum_{j \in J} j^{\sigma+1}$, which is not necessary in [12]. At first glance, it appears that this new restriction combined with $\hat{j} = o\left(m^{\frac{1}{1+\sigma}}\right)$ implies that

$$\sum_{j \in J} j^\sigma = o\left(m^{\frac{1}{1+\sigma}}\right),$$

but this is not the case. These results are summarized in Table 1.

Expression	Original Bound	New Bound	Improvement Factor
$\hat{j}$	$o\left(m^{\frac{1}{1+\sigma}}\right)$	$o\left(m^{\frac{1}{1+\sigma}}\right)$	None
$\sum_{j \in J} j^{\sigma-1}$	$o\left(m^{\frac{\sigma}{6(1+\sigma)}}\right)$	$o\left(m^{\frac{\sigma}{6(1+\sigma)}}\right)$	None
$\sum_{j \in J} j^\sigma$	$o\left(m^{\frac{1}{1+\sigma}}\right)$	$o\left(m^{\frac{6+\sigma}{6(1+\sigma)}}\right)$	$o\left(m^{\frac{\sigma}{6(1+\sigma)}}\right)$
$\sum_{j \in J} j^{\sigma+1}$	-	$o\left(m^{\frac{2}{1+\sigma}}\right)$	N/A

Table 1: Restrictions comparison for negative algebraic exponent.

Theorem 6. *Suppose the pattern S satisfies conditions in (24), and let r be defined*

as in (23). Then as $m \rightarrow \infty$,

$$[z^n]F(z; S) \sim \mathcal{S} \frac{(r\sigma)^{-m}}{\sqrt{2\pi\sigma(1+\sigma)d}} \left(\frac{m}{\sigma d}\right)^{-\frac{\sigma+2}{2(\sigma+1)}} \exp\left(C(r\rho) - C(\rho; J) + \left(\frac{\sigma d}{m}\right)^{\frac{1}{1+\sigma}} \rho C'(\rho; J)\right).$$

Proof. Let S be a restricted pattern. From Lemma 3, we have

$$[z^n]F(z; S) = \mathcal{S} \rho^{-m} [z^m] \exp(h(z) - C(\rho z; J) + c + o(1)). \quad (26)$$

From the Cauchy integral formula, we determine

$$\begin{aligned} [z^m] \exp(h(z) - C(\rho z; J) + c + o(1)) &= \\ \frac{r^{-m}}{2\pi} \int_{-\pi}^{\pi} \exp(h(re^{i\theta} - C(r\rho e^{i\theta}; J) - im\theta + c + o(1)) \, d\theta. \end{aligned} \quad (27)$$

We apply Lagrange inversion [43] to (23) by first rearranging (23):

$$1 - r = r^{\frac{1}{1+\sigma}} \left(\frac{d\sigma}{m}\right)^{\frac{1}{1+\sigma}}. \quad (28)$$

Let $s = \left(\frac{d\sigma}{m}\right)^{\frac{1}{1+\sigma}}$, and let $t = 1 - r$. Then (28) becomes

$$t = s(1 - t)^{\frac{1}{1+\sigma}},$$

to which we may apply Lagrange inversion, which yields.

$$[s^k]t = \frac{1}{k} [t^{k-1}] \left((1 - t)^{\frac{k}{1+\sigma}} \right) = \frac{1}{k} \left(\frac{\frac{k}{1+\sigma}}{k-1} \right) (-1)^{k-1}.$$

Therefore,

$$1 - r = \sum_{k \geq 1} \frac{1}{k} \binom{\frac{k}{1+\sigma}}{k-1} (-1)^{k-1} \left(\frac{\sigma d}{m} \right)^{\frac{k}{1+\sigma}} = \left(\frac{\sigma d}{m} \right)^{\frac{1}{1+\sigma}} + O\left(m^{-\frac{2}{1+\sigma}}\right). \quad (29)$$

As is usual in the saddle point method, let $b(r) = rh'(r) + r^2 h''(r)$. Then

$$\begin{aligned} b(r) &= rd\sigma(1-r)^{-\sigma-1} + r^2 d\sigma(\sigma+1)(1-r)^{-\sigma-2} \\ &= d\sigma \left(1 + O\left(m^{-\frac{1}{1+\sigma}}\right)\right) \left(\left(\frac{\sigma d}{m} \right)^{\frac{1}{1+\sigma}} + O\left(m^{-\frac{2}{1+\sigma}}\right) \right)^{-\sigma-1} \\ &\quad + d\sigma(\sigma+1) \left(1 + O\left(m^{-\frac{1}{1+\sigma}}\right)\right)^2 \left(\left(\frac{\sigma d}{m} \right)^{\frac{1}{1+\sigma}} + O\left(m^{-\frac{2}{1+\sigma}}\right) \right)^{-\sigma-2} \\ &= m \left(1 + O\left(m^{-\frac{1}{1+\sigma}}\right)\right) \left(1 + O\left(m^{-\frac{3}{1+\sigma}}\right)\right) \\ &\quad + d\sigma(\sigma+1) \left(\frac{m}{d\sigma} \right)^{\frac{2+\sigma}{1+\sigma}} \left(1 + O\left(m^{-\frac{1}{1+\sigma}}\right)\right) \left(1 + O\left(m^{-\frac{3}{1+\sigma}}\right)\right) \\ &= d\sigma(\sigma+1) \left(\frac{m}{d\sigma} \right)^{\frac{2+\sigma}{1+\sigma}} + O(m) \\ &\sim d\sigma(1+\sigma) \left(\frac{m}{d\sigma} \right)^{\frac{2+\sigma}{1+\sigma}}. \end{aligned} \quad (30)$$

In order to use the saddle point method, we must find an angle θ_0 such that

$$r^2 \theta_0^2 h''(r) \rightarrow \infty \quad \text{and} \quad r^3 \theta_0^3 h^{(3)}(r) \rightarrow 0$$

as $m \rightarrow \infty$. If we can find such a θ_0 , then for $|\theta| < \theta_0$ the behaviour of the Taylor series for the expression $h(re^{i\theta}) - im\theta$ is dominated by the term $h''(r)\theta_0^2$. Dong *et al.* found that the required condition is satisfied by letting

$$\theta_0 = m^{-\frac{1}{1+\sigma} - \frac{5\sigma}{12(1+\sigma)}} = m^{-\frac{12+5\sigma}{12(1+\sigma)}}. \quad (31)$$

For $\theta_0 \leq |\theta| \leq \pi$, we have

$$|1 - re^{i\theta}| = (1 + r^2 - 2r \cos \theta)^{1/2} \geq (1 + r^2 - 2r \cos \theta_0)^{1/2} = |1 - re^{i\theta_0}|.$$

Hence,

$$|1 - re^{i\theta}|^{-\sigma} \leq |1 - re^{i\theta_0}|^{-\sigma} = |1 - 2r \cos \theta_0 + r^2|^{-\sigma/2}.$$

From the Taylor expansion for $\cos \theta_0$, we obtain

$$|1 - re^{i\theta}|^{-\sigma} \leq (1 - r)^{-\sigma} - \frac{\sigma}{2} r (1 - r)^{-\sigma-2} \theta_0^2 + O(\theta_0^4).$$

Observe that

$$(1 - r)^{-\sigma-2} \theta_0^2 \sim \left(\frac{\sigma d}{m} \right)^{-\frac{\sigma+2}{\sigma+1}} m^{-\frac{12+5\sigma}{6(1+\sigma)}} = B m^{\frac{\sigma}{6(1+\sigma)}},$$

for some constant $B > 0$. Therefore, we have

$$|\exp(h(re^{i\theta}))| \leq \exp \left(h(r) - B m^{\frac{\sigma}{6(1+\sigma)}} \right), \quad (32)$$

as $m \rightarrow \infty$, uniformly for $\theta_0 \leq |\theta| \leq \pi$.

For $|\theta| \leq \theta_0$, we have

$$\begin{aligned} h(re^{i\theta}) &= d(1 - r)^{-\sigma} + i\sigma dr(1 - r)^{-\sigma-1}\theta - \frac{b(r)}{2}\theta^2 + O((1 - r)^{-\sigma-3}\theta^3) \\ &= d(1 - r)^{-\sigma} + i\sigma dr(1 - r)^{-\sigma-1}\theta - \frac{b(r)}{2}\theta^2 + o(1). \end{aligned}$$

We next separate the integral (27) into the following two parts:

$$I_1 = \int_{-\theta_0}^{\theta_0} \exp(h(re^{i\theta}) - C(r\rho e^{i\theta}; J) - im\theta + c + o(1)) \, d\theta, \text{ and}$$

$$I_2 = \int_{|\theta| \geq \theta_0} \exp(h(re^{i\theta}) - C(r\rho e^{i\theta}; J) - im\theta + c + o(1)) \, d\theta.$$

Consider the term $C(r\rho e^{i\theta}; J)$ in the above integrals. If we assume that $\hat{j} = o\left(m^{\frac{1}{1+\sigma}}\right)$, then from (29), we obtain

$$\begin{aligned} C(r\rho e^{i\theta}; J) &= \sum_{j \in J} \frac{C_j}{j!} (r\rho)^j e^{ij\theta} \\ &= \sum_{j \in J} \frac{C_j}{j!} \rho^j \left(1 - \left(\frac{\sigma d}{m}\right)^{\frac{1}{1+\sigma}} + O\left(m^{-\frac{2}{1+\sigma}}\right)\right)^j (1 + O(j\theta_0)) \\ &= \sum_{j \in J} \frac{C_j}{j!} \rho^j \left(1 - j \left(\frac{\sigma d}{m}\right)^{\frac{1}{1+\sigma}} + O\left(jm^{-\frac{2}{1+\sigma}}\right) + O\left(j^2 m^{-\frac{2}{1+\sigma}}\right)\right) (1 + O(j\theta_0)) \\ &= \sum_{j \in J} \frac{C_j}{j!} \rho^j \left(1 - j \left(\frac{\sigma d}{m}\right)^{\frac{1}{1+\sigma}} + O\left(j^2 m^{-\frac{2}{1+\sigma}}\right)\right) \\ &\quad + \sum_{j \in J} \frac{C_j}{j!} \rho^j \left(O(j\theta_0) + O\left(j^2 \theta_0 m^{-\frac{1}{1+\sigma}}\right) + O\left(j^3 \theta_0 m^{-\frac{2}{1+\sigma}}\right)\right) \\ &= \sum_{j \in J} \frac{C_j}{j!} \rho^j \left(1 - j \left(\frac{\sigma d}{m}\right)^{\frac{1}{1+\sigma}}\right) \\ &\quad + \sum_{j \in J} \frac{C_j}{j!} \rho^j \left(O\left(j^2 m^{-\frac{2}{1+\sigma}}\right) + O(j\theta_0) + O\left(j^2 \theta_0 m^{-\frac{1}{1+\sigma}}\right) + O\left(j^3 \theta_0 m^{-\frac{2}{1+\sigma}}\right)\right) \\ &= \sum_{j \in J} \frac{C_j}{j!} \rho^j \left(1 - j \left(\frac{\sigma d}{m}\right)^{\frac{1}{1+\sigma}}\right) \\ &\quad + \sum_{j \in J} \frac{C_j}{j!} \rho^j \left(O\left(j^2 m^{-\frac{2}{1+\sigma}}\right) + O\left(jm^{-\frac{12+5\sigma}{12(1+\sigma)}}\right)\right) \end{aligned} \tag{33}$$

$$+ \sum_{j \in J} \frac{C_j}{j!} \rho^j \left(O\left(j^2 m^{-\frac{24+5\sigma}{12(1+\sigma)}}\right) + O\left(j^3 m^{-\frac{36+5\sigma}{12(1+\sigma)}}\right)\right). \tag{34}$$

In order to evaluate I_1 , we require (33) and (34) to be $o(1)$. First, recall from (9)

that

$$\rho^j \frac{C_j}{j!} \sim \frac{d}{\Gamma(\sigma)} j^{\sigma-1}.$$

Then we are able to determine the required conditions of the pattern:

$$\begin{aligned} \sum_{j \in J} \frac{C_j}{j!} \rho^j O\left(j^2 m^{-\frac{2}{1+\sigma}}\right) &= O\left(\sum_{j \in J} j^{\sigma+1} m^{-\frac{2}{1+\sigma}}\right) = o(1) \iff \sum_{j \in J} j^{\sigma+1} = o\left(m^{\frac{2}{1+\sigma}}\right), \\ \sum_{j \in J} \frac{C_j}{j!} \rho^j O\left(j m^{-\frac{12+5\sigma}{12(1+\sigma)}}\right) &= O\left(\sum_{j \in J} j^{\sigma} m^{-\frac{12+5\sigma}{12(1+\sigma)}}\right) = o(1) \iff \sum_{j \in J} j^{\sigma} = o\left(m^{\frac{12+5\sigma}{12(1+\sigma)}}\right), \\ \sum_{j \in J} \frac{C_j}{j!} \rho^j O\left(j^2 m^{-\frac{24+5\sigma}{12(1+\sigma)}}\right) &= O\left(\sum_{j \in J} j^{\sigma+1} m^{-\frac{24+5\sigma}{12(1+\sigma)}}\right) = o(1) \iff \sum_{j \in J} j^{\sigma+1} = o\left(m^{\frac{24+5\sigma}{12(1+\sigma)}}\right), \\ \sum_{j \in J} \frac{C_j}{j!} \rho^j O\left(j^3 m^{-\frac{36+5\sigma}{12(1+\sigma)}}\right) &= O\left(\sum_{j \in J} j^{\sigma+2} m^{-\frac{36+5\sigma}{12(1+\sigma)}}\right) = o(1) \iff \sum_{j \in J} j^{\sigma+2} = o\left(m^{\frac{36+5\sigma}{12(1+\sigma)}}\right). \end{aligned}$$

Therefore we can summarize the conditions required of the pattern as

$$\begin{aligned} \hat{j} &= o\left(m^{\frac{1}{1+\sigma}}\right), \quad \sum_{j \in J} j^{\sigma} = o\left(m^{\frac{12+5\sigma}{12(1+\sigma)}}\right), \\ \sum_{j \in J} j^{\sigma+1} &= o\left(\min\left\{m^{\frac{2}{1+\sigma}}, m^{\frac{24+5\sigma}{12(1+\sigma)}}\right\}\right), \quad \text{and} \quad \sum_{j \in J} j^{\sigma+2} = o\left(m^{\frac{36+5\sigma}{12(1+\sigma)}}\right). \end{aligned}$$

However, since $\hat{j} = o\left(m^{\frac{1}{1+\sigma}}\right)$,

$$\begin{aligned} \sum_{j \in J} j^{\sigma} &= o\left(m^{\frac{12+5\sigma}{12(1+\sigma)}}\right) \Rightarrow \sum_{j \in J} j^{\sigma+2} = o\left(m^{\frac{36+5\sigma}{12(1+\sigma)}}\right), \\ \sum_{j \in J} j^{\sigma} &= o\left(m^{\frac{12+5\sigma}{12(1+\sigma)}}\right) \Rightarrow \sum_{j \in J} j^{\sigma+1} = o\left(m^{\frac{24+5\sigma}{12(1+\sigma)}}\right), \end{aligned}$$

and the required conditions reduce to

$$\hat{j} = o\left(m^{\frac{1}{1+\sigma}}\right), \quad \sum_{j \in J} j^{\sigma} = o\left(m^{\frac{12+5\sigma}{12(1+\sigma)}}\right), \quad \text{and} \quad \sum_{j \in J} j^{\sigma+1} = o\left(m^{\frac{2}{1+\sigma}}\right).$$

We next note that

$$\begin{aligned} \sum_{j \in J} \frac{C_j}{j!} \rho^j \left(1 - j \left(\frac{\sigma d}{m} \right)^{\frac{1}{1+\sigma}} \right) &= \sum_{j \in J} \frac{C_j}{j!} \rho^j - \sum_{j \in J} \frac{j C_j}{j!} \rho^j \left(\frac{\sigma d}{m} \right)^{\frac{1}{1+\sigma}} \\ &= C(\rho; J) - \left(\frac{\sigma d}{m} \right)^{\frac{1}{1+\sigma}} \rho C'(\rho; J). \end{aligned}$$

Hence, under the conditions (24), we have

$$\begin{aligned} I_1 &= \int_{-\theta_0}^{\theta_0} \exp \left(d(1-r)^{-\sigma} - \frac{b(r)}{2} \theta^2 - C(\rho; J) + \left(\frac{\sigma d}{m} \right)^{\frac{1}{1+\sigma}} \rho C'(\rho; J) + c + o(1) \right) d\theta \\ &\sim \sqrt{\frac{2\pi}{b(r)}} \exp \left(C(r\rho) - C(\rho; J) + \left(\frac{\sigma d}{m} \right)^{\frac{1}{1+\sigma}} \rho C'(\rho; J) \right). \end{aligned} \quad (35)$$

Next we prove that I_2 is negligible compared with I_1 . From (32), we have

$$|I_2| = O \left(\exp(C(r\rho) - Bm^{\frac{\sigma}{6(1+\sigma)}} + C(\rho; J)) \right),$$

which is negligible compared with I_1 if and only if

$$C(\rho; J) = O \left(\sum_{j \in J} j^{\sigma-1} \right) = o \left(m^{\frac{\sigma}{6(1+\sigma)}} \right).$$

Finally, we note that, since $\hat{j} = o \left(m^{\frac{1}{1+\sigma}} \right)$ and $\sigma > 0$,

$$\sum_{j \in J} j^{\sigma-1} = o \left(m^{\frac{\sigma}{6(1+\sigma)}} \right) \Rightarrow \sum_{j \in J} j^{\sigma} = o \left(m^{\frac{12+2\sigma}{12(1+\sigma)}} \right) = o \left(m^{\frac{12+5\sigma}{12(1+\sigma)}} \right).$$

□

Equation (29) implies that

$$\begin{aligned}
(r(m))^m &= \exp(m \log(r(m))) \\
&= \exp \left(m \log \left(1 - \sum_{k \geq 1} \frac{1}{k} \left(\frac{1}{1+\sigma} \right) (-1)^{k-1} \left(\frac{\sigma d}{m} \right)^{\frac{k}{1+\sigma}} \right) \right) \\
&= \exp \left(m \log \left(1 - \sum_{k=1}^{1+[\sigma]} \frac{1}{k} \left(\frac{1}{1+\sigma} \right) (-1)^{k-1} \left(\frac{\sigma d}{m} \right)^{\frac{k}{1+\sigma}} + O \left(m^{-\frac{2+[\sigma]}{1+\sigma}} \right) \right) \right) \\
&= \exp \left(m \left[\sum_{\ell=1}^{\infty} \frac{-1}{\ell} \left(\sum_{k=1}^{1+[\sigma]} \frac{1}{k} \left(\frac{1}{1+\sigma} \right) (-1)^{k-1} \left(\frac{\sigma d}{m} \right)^{\frac{k}{1+\sigma}} + O \left(m^{-\frac{2+[\sigma]}{1+\sigma}} \right) \right)^{\ell} \right] \right) \\
&= \exp \left(m \left[\sum_{j=1}^{1+[\sigma]} B_j \left(\frac{\sigma d}{m} \right)^{\frac{j}{1+\sigma}} + O \left(m^{-\frac{2+[\sigma]}{1+\sigma}} \right) \right] \right),
\end{aligned}$$

where B_j can be calculated for any $j \geq 1$. Therefore, if $n - m = o \left(n^{\frac{1}{1+\sigma}} \right)$,

$$\begin{aligned}
(r(m))^m &= \exp \left(n \left(1 + o \left(n^{-\frac{\sigma}{1+\sigma}} \right) \right) \left[\sum_{j=1}^{1+[\sigma]} B_j \left(\frac{\sigma d}{n} \right)^{\frac{j}{1+\sigma}} \left(1 + o \left(n^{-\frac{\sigma}{1+\sigma}} \right) \right) + O \left(n^{-\frac{2+[\sigma]}{1+\sigma}} \right) \right] \right) \\
&= \exp \left(n \left(1 + o \left(n^{-\frac{\sigma}{1+\sigma}} \right) \right) \left[\sum_{j=1}^{1+[\sigma]} B_j \left(\frac{\sigma d}{n} \right)^{\frac{j}{1+\sigma}} + o(n^{-1}) + O \left(n^{-\frac{2+[\sigma]}{1+\sigma}} \right) \right] \right) \\
&= \exp \left(n \left[\sum_{j=1}^{1+[\sigma]} B_j \left(\frac{\sigma d}{n} \right)^{\frac{j}{1+\sigma}} + O \left(n^{-\frac{2+[\sigma]}{1+\sigma}} \right) \right] + o(1) \right) \\
&\sim (r(n))^n.
\end{aligned}$$

Similarly, $\exp(C(r(m)\rho)) \sim \exp(C(r(n)\rho))$, and the following corollary holds.

Corollary 2. *Suppose that, in addition to the conditions (24), the restricted pattern*

$$S = \left[\prod_{j \in J} j^{S(j)} \right]$$

also satisfies

$$n - m = \sum_{j \in J} j S(j) = o \left(n^{\frac{1}{1+\sigma}} \right). \quad (36)$$

Then, as $n \rightarrow \infty$,

$$P(S, n) \sim \prod_{j \in J} P([j^{S(j)}], n) \sim \prod_{j \in J} \frac{1}{S(j)!} (C_j \rho^j / j!)^{S(j)} \exp(-C_j \rho^j / j!).$$

Chapter 4

Negative Algebraic Exponent and Nonzero Logarithmic Exponent

Figure 7: An illustration of the range of α and β examined in Chapter 4: $\alpha < 0$, $\beta \in \mathbb{R} \setminus \{0\}$.

This chapter is an extension of [22] with the addition of restricted patterns and consideration of the unlabeled case. In this chapter, let $\sigma > 0$ and assume that $C(z)$ is of alg-log type with algebraic exponent $\alpha = -\sigma$ and logarithmic exponent $\beta \neq 0$. As in the previous chapter, we use $-\sigma$ instead of α to emphasize the fact that the

algebraic exponent in $C(z)$ is negative. For the sake of clarity, we now restate the definition of alg-log.

Definition 7. Let

$$\Delta(\nu, \theta) = \{z : |z| \leq \rho(1 + \nu), z \neq \rho, |\arg(z - \rho)| \geq \theta\}$$

for some constants $\rho > 0$, $\nu > 0$, and $0 < \theta < \pi/2$. Then $C(z)$ is of *alg-log type* at the singularity ρ with algebraic exponent $-\sigma$ and logarithmic exponent β if $C(z)$ is analytic in $\Delta(\nu, \theta)$, and for some constants c and d ,

$$C(z) = c + d(1 - z/\rho)^{-\sigma} \left(\log \frac{1}{1 - z/\rho} \right)^\beta + o(1), \quad (37)$$

as $z \rightarrow \rho$ in $\Delta(\nu, \theta)$.

Note that if $C(z)$ is alg-log in the sense of Definition 7, then it also satisfies (37) as $z \rightarrow \rho$ in the open circle $|z| < \rho$.

Again, let $\hat{j} = \max\{j : j \in J\}$. Throughout this section, we assume the following conditions:

$$\begin{aligned} \hat{j} &= o(m^{\frac{1}{1+\sigma}} (\log m)^{\frac{-\beta}{1+\sigma}}), \quad \sum_{j \in J} j^{\sigma-1} (\log j)^\beta = o\left((m^{\sigma/6} (\log m)^{-\beta})^{\frac{1}{1+\sigma}}\right), \\ \text{and} \quad \sum_{j \in J} j^\sigma (\log j)^\beta &= o\left(m^{\frac{1}{1+\sigma}} (\log m)^{\frac{-\beta}{1+\sigma}}\right). \end{aligned} \quad (38)$$

4.1 Labeled Case

In this section we consider labeled structures. Let $h(z) = d(1 - z)^{-\sigma} \left(\log \frac{1}{1-z} \right)^\beta$, and let $r = r(m)$ be defined by the saddle point equation

$$rh'(r) = m, \quad 0 < r < 1. \quad (39)$$

Theorem 7. *Suppose the pattern S satisfies Condition (38). Then as $m \rightarrow \infty$,*

$$[z^n]F(z; S) \sim \mathcal{S}(\rho)^{-m} e^{C(\rho) - C(\rho; J)} \sqrt{\frac{1}{2\pi\gamma} m^{-\frac{\sigma+2}{\sigma+1}}},$$

where $\gamma = d\sigma^{-\frac{1}{1+\sigma}}(1+\sigma)^{1+\frac{\beta(\sigma+2)}{\sigma+1}}$, and $r \sim 1 - \left(\frac{d\sigma}{m}\right)^{\frac{1}{1+\sigma}} \left(\frac{\log m}{1+\sigma}\right)^{\frac{\beta}{1+\sigma}}$. Here r is defined by the saddle point equation (39).

Proof. Let $z = re^{i\theta}$, where $|\theta| \leq \pi$. Let S be a restricted pattern. Then by Lemma 3,

$$[z^n]F(z; S) = \mathcal{S}\rho^{-m}[z^m] \exp(h(z) - C(\rho z; J) + c + o(1)). \quad (40)$$

Let $\mathcal{C}$ be contour obtained by tracing a counterclockwise circle of radius r centred at the origin. From the Cauchy integral formula, we obtain

$$\begin{aligned} [z^m] \exp(h(z) - C(\rho z; J) + c + o(1)) \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}} \exp(h(z) - C(\rho z; J) + c + o(1)) \frac{dz}{z^{m+1}} \\ &= \frac{r^{-m}}{2\pi} \int_{-\pi}^{\pi} \exp(h(re^{i\theta}) - C(\rho re^{i\theta}; J) - im\theta + c + o(1)) d\theta. \end{aligned}$$

We cannot solve (39) directly, so we use the bootstrap method (see de Bruijn [9]) to find an asymptotic expression for r . From (39), we have

$$\frac{dr}{(1-r)^{\sigma+1}} \left(\log \frac{1}{1-r} \right)^{\beta-1} \left(\sigma \log \frac{1}{1-r} + \beta \right) = m. \quad (41)$$

Let $t = 1 - r$. Then (41) becomes

$$\frac{m}{d\sigma} = \frac{1-t}{t^{\sigma+1}} \left(\log \frac{1}{t} \right)^{\beta} \left(1 - \frac{\beta}{\sigma \log t} \right),$$

which we rearrange below to find a recursive expression for t :

$$\begin{aligned} t^{\sigma+1} &= \frac{d\sigma}{m}(1-t) \left(\log \frac{1}{t} \right)^{\beta} \left(1 + \frac{\beta}{\sigma \log \frac{1}{t}} \right) \\ t &= \left(\frac{d\sigma}{m} \right)^{\frac{1}{1+\sigma}} (1-t)^{\frac{1}{1+\sigma}} \left(\log \frac{1}{t} \right)^{\frac{\beta}{1+\sigma}} \left(1 + \frac{\beta}{\sigma \log \frac{1}{t}} \right)^{\frac{1}{1+\sigma}}. \end{aligned} \quad (42)$$

It is clear from (41) that as $m \rightarrow \infty$ the value r approaches 1 from the left. Therefore, as $m \rightarrow \infty$, $t \rightarrow 0$ from the right. From this fact and from (42) we deduce that

$$t \sim \left(\frac{d\sigma}{m} \right)^{\frac{1}{1+\sigma}} \left(\log \frac{1}{t} \right)^{\frac{\beta}{1+\sigma}}, \text{ and } \frac{1}{t} \sim \left(\frac{m}{d\sigma} \right)^{\frac{1}{1+\sigma}} \left(\log \frac{1}{t} \right)^{\frac{-\beta}{1+\sigma}}.$$

Therefore,

$$\log \frac{1}{t} \sim \frac{1}{1+\sigma} \log \frac{m}{d\sigma}, \text{ and } t \sim \left(\frac{d\sigma}{m} \right)^{\frac{1}{1+\sigma}} \left(\frac{1}{1+\sigma} \log \frac{m}{d\sigma} \right)^{\frac{\beta}{1+\sigma}}. \quad (43)$$

We next combine (42) and (43) to determine an estimate for $\log \frac{1}{t}$ which includes an error term:

$$\begin{aligned} \log \frac{1}{t} &= \frac{1}{1+\sigma} \left(\log \frac{m}{d\sigma} + \log \frac{1}{1-t} - \beta \log \log \frac{1}{t} + \log \left(\frac{1}{1 + \frac{\beta}{\sigma \log \frac{1}{t}}} \right) \right) \\ &= \frac{1}{1+\sigma} \log m + O(\log \log m) = \frac{1}{1+\sigma} \log m \left(1 + O \left(\frac{\log \log m}{\log m} \right) \right). \end{aligned} \quad (44)$$

Finally, we use (42), (43), and (44) to find an asymptotic estimate for t (and thus r):

$$\begin{aligned} t &= \left(\frac{d\sigma}{m} \right)^{\frac{1}{1+\sigma}} \left(\frac{1}{1+\sigma} \right)^{\frac{\beta}{1+\sigma}} (\log m)^{\frac{\beta}{1+\sigma}} \left(1 + O \left(\frac{\log \log m}{\log m} \right) \right)^{\frac{\beta}{1+\sigma}} \left(1 + O \left(\frac{1}{\log m} \right) \right)^{\frac{1}{1+\sigma}} \\ &= \left(\frac{d\sigma}{m} \right)^{\frac{1}{1+\sigma}} \left(\frac{\log m}{1+\sigma} \right)^{\frac{\beta}{1+\sigma}} \left(1 + O \left(\frac{\log \log m}{\log m} \right) \right). \end{aligned} \quad (45)$$

In order to use the saddle point method we must find an angle θ_0 such that

$$r^2 \theta_0^2 h''(r) \rightarrow \infty \quad \text{and} \quad r^3 \theta_0^3 h^{(3)}(r) \rightarrow 0 \quad (46)$$

as $m \rightarrow \infty$. If we can find such a θ_0 , then for $|\theta| < \theta_0$ the behaviour of the Taylor series for the expression $h(re^{i\theta}) - im\theta$ is dominated by the term $h''(r)\theta_0^2$. First we determine that

$$\begin{aligned} h''(r) &= d(\sigma^2 + \sigma)(1-r)^{-\sigma-2} \left(\log \frac{1}{1-r} \right)^\beta \left(1 + O \left(\left(\log \frac{1}{1-r} \right)^{-1} \right) \right) \\ &= d(\sigma^2 + \sigma)t^{-\sigma-2} \left(\log \frac{1}{t} \right)^\beta \left(1 + O \left(\frac{1}{\log t} \right) \right) \\ &= d(\sigma^2 + \sigma) \left(\frac{m}{d\sigma} \right)^{\frac{\sigma+2}{\sigma+1}} \left(\frac{1+\sigma}{\log m} \right)^{\beta \frac{\sigma+2}{\sigma+1}} (\log m)^\beta \left(1 + O \left(\frac{1}{\log m} \right) \right) \\ &\geq Km^{\frac{\sigma+2}{\sigma+1}} (\log m)^{\frac{-\beta}{1+\sigma}}, \end{aligned} \quad (47)$$

where K is some positive constant, and

$$\begin{aligned} h^{(3)}(r) &= d(\sigma^3 + 3\sigma^2 + 2\sigma)t^{-\sigma-3} \left(\log \frac{1}{t} \right)^\beta \left(1 + O \left(\frac{1}{\log t} \right) \right) \\ &= O \left(m^{\frac{\sigma+3}{\sigma+1}} (\log m)^{-\frac{2\beta}{1+\sigma}} \right). \end{aligned}$$

Let $\theta_0 = m^k$, and note that $r = O(1)$. Then in order to satisfy (46), we may choose any constant k which satisfies

$$2k + \frac{\sigma+2}{\sigma+1} > 0 \quad \text{and} \quad 3k + \frac{\sigma+3}{\sigma+1} < 0.$$

In particular, we may choose $k = -\frac{5\sigma+12}{12(\sigma+1)}$, i.e., $\theta_0 = m^{-\frac{5\sigma+12}{12(\sigma+1)}}$.

For $|\theta| \leq \theta_0$ we have, from the Taylor series of $h(z)$ and e^z ,

$$\begin{aligned}
h(re^{i\theta}) &= h(r) + rh'(r)(e^{i\theta} - 1) + \frac{r^2}{2}h''(r)(e^{i\theta} - 1)^2 + O(r^3h'''(r)\theta^3) \\
&= h(r) + i\theta rh'(r) - \frac{r\theta^2 h'(r)}{2} - \frac{r^2\theta^2 h''(r)}{2} + O(r^3h'''(r)\theta^3) \\
&= h(r) + i\theta rh'(r) - \frac{b(r)}{2}\theta^2 + o(1),
\end{aligned} \tag{48}$$

where $b(r) = rh'(r) + r^2h''(r)$.

We next divide the integral from (40) into two parts:

$$I_1 = \int_{-\theta_0}^{\theta_0} \exp(h(re^{i\theta}) - C(re^{i\theta}; J) - im\theta + c + o(1)) \, d\theta$$

and

$$I_2 = \int_{|\theta| \geq \theta_0} \exp(h(re^{i\theta}) - C(re^{i\theta}) - im\theta + c + o(1)) \, d\theta.$$

Then for $|\theta| \leq \theta_0$, and by (38) and (41), we have

$$\begin{aligned}
C(re^{i\theta}; J) &= \sum_{j \in J} \frac{C_j}{j!} r^j e^{ij\theta} \\
&= \sum_{j \in J} \frac{C_j}{j!} \left(1 - \left(\frac{d\sigma}{m} \right)^{\frac{1}{1+\sigma}} \left(\frac{\log m}{1+\sigma} \right)^{\frac{\beta}{1+\sigma}} + O\left(m^{-\frac{1}{1+\sigma}} (\log m)^{\frac{\beta}{1+\sigma}-1} \log \log m \right) \right)^j \\
&\quad \times (1 + O(j\theta_0)).
\end{aligned}$$

By our choice of $\hat{j}$, it is clear that for all $j \in J$,

$$j\theta_0 = m^{-\frac{5\sigma+12}{12(\sigma+1)}} o\left(m^{\frac{1}{1+\sigma}} (\log m)^{-\frac{\beta}{1+\sigma}} \right) = o\left(m^{-\frac{5\sigma}{1+\sigma}} (\log m)^{-\frac{\beta}{1+\sigma}} \right) = o(1).$$

Since

$$m^{-\frac{1}{1+\sigma}} (\log m)^{\frac{\beta}{1+\sigma}-1} \log \log m = o\left(m^{-\frac{1}{1+\sigma}} (\log m)^{\frac{\beta}{1+\sigma}} \right),$$

and

$$\hat{j} m^{-\frac{1}{1+\sigma}} (\log m)^{\frac{\beta}{1+\sigma}} = o(1),$$

it follows that

$$\left(1 - \left(\frac{d\sigma}{m} \right)^{\frac{1}{1+\sigma}} \left(\frac{\log m}{1+\sigma} \right)^{\frac{\beta}{1+\sigma}} + O \left(m^{-\frac{1}{1+\sigma}} (\log m)^{\frac{\beta}{1+\sigma}-1} \log \log m \right) \right)^j = 1 + j O \left(m^{-\frac{1}{1+\sigma}} (\log m)^{\frac{\beta}{1+\sigma}} \right).$$

Therefore,

$$\begin{aligned} C(re^{i\theta}; J) &= \sum_{j \in J} \frac{C_j}{j!} \left(1 + j O \left(m^{-\frac{1}{1+\sigma}} (\log m)^{\frac{\beta}{1+\sigma}} \right) \right) \\ &= \sum_{j \in J} \frac{C_j}{j!} + \sum_{j \in J} \frac{j C_j}{j!} O \left(m^{-\frac{1}{1+\sigma}} (\log m)^{\frac{\beta}{1+\sigma}} \right) \\ &= C(\rho; J) + o(1), \end{aligned} \tag{49}$$

where (49) follows by Theorems 2 and 3 with $\alpha = -\sigma$.

From (41), (48), and (49) we have

$$I_1 = \int_{-\theta_0}^{\theta_0} \exp \left(h(r) - \frac{\theta^2}{2} b(r) - C(\rho; J) + o(1) \right) d\theta.$$

Let $\gamma = d\sigma^{-\frac{1}{1+\sigma}} (1+\sigma)^{1+\frac{\beta(\sigma+2)}{\sigma+1}}$. Then from (47), $b(r) \sim \gamma m^{\frac{\sigma+2}{\sigma+1}} (\log m)^{\frac{-\beta}{1+\sigma}}$. Letting $t = \theta \sqrt{b(r)}$, we see that $dt = \sqrt{b(r)} d\theta$, and

$$I_1 = \frac{e^{C(\rho)}}{\sqrt{b(r)}} \int_{-\sqrt{b(r)}\theta_0}^{\sqrt{b(r)}\theta_0} \exp \left(-\frac{t^2}{2} - C(\rho; J) + o(1) \right) dt,$$

which is asymptotic to a Gaussian integral, since $\sqrt{b(r)}\theta_0 \sim \sqrt{\gamma} m^{\frac{\sigma}{12(\sigma+1)}} (\log m)^{\frac{-\beta}{2(1+\sigma)}}$ as $n \rightarrow \infty$. Therefore,

$$I_1 \sim \exp(C(r\rho) - C(\rho; J)) \sqrt{\frac{2\pi}{b(r)}} \sim \exp(C(r\rho) - C(\rho; J)) \sqrt{\frac{2\pi}{\gamma} m^{-\frac{\sigma+2}{\sigma+1}} (\log m)^{\frac{\beta}{1+\sigma}}}.$$

Finally, we must prove that the contribution of I_2 to the integral from (40) is negligible. First we examine the function $1 - re^{i\theta}$. As θ ranges from $-\pi$ to π , the function $1 - re^{i\theta}$ traces out a circle of radius r centred at $z = 1$. This circle begins at the point $1 - r$ and proceeds clockwise. The function $|1 - re^{i\theta}|$ is unimodal for $-\pi \leq \theta \leq \pi$ and achieves a unique minimum of $1 - r$ at $\theta = 0$. Therefore $|1 - re^{i\theta}|^{-\sigma}$ has a unique maximum at $\theta = 0$ and a unique minimum at $\theta = \pi$.

Furthermore, for $|\theta| \geq \theta_0$ we have

$$|1 - re^{i\theta}|^{-\sigma} \leq |1 - re^{i\theta_0}|^{-\sigma} = |1 - 2r \cos \theta_0 + r^2|^{-\sigma/2}.$$

We next apply the Taylor series expansion for $\cos \theta_0$ to obtain

$$|1 - re^{i\theta}|^{-\sigma} \leq (1 - r)^{-\sigma} - \frac{\sigma}{2} r (1 - r)^{-\sigma-2} \theta_0^2 + O((1 - r)^{-\sigma-4} \theta_0^4).$$

From our choice of θ_0 ,

$$\frac{\sigma}{2} r (1 - r)^{-\sigma-2} \theta_0^2 = \frac{\sigma}{2} \left[\left(\frac{d\sigma}{m} \right)^{\frac{1}{1+\sigma}} \left(\frac{\log m}{1 + \sigma} \right)^{\frac{\beta}{1+\sigma}} \left(1 + O \left(\frac{\log \log m}{\log m} \right) \right) \right]^{-\sigma-2} m^{-\frac{5\sigma+12}{6(\sigma+1)}},$$

which implies that

$$|1 - re^{i\theta}|^{-\sigma} \leq (1 - r)^{-\sigma} - K m^{\frac{\sigma}{6(\sigma+1)}} (\log m)^{-\frac{\beta(\sigma+2)}{(1+\sigma)}},$$

where K is some positive constant.

It follows that, for $\theta_0 \leq |\theta| \leq \pi$,

$$\begin{aligned} |h(re^{i\theta})| &\leq |1 - re^{i\theta}|^{-\sigma} \left| \log \frac{1}{1 - re^{i\theta}} \right|^\beta \\ &\leq \left((1 - r)^{-\sigma} - K m^{\frac{\sigma}{6(\sigma+1)}} (\log m)^{-\frac{\beta(\sigma+2)}{(1+\sigma)}} \right) \left(\log \frac{1}{1 - r} \right)^\beta \\ &\leq h(r) - K' m^{\frac{\sigma}{6(\sigma+1)}} (\log m)^{-\frac{\beta}{(1+\sigma)}} \end{aligned}$$

for some positive constant K' . Thus

$$|\exp(h(re^{i\theta}))| \leq \exp \left(h(r) - K' m^{\frac{\sigma}{6(\sigma+1)}} (\log m)^{-\frac{\beta}{(1+\sigma)}} \right), \quad (50)$$

as $m \rightarrow \infty$ for $\theta_0 \leq |\theta| \leq \pi$.

From (50) we immediately obtain

$$\begin{aligned} \frac{|I_2|}{|I_1|} &= \frac{O \left(\exp \left(C(r\rho) - (K' m^{\sigma/6} (\log m)^{-\beta})^{\frac{1}{1+\sigma}} + C(\rho; J) \right) \right)}{\sqrt{2\pi} \exp(C(r\rho) - C(\rho; J)) / \sqrt{b(r)}} \\ &= O \left(\exp \left(- (K' m^{\sigma/6} (\log m)^{-\beta})^{\frac{1}{1+\sigma}} \right) m^{\frac{\sigma+2}{2(\sigma+1)}} (\log m)^{\frac{-\beta}{2(\sigma+1)}} \right), \end{aligned}$$

where the first condition in (38) implies

$$C(\rho; J) = \sum_{j \in J} \frac{C_j \rho^j}{j!} = O \left(\sum_{j \in J} j^{\sigma-1} (\log j)^\beta \right) = o \left((m^{\sigma/6} (\log m)^{-\beta})^{\frac{1}{1+\sigma}} \right).$$

This implies that

$$\lim_{n \rightarrow \infty} \frac{|I_2|}{|I_1|} = 0.$$

Therefore,

$$|I_2| = o \left(\frac{\exp(C(r\rho) - C(\rho; J))}{\sqrt{b(r)}} \right),$$

so the contribution of I_2 to (40) is negligible compared to that of I_1 . □

Let $P(S, n)$ denote that a uniformly chosen decomposable structure of size n has a restricted pattern S . Then

$$P(S, n) = \frac{[z^n]F(z; S)}{[z^n]F(z; \emptyset)}, \quad (51)$$

and we obtain the following corollary.

Corollary 3. *Consider labeled decomposable combinatorial structures whose component generating function $C(z)$ is of alg-log type with algebraic exponent $-\sigma < 0$ and logarithmic exponent $\beta \neq 0$, and suppose that the pattern S satisfies Condition (38). Then, as $m \rightarrow \infty$,*

$$P(S, n) \sim \mathcal{S} \rho^{n-m} \frac{r(n)^n}{r(m)^m} \exp(C(\rho r(m)) - C(\rho r(n) - C(\rho; J))) \left(\frac{n}{m}\right)^{\frac{\sigma+2}{2(\sigma+1)}},$$

where $\gamma = d\sigma^{-\frac{1}{1+\sigma}}(1+\sigma)^{1+\frac{\beta(\sigma+2)}{\sigma+1}}$, and $r(n) \sim 1 - \left(\frac{d\sigma}{n}\right)^{\frac{1}{1+\sigma}} \left(\frac{\log n}{1+\sigma}\right)^{\frac{\beta}{1+\sigma}}$.

We close this section by noting that if $m \sim n$ as $n \rightarrow \infty$, then $\left(\frac{n}{m}\right)^{\frac{\sigma+2}{2(\sigma+1)}} \sim 1$, and we may eliminate that term in Corollary 3.

4.2 Unlabeled Case

In the unlabeled case, we assume that $0 < \rho < 1$. The generating function of a decomposable structure with no pattern is

$$F(z) = F(z, \emptyset) = \exp\left(C(z) + \frac{C(z^2)}{2} + \frac{C(z^3)}{3} + \dots\right),$$

and, by Lemma 3, the generating function of a structure with a restricted pattern S is

$$F(z; S) = \prod_{j \in J} \binom{C_j + S(j) - 1}{S(j)} z^{m'} \exp\left(\sum_{k \geq 1} \frac{C(z^k) - C(z^k; J)}{k}\right),$$

where $m' = n - \sum_{j \in J} jS(j)$ is the size of the unrestricted part.

Since $0 < \rho < 1$, if $|z| < \rho$ then

$$\left| \frac{C(z)}{z} \right| \leq K$$

for some fixed constant K . Therefore, for all $j \geq 2$, $C(\rho^j) < K\rho^j$, which implies that

$$r_0 = \sum_{k \geq 2} \frac{C(\rho^k)}{k}$$

converges. Thus, as $z \rightarrow \rho$,

$$\begin{aligned} F(z; \emptyset) &= \exp(C(z) + r_0(1 + o(1))) \\ &\sim e^{r_0} \exp(C(z)), \end{aligned}$$

so by Theorem 7,

$$[z^n]F(z; \emptyset) \sim (r\rho)^{-n} e^{C(r\rho) + r_0} \sqrt{\frac{1}{2\pi\gamma} n^{-\frac{\sigma+2}{\sigma+1}}},$$

where r is defined by (39) and $\gamma = d\sigma^{-\frac{1}{1+\sigma}}(1+\sigma)^{1+\frac{\beta(\sigma+2)}{\sigma+1}}$.

We now prove an analog to Theorem 7.

Theorem 8. *Suppose the pattern S satisfies Condition (38). Then as $m \rightarrow \infty$,*

$$[z^n]F(z; S) \sim \left(\prod_{j \in J} \binom{C_j + S(j) - 1}{S(j)} (1 - \rho^j)^{C_j} \right) (r\rho)^{-m} e^{C(r\rho) + r_0} \sqrt{\frac{1}{2\pi\gamma} m^{-\frac{\sigma+2}{\sigma+1}}},$$

where $\gamma = d\sigma^{-\frac{1}{1+\sigma}}(1+\sigma)^{1+\frac{\beta(\sigma+2)}{\sigma+1}}$, and $r \sim 1 - \left(\frac{d\sigma}{m}\right)^{\frac{1}{1+\sigma}} \left(\frac{\log m}{1+\sigma}\right)^{\frac{\beta}{1+\sigma}}$. Here r is defined by the saddle point equation (39).

Proof. By Lemma 3,

$$[z^n]F(z; S) = \prod_{j \in J} \binom{C_j + S(j) - 1}{S(j)} [z^m] \exp \left(\sum_{k \geq 1} \frac{C(z)}{k} - \frac{C(z^k; J)}{k} \right),$$

so we must determine

$$\begin{aligned} [z^m] \exp \left(\sum_{k \geq 1} \frac{C(z)}{k} - \frac{C(z^k; J)}{k} \right) &= \rho^{-m} [z^m] \exp \left(\sum_{k \geq 1} \frac{C((\rho z)^k)}{k} - \frac{C((\rho z)^k; J)}{k} \right) \\ &= \rho^{-m} e^{r_0} [z^m] \exp(C(\rho z)) \exp \left(- \sum_{k \geq 1} \frac{C((\rho z)^k; J)}{k} \right). \end{aligned} \quad (52)$$

We first examine the sum in the second exponential term in (52):

$$\begin{aligned} \sum_{k \geq 1} \frac{C((\rho z)^k; J)}{k} &= \sum_{k \geq 1} \sum_{j \in J} \frac{C_j(\rho z)^{jk}}{k} \\ &= \sum_{j \in J} C_j \sum_{k \geq 1} \frac{(\rho z)^{jk}}{k} \\ &= \sum_{j \in J} C_j \sum_{k \geq 1} \frac{\rho^{jk}}{k} + \sum_{j \in J} C_j \sum_{k \geq 1} \frac{\rho^{jk}}{k} (1 - z^{jk}) \\ &= \sum_{j \in J} \log(1 - \rho^j)^{C_j} + \sum_{j \in J} C_j \rho^j (1 - z^j) \sum_{k \geq 0} \frac{\rho^{jk}}{k+1} (1 + z^j + \cdots + z^{jk}). \end{aligned} \quad (53)$$

Next, consider the circle $\gamma = \{z : |z| = r\}$, and let $z = 1 - \frac{w}{m^{\frac{1}{1+\sigma}}}$. Then $w = O\left((\log m)^{\frac{\beta}{1+\sigma}}\right)$, and

$$z^j = 1 + O\left(\frac{jw}{m^{\frac{1}{1+\sigma}}}\right),$$

which implies

$$1 - z^j = O\left(j \left(\frac{(\log m)^\beta}{m}\right)^{\frac{1}{1+\sigma}}\right).$$

Therefore, by the transfer theorems and (38),

$$\sum_{j \in J} C_j \rho^j (1 - z^j) = \sum_{j \in J} O\left(\frac{(\log j)^\beta}{j^{1+\sigma}}\right) O\left(j \left(\frac{(\log m)^\beta}{m}\right)^{\frac{1}{1+\sigma}}\right) = o(1). \quad (54)$$

On the contour γ , $|z| < 1$, and $\rho < 1$, so

$$\begin{aligned} \left| \sum_{k \geq 0} \frac{\rho^{jk}}{k+1} (1 + z^j + z^{2j} + \dots + z^{kj}) \right| &\leq \sum_{k \geq 0} \frac{\rho^{jk}}{k+1} |1 + z^j + z^{2j} + \dots + z^{kj}| \\ &\leq \sum_{k \geq 0} \frac{\rho^{jk}}{k+1} (k) \leq \sum_{k \geq 0} \rho^{jk} = \frac{1}{1 - \rho^j} = O(1). \end{aligned} \quad (55)$$

Combining (52), (53), (54), and (55), we have

$$[z^m] \exp\left(\sum_{k \geq 1} \frac{C(z)}{k} - \frac{C(z^k; J)}{k}\right) = \rho^{-m} e^{r_0} \prod_{j \in J} (1 - \rho^j)^{C_j} [z^m] \exp(C(\rho z)). \quad (56)$$

We apply Theorem 7 to (56) to obtain

$$[z^m] \exp(C(\rho z)) \sim (r\rho)^{-m} e^{C(r\rho)} \sqrt{\frac{1}{2\pi\gamma} m^{-\frac{\sigma+2}{\sigma+1}}},$$

which completes the proof. $\square$

As in the labeled case, we have the following corollary.

Corollary 4. *Consider unlabeled decomposable combinatorial structures whose component generating function $C(z)$ is of alg-log type with algebraic exponent $-\sigma < 0$ and*

logarithmic exponent $\beta \neq 0$, and suppose that the pattern S satisfies Condition (38).

Then, as $m \rightarrow \infty$,

$$P(S, n) \sim \left(\prod_{j \in J} \binom{C_j + S(j) - 1}{S(j)} (1 - \rho^j)^{C_j} \right) \rho^{n-m} \frac{r(n)^n}{r(m)^m} \\ \times \exp(C(\rho r(m)) - C(\rho r(n))) \left(\frac{n}{m} \right)^{\frac{\sigma+2}{2(\sigma+1)}},$$

where $\gamma = d\sigma^{-\frac{1}{1+\sigma}}(1+\sigma)^{1+\frac{\beta(\sigma+2)}{\sigma+1}}$, and $r(n) \sim 1 - \left(\frac{d\sigma}{n}\right)^{\frac{1}{1+\sigma}} \left(\frac{\log n}{1+\sigma}\right)^{\frac{\beta}{1+\sigma}}$.

4.3 The r th Smallest Component Size

Let $X_n^{[r]}(S)$ denote the size of the r th smallest component in a random structure of size n with restricted pattern S . In [11], Dong *et al.* prove a theorem (Theorem 4.3) that gives an asymptotic expression for the probability that $X_n^{[r]}(S)$ is greater than k as $n \rightarrow \infty$ in the case $\alpha < 0$, $\beta = 0$. Unfortunately, we cannot prove a result analogous to Theorem 4.3 in [11] because the proof of that theorem requires $r(m)^m \sim r(n)^n$, which we cannot show for *any* $m \neq n$ due to the asymptotic scale of $r(n)$. In the case of Corollary 2, we have an exact expansion for $r(n)$ whose asymptotic scale is descending powers of $n^{1/(1+\sigma)}$ and are able to prove that $r(m)^m \sim r(n)^n$ under certain conditions. However, for Theorem 7, the scale of $r(n)$ is much more complicated and descends in powers of $\log n$.

Chapter 5

Positive Logarithmic Exponent

Figure 8: An illustration of the ranges of α and β examined in Chapter 5: $\alpha = 0$, $\beta \geq 2$ and $\alpha = 0$, $0 < \beta < 1$.

In this chapter we consider the case where $C(z)$ has only a logarithmic exponent. This case may seem redundant in light of Chapter 4, but the behaviour of $\exp(C(z))$ near the singularity $z = \rho$ in the case $\alpha \neq 0$ is so dominated by the algebraic term in $C(z)$ that the case $\alpha = 0$ requires its own analysis. We further divide this case into three sub-cases: $\beta \in (0, 1)$, $\beta \in (1, 2)$, and $\beta \geq 2$. We do so because different analyses are required for each case.

5.1 Alg-Log Components with $\alpha = 0$ and $\beta \geq 2$

This section extends some of the results in [22]. The main technique used is the saddle point method.

5.1.1 Labeled Case

Throughout this section assume the following conditions hold on a given pattern S :

$$\hat{j} = o\left(m(\log m)^{\frac{2(1-\beta)}{3}}\right), \quad \sum_{j \in J} (\log j)^\beta = o\left(m(\log m)^{1-\beta}\right). \quad (57)$$

Let $h(z) = d\left(\log \frac{1}{1-z}\right)^\beta$, and let $r = r(m)$ be defined by the saddle point equation

$$rh'(r) = m, \quad 0 < r < 1. \quad (58)$$

Theorem 9. *Suppose the pattern S satisfies Condition (57). Then as $m \rightarrow \infty$,*

$$[z^n]F(z; S) \sim \mathcal{S} \frac{e^{C(\rho r) - C(\rho; J)} (\rho r)^{-m}}{m} \sqrt{\frac{\beta}{2\pi} (\log m)^{\beta-1}},$$

where $r \sim 1 - \beta m^{-1} (\log m)^{\beta-1}$. Here r is defined by the saddle point equation (58).

Proof. By Lemma 3,

$$[z^n]F(z; S) = \mathcal{S} \rho^{-m} [z^m] \exp(h(z) - C(\rho z; J) + c + o(1)). \quad (59)$$

Let $\mathcal{C}$ be the contour obtained by tracing a counterclockwise circle of radius r

centred at the origin, and let $z = re^{i\theta}$. From the Cauchy integral formula,

$$\begin{aligned} [z^m] \exp(h(z) - C(\rho z; J) + c + o(1)) &= \frac{1}{2\pi i} \int_{\mathcal{C}} \exp(h(z) - C(\rho z; J) + c + o(1)) \frac{dz}{z^{m+1}} \\ &= \frac{r^{-m}}{2\pi} \int_{-\pi}^{\pi} \exp(h(re^{i\theta}) - C(r\rho e^{i\theta}; J) - im\theta + c + o(1)) d\theta. \end{aligned} \quad (60)$$

Next, we expand the saddle point equation (58):

$$\frac{r\beta \left(\log \frac{1}{1-r}\right)^{\beta-1}}{1-r} = m. \quad (61)$$

Equation (61) cannot be solved exactly, but we can find an asymptotic expression for r . Note that $r \rightarrow 1$ as $m \rightarrow \infty$, and let $r = 1 - t$. Let $x = \beta/m$. Then

$$t = x \left(\log \frac{1}{t}\right)^{\beta-1} (1-t) \sim x \left(\log \frac{1}{t}\right)^{\beta-1}, \quad (62)$$

$$\log \frac{1}{t} = \log \frac{1}{x} - (\beta-1) \log \log \frac{1}{t} + \log \left(\frac{1}{1-t}\right) \sim \log \frac{1}{x}. \quad (63)$$

We substitute (63) into the right hand side of (62) and find $t \sim x \left(\log \frac{1}{x}\right)^{\beta-1}$. Therefore,

$$\begin{aligned} t &= x \left(\log \frac{1}{t}\right)^{\beta-1} (1-t) = x \left(\log \frac{1}{t}\right)^{\beta-1} \left(1 - O\left(x \left(\log \frac{1}{x}\right)^{\beta-1}\right)\right), \\ \log \frac{1}{t} &= \log \frac{1}{x} + (1-\beta) \log \log \frac{1}{t} + \log \frac{1}{1-t}, \\ \log \log \frac{1}{t} &= \log \log \frac{1}{x} + O\left(\frac{\log \log \frac{1}{t}}{\log \frac{1}{x}}\right) = \log \log \frac{1}{x} + O\left(\frac{\log \log \frac{1}{x}}{\log \frac{1}{x}}\right), \end{aligned}$$

and

$$t = \frac{\beta}{m} (\log m)^{\beta-1} \left(1 + O\left(\frac{\log \log m}{\log m}\right)\right).$$

Once again, in order to use the saddle point method we must find an angle θ_0

such that

$$r^2\theta_0^2h''(r) \rightarrow \infty \quad \text{and} \quad r^3\theta_0^3h^{(3)}(r) \rightarrow 0 \quad (64)$$

as $m \rightarrow \infty$. In order to find such an angle, we expand $r^2h''(r)$ and $r^3h'''(r)$:

$$r^2h''(r) \sim \frac{\beta \left(\log \frac{1}{1-r}\right)^{\beta-1}}{(1-r)^2} \sim \frac{\beta \left(\log \frac{1}{t}\right)^{\beta-1}}{t^2} \sim \frac{m^2}{\beta(\log m)^{\beta-1}},$$

and

$$r^3h'''(r) = O\left(\frac{r^3 \left(\log \frac{1}{1-r}\right)^{\beta-1}}{(1-r)^3}\right) = O\left(\frac{\left(\log \frac{1}{t}\right)^{\beta-1}}{t^3}\right) = O\left(\frac{m^3}{(\log m)^{2(\beta-1)}}\right).$$

In order to satisfy (64), we choose $\theta_0 = m^{-1}(\log m)^k$ such that

$$\frac{\beta-1}{2} < k < \frac{2(\beta-1)}{3}.$$

As in Chapter 4, for $|\theta| \leq \theta_0$ we have

$$h(re^{i\theta}) = h(r) + i\theta rh'(r) - \frac{b(r)}{2}\theta^2 + o(1),$$

where

$$b(r) = rh'(r) + r^2h''(r) \sim \frac{1}{\beta}m^2(\log m)^{1-\beta}.$$

We next divide the integral from (60) into two parts:

$$I_1 = \int_{-\theta_0}^{\theta_0} \exp\left(h(r) - C(r\rho e^{i\theta}; J) - \frac{\theta^2}{2}b(r) + c + o(1)\right) d\theta,$$

and

$$I_2 = \int_{\theta_0 < |\theta| < \pi} \exp\left(h(r) - C(r\rho e^{i\theta}; J) - \frac{\theta^2}{2}b(r) + c + o(1)\right) d\theta.$$

First, we evaluate I_1 . For $|\theta| \leq \theta_0$, we have

$$\begin{aligned} C(re^{i\theta}; J) &= \sum_{j \in J} \frac{C_j}{j!} r^j e^{ij\theta} \\ &= \sum_{j \in J} \frac{C_j}{j!} (1 - \beta(m^{-1}(\log m)^{\beta-1}) + O(m^{-1} \log \log m (\log m)^{\beta-2}))^j (1 + O(j\theta_0)). \end{aligned}$$

By the conditions in (57) on $\hat{j}$, it is clear that for all $j \in J$,

$$j\theta_0 = O\left(m^{-1}(\log m)^{\frac{2(\beta-1)}{3}}\right) o\left(m(\log m)^{\frac{2(1-\beta)}{3}}\right) = o(1),$$

and

$$\begin{aligned} jO\left(m^{-1} \log \log m (\log m)^{\beta-2}\right) &= o\left(m(\log m)^{\frac{2(1-\beta)}{3}}\right) O\left(m^{-1} \log \log m (\log m)^{\beta-2}\right) \\ &= o(1). \end{aligned}$$

It follows that

$$(1 - \beta(m^{-1}(\log m)^{\beta-1}) + O(m^{-1} \log \log m (\log m)^{\beta-2}))^j = 1 + jO(m^{-1}(\log m)^{\beta-1}),$$

and from (57),

$$\begin{aligned} C(re^{i\theta}; J) &= \sum_{j \in J} \frac{C_j}{j!} (1 + jO(m^{-1}(\log m)^{\beta-1})) \\ &= \sum_{j \in J} \frac{C_j}{j!} + \sum_{j \in J} \frac{jC_j}{j!} O(m^{-1}(\log m)^{\beta-1}) \\ &= C(\rho; J) + o(1). \end{aligned}$$

Therefore,

$$I_1 = \int_{-\theta_0}^{\theta_0} \exp\left(h(r) - \frac{\theta^2}{2}b(r) - C(\rho; J) + c + o(1)\right) d\theta.$$

Recall that $b(r) \sim \beta^{-1}m^2(\log m)^{1-\beta}$, and let $t = \theta\sqrt{b(r)}$. Then $dt = \sqrt{b(r)}d\theta$,

and

$$I_1 = \frac{e^{c+h(r)}}{\sqrt{b(r)}} \int_{-\sqrt{b(r)}\theta_0}^{\sqrt{b(r)}\theta_0} \exp\left(-\frac{t^2}{2} - C(\rho; J) + o(1)\right) dt,$$

which is asymptotic to a Gaussian integral, since $\sqrt{b(r)}\theta_0 > K(\log m)^M$ as $m \rightarrow \infty$, where K and M are positive constants. Therefore

$$I_1 \sim \exp(C(r\rho) - C(\rho; J)) \sqrt{\frac{2\pi}{b(r)}} \sim \exp(C(r\rho) - C(\rho; J)) \frac{\sqrt{2\pi}\beta}{m} (\log m)^{\frac{b-1}{2}}.$$

Now we consider the integral I_2 . For convenience, we write

$$\frac{1}{1 - re^{i\theta}} = \xi e^{i\phi}, \quad \text{where } |\phi| \leq \pi, \quad \text{and } \xi = \frac{1}{|1 - re^{i\theta}|}.$$

To show that the contribution of I_2 is negligible compared to I_1 , we further divide the tail region into the subregions $\theta_0 \leq |\theta| \leq \theta_1$ and $\theta_1 \leq |\theta| \leq \pi$, where

$$\theta_1 = \frac{1}{m}(\log m)^\beta.$$

Let I'_2 and I''_2 denote the respective integrals associated with those regions.

Recall that for $|\theta| \leq \pi$, the function $|1 - re^{i\theta}|$ is unimodal with its minimum at $\theta = 0$. For $|\theta| \geq \theta_1$, we have

$$\Re\left(\left(\log \frac{1}{1 - re^{i\theta}}\right)^\beta\right) \leq \left|\log \frac{1}{1 - re^{i\theta}}\right|^\beta \leq (\log \xi + \pi)^\beta \leq \left(\log \frac{1}{|1 - re^{i\theta_1}|} + \pi\right)^\beta,$$

where the last inequality is due to the unimodality of $|1 - re^{i\theta}|$. Because θ_1 is a very small angle, we may easily express $\cos(\theta_1)$ in terms of its Taylor series, *i.e.*,

$\cos(\theta_1) = 1 - \frac{\theta_1^2}{2} + O(\theta_1^4)$. Therefore $\cos(\theta_1) \leq 1 - \frac{\theta_1^2}{4}$, which implies that

$$\begin{aligned} \Re \left(\left(\log \frac{1}{1 - re^{i\theta}} \right)^\beta \right) &\leq \left(\frac{1}{2} \log \frac{1}{1 + r^2 - 2r \cos(\theta_1)} + \pi \right)^\beta \\ &\leq \left(\frac{1}{2} \log \frac{1}{1 + r^2 - 2r(1 - \theta_1^2/4)} + \pi \right)^\beta \\ &\leq \left(\log \frac{1}{1 - r} + \frac{1}{2} \log \left(\frac{1}{1 + \frac{r\theta_1^2}{2(1-r)^2}} \right) + \pi \right)^\beta. \end{aligned}$$

By our choice of θ_1 and our expression of the value r , it is clear that $\frac{r\theta_1^2}{2t^2} \rightarrow \infty$ as $m \rightarrow \infty$. Furthermore, $\theta_1 \geq \sqrt{2}e^9(1-r)$ for sufficiently large m . Therefore,

$$\begin{aligned} \frac{1}{1 + \frac{r\theta_1^2}{2(1-r)^2}} &\leq \frac{2(1-r)^2}{r\theta_1^2}, \text{ and} \\ \frac{1}{2} \log \frac{1}{1 + \frac{r\theta_1^2}{2(1-r)^2}} &\leq \frac{1}{2} \log \frac{2(1-r)^2}{r\theta_1^2} \leq \frac{1}{2} \log \frac{2(1-r)^2}{2e^{18}(1-r)^2} = -9, \end{aligned}$$

which implies that

$$\begin{aligned} \Re \left(\left(\log \frac{1}{1 - re^{i\theta}} \right)^\beta \right) &\leq \left(\log \frac{1}{1 - r} - 9 + \pi \right)^\beta \leq \left(\log \frac{1}{1 - r} \right)^\beta - 5 \left(\log \frac{1}{1 - r} \right)^{\beta-1} \\ &\leq \left(\log \frac{1}{1 - r} \right)^\beta - 4 \log m, \end{aligned}$$

where the last inequality holds because $\beta \geq 2$. This implies that

$$\frac{|I_2''|}{I_1} = \frac{O \left(\exp \left(\left(\log \frac{1}{1-r} \right)^\beta - 4 \log m \right) \right)}{\frac{\exp(C(r))}{m} \sqrt{2\pi\beta} (\log m)^{\beta-1}} = O \left(m^{-3} (\log m)^{(1-\beta)/2} \right),$$

i.e., $I_2'' = o(I_1)$.

For $\theta_0 \leq |\theta| \leq \theta_1$, we note that $\log \xi \rightarrow \infty$ as $n \rightarrow \infty$, so

$$0 < \Re \left(1 + \frac{i\phi}{\log \xi} \right)^\beta \leq 1.$$

Therefore,

$$\begin{aligned} \Re \left(\left(\log \frac{1}{1 - re^{i\theta}} \right)^\beta \right) &= \Re \left((\log \xi + i\phi)^\beta \right) \\ &= \Re \left((\log \xi)^\beta \left(1 + \frac{i\phi}{\log \xi} \right)^\beta \right) \\ &\leq \left(\log \frac{1}{|1 - re^{i\theta_0}|} \right)^\beta \\ &\leq \left(\log \frac{1}{1 - r} - \left(\frac{r\theta_0}{2(1 - r)} \right)^2 + O \left(\left(\frac{\theta_0}{1 - r} \right)^4 \right) \right)^\beta \\ &\leq \left(\log \frac{1}{1 - r} \right)^\beta \left(1 + \frac{1}{\log(1 - r)} \left(\frac{r\theta_0}{2(1 - r)} \right)^2 + O \left((\log m)^{4k - 4\beta + 3} \right) \right)^\beta \\ &\leq \left(\log \frac{1}{1 - r} \right)^\beta - \frac{1}{2} (\log m)^{2k - \beta + 1}. \end{aligned}$$

It follows that

$$\begin{aligned} |I'_2| &\leq \theta_1 \exp \left(\left(\log \frac{1}{1 - r} \right)^\beta - \frac{1}{2} (\log m)^{2k - \beta + 1} \right) \\ &\leq \frac{1}{m} (\log m)^\beta \exp \left(\left(\log \frac{1}{1 - r} \right)^\beta - \frac{1}{2} (\log m)^{2k - \beta + 1} \right) = o(I_1). \end{aligned}$$

□

As in Chapter 4, we let $P(S, n)$ denote the probability that a uniformly chosen decomposable structure of size n has restricted pattern S , and we have the following corollary.

Corollary 5. *Consider labeled decomposable combinatorial structures whose component generating function $C(z)$ is of alg-log type with algebraic exponent $\alpha = 0$ and*

logarithmic exponent $\beta \geq 2$, and suppose that the pattern S satisfies Condition (57). Then, as $m \rightarrow \infty$,

$$P(S, n) \sim \mathcal{S} \rho^{n-m} \frac{r(n)^n}{r(m)^m} \exp(C(\rho r(m)) - C(\rho r(n)) - C(\rho; J)),$$

where $r(n) \sim 1 - \frac{\beta}{n}(\log n)^{\beta-1}$. Here $r(n)$ is defined by the saddle point equation (58).

5.1.2 Unlabeled Case

The goal of this section is to prove a theorem equivalent to Theorem 9. As in Chapter 4, we assume $0 < \rho < 1$. Using an identical argument to that chapter, we see that as $z \rightarrow \rho$,

$$F(z; \emptyset) \sim e^{r_0} \exp(C(z)),$$

to which we may apply Theorem 9 to obtain

$$[z^n]F(z; \emptyset) \sim \frac{e^{r_0+C(r\rho)-C(\rho;J)}(r\rho)^{-n}}{n} \sqrt{\frac{\beta}{2\pi}(\log n)^{\beta-1}},$$

where $r \sim 1 - \beta n^{-1}(\log n)^{\beta-1}$.

We now prove an analog to Theorem 9.

Theorem 10. *Suppose the pattern S satisfies Condition (57). Then, as $m \rightarrow \infty$,*

$$[z^n]F(z; S) \sim \left(\prod_{j \in J} \binom{C_j + S(j) - 1}{S(j)} (1 - \rho^j)^{C_j} \right) \frac{e^{r_0+C(\rho r)}(r\rho)^{-m}}{m} \sqrt{\frac{\beta}{2\pi}(\log m)^{\beta-1}},$$

where $r \sim 1 - \beta m^{-1}(\log m)^{\beta-1}$. Here r is defined by the saddle point equation (58).

Proof. The proof is very similar to that of Theorem 8. We begin again by using

Lemma 3 to show that

$$[z^n]F(z; S) = \prod_{j \in J} \binom{C_j + S(j) - 1}{S(j)} [z^m] \exp \left(\sum_{k \geq 0} \frac{C(z)}{k} - \frac{C(z^k; J)}{k} \right),$$

so we must determine

$$[z^m] \exp \left(\sum_{k \geq 1} \frac{C(z)}{k} - \frac{C(z^k; J)}{k} \right) = \rho^{-m} e^{r_0} [z^m] \exp(C(\rho z)) \exp \left(- \sum_{k \geq 1} \frac{C((\rho z)^k)}{k} \right). \quad (65)$$

As before, we examine the summation term:

$$\sum_{k \geq 1} \frac{C((\rho z)^k; J)}{k} = \sum_{j \in J} \log(1 - \rho^j)^{C_j} + \sum_{j \in J} C_j \rho^j (1 - z^j) \sum_{k \geq 0} \frac{\rho^{jk}}{k+1} (1 + z^k + z^{2k} + \dots + z^{jk}). \quad (66)$$

Consider the circle $\gamma = \{z : |z| = r\}$, and let $z = 1 - w/m$. Then $w = O((\log m)^{\beta-1})$, and

$$1 - z^j = O \left(j \left(\frac{(\log m)^{\beta-1}}{m} \right) \right),$$

which implies by the transfer theorems and Condition (57) that

$$\sum_{j \in J} C_j \rho^j (1 - z^j) = \sum_{j \in J} O \left(\frac{(\log j)^{\beta-1}}{j} \right) O \left(j \left(\frac{(\log m)^{\beta-1}}{m} \right) \right) = o(1). \quad (67)$$

On the contour γ , $|z| < 1$, and we have assumed $\rho < 1$ so

$$\begin{aligned} \left| \sum_{k \geq 0} \frac{\rho^{jk}}{k+1} (1 + z^j + z^{2j} + \dots + z^{jk}) \right| &\leq \sum_{k \geq 0} \frac{\rho^{jk}}{k+1} |1 + z^j + z^{2j} + \dots + z^{jk}| \\ &\leq \sum_{k \geq 0} \frac{\rho^{jk}}{k+1} (k) \leq \sum_{k \geq 0} \rho^{jk} = \frac{1}{1 - \rho^j} = O(1). \end{aligned} \quad (68)$$

Combining (65), (66), (67), and (68), we have

$$[z^m] \exp \left(\sum_{k \geq 1} \frac{C(z)}{k} - \frac{C(z^k; J)}{k} \right) = \rho^{-m} e^{r_0} \prod_{j \in J} (1 - \rho^j)^{C_j} [z^m] \exp(C(\rho z)),$$

to which we may apply Theorem 9 to obtain

$$[z^m] \exp(C(\rho z)) \sim \frac{e^{C(r\rho) - C(\rho; J)} (r\rho)^{-m}}{m} \sqrt{\frac{\beta}{2\pi}} (\log m)^{\beta-1},$$

which completes the proof. $\square$

As in the labeled case, we immediately obtain the following corollary.

Corollary 6. *Consider unlabeled decomposable combinatorial structures whose component generating function $C(z)$ is of alg-log type with algebraic exponent $\alpha = 0$ and logarithmic exponent $\beta \geq 2$, and suppose that the pattern S satisfies Condition (57). Then, as $m \rightarrow \infty$,*

$$P(S, n) \sim (\mathcal{S}(1 - \rho^j)^{C_j}) \frac{n}{m} \left(\frac{\log m}{\log n} \right)^{\frac{\beta-1}{2}} \frac{\rho^n r(n)^n}{\rho^m r(m)^m} e^{C(\rho r(m)) - C(\rho r(n)) - C(\rho; J)},$$

where $r(n) \sim 1 - \frac{\beta}{n} (\log n)^{\beta-1}$. Here $r(n)$ is defined by the saddle point equation (58).

5.1.3 Example

Example 1 (Sets of Bicycles). Let $\mathcal{C}$ be the labeled class of ordered pairs of non-empty cycles, which we will henceforth refer to as *bicycles*. Let $\mathcal{C}_n$ denote the subset of $\mathcal{C}$ which contains all elements of size n , and let $C_n = |\mathcal{C}_n|$. Table 2 shows C_i and C_i for $i \in \{1, 2, 3, 4\}$.

Since the EGF for a single cycle is $\log \frac{1}{1-z}$, the EGF for bicycles is

$$C(z) = \left(\log \frac{1}{1-z} \right)^2,$$

$\mathcal{C}_i$	C_i
$\mathcal{C}_0 = \mathcal{C}_1 = \emptyset$	$C_0 = C_1 = 0$
$\mathcal{C}_2 = \{((1), (2)), ((2), (1))\}$	$C_2 = 2$
$\mathcal{C}_3 = \left\{ \begin{array}{l} ((12), (3)), ((13), (2)), ((23), (1)), \\ ((1), (23)), ((2), (13)), ((3), (12)) \end{array} \right\}$	$C_3 = 6$
$\mathcal{C}_4 = \left\{ \begin{array}{l} ((123), (4)), ((124), (3)), ((134), (2)), ((132), (4)), \\ ((142), (3)), ((143), (2)), ((234), (1)), ((243), (1)), \\ ((12), (34)), ((13), (24)), ((14), (23)), \\ ((23), (14)), ((24), (13)), ((34), (12)), \\ ((1), (234)), ((1), (243)), ((2), (134)), ((2), (143)), \\ ((3), (124)), ((3), (142)), ((4), (123)), ((4), (132)) \end{array} \right\},$	$C_4 = 22$

Table 2: Bicycles of order 1, 2, 3 and 4.

i.e., $C(z)$ is alg-log with $\beta = 2$, and $\rho = 1$. By Theorem 2,

$$[z^n]C(z) \sim 2 \frac{\log n}{n} \quad (69)$$

as $n \rightarrow \infty$.

Let $\mathcal{F}$ be the class of sets of bicycles, i.e., $\mathcal{F} = \text{SET}(\mathcal{C})$. Then the EGF for $\mathcal{F}$ is

$$F(z) = \exp(C(z)) = \exp\left(\left(\log \frac{1}{1-z}\right)^2\right),$$

which implies that Theorem 9 applies to $F(z; \emptyset)$.

Finally, we define the restricted pattern $S = 3^{\lfloor \log n \rfloor} \lfloor \sqrt{n} \rfloor^5$. Then

$$m = n - 5\lfloor \sqrt{n} \rfloor - 3\lfloor \log n \rfloor$$

$$= n \left(1 + O(n^{-1/2})\right),$$

$$\log m = \log n + O(n^{-1/2}),$$

so S satisfies (57), and Theorem 9 implies that

$$[z^n]F(z; S) \sim \mathcal{S} \frac{\exp\left(\left(\log \frac{1}{1-r}\right)^2 - C(1; J)\right) r^{-m}}{m} \sqrt{\frac{1}{\pi} \log m},$$

where $r \sim 1 - \frac{2}{m} \log m$,

$$\mathcal{S} \sim \frac{1}{[\log n]!} \left(2 \frac{\log \lceil \sqrt{n} \rceil}{\lceil \sqrt{n} \rceil}\right)^5 \frac{1}{5!},$$

and

$$C(1; J) = C_3 + C_{\lceil \sqrt{n} \rceil} \sim 1 + 2 \frac{\log \lceil \sqrt{n} \rceil}{\lceil \sqrt{n} \rceil}.$$

For example, if $n = 10^3$, then

$$\lceil \sqrt{n} \rceil = 32,$$

$$[\log n] = 6,$$

$$m = 822,$$

$$r \approx 0.989120042,$$

$$\frac{C_3}{3!} = 1,$$

$$\frac{C_{32}}{32!} \approx 0.251702824,$$

$$\mathcal{S} \approx 1.16930203 \times 10^{-8},$$

and, by Theorem 9,

$$[z^n]F(z; S) \sim 35.949318.$$

For this value of n , we can use MAPLE to find an exact value for $[z^n]F(z; S)$:

$$[z^n]F(z; S) = 28.968008\dots,$$

and we see that the approximate value differs from the exact value by less than 25%.

5.2 Alg-Log Components with $\alpha = 0$ and $0 < \beta < 1$

In this section we consider component generating functions that are of alg-log type in the sense of Definition 5 with algebraic component equal to zero and positive logarithmic component. That is, suppose

$$C(z) = c + d \left(\log \frac{1}{1 - z/\rho} \right)^\beta (1 + O(1 - z/\rho)),$$

in some delta domain $\Delta(\nu, \theta)$, with c and d constant and $\beta \in (0, 1)$. We do not consider the case $\beta = 1$, since this is the so-called exp-log case, which was first considered by Flajolet and Soria [20]. In Chapter 5 of his doctoral thesis, Gourdon [24] studied the size of the r th largest components for objects of exp-log type. Panario and Richmond [32] then considered the problem of the distribution of the r th smallest components in this class.

For the range $\beta \in (0, 1)$, $n![z^n]F(z)$ is not an integer, *i.e.*, $F(z)$ is not a generating function for any combinatorial object. However, it is still worth considering from a theoretical point of view in the spirit of Wright [44, 45], but we do not include restricted patterns. Before we can prove the main result of this section, we require the following lemma.

Lemma 4. *Let $0 < \beta < 1$, let d be a constant, and suppose that $t = o(n)$ as $n \rightarrow \infty$.*

Then

$$\exp \left(d \left(\log \frac{n}{-t} \right)^\beta \right) = e^{d(\log n)^\beta} \left(\sum_{m \geq 0} E_m (-t)^{-\beta d(\log n)^{\beta-1}} \log(-t)^m \right),$$

where

$$E_m = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{d^j (\log n)^{\beta j - m}}{j!} \sum_{k_2, \dots, k_m} \binom{j}{k_2, \dots, k_m} \prod_{i=2}^m \left((-1)^i \binom{\beta}{i} \right)^{k_i},$$

with $k_2 + \dots + k_m = j$ and $2k_2 + 3k_3 + \dots + mk_m = m$. Furthermore, for fixed m ,

$$E_m \sim \begin{cases} \binom{\beta}{2}^{m/2} \frac{d^{m/2}}{(m/2)!} (\log n)^{\beta m/2 - m} & \text{if } m \text{ even,} \\ -\binom{\beta}{3} \binom{\beta}{2}^{\lfloor m/2 \rfloor - 1} \frac{d^{\lfloor m/2 \rfloor}}{(\lfloor m/2 \rfloor - 1)!} (\log n)^{\lfloor m/2 \rfloor \beta - m} & \text{if } m \text{ odd.} \end{cases}$$

Proof. Since $t = o(n)$,

$$\begin{aligned} \exp \left(d \left(\log \frac{n}{-t} \right)^\beta \right) &= \exp \left(d(\log n)^\beta \left(1 - \frac{\log(-t)}{\log n} \right)^\beta \right) \\ &= \exp \left(d(\log n)^\beta \left(\sum_{k \geq 0} \binom{\beta}{k} (-1)^k \left(\frac{\log(-t)}{\log n} \right)^k \right) \right) \\ &= \exp \left(d(\log n)^\beta \left(1 - \beta \frac{\log(-t)}{\log n} + \sum_{k \geq 2} \binom{\beta}{k} (-1)^k \left(\frac{\log(-t)}{\log n} \right)^k \right) \right) \\ &= \exp \left(d(\log n)^\beta \right) \exp \left(-\beta d \log(-t) (\log n)^{\beta-1} \right) \\ &\quad \times \exp \left(d(\log n)^\beta \left(\sum_{k \geq 2} \binom{\beta}{k} (-1)^k \left(\frac{\log(-t)}{\log n} \right)^k \right) \right). \end{aligned} \quad (70)$$

It is useful to rewrite (70) in terms of a power of $(-t)$. Therefore, we have

$$\begin{aligned} r \exp \left(d \left(\log \frac{n}{-t} \right)^\beta \right) &= e^{d(\log n)^\beta} (-t)^{-\beta d(\log n)^{\beta-1}} \\ &\quad \times \left(\sum_{j \geq 0} \frac{d^j (\log n)^{j\beta}}{j!} \left(\sum_{k \geq 2} \binom{\beta}{k} (-1)^k \left(\frac{\log(-t)}{\log n} \right)^k \right)^j \right) \\ &= e^{d(\log n)^\beta} \left(\sum_{m \geq 0} E_m (-t)^{-\beta d(\log n)^{\beta-1}} \log(-t)^m \right), \end{aligned}$$

where E_m can be calculated exactly for any m . Specifically,

$$\begin{aligned} E_m &= [(\log(-t))^m] \sum_{j \geq 0} \frac{d^j (\log n)^{\beta j}}{j!} \left(\sum_{k \geq 2} \binom{\beta}{k} (-1)^k \left(\frac{\log(-t)}{\log n} \right)^k \right)^j \\ &= [(\log(-t))^m] \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{d^j (\log n)^{\beta j}}{j!} \left(\sum_{k=2}^m \binom{\beta}{k} (-1)^k \left(\frac{\log(-t)}{\log n} \right)^k \right)^j \\ &= [(\log(-t))^m] \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{d^j (\log n)^{\beta j}}{j!} \\ &\quad \times \sum_{k_2, k_3, \dots, k_m} \binom{j}{k_2, k_3, \dots, k_m} \prod_{i=2}^m \left((-1)^i \binom{\beta}{i} \left(\frac{\log(-t)}{\log n} \right)^i \right)^{k_i}, \end{aligned}$$

where the second summation is taken over all sequences of nonnegative integer indices k_2 through k_m such that $k_2 + k_3 + \dots + k_m = j$. Therefore,

$$E_m = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{d^j (\log n)^{\beta j - m}}{j!} \sum_{k_2, \dots, k_m} \binom{j}{k_2, \dots, k_m} \prod_{i=2}^m \left((-1)^i \binom{\beta}{i} \right)^{k_i}, \quad (71)$$

where both $k_2 + \dots + k_m = j$ and $2k_2 + 3k_3 + \dots + mk_m = m$ are satisfied. From (71), we see that the leading term of an asymptotic expression for E_m in terms of n as $n \rightarrow \infty$ is the term with the largest power of $\log n$, i.e., the term corresponding to $j = \lfloor m/2 \rfloor$. To find this term, we consider even m and odd m separately.

If m is even, then we must find coefficients $k_2, k_3, \dots, k_m$ such that both the following equations are satisfied simultaneously:

$$k_2 + \dots + k_m = \frac{m}{2}, \quad (72)$$

$$2k_2 + 3k_3 + \dots + mk_m = m. \quad (73)$$

Multiplying (72) by two and subtracting the result from (73) yields

$$k_3 + 2k_4 + \dots + (m-2)k_m = 0,$$

i.e., the only solution to the system of equations is $k_2 = m/2, k_3 = k_4 = \dots = k_m = 0$, which implies that

$$E_m \sim \frac{d^{m/2}(\log n)^{\beta m/2 - m}}{(m/2)!} \left(\frac{\beta}{2}\right)^{m/2}.$$

If m is odd, then $k_2, k_3, \dots, k_m$ must satisfy both (73) and

$$k_2 + \dots + k_m = \frac{m-1}{2}. \quad (74)$$

Once again, we multiply (74) by two and subtract the result from (73) to see that

$$k_3 + 2k_4 + \dots + (m-2)k_m = 1. \quad (75)$$

The only solution that satisfies (73), (74), and (75) is $k_2 = \frac{m-3}{2}, k_3 = 1, k_4 = k_5 = \dots = k_m = 0$. Therefore, if m is odd,

$$\begin{aligned} E_m &\sim -\frac{d^{\lfloor m/2 \rfloor}(\log n)^{\beta \lfloor m/2 \rfloor - m}}{\lfloor m/2 \rfloor!} \frac{\lfloor m/2 \rfloor!}{(\lfloor m/2 \rfloor - 1)!1!} \left(\frac{\beta}{2}\right)^{\lfloor m/2 \rfloor - 1} \left(\frac{\beta}{3}\right) \\ &\sim -\frac{d^{\lfloor m/2 \rfloor}(\log n)^{\beta \lfloor m/2 \rfloor - m}}{(\lfloor m/2 \rfloor - 1)!} \left(\frac{\beta}{2}\right)^{\lfloor m/2 \rfloor - 1} \left(\frac{\beta}{3}\right). \end{aligned}$$

Therefore, for fixed m ,

$$E_m \sim \begin{cases} \binom{\beta}{2}^{m/2} \frac{d^{m/2}}{(m/2)!} (\log n)^{\beta m/2 - m} & \text{if } m \text{ even,} \\ -\binom{\beta}{3} \binom{\beta}{2}^{\lfloor m/2 \rfloor - 1} \frac{d^{\lfloor m/2 \rfloor}}{(\lfloor m/2 \rfloor - 1)!} (\log n)^{\lfloor m/2 \rfloor \beta - m} & \text{if } m \text{ odd.} \end{cases}$$

□

Table 3 gives E_m for $0 \leq m \leq 6$.

m	E_m
0	1
1	0
2	$\binom{\beta}{2} d (\log n)^{\beta-2}$
3	$-\binom{\beta}{3} d (\log n)^{\beta-3}$
4	$\binom{\beta}{2}^2 d^2 \frac{(\log n)^{2\beta-4}}{2!} + \binom{\beta}{4} d (\log n)^{\beta-4}$
5	$-\binom{\beta}{2} \binom{\beta}{3} d^2 (\log n)^{2\beta-5} - \binom{\beta}{5} d (\log n)^{\beta-5}$
6	$\binom{\beta}{2}^3 \frac{d^3 (\log n)^{3\beta-6}}{6} + \left(\binom{\beta}{2} \binom{\beta}{4} + \binom{\beta}{3}^2 \right) \frac{d^2 (\log n)^{2\beta-6}}{2} + \binom{\beta}{6} d (\log n)^{\beta-6}$

Table 3: E_m for $0 \leq m \leq 6$

We now state the main theorem of this section.

Theorem 11. *Let $F(z) = \exp(C(z))$, and let $k \geq 2$ be a fixed integer. Then, as $n \rightarrow \infty$,*

$$[z^n]F(z) \sim \rho^{-n} \sum_{m=0}^k I_m,$$

where

$$I_0 \sim \frac{-\beta d (\log n)^{\beta-1} e^{c+d(\log n)^\beta}}{n},$$

$$I_1 = 0,$$

and, for $2 \leq m \leq k$,

$$I_m \sim \frac{e^{c+d(\log n)^\beta}}{n} E_m \frac{d^m}{ds^m} \left(\frac{1}{\Gamma(-s)} \right) \Big|_{s=-\beta d (\log n)^{\beta-1}} = \Theta \left(\frac{e^{d(\log n)^\beta}}{n} (\log n)^{\beta \lfloor m/2 \rfloor - m} \right),$$

with

$$E_m \sim \begin{cases} \binom{\beta}{2}^{m/2} \frac{d^{m/2}}{(m/2)!} (\log n)^{\beta m/2 - m} & \text{if } m \text{ even,} \\ -\binom{\beta}{3} \binom{\beta}{2}^{\lfloor m/2 \rfloor - 1} \frac{d^{\lfloor m/2 \rfloor}}{(\lfloor m/2 \rfloor - 1)!} (\log n)^{\lfloor m/2 \rfloor \beta - m} & \text{if } m \text{ odd.} \end{cases}$$

Proof. From the Cauchy integration formula,

$$[z^n] F(z) = [z^n] \frac{F(\rho z)}{\rho^n} = \frac{1}{\rho^n 2\pi i} \int_\gamma \exp \left(c + d \left(\log \frac{1}{1-z} \right)^\beta \right) (1 + O(1-z)) \frac{dz}{z^{n+1}},$$

where $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ is defined as

$$\begin{aligned} \gamma_1 &= \left\{ z : |z-1| = \frac{1}{n}, \theta \leq \arg(z-1) \leq 2\pi - \theta \right\} \\ \gamma_2 &= \left\{ z : \frac{1}{n} \leq |z-1| \leq \frac{A \log n}{n}, \arg(z-1) = \theta \right\} \\ \gamma_3 &= \left\{ z : |z| = 1 + \frac{2 \log n}{n}, \theta \leq \arg(z-1) \leq 2\pi - \theta \right\} \\ \gamma_4 &= \left\{ z : \frac{1}{n} \leq |z-1| \leq \frac{A \log n}{n}, \arg(z-1) = 2\pi - \theta \right\}, \end{aligned}$$

and $A \in \mathbb{R}_{>0}$ satisfies

$$\left| 1 + A \frac{\log n}{n} e^{i\theta} \right| = 1 + \frac{2 \log n}{n}.$$

For $z \in \gamma$, let $z = 1 + \frac{t}{n}$. Next, define γ' as $\gamma'_1 \cup \gamma'_2 \cup \gamma'_4$, where

$$\gamma'_1 = \{t : |t| = 1, \theta \leq \arg(t) \leq 2\pi - \theta\}$$

$$\gamma'_2 = \{t : 1 \leq |t| \leq A \log n, \arg(t) = \theta\}$$

$$\gamma'_4 = \{t : 1 \leq |t| \leq A \log n, \arg(t) = 2\pi - \theta\}.$$

Then

$$[z^n]F(\rho z) = \frac{e^c}{2\pi i} \int_{\gamma'} \exp \left(d \left(\log \frac{n}{-t} \right)^\beta \right) \left(1 + \frac{t}{n} \right)^{-n-1} \left(1 + O \left(\frac{\log n}{n} \right) \right) \frac{dt}{n} \quad (76)$$

$$+ \frac{1}{2\pi i} \int_{\gamma_3} F(\rho z) \left(1 + O \left(\frac{\log n}{n} \right) \right) \frac{dz}{z^{n+1}}. \quad (77)$$

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ denote the right hand side of (76) and (77), respectively. We focus first on the integrand of $\mathcal{I}_1$. From Lemma 4,

$$\mathcal{I}_1 = \sum_{m \geq 0} \frac{e^{c+d(\log n)^\beta}}{2n\pi i} \int_{\gamma'} E_m(-t)^{-\beta d(\log n)^{\beta-1}} (\log(-t))^m \left(1 + \frac{t}{n} \right)^{-n-1} \left(1 + O \left(\frac{\log n}{n} \right) \right) dt,$$

where we are able to move the summation outside the integral because

$$\sum_{m \geq 0} E_m(\log(-t))^m$$

converges absolutely for all $t \in \gamma'$.

Next, we must expand $(1 + \frac{t}{n})^{-n-1}$. On γ' , $t = O(\log n)$, so

$$\begin{aligned} \left(1 + \frac{t}{n}\right)^{-n-1} &= \exp\left((-n-1) \log\left(1 + \frac{t}{n}\right)\right) \\ &= \exp\left((-n-1) \left(\frac{t}{n} + O\left(\frac{(\log n)^2}{n}\right)\right)\right) \\ &= e^{-t} \exp\left(-\frac{t}{n} + O\left(\frac{(\log n)^2}{n}\right)\right) \\ &= e^{-t} \left(1 + O\left(\frac{(\log n)^2}{n}\right)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{I}_1 &= \frac{e^c}{2\pi i} \int_{\gamma'} \exp\left(d \left(\log\left(\frac{n}{-t}\right)\right)^\beta\right) \left(1 + \frac{t}{n}\right)^{-n-1} \left(1 + O\left(\frac{(\log n)^2}{n}\right)\right) \frac{dt}{n} \\ &= \sum_{m \geq 0} \frac{e^{c+d(\log n)^\beta}}{2n\pi i} \int_{\gamma'} E_m e^{-t} (-t)^{-\beta d(\log n)^{\beta-1}} (\log(-t))^m dt \\ &\quad + \frac{e^c}{2\pi i} \int_{\gamma'} e^{-t} \exp\left(d \left(\log\left(\frac{n}{-t}\right)\right)^\beta\right) O\left(\frac{(\log n)^2}{n^2}\right) dt. \end{aligned} \quad (78)$$

Let

$$I_m = \frac{e^{c+d(\log n)^\beta}}{2n\pi i} \int_{\gamma'} E_m e^{-t} (-t)^{-\beta d(\log n)^{\beta-1}} (\log(-t))^m dt. \quad (79)$$

In order to evaluate (79), we define a new contour $\gamma'' = \gamma'_1 \cup \gamma''_2 \cup \gamma''_4$, where

$$\gamma''_2 = \{t : 1 \leq |t|, \arg(t) = \theta\}$$

$$\gamma''_4 = \{t : 1 \leq |t|, \arg(t) = 2\pi - \theta\},$$

i.e., γ'' is the contour obtained from γ' by extending the rectilinear portions to infinity.

It was shown by Hankel (see [28, 42]) that the following definition of the reciprocal

Gamma function holds:

$$\frac{1}{\Gamma(a)} = \frac{1}{2\pi i} \int_{\gamma''} (-t)^{-a} \exp(-t) dt. \quad (80)$$

Similarly, we have for all $m \in \mathbb{Z}_{\geq 0}$

$$\frac{1}{2\pi i} \int_{\gamma''} (-t)^{-a} \exp(-t) \log(-t)^m dt = \frac{d^m}{ds^m} \left(\frac{1}{\Gamma(-s)} \right) \Big|_{s=-a}, \quad (81)$$

which we obtain simply by differentiating the integrand of (80) with respect to $-a$.

Let

$$I_m'' = \frac{e^{c+d(\log n)^\beta}}{2n\pi i} \int_{\gamma''} E_m e^{-t} (-t)^{-\beta d(\log n)^{\beta-1}} (\log(-t))^m dt.$$

Then, applying the Hankel contour,

$$I_m'' = \frac{e^{c+d(\log n)^\beta}}{n} E_m \frac{d^m}{ds^m} \left(\frac{1}{\Gamma(-s)} \right) \Big|_{s=-\beta d(\log n)^{\beta-1}}.$$

In the case $\beta \in (0, 1)$, the argument $-\beta d(\log n)^{\beta-1} \rightarrow 0$ as $n \rightarrow \infty$, so we may apply the Taylor series expansion for the reciprocal gamma function [7]:

$$\frac{1}{\Gamma(z)} = z + \gamma z^2 + \left(\frac{\gamma^2}{2} - \frac{\pi^2}{2} \right) z^3 + O(z^4) \quad \text{as } z \rightarrow 0,$$

where γ is the Euler-Mascheroni constant. For $k > 2$, $a_k = [z^k] \frac{1}{\Gamma(z)}$ can be determined recursively by

$$a_k = k a_1 a_k - a_2 a_{k-1} + \sum_{j=2}^{k-1} (-1)^j \zeta(j) a_{k-j},$$

where $\zeta(s)$ is the Riemann zeta function. For small z , we therefore have $\Gamma(z)^{-1} \sim z$ and, for all $m \geq 1$,

$$\frac{d^m}{ds^m} \frac{1}{\Gamma(-s)} \Big|_{s=-z} \sim a_m m! = \Theta(1).$$

In order to make use of I_m'' , we need to show $I_m'' \sim I_m$. Let $t = re^{i\theta}$, and let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where

$$\mathcal{C}_1 = \{t : |t| \geq A \log n, \arg(t) = \theta\},$$

$$\mathcal{C}_2 = \{t : |t| \geq A \log n, \arg(t) = 2\pi - \theta\}.$$

Then

$$\begin{aligned} |I_m'' - I_m| &= \left| \frac{e^{c+d(\log n)^\beta}}{2n\pi i} \int_{\mathcal{C}} E_m e^{-t} (-t)^{-\beta d(\log n)^{\beta-1}} \log(-t)^m dt \right| \\ &\leq \frac{e^{c+d(\log n)^\beta} |E_m|}{n\pi} \int_{r \geq A \log n} e^{-r \cos \theta} r^{-\beta d(\log n)^{\beta-1}} (\log r + \pi)^m dr. \end{aligned} \quad (82)$$

We focus on the integral in (82):

$$\begin{aligned} &\int_{r \geq A \log n} \exp \left(-r \cos \theta - \beta d(\log n)^{\beta-1} \log r + m \log(\log r + \pi) \right) dr \\ &= \int_{r \geq A \log n} \exp \left(-\frac{r \cos \theta}{2} \right) \exp \left(-\frac{r \cos \theta}{2} - \beta d(\log n)^{\beta-1} \log r + m \log(\log r + \pi) \right) dr. \end{aligned} \quad (83)$$

We next show that the integrand in (83) is a strictly decreasing function of r . We do so by taking its derivative with respect to r :

$$\begin{aligned} &\frac{d}{dr} \exp \left(-r \cos \theta - \beta d(\log n)^{\beta-1} \log r + m \log(\log r + \pi) \right) \\ &= \exp \left(-r \cos \theta - \beta d(\log n)^{\beta-1} \log r + m \log(\log r + \pi) \right) \\ &\quad \times \left(-\cos \theta - \frac{\beta d(\log n)^{\beta-1}}{r} + \frac{m}{r(\log r + \pi)} \right). \end{aligned}$$

Since $0 < \theta < \pi/2$, the quantity $\cos \theta$ is strictly positive. Therefore, for large enough

n , it is clear that the following inequality holds for all $r \geq A \log n$:

$$-\cos \theta - \frac{\beta d(\log n)^{\beta-1}}{r} + \frac{m}{r(\log r + \pi)} < 0.$$

This implies that, for large enough n , the integrand in (83) is strictly decreasing with its maximum at $r = A \log n$. Therefore,

$$\begin{aligned} & \int_{r \geq A \log n} \exp\left(-\frac{r \cos \theta}{2}\right) \exp\left(\frac{-r \cos \theta}{2} + (-\beta d(\log n)^{\beta-1}) \log r + m \log(\log r + \pi)\right) dr \\ & \leq \exp\left(-\frac{A \log n \cos \theta}{2} + (-\beta d(\log n)^{\beta-1}) (\log(A \log n)) + m \log(\log(A \log n) + \pi)\right) \\ & \quad \times \int_{r \geq A \log n} \exp\left(-r \frac{\cos \theta}{2}\right) dr \\ & \leq n^{-K} (A \log n)^{-\beta d(\log n)^{\beta-1}} (\log A + 2 \log \log n + \pi)^m \frac{2}{A \cos \theta} n^{-L} \\ & = o(n^{-K}), \end{aligned} \tag{84}$$

where L and K are positive constants. Therefore, from (82) and (84), we have

$$\begin{aligned} \frac{|I_m'' - I_m|}{|I_m''|} & \leq \frac{|E_m| \times o\left(e^{d(\log n)^\beta} n^{-K-1}\right)}{|E_m K e^{d(\log n)^\beta} n^{-1}|} \\ & = o(n^{-K}) = o(1), \end{aligned}$$

and we may conclude that, for any $m \geq 0$, $I_m'' \sim I_m$.

Next, we show that (78) is negligible compared to I_m for any $2 \leq m \leq k$. On γ'_1 , $1 \leq |t| \leq A \log n$. Therefore, $e^{-t} = O(1)$, and

$$\begin{aligned} \left| \exp\left(d(\log n)^\beta \left(1 - \frac{\log(-t)}{\log n}\right)^\beta\right) \right| & = |\exp(d(\log n)^\beta (1 + o(1)))| \\ & = O\left(n^{d(\log n)^{\beta-1}}\right). \end{aligned}$$

On γ'_2 , t can be written as $t = r e^{i\theta}$, where $1 \leq r \leq A \log n$. Therefore, $e^{-t} = O(1)$,

and

$$\begin{aligned}
\left| \log \left(\frac{n}{-t} \right) \right| &\leq \log n + |\log(-t)| \\
&\leq \log n + |\log r + i(\theta + \pi)| \\
&\leq \log n + \log r + \pi + \theta \\
&\leq \log n \left(1 + \frac{\log \log n}{\log n} + \frac{\pi + \theta + \log A}{\log n} \right),
\end{aligned}$$

which implies that

$$\begin{aligned}
d \left| \log \left(\frac{n}{-t} \right) \right|^\beta &\leq d(\log n)^\beta \left(1 + \frac{\log \log n}{\log n} + \frac{\pi + \theta + \log A}{\log n} \right)^\beta \\
&\leq d(\log n)^\beta \left(1 + O \left(\frac{\log \log n}{\log n} \right) \right) \\
&\leq d(\log n)^\beta + o(1),
\end{aligned}$$

since $0 < \beta < 1$. Therefore, on γ'_2 we have

$$\left| \exp \left(d \left(\log \left(\frac{n}{-t} \right)^\beta \right) \right) \right| = O \left(e^{d(\log n)^\beta} \right).$$

A nearly identical argument gives the same bounds for e^{-t} and $\exp \left(d \left(\log \frac{n}{-t} \right)^\beta \right)$ on γ'_4 , which implies that

$$\begin{aligned}
\frac{1}{|I_m''|} \frac{e^c}{2\pi i} \int_{\gamma'} e^{-t} \exp \left(d \left(\log \frac{n}{-t} \right)^\beta \right) O \left(\frac{(\log n)^2}{n^2} \right) dt &= \frac{O \left(e^{d(\log n)^\beta} (\log n)^2 n^{-2} \right)}{|E_m K e^{d(\log n)^\beta} n^{-1}|} \\
&= O \left(\frac{(\log n)^2}{E_m n} \right) = o(1),
\end{aligned}$$

where the last inequality holds because, for $2 \leq m \leq k$, $E_m = \Theta \left((\log n)^{\beta \lfloor \frac{m}{2} \rfloor - m} \right)$.

Finally, we show that the contribution of $\mathcal{I}_2$ is negligible. First, we examine z^{-n-1}

on γ_3 :

$$\begin{aligned}
|z^{-n-1}| &= \left(1 + \frac{2 \log n}{n}\right)^{-n-1} \\
&= \exp \left(-(n+1) \log \left(1 + \frac{2 \log n}{n}\right) \right) \\
&= \exp \left(-(n+1) \left(\frac{2 \log n}{n} + O \left(\frac{(\log n)^2}{n^2} \right) \right) \right) \\
&= \exp (-2 \log n + o(1)) \\
&\sim n^{-2}.
\end{aligned} \tag{85}$$

The function $\left| \frac{1}{1-z} \right|$ is unimodal on the circle $|z| = 1 + \frac{\log n}{n}$ with its maximum at $z = 1 + \frac{\log n}{n}$ and its minimum at $z = -1 - \frac{\log n}{n}$. Therefore,

$$\frac{n}{2n + \log n} \leq \left| \frac{1}{1-z} \right| \leq \frac{n}{\log n},$$

which implies that

$$\begin{aligned}
\left| \log \left(\frac{1}{1-z} \right) \right| &\leq \log \left(\frac{n}{\log n} \right) + \pi = \log n - \log \log n + \pi \\
d \left| \log \left(\frac{1}{1-z} \right) \right|^\beta &\leq d(\log n)^\beta \left(1 - \frac{\log \log n - \pi}{\log n} \right)^\beta \\
d \left| \log \left(\frac{1}{1-z} \right) \right|^\beta &\leq d(\log n)^\beta \left(1 - \beta \frac{\log \log n - \pi}{\log n} + O \left(\frac{(\log \log n)^2}{(\log n)^2} \right) \right) \\
d \left| \log \left(\frac{1}{1-z} \right) \right|^\beta &\leq d(\log n)^\beta - d\beta(\log n)^{\beta-1}(\log \log n - \pi) + o(1).
\end{aligned}$$

Therefore, it is clear that

$$\begin{aligned}
\left| \exp \left(d \left(\log \left(\frac{1}{1-z} \right) \right)^\beta \right) \right| &\leq e^{d(\log n)^\beta} \exp(-d\beta(\log n)^{\beta-1}(\log \log n - \pi))(1 + o(1)) \\
&= O \left(n^{d(\log n)^{\beta-1}} \right).
\end{aligned} \tag{86}$$

Finally, from (85) and (86), we conclude that

$$\frac{1}{2\pi i} \int_{\gamma_3} F(z) \frac{dz}{z^{n+1}} = O\left(n^{d(\log n)^{\beta-1}-2}\right) = o(I_m''),$$

for any $2 \leq m \leq k$, $0 < \beta < 1$.

In summary, we have shown that

$$\frac{f_n}{\rho^{-n}} = \sum_{m=0}^k I_m + O(I_{k+1}) + O\left(\frac{e^{d(\log n)^\beta} (\log n)^2}{n^2}\right),$$

where, for any m ,

$$I_m = I_m'' + o\left(E_m e^{d(\log n)^\beta} n^{-K-1}\right),$$

for some positive K , with

$$I_m'' = \frac{e^{c+d(\log n)^\beta}}{n} E_m \frac{d^m}{ds^m} \left(\frac{1}{\Gamma(-s)} \right) \Big|_{s=-\beta d(\log n)^{\beta-1}},$$

and

$$E_m \sim \begin{cases} \binom{\beta}{2}^{m/2} \frac{d^{m/2}}{(m/2)!} (\log n)^{\beta m/2 - m} & \text{if } m \text{ even,} \\ -\binom{\beta}{3} \binom{\beta}{2}^{\lfloor m/2 \rfloor - 1} \frac{d^{\lfloor m/2 \rfloor}}{(\lfloor m/2 \rfloor - 1)!} (\log n)^{\lfloor m/2 \rfloor \beta - m} & \text{if } m \text{ odd.} \end{cases}$$

Thus, we may conclude that, for any fixed k ,

$$\frac{f_n}{\rho^{-n}} \sim \sum_{m=0}^k I_m,$$

with $I_m \sim I_m''$. □

Note that Theorem 11 falls short of giving a complete asymptotic expansion for $[z^n]F(z)$. The reason for this limitation is that the asymptotic expression for E_m

given by Lemma 4 is only valid for fixed m . For a complete asymptotic expansion, we would need an expression for E_m that is valid as *both* m and $n \rightarrow \infty$.

5.3 Alg-Log Components with $\alpha = 0$ and $1 < \beta < 2$

Assume that $C(z)$ is as defined in Definition 5 with $\alpha = 0$ and $1 < \beta < 2$. At first glance, this case appears to be amenable to a similar analysis as the case $0 < \beta < 1$. However, there are substantial differences, which makes such an analysis much more difficult. We were unable to prove a theorem analogous to Theorem 11. The purpose of this section is to outline the progress we made towards such a theorem, as well as to provide a conjecture.

The main reason we cannot simply apply the method of Theorem 11 to prove an analogue for the case $1 < \beta < 2$ is that the behaviour of the expression

$$\left. \frac{d^m}{ds^m} \left(\frac{1}{\Gamma(-s)} \right) \right|_{s=-\beta d(\log n)^{\beta-1}} \quad (87)$$

as $n \rightarrow \infty$ is substantially different depending on if β is less than or greater than 1. If $\beta < 1$, then the argument $-\beta d(\log n)^{\beta-1} \rightarrow 0$ as $n \rightarrow \infty$, and (87) has a Taylor series expansion in the neighbourhood of $z = 0$. However, if $\beta > 1$, then $-\beta d(\log n)^{\beta-1} \rightarrow -\infty$ as $n \rightarrow \infty$, and the Taylor series for the reciprocal Gamma function that we used in the previous section no longer applies. Instead, we must find an asymptotic expansion for (87).

In the proof of Theorem 11, we showed that the contribution of the integral (78) was negligible compared to the expression

$$I_m \sim \frac{e^{c+d(\log n)^\beta}}{n} E_m \left. \frac{d^m}{ds^m} \left(\frac{1}{\Gamma(-s)} \right) \right|_{s=-\beta d(\log n)^{\beta-1}}.$$

However, when $\beta > 1$, (87) approaches 0 as $n \rightarrow \infty$ much more quickly than (78), and hence (78) is no longer negligible.

We attempted to account for this change by using a complete expansion for the function

$$\left(1 + \frac{t}{n}\right)^{-n-1},$$

which we determined in the following lemma.

Lemma 5. *Suppose $t = o(n)$ as $n \rightarrow \infty$. Then*

$$\left(1 + \frac{t}{n}\right)^{-n-1} = e^{-t} \sum_{j \geq 0} a_j (-t)^j,$$

where

$$a_j = \frac{1}{n^j} \sum_{\ell=1}^j \frac{1}{\ell!} \sum_{k_1, \dots, k_j} \binom{\ell}{k_1, \dots, k_j} \prod_{m=2}^j \frac{(n+1)^{k_m}}{m^{k_m}},$$

and $k_1, \dots, k_j \in \mathbb{N}$ satisfy the following conditions:

$$k_1 + k_2 + \dots + k_j = \ell, \text{ and}$$

$$k_1 + 2k_2 + \dots + jk_j = j.$$

Furthermore, as $n \rightarrow \infty$,

$$a_j \sim \begin{cases} \frac{1}{(j/2)!} \cdot \frac{1}{(2n)^{j/2}} & \text{if } j \text{ is even,} \\ \frac{1}{(2n)^{\lceil j/2 \rceil}} \cdot \frac{1}{\lceil j/2 \rceil!} \cdot \frac{j^2 + 3j + 2}{3} & \text{if } j \text{ is odd.} \end{cases} \quad (88)$$

Proof. We first expand

$$\begin{aligned}
\left(1 + \frac{t}{n}\right)^{-n-1} &= \exp\left((-n-1) \log\left(1 + \frac{t}{n}\right)\right) \\
&= \exp\left((n+1) \left(\sum_{k \geq 1} \frac{1}{k} \left(\frac{-t}{n}\right)^k\right)\right) \\
&= \exp\left((n+1) \left(\frac{-t}{n} + \sum_{k \geq 2} \frac{1}{k} \left(\frac{-t}{n}\right)^k\right)\right) \\
&= e^{-t} \exp\left(\frac{-t}{n} + (n+1) \sum_{k \geq 2} \frac{1}{k} \left(\frac{-t}{n}\right)^k\right) \\
&= e^{-t} \sum_{j \geq 0} a_j (-t)^j,
\end{aligned}$$

where

$$\begin{aligned}
a_j &= [(-t)^j] \sum_{\ell \geq 0} \frac{1}{\ell!} \left(\frac{-t}{n} + \sum_{k \geq 2} \frac{n+1}{k} \left(\frac{-t}{n}\right)^k\right)^\ell \\
&= [(-t)^j] \sum_{\ell=1}^j \frac{1}{\ell!} \left(\frac{-t}{n} + \sum_{k=2}^j \frac{n+1}{k} \left(\frac{-t}{n}\right)^k\right)^\ell \\
&= [(-t)^j] \sum_{\ell=1}^j \frac{1}{\ell!} \sum_{k_1, \dots, k_j} \binom{\ell}{k_1, \dots, k_j} \left(\frac{-t}{n}\right)^{k_1} \prod_{m=2}^j \left(\frac{n+1}{m} \left(\frac{-t}{n}\right)^m\right)^{k_m} \\
&= \frac{1}{n^j} \sum_{\ell=1}^j \frac{1}{\ell!} \sum_{k_1, \dots, k_j} \binom{\ell}{k_1, \dots, k_j} \prod_{m=2}^j \frac{(n+1)^{k_m}}{m^{k_m}}, \tag{89}
\end{aligned}$$

and $k_1, \dots, k_j \in \mathbb{N}$ satisfy the conditions

$$k_1 + k_2 + \dots + k_j = \ell, \tag{90}$$

$$\text{and } k_1 + 2k_2 + \dots + jk_j = j. \tag{91}$$

We see from (89) that $n^j a_j$ is a polynomial over n whose degree must be no greater than j , and we use (89), (90), and (91) to deduce what the degree of $n^j a_j$ actually

is. From the product term (89), we can see that the highest order term in $n^j a_j$ must be of degree $k_2 + k_3 + \cdots + k_j$. Therefore the degree of $n^j a_j$ is simply the maximum possible value of $k_2 + k_3 + \cdots + k_j$ given (90) and (91).

From (91), we see that the k_i s give a partition of j with k_i equal to the number of times the integer i appears in the partition of j . The sum $k_2 + k_3 + \cdots + k_j$ is maximized by a partition of j with the most possible terms that are not equal to 1. Therefore the partition must have as many 2s as possible, *i.e.*, k_2 as large as possible.

If j is even, then we see that $k_2 + \cdots + k_j$ is maximized when $k_2 = j/2$, and $k_1 = k_3 = \cdots = k_j = 0$. If j is odd, then $k_2 + \cdots + k_j$ is maximum when either

- $k_2 = \frac{j-1}{2}$ and $k_1 = 1$ with all other k_i 's equal to zero, or
- $k_2 = \frac{j-3}{2}$ and $k_3 = 1$ with all other k_i 's equal to zero.

In either case, we have that the degree of $n^j a_j$ is $\lfloor j/2 \rfloor$.

Next, we find the leading coefficient of $n^j a_j$, which we determine from (89):

$$[n^{\lfloor j/2 \rfloor}] n^j a_j = \begin{cases} \frac{1}{(j/2)!} \cdot \frac{1}{2^{j/2}} & \text{if } j \text{ is even,} \\ \frac{1}{2^{\lfloor j/2 \rfloor}} \cdot \frac{1}{\lfloor j/2 \rfloor!} \cdot \frac{j^2+3j+2}{3} & \text{if } j \text{ is odd.} \end{cases}$$

Since $n^j a_j$ is a polynomial over n , for large enough n we have

$$a_j \sim \begin{cases} \frac{1}{(j/2)!} \cdot \frac{1}{(2n)^{j/2}} & \text{if } j \text{ is even,} \\ \frac{1}{(2n)^{\lfloor j/2 \rfloor}} \cdot \frac{1}{\lfloor j/2 \rfloor!} \cdot \frac{j^2+3j+2}{3} & \text{if } j \text{ is odd.} \end{cases} \quad (92)$$

□

Table 4 gives a_j for $0 \leq j \leq 6$. It is also worth noting that a_j can also be expressed as a finite sum involving the binomial coefficient and Stirling numbers of

the first kind. We discuss this in Appendix A.

j	a_j
0	1
1	$\frac{1}{n}$
2	$\frac{n+2}{2n^2}$
3	$\frac{5n+6}{6n^3}$
4	$\frac{3n^2+26n+24}{24n^4}$
5	$\frac{35n^2+154n+120}{120n^5}$
6	$\frac{15n^3+340n^2+1044n+720}{720n^6}$

Table 4: a_j for $0 \leq j \leq 6$

We also were able to find an asymptotic expression for the derivatives of the reciprocal Gamma function, which we give in the following lemma.

Lemma 6. As $-s \rightarrow \infty$,

$$\frac{d^m}{ds^m} \frac{1}{\Gamma(-s)} \sim \frac{(\log -s)^m}{\Gamma(-s)},$$

for any fixed integer $m \geq 0$.

Proof. As $z \rightarrow \infty$, the reciprocal Gamma function $\frac{1}{\Gamma(z)}$ has the following asymptotic expansion [1]:

$$\frac{1}{\Gamma(z)} = \sqrt{\frac{z}{2\pi}} \left(\frac{z}{e}\right)^{-z} \left(1 + O\left(\frac{1}{z}\right)\right).$$

The asymptotic expansion for the m -th derivative of the reciprocal Gamma function

is slightly more difficult to calculate. We begin with the first derivative:

$$\frac{d}{ds} \frac{1}{\Gamma(-s)} = \frac{\psi^{(0)}(-s)}{\Gamma(-s)},$$

where $\psi^{(k)}(z) = \frac{d^{k+1}}{dz^{k+1}} \log \Gamma(z)$ is the polygamma function. From [1] (p. 260), we know the following asymptotic expansions for the polygamma function for large z :

$$\psi^{(0)}(z) \sim \log z - \frac{1}{2z} + \sum_{n=1}^{\infty} \frac{B(2n)}{2nz^{2n}}, \quad (93)$$

$$\psi^{(n)}(z) \sim \frac{(-1)^{n-1}(n-1)!}{z^n} + \frac{(-1)^{n-1}n!}{2z^{n+1}} + \frac{(-1)^n(n+1)!}{12z^{n+2}} (1 + O(z^{-2})), \quad (94)$$

where $B(n)$ is the n -th Bernoulli number.

We calculate higher derivatives of the reciprocal Gamma function:

$$\begin{aligned} \frac{d^2}{ds^2} \frac{1}{\Gamma(-s)} &= \frac{1}{\Gamma(-s)} [\psi^{(0)}(-s)^2 + \psi^{(1)}(-s)], \\ \frac{d^3}{ds^3} \frac{1}{\Gamma(-s)} &= \frac{1}{\Gamma(-s)} [\psi^{(0)}(-s)^3 - 3\psi^{(0)}(-s)\psi^{(1)}(-s) + \psi^{(2)}(-s)], \\ \frac{d^4}{ds^4} \frac{1}{\Gamma(-s)} &= \frac{1}{\Gamma(-s)} [\psi^{(0)}(-s)^4 - 6\psi^{(0)}(-s)^2\psi^{(1)}(-s) \\ &\quad + 4\psi^{(0)}(-s)\psi^{(2)}(-s) + 3\psi^{(1)}(-s)^2 - \psi^{(3)}(-s)]. \end{aligned}$$

By induction, we see that for any fixed m ,

$$\begin{aligned} \frac{d^m}{ds^m} \frac{1}{\Gamma(-s)} &= \frac{1}{\Gamma(-s)} \left[\psi^{(0)}(-s)^m + O\left(\binom{m}{2} \psi^{(0)}(-s)^{m-1} \psi^{(1)}(-s)\right) \right] \\ &= \frac{\psi^{(0)}(-s)^m}{\Gamma(-s)} \left[1 + O\left(\frac{m^2 \psi^{(1)}(-s)}{\psi^{(0)}(-s)}\right) \right]. \end{aligned} \quad (95)$$

Therefore, from (93), (94), and (95), we see that, for large $-s$,

$$\frac{d^m}{ds^m} \frac{1}{\Gamma(-s)} \sim \frac{\psi^{(0)}(-s)^m}{\Gamma(-s)} \sim \frac{(\log -s)^m}{\Gamma(-s)},$$

for any fixed $m \geq 0$.

□

We now state the conjecture of this section.

Conjecture 1. *Let $F(z) = \exp(C(z))$. Then, as $n \rightarrow \infty$, and for any fixed integer $k \geq 0$,*

$$[z^n]F(z) \sim \rho^{-n} \sum_{m=0}^k I_m'', \quad (96)$$

where

$$I_m'' = \frac{e^{d(\log n)^\beta}}{n} E_m \frac{d^m}{ds^m} \left(\frac{1}{\Gamma(-s)} \right) \Big|_{s=-\beta d(\log n)^{\beta-1}},$$

with E_m as defined in Lemma 4.

We initially tried to prove Conjecture 1 using a modified version of the proof of Theorem 11. Unfortunately, using the complete expansion of $(1 + \frac{t}{n})^{-n-1}$ introduced several complications, specifically when considering asymptotics that depend on n , j , and m as all three integers grow to infinity.

Several simulations in MAPLE indicate that this conjecture is correct, but of course such evidence proves nothing. In Appendix B, we illustrate the apparent accuracy of the approximation (96).

Chapter 6

Largest component of an exp-log structure

The exp-log class was introduced by Flajolet and Soria [20, 21], and they studied the behaviour of the number of components of objects in this class. In Chapter 5 of his thesis, Gourdon [24] studied the size of the r th largest component for exp-log objects. The distribution of the r th smallest component in this class was studied by Panario and Richmond [32]. Most recently, Dong *et al.* studied the distribution of the r th smallest component for exp-log objects with restricted patterns [10, 11].

In this chapter we determine the probability distribution of the largest component in an exp-log structure with a given restricted pattern $S : J \mapsto \mathbb{N}$. This is primarily an extension of work done by Gourdon in his doctoral thesis [24].

6.1 Probability Distribution

We begin with the definition the exp-log schema of functions, which is similar to that of the alg-log schema as defined in Chapter 3 with $\alpha = 0$ and $\beta = 1$. The region $\Delta(\nu, \theta)$ defined in Definition 8 is identical to that given by Definition 5.

Definition 8. Let

$$\Delta(\nu, \theta) = \{z : |z| \leq \rho(1 + \nu), z \neq \rho, |\arg(z - \rho)| \geq \theta\},$$

for some constants $\rho > 0$, $\nu > 0$, and $0 < \theta < \pi/2$. Then $C(z)$ is of *logarithmic type* at the singularity ρ with multiplicative constant $0 < a \leq 1$ and error exponent $0 < \alpha \leq 1$ if

$$C(z) = a \log \frac{1}{1 - z/\rho} + R(z),$$

where $R(\rho z)$ is analytic in $\Delta(\nu, \theta)$ and satisfies

$$R(\rho z) = K + O((1 - z)^\alpha),$$

where K is a complex constant. Furthermore, $F(z)$ is of *exp-log* type if $F(z) = \exp(C(z))$.

Under this definition, $F(z) = \exp(C(z))$ satisfies

$$F(z) = \frac{\exp(R(z))}{(1 - z/\rho)^a} = \frac{e^K}{(1 - z/\rho)^a} + O((1 - z/\rho)^{\alpha-a}),$$

which implies by Theorem 1 and Corollary 1 that

$$[z^n]F(z) = \frac{1}{\rho^n} \frac{e^K}{\Gamma(\alpha)} n^{\alpha-1} (1 + O(n^{-\alpha})). \quad (97)$$

Let $L_n(S)$ denote the random variable that counts the size of the largest component in a uniformly chosen exp-log object with restricted pattern $S : J \mapsto \mathbb{N}$. In order to determine the distribution of $L_n(S)$, we first determine the exponential generating function for an exp-log structure with pattern S whose components are all of size no greater than m . Let $F_m(z; S)$ denote this generating function, and let

$$s_m(z) = \sum_{n=1}^m \frac{C_n}{n!} z^n, \text{ and } r_m(z) = \sum_{n>m} \frac{C_n z^n}{n!},$$

i.e., $s_m(z)$ is the truncation of the first m terms of $C(z)$, and $r_m(z)$ is the remainder.

Then it is easy to see in the case $S = \emptyset$ that

$$F_m(z; \emptyset) = \exp(s_m(z)) = F(z; \emptyset) \exp(-r_m(z)),$$

and

$$\Pr(L_n(\emptyset) \leq m) = \frac{[z^n]F_m(z; \emptyset)}{[z^n]F(z; \emptyset)}. \quad (98)$$

In Chapter 3 of his thesis, Gourdon determined an asymptotic expression for (98) as well as the limiting distribution for $L_n(\emptyset)$ as $n \rightarrow \infty$. We adapt his methods to determine an analogous result for the case with a non-empty pattern S . There are actually two quantities that we may calculate:

$$\frac{[z^n]F_m(z; S)}{[z^n]F(z; \emptyset)} \quad (99)$$

and

$$\frac{[z^n]F_m(z; S)}{[z^n]F(z; S)}, \quad (100)$$

where (99) represents the probability that a random object of size n with largest component no greater than m has the restricted pattern S , and (100) represents the conditional probability that a random object of size n with restricted pattern S has largest component no greater than m . The denominator of (99) is given by (97), and we determine the denominator of (100) in this section. Before we proceed, we restate without proof two lemmas from [24], which require the following definition (see [1], p 228).

Definition 9. The *exponential integral* is the integral

$$E(x) = \int_x^{+\infty} \frac{e^{-v}}{v} dv, \quad |\arg(x)| < \pi. \quad (101)$$

In (101), it is assumed that the path of integration excludes the origin and does not cross the negative real axis.

Lemma 7 (p. 129, [24]). *The exponential integral $E(x)$ satisfies the following properties.*

(i) *We may write*

$$E(u) = -\log u - \gamma + \tilde{E}(u),$$

where γ is the Euler-Mascheroni constant and $\tilde{E}(u)$ is an entire function that is null at $u = 0$;

(ii) *we have the estimate*

$$E(u) = \frac{e^{-u}}{u} + O\left(\frac{e^{-u}}{u^2}\right) \quad \text{as } u \rightarrow \infty, \Re(u) > 0;$$

(iii) *for all $c > 0$, the function $e^{-cE(u)}$ is bounded on the half-plane $\Re(u) > 0$.*

Lemma 8 (p. 140 [24]). *Suppose $C(z)$ is a logarithmic function at $z = \rho > 0$ with logarithmic coefficient a and error exponent $0 < \alpha \leq 1$. Then the remainder of the series $C(z)$ satisfies*

$$r_m(\rho e^{-p}) = aE(mp) + O\left(\frac{1}{m^\alpha}\right),$$

where the O is uniform for $\Re(p) \geq 0$ and where $E(x)$ is the exponential integral from Definition 9.

In this chapter, we will also make use of the function $f_a(\mu)$ which is defined by

$$f_a(\mu) = \frac{\Gamma(a)}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{e^{-aE(\mu p)}}{p^a} e^p dp. \quad (102)$$

We now determine $[z^n]F_m(z; S)$. Let η denote the size of the unrestricted part of

an object of size n with pattern S , *i.e.*,

$$\eta = n - \sum_{j \in J} jS(j).$$

Throughout this chapter, we assume that $\eta \rightarrow \infty$ as $n \rightarrow \infty$. For a given pattern $S : J \mapsto \mathbb{N}$, we define the quantities $J_m^{(1)}$, $J_m^{(2)}$, $\hat{j}$, and j^* as follows:

$$\begin{aligned} J_m^{(1)} &= \{j \in J : j \leq m\}, \\ J_m^{(2)} &= \{j \in J : j > m\}, \\ \hat{j} &= \max\{j : j \in J\}, \\ j^* &= \max\{j : j \in J, S(j) \neq 0\}. \end{aligned}$$

Furthermore, we assume that $\hat{j} = O(\eta^{1-a\alpha/(1+\alpha)})$ and that, for all $1 \leq m \leq \eta$, $|J_m^{(1)}| = O(\eta^{1-\alpha})$. Note that the maximum size of an object of size n with restricted pattern S under the above hypothesis is η . That maximum is achieved by an object made up of the components in the pattern and a *single* component of size η . Therefore, if $m > \eta$, then $F_m(z; S) = F_\eta(z; S) = F(z; S)$, so in Theorem 12 we only consider the range $1 \leq m \leq \eta$.

Theorem 12. *Under the above hypotheses,*

$$[z^n]F_m(z; S) = \mathcal{S}_m \frac{1}{\rho^n} \eta^{a-1} e^K \exp(-s_m(\rho; J)) \left(\frac{f_a(m/\eta)}{\Gamma(a)} + O(m^{-a\alpha/(1+\alpha)}) \right),$$

uniformly for $1 \leq m \leq \eta$, where $f_a(\mu)$ is as defined in (102), $s_m(\rho; J) = \sum_{j \in J_m^{(1)}} \frac{C_j \rho^j}{j!}$,

and

$$\mathcal{S}_m = \begin{cases} \prod_{j \in J_m^{(1)}} \frac{1}{S(j)!} \left(\frac{C_j}{j!} \right)^{S(j)} & \text{if } j^* \leq m, \\ 0 & \text{if } j^* > m. \end{cases}$$

Proof. If $j^* > m$, then $F_m(z; S) = 0$ since any object with such a pattern S must have a component of size greater than m . If $j^* \leq m$, then by Lemma 3

$$F_m(z; S) = \prod_{j \in J_m^{(1)}} \left(C_j \frac{z^j}{j!} \right)^{S(j)} \frac{1}{S(j)!} \exp \left(s_m(z) - \sum_{j \in J_m^{(1)}} C_j \frac{z^j}{j!} \right).$$

Note that $\exp(s_m(z)) = F(z; \emptyset) \exp(-r_m(z))$, so

$$\exp \left(s_m(z) - \sum_{j \in J_m^{(1)}} C_j \frac{z^j}{j!} \right) = F(z; \emptyset) \exp \left(-r_m(z) - \sum_{j \in J_m^{(1)}} C_j \frac{z^j}{j!} \right).$$

Furthermore, by definition of an exp-log function

$$F(z; \emptyset) = \frac{e^{R(z)}}{(1 - z/\rho)^a},$$

so by the Cauchy integration formula,

$$\begin{aligned} [z^n] F_m(z; S) &= \mathcal{S}_m [z^n] F(z; \emptyset) \exp \left(-r_m(z) - \sum_{j \in J_m^{(1)}} C_j \frac{z^j}{j!} \right) \\ &= \frac{\mathcal{S}_m}{\rho^\eta} \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\exp \left(-r_m(\rho z) - \sum_{j \in J_m^{(1)}} C_j \frac{(\rho z)^j}{j!} \right)}{(1 - z)^a} \exp(R(\rho z)) \frac{dz}{z^{\eta+1}}, \quad (103) \end{aligned}$$

where the contour $\mathcal{C} = \{z : |z| = \exp(-1/\eta)\}$ is positively oriented.

To evaluate the integral in (103), we divide $\mathcal{C}$ into two contours and consider them

separately. Let

$$\mathcal{C}_0 = \{z : |z| = e^{-1/\eta}, |\arg(z)| \leq \eta^{\delta-1}\},$$

$$\mathcal{C}_1 = \{z : |z| = e^{-1/\eta}, |\arg(z)| > \eta^{\delta-1}\},$$

where $\delta \in (0, 1)$ is a constant which we define as $\delta = \alpha/(1 + \alpha)$. Let

$$I_0 = \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{\exp\left(-r_m(\rho z) - \sum_{j \in J_m^{(1)}} C_j \frac{(\rho z)^j}{j!}\right)}{(1-z)^a} \exp(R(\rho z)) \frac{dz}{z^{\eta+1}},$$

and

$$I_1 = \frac{1}{2\pi i} \int_{\mathcal{C}_1} \frac{\exp\left(-r_m(\rho z) - \sum_{j \in J_m^{(1)}} C_j \frac{(\rho z)^j}{j!}\right)}{(1-z)^a} \exp(R(\rho z)) \frac{dz}{z^{\eta+1}}.$$

We first focus on I_1 . Integrating by parts, we have

$$\begin{aligned} I_1 &= \frac{1}{2\pi i} \int_{\mathcal{C}_1} \exp\left(s_m(\rho z) - \sum_{j \in J_m^{(1)}} C_j \frac{(\rho z)^j}{j!}\right) \frac{dz}{z^{\eta+1}} \\ &= -\frac{1}{\eta} \frac{1}{2\pi i} \left[\frac{\exp\left(s_m(\rho z) - \sum_{j \in J_m^{(1)}} C_j \frac{(\rho z)^j}{j!}\right)}{z^\eta} \right]_{\bar{c}}^c \\ &\quad + \frac{\rho}{\eta} \frac{1}{2\pi i} \int_{\mathcal{C}_1} \left(s'_m(\rho z) - \sum_{j \in J_m^{(1)}} \frac{C_j \rho^{j-1} z^{j-1}}{(j-1)!}\right) \exp\left(s_m(\rho z) - \sum_{j \in J_m^{(1)}} C_j \frac{\rho^j z^j}{j!}\right) \frac{dz}{z^\eta}, \end{aligned}$$

where $c = \exp(-1/\eta + i\eta^{\delta-1})$.

On $\mathcal{C}_1$, $z = e^{-1/\eta + i\theta}$ where $\eta^{\delta-1} < |\theta| \leq \pi$. Therefore, $\rho z = \rho e^{-p}$ with $\Re(p) = 1/\eta$. By Lemma 8, we have $e^{-r_m(\rho e^{-p})} = e^{-aE(mp) + o(1)}$ as $m \rightarrow \infty$, and we deduce via property (iii) of the exponential integral given in Lemma 7 that $e^{-r_m(\rho z)}$ is bounded

on $|z| < 1$ independently of m . Therefore, for $|z| < 1$,

$$e^{s_m(\rho z)} = \frac{e^{-r_m(\rho z)}}{(1-z)^a} e^{R(\rho z)} = O\left(\frac{1}{(1-z)^a}\right) \quad \text{as } z \rightarrow 1.$$

Furthermore, from Gourdon (p. 167, [24]), we also have

$$s'_\eta(\rho z) = O(C'(\rho z)) = O\left(\frac{1}{1-z}\right) \quad \text{as } z \rightarrow 1.$$

Hence, on $\mathcal{C}_1$,

$$e^{s_m(\rho z)} = O\left(\frac{1}{|1-c|^a}\right) = O(\eta^{a(1-\delta)}), \quad (104)$$

and

$$s'_\eta(\rho z) = O\left(\frac{1}{|1-c|}\right) = O(\eta^{1-\delta}).$$

Next, we consider $\sum_{j \in J_m^{(1)}} C_j \frac{\rho^j z^j}{j!}$ and $\sum_{j \in J_m^{(1)}} C_j \frac{\rho^{j-1} z^{j-1}}{(j-1)!}$ on $\mathcal{C}_1$. We focus first on the latter sum:

$$\left| \sum_{j \in J_m^{(1)}} C_j \frac{\rho^{j-1} z^{j-1}}{(j-1)!} \right| \leq \sum_{j \in J_m^{(1)}} C_j \frac{\rho^{j-1} |z|^{j-1}}{(j-1)!} \leq \sum_{j \geq 1} C_j \frac{\rho^{j-1} |z|^{j-1}}{(j-1)!} = s'_\eta(\rho e^{-1/\eta}) = O(\eta^{1-\delta}).$$

Let $s_m(z; J) = \sum_{j \in J_m^{(1)}} \frac{C_j z^j}{j!}$. By the transfer theorems, the fact that $|z| = e^{-1/\eta}$ on $\mathcal{C}_1$,

and the assumption $|J_m^{(1)}| = O(\eta^{1-\alpha})$,

$$\begin{aligned} s_m(z; J) &= \sum_{j \in J_m^{(1)}} C_j \frac{\rho^j z^j}{j!} = \sum_{j \in J_m^{(1)}} \frac{C_j \rho^j}{j!} (1 + O(-j/\eta)) = \sum_{j \in J_m^{(1)}} \left(\frac{C_j \rho^j}{j!} + O(1/\eta) \right) \\ &= \left(\sum_{j \in J_m^{(1)}} \frac{C_j \rho^j}{j!} \right) + o(1). \end{aligned} \quad (105)$$

Then by (104) and (105), we have

$$\begin{aligned} -\frac{1}{\eta} \frac{1}{2\pi i} \left[\frac{\exp(s_m(\rho z) - s_m(\rho z; J))}{z^\eta} \right]_{\bar{c}}^c &= \exp(-s_m(\rho; J)) O(\eta^{-1}) O(1) O\left(\frac{1}{|1-c|^a}\right) \\ &= \exp(-s_m(\rho; J)) O(\eta^{a-1-a\delta}). \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\rho}{\eta} \frac{1}{2\pi i} \int_{c_1} (s'_m(\rho z) - s'_m(\rho z; J)) \exp(s_m(\rho z) - s_m(\rho z; J)) \frac{dz}{z^\eta} \\ = O(\eta^{-1}) O(\eta^{\delta-1}) O\left(\frac{1}{|1-c|}\right) \exp(-s_m(\rho; J)) O\left(\frac{1}{|1-c|^a}\right) \\ = \exp(-s_m(\rho; J)) O(\eta^{a-1-a\delta}), \end{aligned}$$

and we estimate I_1 as

$$I_1 = \exp(-s_m(\rho; J)) O(\eta^{a-1} \eta^{-a\delta}).$$

Next, we estimate the contribution of I_0 . Let $z = e^{-p/\eta}$. Then

$$I_0 = \frac{1}{\eta} \frac{1}{2\pi i} \int_{1-i\eta^\delta}^{1+i\eta^\delta} \frac{\exp\left(-r_m(\rho e^{-p/\eta}) - \sum_{j \in J_m^{(1)}} \frac{C_j \rho^j e^{-jp/\eta}}{j!}\right)}{(1 - e^{-p/\eta})^a} e^{R(p e^{-p/\eta})} e^p dp.$$

On $\mathcal{C}_0$,

$$|z|^j = e^{-j/\eta} = 1 + O(j/\eta),$$

which, when combined with the transfer theorems, implies that

$$\frac{C_j \rho^j z^j}{j!} = \frac{C_j \rho^j}{j!} (1 + O(j/\eta)) = \frac{C_j \rho^j}{j!} + O(\eta^{-1}),$$

From the assumption that $|J_m^{(1)}| = O(\eta^{1-\alpha})$ and the fact that $\eta^{-\alpha} \leq m^{-\alpha}$ for all $1 \leq m \leq \eta$, we conclude that

$$\sum_{j \in J_m^{(1)}} \frac{C_j \rho^j z^j}{j!} = s_m(\rho; J) + O(m^{-\alpha}).$$

Next, let $\mu = m/\eta$. Then by Lemma 8 and the hypothesis on $R(z)$,

$$\begin{aligned} I_0 &= \frac{1}{\eta} \frac{\exp(-s_m(\rho; J))}{2\pi i} \int_{1-i\eta^\delta}^{1+i\eta^\delta} \frac{e^{-aE(\mu p) + O(m^{-\alpha})}}{(p/\eta(1 + O(p/\eta)))^\alpha} e^{K + O((p/\eta)^\alpha)} e^p dp \\ &= \eta^{a-1} \frac{\exp(-s_m(\rho; J))}{2\pi i} \int_{1-i\eta^\delta}^{1+i\eta^\delta} \frac{e^{-aE(\mu p)}}{p^a} e^K \left(1 + O\left(\frac{p}{\eta}\right)^\alpha + O(m^{-\alpha}) \right) e^p dp. \end{aligned}$$

Gourdon (p. 142, [24]) proved that

$$\begin{aligned} &\int_{1-i\eta^\delta}^{1+i\eta^\delta} \frac{e^{-aE(\mu p)}}{p^a} e^K \left(1 + O\left(\frac{p}{\eta}\right)^\alpha + O(m^{-\alpha}) \right) e^p dp \\ &= \int_{1-i\eta^\delta}^{1+i\eta^\delta} \frac{e^{-aE(\mu p)}}{p^a} e^K dp + O(m^{\alpha(\delta-1) + \delta(1-a)}). \end{aligned}$$

Therefore,

$$I_0 = \eta^{a-1} e^K \exp(-s_m(\rho; J)) \left[\frac{1}{2\pi i} \int_{1-i\eta^\delta}^{1+i\eta^\delta} \frac{e^{-aE(\mu p)}}{p^a} e^k dp + O(m^{\alpha(\delta-1) + \delta(1-a)}) \right].$$

We now use integration by parts to estimate the integral

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1-i\eta^\delta} \frac{e^{-aE(\mu p)}}{p^a} e^p dp.$$

Let $u = p^{-a} e^{-aE(\mu p)}$, and let $dv = e^p dp$. Then $v = e^p$, and

$$du = -ap^{-a-1} e^{-aE(\mu p)} + p^{-a} e^{-aE(\mu p)} \frac{d}{dp} (-aE(\mu p)) dp.$$

Recall that $E(x)$ is defined as

$$E(x) = \int_x^{+\infty} \frac{e^{-v}}{v} dv,$$

so

$$\frac{d}{dp} (-aE(\mu p)) = a\mu \frac{e^{-\mu p}}{\mu p} = a \frac{e^{-\mu p}}{p}.$$

Therefore,

$$du = -ap^{-a-1} e^{-aE(\mu p)} + ap^{-a-1} e^{-aE(\mu p)} e^{-\mu p} dp,$$

and

$$\begin{aligned} \frac{1}{2\pi i} \int_{1-i\infty}^{1-i\eta^\delta} \frac{e^{-aE(\mu p)}}{p^a} e^p dp &= \frac{1}{2\pi i} p^{-a} e^{-aE(\mu p)} e^p \Big|_{1-i\infty}^{1-i\eta^\delta} \\ &\quad + \frac{1}{2\pi i} \int_{1-i\infty}^{1-i\eta^\delta} \left(ap^{-a-1} e^{-aE(\mu p)} - ap^{-a-1} e^{-aE(\mu p)} e^{-\mu p} \right) e^p dp. \end{aligned}$$

By Lemma 7, $E(\mu p) = O(1)$, so

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1-i\eta^\delta} \frac{e^{-aE(\mu p)}}{p^a} e^p dp = O(\eta^{-a\delta}) + \int_{1-i\infty}^{1-i\eta^\delta} O(p^{-a-1}) dp = O(\eta^{-a\delta}).$$

A nearly identical argument shows that

$$\frac{1}{2\pi i} \int_{1+i\eta^\delta}^{1+i\infty} \frac{e^{-aE(\mu p)}}{p^a} e^p dp = O(\eta^{-a\delta}),$$

which, by applying (102), finally implies that

$$[z^\eta]F_m(z; S) = \frac{1}{\rho^\eta} \eta^{a-1} e^K \exp(-s_m(\rho; J)) \left[\frac{f_a(\mu)}{\Gamma(a)} + O(m^{\alpha(\delta-1)+\delta(1-a)}) + O(m^{-a\delta}) \right],$$

where I_1 contributes the $O(m^{-a\delta})$ term. Since $\delta = \alpha/(1+\alpha)$, we have

$$[z^\eta]F_m(z; S) = \frac{1}{\rho^\eta} \eta^{a-1} e^K \exp(-s_m(\rho; J)) \left(\frac{f_a(\mu)}{\Gamma(a)} + O(m^{-a\alpha/(1+\alpha)}) \right).$$

Therefore, we may conclude that

$$[z^n]F_m(z; S) = \frac{\mathcal{S}_m}{\rho^\eta} \eta^{a-1} e^K \exp(-s_m(\rho; J)) \left(\frac{f_a(m/\eta)}{\Gamma(a)} + O(m^{-a\alpha/(1+\alpha)}) \right)$$

uniformly for $1 \leq m \leq \eta$. □

Letting $m = \eta$, we see that $F_\eta(z; S) = F(z; S)$, and we have

$$[z^n]F(z; S) = \frac{\mathcal{S}}{\rho^\eta} \eta^{a-1} e^K \exp(-C(\rho; J)) \left(\frac{1}{\Gamma(a)} + O(\eta^{-a\alpha/(1+\alpha)}) \right), \quad (106)$$

because $f_a(1) = 1$. This matches a result by Dong, albeit under different conditions on the pattern S . Specifically, we require that $\eta \rightarrow \infty$, $\hat{j} = O(\eta^{1-a\alpha/(1+\alpha)})$, and $|J_\eta^{(1)}| = O(\eta^{1-\alpha})$ as $n \rightarrow \infty$. Dong requires that $\eta \rightarrow \infty$, $j^* = O(\eta/\log \eta)$, and $|J_\eta^{(1)}| = o(\eta/\log \eta)$ as $\eta \rightarrow \infty$, which allows for larger quantities in the case of both j^* and $|J_\eta^{(1)}|$. However, the result of Dong only applies to the case $m = \eta$.

Let $L_n(S)$ denote the size of the largest component in an exp-log object of size n

with restricted pattern S . Then

$$\Pr(L_n(S) \leq m) = \frac{[z^n]F_m(z; S)}{[z^n]F(z; S)},$$

which we may determine using Theorem 12 and (106). However, we must take care to consider all the possible cases.

If $j^* > m$, then $F_m(z; S) = 0$, i.e., $\Pr(L_n(S) \leq m) = 0$. If $j^* \leq m < \hat{j}$, then

$$\begin{aligned} \Pr(L_n(S) \leq m) &= \prod_{\substack{j \in J \\ j > m}} S(j)! \left(\frac{j!}{C_j} \right)^{S(j)} \exp(C(\rho; J) - s_m(\rho; J)) \left(f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)}) \right) \\ &\quad \times \left(1 + O(\eta^{-a\alpha/(1+\alpha)}) \right). \end{aligned}$$

However, if $j^* \leq m$, then for $j > m$, $S(j) = 0$. Therefore, letting $r_m(z; J) = C(z; J) - s_m(z; J)$,

$$\begin{aligned} \Pr(L_n(S) \leq m) &= \exp(C(\rho; J) - s_m(\rho; J)) \left(f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)}) \right) \left(1 + O(\eta^{-a\alpha/(1+\alpha)}) \right) \\ &= \exp(r_m(\rho; J)) \left(f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)}) \right) \left(1 + O(\eta^{-a\alpha/(1+\alpha)}) \right). \end{aligned}$$

Finally, if $m \geq \hat{j}$, then

$$\Pr(L_n(S) \leq m) = \left(f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)}) \right) \left(1 + O(\eta^{-a\alpha/(1+\alpha)}) \right),$$

and we have proven the following corollary.

Corollary 7. *Let $L_n(S)$ denote the size of the largest component in an exp-log object of size n with restricted pattern S , and assume that S satisfies the conditions of Theorem 12. Then uniformly for $1 \leq m \leq \eta$,*

$$\Pr(L_n(S) \leq m) = \mathcal{P}_m \left(f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)}) \right) \left(1 + O(\eta^{-a\alpha/(1+\alpha)}) \right), \quad (107)$$

where the quantity

$$\mathcal{P}_m = \begin{cases} 0 & \text{if } m < j^*, \\ \exp(r_m(\rho; J)) & \text{if } j^* \leq m < \hat{j}, \\ 1 & \text{otherwise.} \end{cases} \quad (108)$$

Furthermore, if $m > \eta$, then $\Pr(L_n(S) \leq m) = 1$.

6.1.1 Mean and Variance

In this section we deduce $\mathbb{E}(L_n(S))$ and $\text{Var}(L_n(S))$. It is well known that, via an Abel transformation (p. 3, [40]),

$$\mathbb{E}(L_n(S)) = \sum_{m \geq 0} \Pr(L_n(S) > m),$$

which, given that $\Pr(L_n(S) > m) = 1 - \Pr(L_n(S) \leq m)$, we may rewrite as

$$\mathbb{E}(L_n(S)) = n - \sum_{m=1}^n \Pr(L_n(S) \leq m).$$

Similarly, we may calculate the variance $\text{Var}(L_n(S)) = \mathbb{E}(L_n(S)^2) - \mathbb{E}(L_n(S))^2$ via the equation

$$\mathbb{E}(L_n(S)^2) = \sum_{m \geq 0} (2m + 1) \Pr(L_n(S) > m), \quad (109)$$

which is also obtained by an Abel transformation. However, before we calculate the mean and variance of $L_n(S)$, we must outline an alternative definition for the function $f_a(\lambda)$, which was determined by Gourdon in terms of a generalized Dickman function. The description we give below of that function (up to and including Lemma 9) is

primarily a translation from the original French of Gourdon's thesis with some added details.

The Generalized Dickman Function

Let g be a function that is locally integrable on the interval $[0, +\infty)$. The *Laplace transform* (see [40], pp. 107–116) of g is defined by

$$\mathcal{L}[g](p) = \int_0^{+\infty} g(u)e^{-pu} \, du.$$

If the integral exists for $p = \sigma + it$, it exists for all complex numbers that satisfy $\Re(p) \geq \sigma$. The lower bound σ_0 of the values of σ for which $|g(u)|e^{-\sigma u}$ is integrable is called the *abscissa of integrability* of the Laplace integral. The Laplace integral defines a function that is analytic in p in the domain $\Re(p) > \sigma_0$, which is called the *domain of integrability*.

One useful property of the Laplace transform is that it is applicable to convolution products. Specifically, consider two functions g and h , both of which are locally integrable on $[0, +\infty)$. We define their convolution by

$$(g \star h)(u) = \int_0^u g(t)h(u-t) \, dt.$$

If the domains of integrability of g and h contain a semi-plane $\Re(p) > \sigma_0$, then so does the domain of integrability of $g \star h$, and in that domain,

$$\mathcal{L}[g \star h] = \mathcal{L}[g] \cdot \mathcal{L}[h].$$

The *inverse Laplace transform* of an analytic function $F(p)$ in a semi-plane

$\Re(p) > \sigma_0$ is defined for all $u > 0$ by the integral

$$\mathcal{L}^{-1}[F](u) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(p)e^{up} dp, \quad \sigma > \sigma_0.$$

If $F(p)$ is bounded on the semi-plane $\Re(p) \geq \sigma$ for all $\sigma > \sigma_0$, and if the integral along the right abscissa σ is defined, then the value of the above integral is independent of $\sigma > \sigma_0$. If $F(p) = \mathcal{L}[g](p)$ satisfies the above properties, then we have the relation

$$\mathcal{L}^{-1}[\mathcal{L}[g]] = g.$$

In the case $a = 1$, $f_1(\mu) = \rho(1/\mu)$, where $\rho(1/\mu)$ is the *Dickman function*, which we define below. Figure 9 [41] illustrates the Dickman function $\rho(u)$ for $1 \leq u \leq 5$.

Definition 10. The *Dickman function* is the continuous solution to the delay differential equation

$$\lambda \rho'(\lambda) + \rho(\lambda - 1) = 0, \quad \lambda \geq 1,$$

with the initial condition $\rho(\lambda) = 1$ for $0 \leq \lambda \leq 1$ (see [40], pp. 371–377).

Figure 9: The Dickman function $\rho(u)$ for $1 \leq u \leq 5$

Goncharov [23] proved that $\rho(\lambda)$ can be calculated as an inverse Laplace transform resulting in the expression

$$f_1(1/\lambda) = \rho(\lambda) = 1 + \sum_{k=1}^{[\lambda]} \frac{(-1)^k}{k!} \int_{\substack{t_1+\dots+t_k \leq \lambda \\ t_1, \dots, t_k \geq 1}} \frac{dt_1 dt_2 \cdots dt_k}{t_1 t_2 \cdots t_k}. \quad (110)$$

When $0 \leq \lambda \leq 1$, there is only one term in the integrand of (110), so $\rho(\lambda) = 1$.

When $1 \leq \lambda \leq 2$,

$$\rho(\lambda) = 1 - \int_0^\lambda H_1(u) du = 1 - \log \lambda,$$

where

$$H_1(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1, \\ 1/u & \text{if } u > 1. \end{cases}$$

For $2 \leq \lambda \leq 3$,

$$\rho(\lambda) = 1 - (1 - \log(\lambda - 1)) \log \lambda + \text{Li}_2(1 - \lambda) + \frac{\pi^2}{2},$$

where $\text{Li}_2(z)$ is the dilogarithm function defined by

$$\text{Li}_2(\pm z) \equiv - \int_0^z \frac{\log |1 \mp x|}{x} dx.$$

This technique can be continued for the intervals $[3, 4]$, $[4, 5]$, *etc.*, but past $[3, 4]$, (110) does not yield a closed form for $\rho(\lambda)$.

An asymptotic expression for $\rho(\lambda)$ in terms of elementary functions is

$$\rho(\lambda) = \exp \left(-\lambda \left(\log \lambda + \log \log \lambda - 1 + \frac{\log \log \lambda}{\log \lambda} + O \left(\frac{1}{\log \lambda} \right) \right) \right). \quad (111)$$

The asymptotic formula (111) was determined by Hua [29], although it is in fact a slightly weaker version of a result proven by de Bruijn [8].

Gourdon [24] defined a generalized Dickman function $\rho_a(\lambda) = f_a(1/\lambda)$, which is given by

$$\rho_a(\lambda) = \lambda^{1-a} \frac{\Gamma(a)}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{e^{-aE(p)}}{p^a} e^{\lambda p} dp. \quad (112)$$

The integral that appears in (112) is an inverse Laplace transform, and may be evaluated to determine the expression

$$f_a(1/\lambda) = \rho_a(\lambda) = \lambda^{1-a} \Gamma(a) \mathcal{L}^{-1} \left[\frac{e^{-aE(p)}}{p^a} \right] (\lambda) = \lambda^{1-a} \Gamma(a) \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \mathcal{L}^{-1} \left[\frac{E(p)^k}{p^a} \right] (\lambda).$$

Next, we note that

$$\begin{aligned} E(p) &= \int_1^{+\infty} \frac{e^{-up}}{u} du = \mathcal{L}[H_1](p), \\ \frac{1}{p^a} &= \frac{1}{\Gamma(a)} \int_0^{+\infty} u^{a-1} e^{-up} du = \frac{1}{\Gamma(a)} \mathcal{L}[u^{a-1}](p), \end{aligned}$$

from which we deduce that

$$\Gamma(a) \mathcal{L} \left[H_1^{(k)} \star u^{a-1} \right] (p) = \frac{E(p)^k}{p^a},$$

where $H_1^{(k)}$ is the k -times convolution of H_1 with itself. Since $H_1(u) = 0$ for $u \in [0, 1]$, $H_1^{(k)}(u) = 0$ on $[0, k]$. Therefore,

$$f_a(1/\lambda) = \rho_a(\lambda) = 1 + \lambda^{1-a} \sum_{k=1}^{\lfloor \lambda \rfloor} \frac{(-a)^k}{k!} \int_{\substack{t_1 + \dots + t_k \leq \lambda \\ t_1, \dots, t_k \geq 1}} \frac{dt_1 \cdots dt_k}{t_1 \cdots t_k (\lambda - t_1 - \dots - t_k)^{1-a}}.$$

When $0 \leq \lambda \leq 1$, $\rho(\lambda) = 1$. When $1 \leq \lambda \leq 2$, there are two terms:

$$\rho_a(\lambda) = 1 - a\lambda^{1-a} \int_1^\lambda \frac{dt}{t(\lambda - t)^{1-a}}.$$

In general, this expression for $\rho_a(\lambda)$ on $[1, 2]$ does not yield a closed form. However, many combinatorial examples are given by $a = 1$ or $a = \frac{1}{2}$. In the case $a = \frac{1}{2}$, we have

$$f_{1/2}(1/\lambda) = \rho_{1/2}(\lambda) = 1 - \log(\sqrt{\lambda} + \sqrt{\lambda - 1}), \quad 1 \leq \lambda \leq 2.$$

Furthermore, Omar *et al.* [31] determined the following upper bound for $\lambda > 1$:

$$f_a(1/\lambda) = O(a^{1/\lambda}).$$

They also proved that $\rho'_a(\lambda) < 0$, *i.e.*, it is decreasing, which implies that $f_a(u)$ is an increasing function.

Gourdon also proved the following lemma which provides a functional equation for $\rho_a(\lambda)$.

Lemma 9 (p. 144, [24]). *The generalized Dickman function $\rho_a(\lambda)$ is continuous on $[0, +\infty)$, is identically equal to 1 on $[0, 1]$, is differentiable on $[1, +\infty)$, and satisfies the differential equation of differences*

$$\lambda^a \rho'_a(\lambda) + a(\lambda - 1)^{a-1} \rho_a(\lambda - 1) = 0, \quad \lambda \geq 1.$$

Lemma 9 proves that, on the interval $[0, 1]$, the function $f_a(\lambda)$ is in the class of functions $\mathcal{C}_1$, *i.e.*, the Riemann sum of f approaches its integral to within $O(1/n)$.

Mean and Variance

Before we state the following corollary, we recall the quantities and conditions associated with the restricted pattern S in Theorem 12. We assume that $n - \sum_{j \in J} jS(j) = \eta \rightarrow \infty$ as $n \rightarrow \infty$, and we define the quantities $J_m^{(1)}, J_m^{(2)}, \hat{j}$,

and j^* as follows:

$$\begin{aligned} J_m^{(1)} &= \{j \in J : j \leq m\}, \\ J_m^{(2)} &= \{j \in J : j > m\}, \\ \hat{j} &= \max\{j : j \in J\}, \\ j^* &= \max\{j : j \in J, S(j) \neq 0\}. \end{aligned}$$

Finally, we assume that, for all $1 \leq m \leq \eta$, $|J_m^{(1)}| = O(\eta^{1-\alpha})$, and we assume that $\hat{j} = O(\eta^{1-a\alpha/(1+\alpha)})$.

Corollary 8. *As $n \rightarrow \infty$, the mean size of the largest component in an exp-log structure of size n with restricted pattern S which satisfies the conditions of Theorem 12 asymptotically satisfies*

$$\mathbb{E}(L_n(S)) = G_a \eta + O(\eta^{1-a\alpha/(1+\alpha)}),$$

where

$$G_a = \frac{1}{a} \int_0^\infty (1 - e^{-aE(u)}) \, du.$$

For the variance, we have the estimate

$$\text{Var}(L_n(S)) = H_a \eta^2 + O(\eta^{2-a\alpha/(1+\alpha)}),$$

where

$$H_a = \frac{2}{a(a+1)} \int_0^\infty u (1 - e^{-aE(u)}) \, du - G_a^2.$$

Table 5 gives examples of specific values of G_a and H_a for some values of a .

a	G_a	H_a
1	0.6243299855...	0.03690783...
1/2	0.7578230112...	0.0370072168...

Table 5: G_a and H_a for $a \in \{1/2, 1\}$.

Proof. To find the mean, we use the relation

$$\mathbb{E}(L_n(S)) = \sum_{m \geq 0} (1 - \Pr(L_n(S) \leq m)) = n - \mu_n, \quad \mu_n = \sum_{m=1}^n \Pr(L_n(S) \leq m), \quad (113)$$

which is a consequence of the fact that $\Pr(L_n(S) > m) = 0$ for $m \geq n$, and the fact that $\Pr(L_n(S) = 0) = 0$. Recall that, by Corollary 7,

$$\Pr(L_n(S) \leq m) = \mathcal{P}_m \left(f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)}) \right) \left(1 + O(\eta^{-a\alpha/(1+\alpha)}) \right),$$

for $1 \leq m \leq \eta$, where the multiplicative quantity $\mathcal{P}_m$ is defined by (108), and for $\eta < m \leq n$, $\Pr(L_n(S) \leq m) = 1$. Therefore,

$$\begin{aligned} \mu_n &= \sum_{m=1}^{j^*-1} 0 \\ &\quad + \sum_{m=j^*}^{\hat{j}-1} \exp(r_m(\rho; J)) \left(f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)}) \right) \left(1 + O(\eta^{-a\alpha/(1+\alpha)}) \right) \\ &\quad + \sum_{m=\hat{j}}^{\eta} \left(f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)}) \right) \left(1 + O(\eta^{-a\alpha/(1+\alpha)}) \right) + \sum_{m=\eta+1}^n 1, \end{aligned}$$

which implies that

$$\begin{aligned}
n - \mu_n &= \eta - \sum_{m=j^*}^{\hat{j}-1} \exp(r_m(\rho; J)) (f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)})) (1 + O(\eta^{-a\alpha/(1+\alpha)})) \\
&\quad - \sum_{m=\hat{j}}^{\eta} (f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)})) (1 + O(\eta^{-a\alpha/(1+\alpha)})). \tag{114}
\end{aligned}$$

In his thesis, Gourdon (p.144, [24]) proved that as $n \rightarrow \infty$,

$$n - \sum_{m=1}^n f_a(m/n) = nG_a + O(\eta^{1-a\alpha/(1+\alpha)}), \tag{115}$$

where

$$G_a = \frac{1}{a} \int_0^\infty (1 - e^{-aE(u)}) \, du$$

is a positive number that depends on a . In order to take advantage of this fact, we change the summation in (114) so that it runs from $m = 1$ to $m = \eta$, and we subtract the extra terms from $m = 1$ to $m = \hat{j} - 1$, *i.e.*

$$\begin{aligned}
n - \mu_n &= \eta - \sum_{m=j^*}^{\hat{j}-1} \exp(r_m(\rho; J)) (f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)})) (1 + O(\eta^{-a\alpha/(1+\alpha)})) \\
&\quad - \sum_{m=1}^{\eta} (f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)})) (1 + O(\eta^{-a\alpha/(1+\alpha)})) \tag{116}
\end{aligned}$$

$$+ \sum_{m=1}^{\hat{j}-1} (f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)})) (1 + O(\eta^{-a\alpha/(1+\alpha)})). \tag{117}$$

We may apply (115) directly to (116) but not to (117). However, since $f_a(m/\eta)$ is an increasing function (see p. 114), $f_a(m/\eta) \leq f_a(m/(\hat{j} - 1))$ for all $1 \leq m \leq \hat{j} - 1$.

Therefore,

$$\begin{aligned}
n - \mu_n &= \eta G_a + O(\eta^{1-a\alpha/(1+\alpha)}) + O(\hat{j}) \left(1 - G_a + O(\hat{j}^{-a\alpha/(1+\alpha)})\right) \\
&\quad - \sum_{m=j^*}^{\hat{j}-1} \exp(r_m(\rho; J)) (f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)})) (1 + O(\eta^{-a\alpha/(1+\alpha)})) \\
&= \eta G_a + O(\eta^{1-a\alpha/(1+\alpha)}) \\
&\quad - \sum_{m=j^*}^{\hat{j}-1} \exp(r_m(\rho; J)) (f_a(m/\eta) + O(m^{-a\alpha/(1+\alpha)})) (1 + O(\eta^{-a\alpha/(1+\alpha)})), \tag{118}
\end{aligned}$$

where the last line follows because $\hat{j} = O(\eta^{1-a\alpha/(1+\alpha)})$.

All that remains is to prove that the summation term in (118) is of order $O(\eta^{1-a\alpha/(1+\alpha)})$. If $\hat{j} = j^*$, then the summation is identically zero, and we are done. Otherwise, since $f_a(m/\eta) = O(a^{m/\eta})$ and $\alpha > 0$,

$$\begin{aligned}
\sum_{m=j^*}^{\hat{j}-1} \exp(r_m(\rho; J)) f_a(m/\eta) &= \sum_{m=j^*}^{\hat{j}-1} \exp\left(\sum_{j \in J_m^{(2)}} \frac{C_j \rho^j}{j!}\right) f_a(m/\eta) \\
&= O(a^{j^*/\eta}) \exp\left(\sum_{j \in J_m^{(2)}} \frac{C_j \rho^j}{j!}\right) \\
&= O(1) \exp\left(\sum_{j \in J_m^{(2)}} \frac{a}{j} + O(j^{-\alpha-1})\right), \tag{119}
\end{aligned}$$

where (119) follows from the transfer theorems and the definition of $C(z)$. Therefore,

$$\begin{aligned}
\sum_{m=j^*}^{\hat{j}-1} \exp(r_m(\rho; J)) f_a(m/\eta) &= O(\exp(aH_{\hat{j}})) \\
&= O(\hat{j}^a) = O(\hat{j}) = O(\eta^{1-a\alpha/(1+\alpha)}), \tag{120}
\end{aligned}$$

where H_n is the n -th harmonic number.

For the variance, we proceed in a similar manner. We start with (109) and the estimate from Theorem 12. Since $\Pr(L_n(S) \leq m) = 1$ for $m > \eta$,

$$\begin{aligned}
\mathbb{E} (L_n(S)^2) &= \sum_{m \geq 0} (2m + 1) \Pr(L_n(S) > m) \\
&= \sum_{m=1}^{\eta} (2m + 1) (1 - \Pr(L_n(S) \leq m)) \\
&= \sum_{m=1}^{\eta} (2m + 1) - \sum_{m=1}^{\eta} (2m + 1) \Pr(L_n(S) \leq m) \\
&= \eta^2 + 2\eta - G_a \eta + O(\eta^{1-a\alpha/(1+\alpha)}) - 2 \sum_{m=1}^{\eta} m \Pr(L_n(S) \leq m) \\
&= \eta^2 \left(1 - \frac{2}{\eta^2} \sum_{m=1}^{\eta} m \Pr(L_n(S) \leq m) + O(\eta^{-1}) \right). \tag{121}
\end{aligned}$$

Similarly to the mean, Gourdon proved that as $n \rightarrow \infty$,

$$1 - \frac{2}{n^2} \sum_{m=1}^n m f_a(m/n) = H_a + G_a^2 + O(n^{-a\alpha/(1+\alpha)}), \tag{122}$$

where

$$H_a = \frac{2}{a(a+1)} \int_0^{\infty} u (1 - e^{-aE(u)}) \, du - G_a^2. \tag{123}$$

Here H_a is a positive number that depends on a .

Combining (121), (123), and Corollary 7 we have

$$\begin{aligned}
\frac{\mathbb{E}(L_n(S)^2)}{\eta^2} &= 1 - \frac{2}{\eta^2} \sum_{m=1}^{\eta} m \Pr(L_n(S) \leq m) + O(\eta^{-1}) \\
&= 1 - \left[\frac{2}{\eta^2} \sum_{m=1}^{\eta} \left(m f_a(m/\eta) + O(m^{1-a\alpha/(1+\alpha)}) \right) \right. \\
&\quad - \frac{2}{\eta^2} \sum_{m=1}^{\hat{j}-1} \left(m f_a(m/\eta) + O(m^{1-a\alpha/(1+\alpha)}) \right) \\
&\quad \left. + \frac{2}{\eta^2} \sum_{m=j^*}^{\hat{j}-1} \exp(r_m(\rho; J)) \left(m f_a(m/\eta) + O(m^{1-a\alpha/(1+\alpha)}) \right) \right] \\
&\quad \times \left(1 + O(n^{-a\alpha/(1+\alpha)}) \right). \tag{124}
\end{aligned}$$

The summation terms in (124) are quite complicated, so we treat them all separately. First, by (122),

$$\frac{2}{\eta^2} \sum_{m=1}^{\eta} \left(m f_a(m/\eta) + O(m^{1-a\alpha/(1+\alpha)}) \right) = 1 - H_a + G_a^2 + O(\eta^{-a\alpha/(1+\alpha)}). \tag{125}$$

Next, we use the fact that $f_a(m/\eta)$ is an increasing function (see p. 114) to show that

$$\begin{aligned}
\frac{2}{\eta^2} \sum_{m=1}^{\hat{j}-1} \left(m f_a(m/\eta) + O(m^{1-a\alpha/(1+\alpha)}) \right) &\leq \frac{2}{\eta^2} \sum_{m=1}^{\hat{j}} \left(m f_a(m/\hat{j}) + O(m^{1-a\alpha/(1+\alpha)}) \right) \\
&\leq \frac{2\hat{j}^2}{\eta^2} \left(1 - H_a - G_a^2 + O(\hat{j}^{-a\alpha/(1+\alpha)}) \right) \\
&= O\left(\frac{\hat{j}^2}{\eta^2}\right) = O(\eta^{-2a\alpha/(1+\alpha)}). \tag{126}
\end{aligned}$$

Finally, since $f_a(m/\eta) = O(1)$,

$$\begin{aligned}
& \frac{2}{\eta^2} \sum_{j=j^*}^{\hat{j}-1} \exp(r_m(\rho; J)) \left(m f_a(m/\eta) + O\left(m^{1-a\alpha/(1+\alpha)}\right) \right) \\
&= O\left(\frac{\hat{j}}{\eta^2}\right) \sum_{m=j^*}^{\hat{j}-1} \exp(r_m(\rho; J)) = O\left(\frac{\hat{j}}{\eta^2}\right) \sum_{m=j^*}^{\hat{j}-1} \exp(aH_{\hat{j}}) = O\left(\frac{\hat{j}^2}{\eta^2}\right) = O\left(\eta^{-2a\alpha/(1+\alpha)}\right).
\end{aligned} \tag{127}$$

Substituting (125), (126), and (127), into (122) yields,

$$\begin{aligned}
\frac{\mathbb{E}(L_n(S)^2)}{\eta^2} &= 1 - [1 - H_a + G_a^2 + O(\eta^{-a\alpha/(1+\alpha)}) + O(\eta^{-2a\alpha/(1+\alpha)}) + O(\eta^{-2a\alpha/(1+\alpha)})] \\
&\quad \times (1 + O(\eta^{-a\alpha/(1+\alpha)})) \\
&= H_a + G_a^2 + O(\eta^{-a\alpha/(1+\alpha)}).
\end{aligned}$$

Therefore, as $n \rightarrow \infty$,

$$\text{Var}(L_n(S)) = \mathbb{E}(L_n(S)^2) - \mathbb{E}(L_n(S))^2 = H_a \eta^2 + O(\eta^{2-a\alpha/(1+\alpha)}).$$

□

6.1.2 Maximum Pattern Size

At this point, a natural question to ask is, “Just how ‘big’ can the pattern S be?” For the proofs of Theorem 12 and Corollary 7, we only require that $\eta \rightarrow \infty$ and $|J_m^{(1)}| = O(\eta^{1-\alpha})$ for all $1 \leq m \leq \eta$. In fact, it is not until the proof of Corollary 8 that we use the condition $\hat{j} = O(\eta^{1-a\alpha/(1+\alpha)})$.

However, this last condition is only required to obtain the respective error terms in the expressions

$$\mathbb{E}(L_n(S)) = G_a \eta + O(\eta^{-a\alpha/(1+\alpha)}),$$

and

$$\text{Var}(L_n(S)) = H_a \eta^2 + O(\eta^{2-a\alpha/(1+\alpha)}).$$

If we instead use the restriction $\hat{j} = o(\eta)$, the proof technique used for Corollary 8 still applies, with minor modifications. If $\hat{j} = o(\eta)$, then from (118) and (120) we obtain expression

$$\mathbb{E}(L_n(S)) = G_a \eta + o(\eta),$$

and from (125), (126), and (127), we have

$$\text{Var}(L_n(S)) = H_a \eta^2 + o(\eta^2).$$

In plain English, we are left with the following answer to the question posed at the beginning of this subsection: if the unrestricted component of a structure with restricted pattern $S : J \mapsto \mathbb{N}$ grows much faster than the largest component in J , and if $|J|$ is appropriately small then Theorem 12, Corollary 7, and Corollary 8 may be applied. Formally, if we do not care about error terms, we require that

$$\hat{j} = o\left(n - \sum_{j \in J} jS(j)\right) \quad \text{and} \quad |J| = O\left(\left(n - \sum_{j \in J} jS(j)\right)^{1-\alpha}\right).$$

If we do care about error terms, we require that

$$\hat{j} = O\left(\left(n - \sum_{j \in J} jS(j)\right)^{1-a\alpha/(1+\alpha)}\right) \quad \text{and} \quad |J| = O\left(\left(n - \sum_{j \in J} jS(j)\right)^{1-\alpha}\right).$$

In Section 6.1.3, we include error terms to remain consistent with the examples used by Gourdon [24].

6.1.3 Examples

We illustrate our results with combinatorial examples from [20]. Throughout this section G_1 , $G_{1/2}$, H_1 , and $H_{1/2}$ are the constants defined in Table 5.

Example 2 (Largest cycles in permutations). Consider the class of permutations, which are of exp-log type with $\alpha = a = 1$, and let

$$S = \left\{ \left(\lfloor (0.9n)^{1/2} \rfloor, \left\lfloor \frac{0.1}{0.9^{1/2}} n^{1/2} \right\rfloor \right) \right\}.$$

That is, the pattern S consists of $\lfloor \frac{0.1}{0.9^{1/2}} n^{1/2} \rfloor$ elements of size $\lfloor (0.9n)^{1/2} \rfloor$. Then $\hat{j} = j^* = \lfloor \frac{0.1}{0.9^{1/2}} n^{1/2} \rfloor$, and

$$\eta' = \sum_{j \in J} jS(j) = \left\lfloor \frac{0.1}{0.9^{1/2}} n^{1/2} \right\rfloor \times \lfloor (0.9n)^{1/2} \rfloor \approx 0.1n. \quad (128)$$

Note that the quantity in (128) is not necessarily equal to $0.1n$ because of the floor functions. Then $\eta = n - \eta' \approx 0.9n$, and the pattern S satisfies the hypotheses of Theorem 12. Therefore Corollary 7 implies that, for $1 \leq m < \hat{j}$, $\Pr(L_n(S) \leq m) = 0$, and, for $\hat{j} \leq m \leq \eta$

$$\Pr(L_n(S) \leq m) = (f_1(m/\eta) + O(m^{-1/2}))(1 + O(n^{-1/2})).$$

Corollary 8 implies that the mean and variance of the variable $L_n(S)$ satisfy

$$\mathbb{E}(L_n(S)) = G_1\eta + O(\eta^{1/2}) \approx 0.56189691n + O(n^{1/2}),$$

and

$$\text{Var}(L_n(S)) = H_1\eta^2 + O(\eta^{3/2}) \approx 0.029891187n^2 + O(n^{3/2}),$$

respectively.

Conversely, we apply Corollary 8 to see that

$$\mathbb{E}(L_n(\emptyset)) = G_1 n + O(n^{1/2}) \approx 0.6243299855n + O(n^{1/2}),$$

and

$$\text{Var}(L_n(\emptyset)) = H_1 n^2 + O(n^{3/2}) \approx 0.03690783n^2 + O(n^{3/2}),$$

which means that the restricted pattern S has a definite impact on the expected size and variance of the largest cycle in a random permutation, even if the elements in that restricted pattern are themselves asymptotically small compared to the size of the permutation.

Example 3 (Largest centred cycle in a children's round). Stanley [37] introduced a class of combinatorial objects that he called “children's rounds”, which he used to illustrate the use of symbolic methods in analytic combinatorics. A round is a labeled set of directed cycles, where each cycle has a centre attached. The EGF for children's rounds is

$$F(z) = (1 - z)^{-z} = e^{C(z)}, \quad C(z) = \log \frac{1}{1 - z} - (1 - z) \log \frac{1}{1 - z}.$$

Hence, we may apply Theorem 12 with $a = 1$ and $\alpha = 1 - \epsilon$ for any $0 < \epsilon < 1$. Let $S = \{(3, \lfloor n/20 \rfloor), (\lceil \log n \rceil, 5)\}$. Then $\hat{j} = j^* = \lceil \log n \rceil$,

$$\eta' = \sum_{j \in J} j S(j) = 3 \left\lfloor \frac{n}{20} \right\rfloor + 5 \lceil \log n \rceil,$$

and $\eta \approx \frac{7n}{20} + O(\log n)$. Once again, the floor and ceiling functions keep us from saying η is exactly equal to $\frac{7n}{20} + O(\log n)$. The pattern S satisfies the hypotheses of Theorem 12. Therefore Corollary 7 implies that, for $1 \leq m < \hat{j}$, $\Pr(L_n(S) \leq m) = 0$,

and, for $\hat{j} \leq m \leq \eta$,

$$\Pr(L_n(S) \leq m) = f_1\left(\frac{m}{\eta}\right) + O\left(m^{-(1-\epsilon)/2}\right).$$

By Corollary 8, the mean and variance of $L_n(S)$ satisfy

$$\mathbb{E}(L_n(S)) = G_1\eta + O(\eta^{(1-\epsilon)/2}) \approx \frac{17}{20}G_1n + O(n^{(1-\epsilon)/2}),$$

and

$$\text{Var}(L_n(S)) = H_1\eta^2 + O(\eta^{3/2-\epsilon/2}) \approx \frac{289}{400}H_1n^2 + O(n^{3/2-\epsilon/2}),$$

respectively. Once again, these quantities differ from $\mathbb{E}(L_n(\emptyset))$ and $\text{Var}(L_n(\emptyset))$, which illustrates the influence the pattern has on the size of the largest component.

Example 4 (Largest connected components in 2-regular graphs). A 2-regular graph of size n is a graph with n edges such that each vertex has degree 2. A 2-regular graph is therefore a collection of non-directed cycles of length at least 3, and each 2-regular graph can be decomposed into these cycles. The EGF of these cycles is

$$C(z) = \frac{1}{2} \left(\log \frac{1}{1-z} - z - \frac{z^2}{2} \right),$$

and the EGF for 2-regular graphs is $G(z) = e^{C(z)}$. Therefore, 2-regular graphs are of exp-log type with $a = 1/2$ and $\alpha = 1$.

Let $S = \{(10, \lceil \log n \rceil), (\lfloor \sqrt{n} \rfloor, 0)\}$. Then $j^* = 10$, $\hat{j} = \lfloor \sqrt{n} \rfloor$,

$$\eta' = \sum_{j \in J} jS(j) = 10\lceil \log n \rceil,$$

and $\eta = n - \eta' = n + O(\log n) \sim n$. Corollary 7 implies that

$$\Pr(L_n(S) \leq m) = \begin{cases} 0 & \text{if } m \leq 10, \\ \exp(r_m(\rho; J)) f_{1/2}(m/\eta) + O(m^{-1/4}) & \text{if } 10 \leq m < \lfloor \sqrt{n} \rfloor, \\ f_{1/2}(m/\eta) + O(m^{-1/4}) & \text{if } \lfloor \sqrt{n} \rfloor \leq m \leq \eta, \\ 1 & \text{if } m > \eta. \end{cases}$$

By Corollary 8, the mean and variance of $L_n(S)$ satisfy

$$\mathbb{E}(L_n(S)) = G_{1/2}\eta + O(\eta^{3/4}) \sim G_{1/2}n \approx 0.7578230112n,$$

and

$$\text{Var}(L_n(S)) = H_{1/2}\eta^2 + O(\eta^{7/4}) \sim H_{1/2}n^2 \approx 0.0370072168n^2,$$

respectively. Note that these quantities do *not* differ from those for $L_n(\emptyset)$. In this case, the pattern S is not large enough to influence the expected size of the largest disjoint cycle. This is in spite of the fact that $\eta' \rightarrow \infty$ as $n \rightarrow \infty$.

Example 5 (Largest connected components in functional graphs). This example is taken from Harary and Palmer [26]. Consider a function ϕ which maps $N = \{1, 2, \dots, n\}$ to itself. This function can be represented by a directed graph G_ϕ which is called the *functional graph* where the vertices are labeled by the integers in N and the edges are the ordered pairs $(i, f(i))$ for $i \in N$. However, for simplicity, if $f(i) = i$, we do not draw the corresponding loop in the graph. Since functions are well-defined, this graph is characterized by the property that each vertex is the head of exactly one edge.

As illustrated in Figure 10, a functional graph may be decomposed into its connected components, each of which is itself a cycle of Cayley trees. Cayley trees are labeled rooted trees. The EGF $C(z)$ of cycles of Cayley trees is expressed as a function of the EGF $T(z)$ of Cayley trees in the form

$$C(z) = \log \frac{1}{1 - T(z)}.$$

Figure 10: A functional graph representing a function from the set $\{1, 2, \dots, 26\}$ to itself. It is composed of four connected components, each of which is a cycle of Cayley trees.

It is known that $T(z) = ze^{T(z)}$ satisfies the expression

$$T(z) = 1 - \sqrt{2(1 - ez)} - \frac{2}{3}(1 - ez) + O((1 - ez)^{3/2}),$$

as $z \rightarrow 1/e$ in the closed complex half-line $[1/e, +\infty)$. Therefore,

$$C(z) = \frac{1}{2} \log \frac{1}{1 - ez} + O(\sqrt{1 - ez}),$$

which implies that $C(z)$ is of logarithmic type with $a = \alpha = 1/2$.

Let $S = \{(\lfloor n^{5/6}/3 \rfloor, 0), (\lceil n^{5/6}/2 \rceil, \lceil n^{1/6} \rceil)\}$. Then $j^* = \hat{j} = \lceil n^{5/6}/2 \rceil$,

$$\eta' = \sum_{j \in J} jS(j) = \lfloor n^{5/6}/2 \rfloor \times \lceil n^{1/6} \rceil \approx n/2,$$

and $\eta = n - \eta' = n - \lfloor n^{5/6}/2 \rfloor \times \lceil n^{1/6} \rceil \approx n/2$.

By Corollary 7,

$$\Pr(L_n(S) \leq m) = \begin{cases} 0 & \text{if } m < \lceil n^{5/6}/2 \rceil, \\ f_{1/2}(m/\eta) + O(m^{-1/6}) & \text{if } \lceil n^{5/6}/2 \rceil \leq m \leq \eta, \\ 1 & \text{if } m > \eta. \end{cases}$$

By Corollary 8, the mean and variance of $L_n(S)$ satisfy

$$\mathbb{E}(L_n(S)) = G_{1/2}\eta + O(\eta^{5/6}) \sim \frac{G_{1/2}}{2}n \approx 0.3789115056n,$$

and

$$\text{Var}(L_n(S)) = H_{1/2}\eta^2 + O(\eta^{11/6}) \sim \frac{H_{1/2}}{4}n^2 \approx 0.0092518042n^2.$$

6.2 The r th Largest Component

In this section we consider a further generalization of the results from the previous section. We again consider exp-log structures, but we now consider the random

variable $L_n^{[r]}(S)$ which counts the size of the r th largest component in a structure of size n with restricted pattern S . If there are fewer than r components, then $L_n^{[r]}(S) = 0$. Once again, the results of this section generalize those of Gourdon [24].

For a given $m \geq 1$ and restricted pattern $S : J \mapsto \mathbb{N}$, we define the following quantities:

$$\begin{aligned} J_m^{(1)} &= \{j : j \in J, j \leq m\}, & J_m^{(2)} &= \{j : j \in J, j > m\}, \\ \kappa_1 &= \sum_{j \in J_m^{(1)}} S(j), & \eta'_1 &= \sum_{j \in J_m^{(1)}} jS(j), \\ \kappa_2 &= \sum_{j \in J_m^{(2)}} S(j), & \eta'_2 &= \sum_{j \in J_m^{(2)}} jS(j), \\ \kappa &= \kappa_1 + \kappa_2, & \eta' &= \eta'_1 + \eta'_2. \end{aligned}$$

In plain English, κ_1 is the number of components in the pattern S of size no greater than m , κ_2 is the number of components in S of size greater than m , and κ is the total number of components in S .

Lemma 10. *The exponential generating function for the number of exp-log structures of size n for whom $L_n^{[r]}(S) \leq m$ is given by*

$$F_m^{[r]}(z; S) = \begin{cases} \prod_{j \in J} \left(\frac{C_j}{j!} \right)^{S(j)} \frac{z^{\eta'}}{S(j)!} \exp \left(s_m(z) - \sum_{j \in J_m^{(1)}} \frac{C_j z^j}{j!} \right) \\ \quad \times \left[\sum_{i=\kappa_2+1}^r \left(\frac{\left(r_m(z) - \sum_{j \in J_m^{(2)}} \frac{C_j z^j}{j!} \right)^{i-1-\kappa_2}}{(i-1-\kappa_2)!} \right) \right] & \text{if } \kappa_2 < r, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Given that $L_n^{[r]}(S) \leq L_n^{[r-1]}(S) \leq \dots \leq L_n^{[1]}(S)$, an exp-log structure of size n

satisfies $L_n^{[r]}(S) \leq m$ if and only if exactly one of the conditions C_i is satisfied, where

$$\begin{aligned} C_1 : & \quad L_n^{[1]}(S) \leq m, \\ C_2 : & \quad L_n^{[2]}(S) \leq m \text{ and } L_n^{[1]}(S) > m, \\ & \quad \vdots \\ C_r : & \quad L_n^{[r]}(S) \leq m \text{ and } L_n^{[r-1]} > m. \end{aligned}$$

We consider three cases.

Case 1: $j \leq m$ for all $j \in J$. In this case, the exponential generating function for objects that satisfy C_i is given by

$$F_m(z; S) \cdot \frac{r_m(z)^{i-1}}{(i-1)!},$$

and we obtain $F_m^{[r]}(z; S)$ by summing over $1 \leq i \leq r$, *i.e.*,

$$F_m^{[r]}(z; S) = F_m(z; S) \left(1 + r_m(z) + \cdots + \frac{r_m(z)^{r-1}}{(r-1)!} \right). \quad (129)$$

Case 2: $j > m$ for all $j \in J$. In this case the exponential generating function for objects that satisfy C_i is given by

$$\begin{cases} \prod_{j \in J} \left(\frac{C_j}{j!} \right)^{s(j)} \frac{z^{\eta'_2}}{S(j)!} \exp(s_m(z)) \cdot \left(\frac{\left(r_m(z) - \sum_{j \in J_m^{(2)}} \frac{C_j z^j}{j!} \right)^{i-1-\kappa_2}}{(i-1-\kappa_2)!} \right) & \text{if } \kappa_2 < i, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the largest i for which C_i can be satisfied is $i = r$ and the smallest i for which C_i can be satisfied is $i = \kappa_2 + 1$. If $\kappa_2 \geq r$, then no C_i can possibly be satisfied because any such object automatically has at least r components of size

greater than m . Therefore,

$$F_m^{[r]}(z; S) = \begin{cases} \exp(s_m(z)) \cdot \prod_{j \in J} \left(\frac{C_j}{j!} \right)^{S(j)} \frac{z^{\eta'_2}}{S(j)!} \\ \quad \times \left[\sum_{i=\kappa_2+1}^r \left(\frac{\left(r_m(z) - \sum_{j \in J_m^{(2)}} \frac{C_j z^j}{j!} \right)^{i-1-\kappa_2}}{(i-1-\kappa_2)!} \right) \right] & \text{if } \kappa_2 < r, \\ 0 & \text{otherwise.} \end{cases} \quad (130)$$

Case 3: $J_m^{(1)} \neq \emptyset$ and $J_m^{(2)} \neq \emptyset$, i.e., some of the components in S have size no greater than m , and some have size greater than m . In order to satisfy the condition C_i , there must be fewer than i components in the pattern whose size is greater than m , i.e., $\kappa_2 < i$. If $\kappa_2 < i$, then an object satisfying C_i can have any number of components of size no greater than m not in the restricted pattern, and it must have exactly $i-1-\kappa_2$ components of size greater than m not in the pattern. In this case the exponential generating function for objects that satisfy C_i is given by

$$\begin{cases} \prod_{j \in J} \left(\frac{C_j}{j!} \right)^{S(j)} \frac{z^{\eta'_2}}{S(j)!} \exp \left(s_m(z) - \sum_{j \in J_m^{(1)}} \frac{C_j z^j}{j!} \right) \left(\frac{\left(r_m(z) - \sum_{j \in J_m^{(2)}} \frac{C_j z^j}{j!} \right)^{i-1-\kappa_2}}{(i-1-\kappa_2)!} \right) & \text{if } \kappa_2 < i, \\ 0 & \text{otherwise.} \end{cases}$$

As in Case 2, the largest i for which C_i can be satisfied is $i = r$ and the smallest i for which C_i can be satisfied is $i = \kappa_2 + 1$. If $\kappa_2 \geq r$, then no C_i can possibly be satisfied because any such object automatically has at least r components of size

greater than m . Therefore,

$$F_m^{[r]}(z; S) = \begin{cases} \prod_{j \in J} \left(\frac{C_j}{j!} \right)^{S(j)} \frac{z^{\eta'}}{S(j)!} \exp \left(s_m(z) - \sum_{j \in J_m^{(1)}} \frac{C_j z^j}{j!} \right) \\ \quad \times \left[\sum_{i=\kappa_2+1}^r \left(\frac{\left(r_m(z) - \sum_{j \in J_m^{(2)}} \frac{C_j z^j}{j!} \right)^{i-1-\kappa_2}}{(i-1-\kappa_2)!} \right) \right] & \text{if } \kappa_2 < r, \\ 0 & \text{otherwise.} \end{cases} \quad (131)$$

Notice that each of the above three cases is actually covered by the third case. In all three cases, $F_m^{[r]}(z; S) = 0$ unless $\kappa_2 < r$. If, as in Case I, $j \leq m$ for all $j \in J$, then $\eta' = \kappa_1$ and $\kappa_2 = 0$, and (129) is equivalent to (131). Similarly, if $j > m$ for all $j \in J$, then $\eta' = \kappa_2$, and (130) is also equivalent to (131). $\square$

Ideally we would now prove a theorem analogous to Theorem 12. Unfortunately, it is not as simple to adapt Gourdon's techniques to include patterns in the case of the r th largest component as it is in the case of the largest component, and we have as yet been unable to make further progress.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

First, we improved existing results for structures with alg-log components with restricted patterns by extending the range of α considered from $0 < \alpha < 1$ to all $\alpha > 0$ and by reducing the required conditions on the patterns. These results are an extension of work by Dong *et al.*

We then used the saddle point method to determine asymptotic results for the generating functions of structures with alg-log components with negative algebraic exponent and non-zero logarithmic exponent.

Next, we considered the generating functions structures whose components only have a logarithmic term, *i.e.*, their algebraic exponent is zero. The analysis of these functions was technical, and it depended on the range of β . For $\beta \geq 2$, we used the saddle point method and included restricted patterns. For the range $0 < \beta < 1$, the generating functions in question did not correspond to combinatorial structures, so we did not consider restricted patterns. This case required a modified version of singularity analysis. We were unable to determine a solution for the range $1 < \beta < 2$, as we were unable to find an asymptotic expression for the term a_j in Lemma 5 that is valid as both $j \rightarrow \infty$ and $n \rightarrow \infty$. Although the range $0 < \beta < 1$ is not combinatorial

in nature, it does fill a gap in the literature from the point of view of asymptotics of coefficients of power series.

Finally, we extended the results of Gourdon [24] to determine the probability distribution of the largest component of an exp-log structure with a restricted pattern, and we computed several examples. We also determined the exponential generating function for the number of exp-log structures of size n whose r th largest component has size no greater than m , where m is a given integer.

7.2 Future Work

For decomposable structures with alg-log components, enumeration asymptotics for all ranges of α and β have been determined except for the case $\alpha = 0$, $1 < \beta < 2$. If an appropriate asymptotic expression for the term a_j in Lemma 5 can be found, singularity analysis will apply. Failing that, the method of steepest descents may prove useful.

It is desirable to determine asymptotic information about the probability distribution of the r th smallest component of the structures examined in Chapter 4 and Section 5.1. The methods used by Dong *et al.* will likely not be directly applicable, so a new method will be necessary. Furthermore, it may be possible to modify current techniques to relax the conditions required of the restricted patterns in any of the structures examined in this thesis.

Finally, we would like to continue the work of Chapter 6 and extend Gourdon's [24, 25] methods the probability distribution for the r th largest component of structures with both alg-log and exp-log components and restricted patterns.

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Appendix A

The Appearance of Stirling Numbers of the First Kind

In this appendix, we find a complete expansion for $\left(1 + \frac{t}{n}\right)^{-n-1}$ which involves Stirling numbers of the first kind. It is unsurprising that we encounter Stirling numbers here given that the function $\left(\frac{1}{1-z}\right)^u$ has the series expansion

$$\left(\frac{1}{1-z}\right)^u = \sum_{n=0}^{\infty} \sum_{k=0}^n u^k \frac{z^n}{n!} \begin{bmatrix} n \\ k \end{bmatrix}.$$

We begin by expanding

$$\begin{aligned} \left(1 + \frac{t}{n}\right)^{-n-1} &= e^{-t} e^t \left(1 + \frac{t}{n}\right)^{-n-1} \\ &= e^{-t} \left(\sum_{k \geq 0} \frac{t^k}{k!}\right) \left(\sum_{\ell \geq 0} \binom{n+\ell}{\ell} \left(\frac{-t}{n}\right)^\ell\right) \\ &= e^{-t} \sum_{k \geq 0} \frac{(-1)^k (-t)^k}{k!} \sum_{\ell \geq 0} \frac{(n+\ell)!}{n! \ell!} \frac{(-t)^\ell}{n^\ell} \\ &= e^{-t} \sum_{j \geq 0} a_j (-t)^j, \end{aligned}$$

where

$$\begin{aligned}
 a_j &= [(-t)^j] \sum_{k \geq 0} \frac{(-1)^k (-t)^k}{k!} \sum_{\ell \geq 0} \frac{(n+\ell)!}{n! \ell!} \frac{(-t)^\ell}{n^\ell} \\
 &= \frac{1}{n^j} \sum_{k=0}^j \frac{(-1)^k}{k!} \binom{n+j-k}{j-k} n^k.
 \end{aligned} \tag{132}$$

We next rearrange the summand in (132) to find an alternative expression for a_j .

$$\begin{aligned}
 n^{-j} \frac{(-1)^k}{k!} \binom{n+j-k}{j-k} n^k &= (-1)^k \frac{(n+j-k)!}{k! n! (j-k)!} n^{k-j} \\
 &= \frac{(-1)^k}{j!} \binom{j}{k} \frac{(n+j-k)!}{n!} n^{k-j} \\
 &= \frac{(-1)^k}{j!} \binom{j}{k} (n+j-k) \cdot (n+j-k-1) \cdots (n+1) \cdot (n) \cdot n^{k-j-1} \\
 &= \frac{(-1)^k}{j!} \binom{j}{k} \sum_{\ell=0}^{j-k+1} \left[\begin{matrix} j-k+1 \\ \ell \end{matrix} \right] n^\ell n^{k-j-1},
 \end{aligned}$$

where $\left[\begin{matrix} n \\ k \end{matrix} \right]$ is the unsigned Stirling number of the first kind. The last line follows from an alternative definition of the binomial coefficient given in [1].

Therefore,

$$\begin{aligned}
 a_j &= \frac{1}{j!} \sum_{k=0}^j (-1)^k \binom{j}{k} \sum_{\ell=0}^{j-k+1} \left[\begin{matrix} j-k+1 \\ \ell \end{matrix} \right] n^\ell n^{k-j-1} \\
 &= \frac{1}{n^j j!} \sum_{k=0}^j (-1)^k \binom{j}{k} n^k \sum_{\ell=0}^{j-k+1} \left[\begin{matrix} j-k+1 \\ \ell \end{matrix} \right] n^{\ell-1}.
 \end{aligned}$$

This means that $n^j a_j$ is a polynomial in n whose degree is no more than j . We can also find a closed form expression for the coefficient of n^ζ in $n^j a_j$:

$$[n^\zeta] n^j a_j = \frac{1}{j!} \sum_{b=0}^{\zeta} (-1)^b \binom{j}{b} \left[\begin{matrix} j-b+1 \\ \zeta-b+1 \end{matrix} \right]. \tag{133}$$

At first it appears that the $n^j a_j$ actually has degree j , but this is not the case.

Letting $\zeta = j$, (133) shows

$$\begin{aligned} [n^j]n^j a_j &= \frac{1}{j!} \sum_{b=0}^j (-1)^b \binom{j}{b} \begin{bmatrix} j-b+1 \\ j-b+1 \end{bmatrix} \\ &= \frac{1}{j!} \sum_{b=0}^j (-1)^b \binom{j}{b} \\ &= \frac{1}{j!} (1-1)^j = 0, \end{aligned}$$

so $n^j a_j$ must have degree no greater than $j-1$. In fact, $[n^\zeta]n^j a_j$ is identically zero for all $\zeta \geq j/2$, which we proved in Section 5.3. We have yet to discover any reason why this is so.

Appendix B

Applications of Theorem 11 and Conjecture 1

In this appendix we give several tables that calculate estimates of $f_n = [z^n]F(z)$ in the case $C(z) = d \left(\log \frac{1}{1-z}\right)^\beta$ for various values of β and d . Both the exact and the estimated values were calculated in MAPLE, where the values for the gamma function and its derivatives were found numerically.

In each table, the given m refers to how many terms from either Theorem 11 or Conjecture 1 were used. A high degree of accuracy can be achieved with relatively few terms and small n . For Table 6, we apply Theorem 11. For the remaining tables, Conjecture 1 applies.

n	f_n (estimate)	f_n (actual)	% difference
10	1.552×10^{-1}	1.425×10^{-1}	9.0%
10^2	2.116×10^{-2}	2.089×10^{-2}	1.3%
10^3	2.795×10^{-3}	2.791×10^{-3}	0.15%
10^4	3.624×10^{-4}	3.624×10^{-4}	0.02%

Table 6: Estimates of f_n with $\beta = 0.5$, $d = 1$, $m = 4$

n	f_n (estimate)	f_n (actual)	% difference
10	0.3319506	0.3411823	2.7%
10^2	0.2677278	0.2699121	0.81%
10^3	0.2660636	0.2664633	0.15%
10^4	3.132×10^{-1}	3.132×10^{-1}	0.02%

Table 7: Estimates of f_n with $\beta = 1.25$, $d = 0.5$, $m = 5$

n	f_n ($m = 10$)	f_n ($m = 20$)	f_n (actual)	% difference from $m = 20$
10	1.487×10^{-2}	1.485×10^{-2}	1.338×10^{-2}	11%
10^2	1.979×10^{-3}	1.979×10^{-3}	1.953×10^{-3}	1.3%
10^3	2.626×10^{-4}	2.626×10^{-4}	2.622×10^{-4}	0.2%
10^4	3.484×10^{-5}	3.484×10^{-5}	3.483×10^{-5}	0.02%

Table 8: Estimates of f_n with $\beta = 1.05$, $d = 0.1$, $m = 10, 20$

n	$f_n(m = 10)$	$f_n(m = 20)$	$f_n(m = 30)$	f_n (actual)	% difference from $m = 30$
10	1.776×10^0	1.778×10^0	1.772×10^0	2.555×10^0	31%
10^2	8.746×10^0	8.746×10^0	8.746×10^0	9.562×10^0	8.5%
10^3	7.091×10^1	7.091×10^1	7.091×10^1	7.208×10^1	1.6%
10^4	8.454×10^2	8.454×10^2	8.454×10^2	8.475×10^2	0.3%

Table 9: Estimates of f_n with $\beta = 1.25$, $d = 1$, $m = 10, 20, 30$

n	$f_n(m = 10)$	$f_n(m = 20)$	$f_n(m = 30)$	f_n (actual)	% difference from $m = 30$
10	3.943×10^0	3.789×10^0	3.773×10^0	2.110×10^1	82%
10^2	5.744×10^3	5.610×10^3	5.608×10^3	1.512×10^4	63%
10^3	6.162×10^9	6.991×10^9	6.991×10^9	1.010×10^{10}	39%
10^4	9.486×10^{18}	1.500×10^{19}	1.502×10^{19}	1.656×10^{19}	8.7%

Table 10: Estimates of f_n with $\beta = 1.95$, $d = 1$, $m = 10, 20, 30$

n	$f_n(m = 10)$	$f_n(m = 20)$	$f_n(m = 30)$	f_n (actual)	% diff. from $m = 30$
10	7.736×10^{-2}	7.735×10^{-2}	7.730×10^{-2}	7.773×10^{-2}	0.6%
10^2	5.688×10^{-2}	5.688×10^{-2}	5.688×10^{-2}	5.764×10^{-2}	1.3%
10^3	7.597×10^{-2}	7.597×10^{-2}	7.597×10^{-2}	7.633×10^{-2}	0.5%
10^4	2.068×10^{-1}	2.068×10^{-1}	2.068×10^{-1}	2.070×10^{-1}	0.1%

Table 11: Estimates of f_n with $\beta = 1.95$, $d = 0.1$, $m = 10, 20, 30$

n	$f_n(m = 10)$	$f_n(m = 20)$	f_n (actual)	% difference from $m = 20$
10	5.373×10^1	5.377×10^1	6.111×10^2	91%
10^2	1.693×10^5	1.693×10^5	3.623×10^5	53%
10^3	3.854×10^9	3.854×10^9	4.519×10^9	15%
10^4	3.462×10^{14}	3.462×10^{14}	3.555×10^{14}	2.6%

Table 12: Estimates of f_n with $\beta = 1.25$, $d = 3$, $m = 10, 20$

n	f_n (estimated)	f_n (actual)	% difference from $m = 20$
10	2.556×10^1	7.623×10^1	66%
10^2	1.453×10^3	1.839×10^3	21%
10^3	1.009×10^5	1.048×10^5	3.7%
10^4	7.959×10^6	7.999×10^6	0.5%

Table 13: Estimates of f_n with $\beta = 1.05$, $d = 2.5$, $m = 20$