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CONFIDENCE BANDS FOR QUANTILE FUNCTION AND PERCENTILE
RESIDUAL LIFETIME UNDER RANDOM CENSORSHIP

by

CHANG-JO F. CHUNG, B.SC., M.SC.

A thesis submitted to
the Faculty of Graduate Studies and Research

in partial fulfilment of
the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Ontario

May, 1986

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ABSTRACT

Under the random censorship model from the right, we construct confidence bands for the quantile function without estimating the density function, by using a strong approximation theorem which says that the normed PL-quantile process ρ_n can be almost surely approximated by Wiener processes, and by studying the distribution functions of the random variables $\{W(t) \leq ct+d : a \leq t \leq b\}$ and $\{|W(t)| \leq ct+d : a \leq t \leq b\}$, where $\{W(t) : t \geq 0\}$ is a Wiener process.

In order to construct confidence bands for the $(1-p)$ percentile residual lifetime $R^{(p)}(t)$, we first prove that the scaled PL $(1-p)$ percentile residual lifetime process $r_n^{(p)}(t)$ can be almost surely approximated by appropriate Gaussian processes. However, the approximating Gaussian processes for $r_n^{(p)}(t)$ depend on the unknown underlying distribution and quantile functions. This leads us to bootstrapping considerations. We define the bootstrapped PL $(1-p)$ percentile residual lifetime process and discuss approximations of this process by appropriate Gaussian processes. The latter enables us to construct bootstrapped confidence bands for $R^{(p)}(t)$.

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FOREWORD

Let $X_1^0, \dots, X_n^0$ be independent identically distributed(i.i.d.) random variables(rv) with continuous distribution function $F^0(x) = P\{X_1^0 \leq x\}$. The corresponding quantile function Q of F^0 is defined by

$$(1) \quad Q(y) := \inf\{t : F^0(t) \geq y\}, \quad 0 < y < 1.$$

In the random censorship model from the right, the X_i^0 may be censored on the right by the Y_i where $\{Y_i\}$ are i.i.d. rv with distribution function H and the two sequences $\{X_i^0\}$ and $\{Y_i\}$ are independent, so that one observes only the pairs (X_i, δ_i) , $i=1,2,\dots,n$, where X_i is the minimum of X_i^0 and Y_i , and δ_i is the indicator function of the event $X_i^0 \leq Y_i$.

The $(1-p)$ percentile residual lifetime $R^{(p)}(t)$ is defined by

$$(2) \quad R^{(p)}(t) := Q(1-p(1-F^0(t))) - t, \quad t > 0 \text{ and } 0 < p < 1.$$

To estimate $Q(y)$ in (1) under the random censorship model from the right, Aly, M. Csörgö and Horváth(1985) proposed the PL-quantile function $Q_n(y)$ defined by

$$(3) \quad Q_n(y) := \inf\{t : F_n^0(t) \geq y\}, \quad 0 < y < 1,$$

where F_n^0 is the product-limit(PL) estimator of F^0 (cf. Kaplan and Meier (1958)) and defined by

$$(4) \quad 1 - F_n^0(t) = \begin{cases} \prod_{1 \leq i \leq n: X_i < t} \left[\frac{n - R_i}{n - R_i + 1} \right]^{\delta_i} & \text{if } t \leq X_{n:n} \\ 0 & \text{if } t > X_{n:n} \end{cases}$$

with $X_{n:n} = \text{Max} (X_1, \dots, X_n)$ and R_i is the rank of $(X_i, 1 - \delta_i)$ in the lexicographic ordering of $\{ (X_i, 1 - \delta_i) \}$, $i=1, 2, \dots, n$. Further details and various preliminaries are summarized in Chapter II.

Aly, M. Csörgö and Horváth(1985) constructed a confidence band for the quantile function by proving a strong approximation theorem, saying that the PL-normed quantile process ρ_n defined as

$$(5) \quad \rho_n(y) := \sqrt{n} f^0(Q(y)) (Q(y) - Q_n(y)), 0 \leq y < 1,$$

can be almost surely approximated by Wiener processes.

In Chapter III we generalize the Aly, M. Csörgö and Horváth approach by introducing a general class of bands which includes the Aly, M. Csörgö and Horváth result. We will also construct confidence bands for the quantile function on a limited range $[a, b]$ under weaker conditions than those of the Aly, M. Csörgö and Horváth theorem.

However, in order to construct the latter confidence bands on a limited range, we must have the distribution functions of the random variables $\{ W(t) \leq ct+d : a \leq t \leq b \}$ and $\{ |W(t)| \leq ct+d : a \leq t \leq b \}$, where $\{ W(t) : t \geq 0 \}$ is a Wiener process. These are discussed in Chapter I, where we also generalize Csáki's formula of $P\{ W(t) \leq ct+d : 0 \leq t \leq b \}$ and Anderson's formula of $P\{ |W(t)| \leq ct+d : 0 \leq t \leq b \}$ to provide formulae of the distributions when the range of t is $[a,b]$ rather than $[0,b]$.

To estimate $R^{(p)}(t)$, in Chapter V we propose the PL $(1-p)$ percentile empirical residual lifetime

$$(6) \quad R_n^{(p)}(t) := Q_n(1-p(1-F_n^0(t))) - t, \quad t > 0, \quad 0 < p < 1,$$

and prove a strong approximation theorem which says that the scaled PL $(1-p)$ percentile residual lifetime process

$$(7) \quad r_n^{(p)}(t) := \sqrt{n} f^0(Q(1-p(1-F^0(t)))) [R_n^{(p)}(t) - R^{(p)}(t)],$$

can be approximated by appropriate Gaussian processes. Using such a strong approximation theorem, we also construct confidence bands for $R^{(p)}(t)$.

However, the approximating Gaussian processes for $r_n^{(p)}(t)$ also depend on the unknown underlying distribution and quantile functions. Thus the constructed confidence bands for $R^{(p)}(t)$ cannot be immediately utilized in practice, because we do not have analytic formula for the

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distribution function of the desired functional, and even if it were available it would not be distribution free. This leads us to bootstrapping considerations. In Chapter V we also define the bootstrapped PL $(1-p)$ percentile residual lifetime process and discuss approximations of this process by appropriate Gaussian processes. The latter enables us to construct bootstrapped confidence bands for $R^{(p)}(t)$.

In summing up, we discuss formulae for $P\{ |W(t)| \leq ct+d : a \leq t \leq b \}$ and $P\{ W(t) \leq ct+d : a \leq t \leq b \}$, and for several related probabilities associated with Wiener and Brownian bridge processes in Chapter I.

In chapter II we outline the basic setting for the random censorship model from the right and quote known results which are required in our sequel.

In Chapter III we discuss and establish confidence bands for the quantile function under the random censorship model and apply the results to the construction of confidence bands for the quantile function of the length of fractures on the surface of a granitic pluton near Lac du Bonnet, Manitoba, where the Underground Research Laboratory is being built for the nuclear waste disposal program in Canada.

Using a theorem developed in Chapter III, we generalize the Kolmogrov-Smirnov confidence band for the distribution function so that

we can construct a modified confidence band for the distribution function on a limited range in Chapter IV. This modified band is useful when the data observed are truncated at either side or both sides. This result is applied to the distribution function of Manganese in stream waters in the north-eastern quarter of map NTS 1051 (Goodfellow, 1981), Nahanni area, Yukon, Canada.

In Chapter V we first show that the asymptotic distribution of $r_n^{(p)}(t)$ is normal for t and p fixed, and construct a confidence interval for $R^{(p)}(t)$ for t and p fixed using the normality of the limiting distribution function. Then we will view $r_n^{(p)}(t)$ as a stochastic process (i) in t for fixed p , (ii) in p for fixed t and (iii) in both t and p , and study strong approximations of these processes by appropriate Gaussian processes. Using these strong approximation theorems, we construct confidence bands for $R^{(p)}(t)$.

In the latter part of Chapter V, we discuss strong approximations of the bootstrapped PL $(1-p)$ percentile residual lifetime processes by appropriate Gaussian processes, and then we construct bootstrapped confidence bands for $R^{(p)}(t)$. Finally, we apply the methodology developed to the data set described in Chapter III.

I. FORMULAE FOR PROBABILITIES ASSOCIATED WITH WIENER AND BROWNIAN BRIDGE PROCESSES.

1. Introduction.

Weak and strong convergence of empirical, empirical quantile, product limit and quantile-quantile processes to Brownian bridge and Wiener process are quite well known in these days (cf., for example, Breslow and Crowley (1974), M. Csörgö and Révész (1978), S. Csörgö and Horváth (1985), Aly, M. Csörgö and Horváth (1985), Aly and Bleuer (1983), and Doksum and Yandell (1984) in particular for further applications).

When applying these methodologies to statistical problems, the computation of probabilities of functionals of interest of the approximating Gaussian processes plays a crucial role. These required computations are frequently complicated. Without these computations, however, the said recent statistical advancements using weak and strong convergence methodologies can not be utilized in practice.

In this chapter we summarize and generalize some of the available methods for computing probabilities associated with Wiener and Brownian bridge processes of the following forms:

- (I.1.1) $P\{ W(t) \leq \tau : a \leq t \leq b \}, a \geq 0, b < \infty, \tau > 0 ;$
- (I.1.2) $P\{ |W(t)| \leq \tau : a \leq t \leq b \}, a \geq 0, b < \infty, \tau > 0 ;$
- (I.1.3) $P\{ W(t) \leq \tau + \delta t : a \leq t \leq b \}, a \geq 0, b < \infty, \tau > 0, \tau + \delta b \geq 0 ;$
- (I.1.4) $P\{ |W(t)| \leq \tau + \delta t : a \leq t \leq b \}, a \geq 0, b < \infty, \tau > 0, \tau + \delta b \geq 0 ;$
- (I.1.5) $P\{ B(t) \leq \tau : a \leq t \leq b \}, a \geq 0, b \leq 1, \tau > 0 ;$
- (I.1.6) $P\{ |B(t)| \leq \tau : a \leq t \leq b \}, a \geq 0, b \leq 1, \tau > 0 ,$

where $W(\cdot)$ and $B(\cdot)$ are Wiener and Brownian bridge processes, respectively.

Some special cases of the probabilities of (I.1.5) and (I.1.6) have been tabulated by S. Csörgö and Horváth (1981), Barr and Davidson (1973), Koziol and Byar (1975), and Schumacher (1984) among many others. We have developed a computer program package called WIENER PACK (cf. Chung (1986)), to compute the probabilities of the above forms for any given values of the parameters a , b , τ and δ . Inversely, the package also enables one to compute any one of the parameters - a , b and τ in (I.1.1), (I.1.2), (I.1.5) and (I.1.6) for a given probability level and given values for the other two of the parameters a , b and τ . In the last section, twelve tables for the critical value τ of (I.1.6) are computed using WIENER PACK and are provided in Tables I.1 - I.12.

Obviously, (I.1.1) and (I.1.2) are special cases of (I.1.3) and (I.1.4), respectively. It can also be shown that (I.1.5) and (I.1.6) are again special cases of (I.1.3) and (I.1.4), respectively. Hence, we will concentrate our discussions on the formulae for (I.1.3) and (I.1.4) in

detail, and those for the other four (I.1.1), (I.1.2), (I.1.5) and (I.1.6) will be obtained as corollaries.

2. Preliminaries.

A stochastic process $\{ W(t) : 0 \leq t < \infty \}$ is called a Wiener process if :

- (i) $W(t) - W(s)$ is $N(0, t-s)$ for all $0 \leq s \leq t < \infty$ and $W(0)=0$ where $N(\mu, \sigma^2)$ stands for normal random variable with mean μ and variance σ^2 ;
- (ii) $W(t)$ is an independent increment process, that is $W(t_2)-W(t_1)$, $W(t_4)-W(t_3), \dots, W(t_{2i})-W(t_{2i-1})$ are independent random variables for all $0 \leq t_1 < t_2 < \dots < t_{2i-1} < t_{2i} < \infty$ ($i = 2, 3, \dots$);
- (iii) the sample path function $W(t)$ is continuous in t with probability 1.

Remark I.2.1. Note that (i) and (ii) imply that the covariance function of a Wiener process is

$$(I.2.1) \quad E(W(s)W(t)) = s \wedge t,$$

since $W(s \vee t) - W(s \wedge t)$ and $W(s \wedge t) - W(0) = W(s \wedge t)$ are independent random variables. Conversely, a continuous Gaussian process having the covariance function of (I.2.1) is a Wiener process, since the process with the covariance function in (I.2.1) must also satisfy properties (i) and (ii).

A stochastic process $\{ B(t) : 0 \leq t \leq 1 \}$ is called a Brownian bridge if:

(i) the joint distribution of $B(t_1), \dots, B(t_n)$; $0 < t_1 \leq \dots \leq t_n < 1$, is Gaussian with $E(B(t)) = 0$;

(ii) the covariance function is

$$(I.2.2) \quad E(B(t)B(s)) = s \wedge t - st;$$

(iii) the sample path function $B(t)$ is continuous in t with probability 1.

Remark I.2.2. Note that (ii) implies that $B(0) = B(1) = 0$ a.s. and also that, if $B(t) = W(t) - tW(1)$ ($0 \leq t \leq 1$) where $W(t)$ is a Wiener process, then $B(t)$ is a Brownian bridge.

We first state a few well known results which are required in our sequel.

Theorem I.A. ((4.1) of Doob (1949) and (2.5) of Chapter VI of Feller (1966)). Let $\{ W(t) : t \geq 0 \}$ be a Wiener process. Then

$$(I.2.3) \quad P\{ W(t) \geq c : 0 \leq t \leq b \} = 2 P\{ W(b) \geq c \}.$$

The equality of (I.2.3) is often referred to as the reflection principle and (I.2.3) is also the one-sided stable distribution function of index $\frac{1}{2}$ ((4.7) of Chapter II of Feller (1966)).

The next theorem states relationships between Wiener and Brownian bridge processes.

Theorem I.B. (Doob (1949), and (1.4.4) and (1.4.5) of M. Csörgö and Révész (1981)). Let $\{ W(t) : t \geq 0 \}$ be a Wiener process and define

$$(I.2.4) \quad B(s) := \begin{cases} (1-s) W(s/(1-s)) & \text{if } 0 \leq s < 1, \\ 0 & \text{if } s = 1. \end{cases}$$

Then $\{ B(s) : 0 \leq s \leq 1 \}$ is a Brownian bridge process.

Conversely, let $\{ B(s) : 0 \leq s \leq 1 \}$ be a Brownian bridge process and define

$$(I.2.5) \quad W(t) := (1+t) B(t/(1+t)), \quad t \geq 0.$$

Then $\{ W(t) : t \geq 0 \}$ is a Wiener process.

The transformations in (I.2.4) and (I.2.5) are referred to as Doob's transformation. Using these transformations, it is easy to observe that (I.1.5) and (I.1.6) are special cases of (I.1.3) and (I.1.4), respectively.

The next theorem states that the uniform empirical process $\{ \alpha_n(y) : 0 \leq y \leq 1 \}$, where $\alpha_n(y) = \sqrt{n} (y - U_n(y))$ and $U_n(y)$ is the uniform empirical distribution function on $[0,1]$, can be approximated by a sequence of Brownian bridge processes.

Theorem I.C. (Komlós, Major and Tusnády (1975), Theorem 4.4.1 of M. Csörgö and Révész (1981)). Given $\{ \alpha_n(y) : 0 \leq y \leq 1 \}$, there exists a sequence of Brownian bridge processes $\{ B_n(y) : 0 \leq y \leq 1 \}$ such that

$$(I.2.6) \quad \sup_{0 \leq y \leq 1} | \alpha_n(y) - B_n(y) | \stackrel{\text{a.s.}}{=} O(n^{-1/2} \log n).$$

Remark I.2.3. Using Theorem I.C, we can state that the distributional properties of $\{ \alpha_n(y) : 0 \leq y \leq 1 \}$ coincide with those of $\{ B(y) : 0 \leq y \leq 1 \}$ as $n \rightarrow \infty$. Doob (1949) is the first one who suggested that in order to study asymptotic distributional properties of $\alpha_n(y)$ as $n \rightarrow \infty$, we may simply replace $\alpha_n(\cdot)$ by $B(\cdot)$. Later Donsker (1952) justified Doob's approach. For a more detailed exposition of weak convergence we refer to Billingsley (1968).

Now, we quote a result on the limiting distribution of a functional of the empirical process by Csáki (1981), which will be used to study the distribution of the corresponding functionals of Brownian bridge process in the light of (I.2.6).

Theorem I.D. (Theorem 2.1 of Csáki (1981)). Let $0 < a < 1$, $v > 0$ and $v - ua \geq 0$, and let $U_n(\cdot)$ be the uniform empirical distribution function on $[0,1]$. Then

$$(I.2.7) \quad \lim_{n \rightarrow \infty} P\{ U_n(x) \geq (1 + u/\sqrt{n})x - v/\sqrt{n} : 0 \leq x \leq a \}$$

$$= \Phi \left[\frac{v - ua}{\sqrt{a(1-a)}} \right] - e^{-2v(v-u)} \Phi \left[\frac{2av - v - ua}{\sqrt{a(1-a)}} \right],$$

where

$$(I.2.8) \quad \Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{1}{2}h^2} dh,$$

is the standard normal distribution function.

The following theorem is due to Anderson (1960) and it is one of the most frequently quoted formulae when computing probabilities of functionals of Wiener process.

Theorem I.B. (Theorem 4.3 of Anderson (1960)). Let $\tau_1 > 0$, $\tau_2 < 0$, $b > 0$, $\tau_2 + \delta_2 b \leq \tau_1 + \delta_1 b$. Then we have

$$(I.2.9) \quad P\{ \tau_2 + \delta_2 t \leq W(t) \leq \tau_1 + \delta_1 t : 0 \leq t \leq b \}$$

$$= 1 - \Phi(w/\sqrt{b}) - \Phi(-v/\sqrt{b})$$

$$+ \sum_{k=1}^{\infty} \left[e^{s_1} [\Phi(q_1) - \Phi(p_1)] + e^{s_2} [\Phi(p_2) - \Phi(q_2)] \right. \\ \left. - e^{s_3} [\Phi(q_3) - \Phi(p_3)] - e^{s_4} [\Phi(p_4) - \Phi(q_4)] \right],$$

where

$$\begin{aligned}
\tau' &= \tau_1 - \tau_2, \quad \delta' = \delta_1 - \delta_2, \quad v = \tau_1 + \delta_1 b, \quad w = \tau_2 + \delta_2 b, \\
s_1 &= -2 [k^2 \tau' \delta' + k(\tau_1 \delta_2 - \tau_2 \delta_1)], \\
s_2 &= -2 [k^2 \tau' \delta' - k(\tau_1 \delta_2 - \tau_2 \delta_1)], \\
s_3 &= -2 [k^2 \tau' \delta' + k(\tau' \delta_2 + \delta' \tau_2) + \tau_2 \delta_2], \\
s_4 &= -2 [k^2 \tau' \delta' - k(\tau' \delta_1 + \delta' \tau_1) + \tau_1 \delta_1], \\
p_1 &= (-2k\tau' + w) / \sqrt{b}, \quad q_1 = (-2k\tau' + v) / \sqrt{b}, \\
p_2 &= (-2k\tau' - w) / \sqrt{b}, \quad q_2 = (-2k\tau' - v) / \sqrt{b}, \\
p_3 &= (-2k\tau' + w - 2\tau_2) / \sqrt{b}, \quad q_3 = (-2k\tau' + v - 2\tau_2) / \sqrt{b}, \\
p_4 &= (-2k\tau' - w + 2\tau_1) / \sqrt{b}, \quad q_4 = (-2k\tau' - v + 2\tau_1) / \sqrt{b},
\end{aligned}$$

and $\Phi(\cdot)$ is defined in (I.2.8).

3. Formulae concerning $P\{W(t) \leq \tau + \delta t : a \leq t \leq b\}$.

We begin with the special case of $a = 0$.

Theorem I.3.1. Let $\{W(t) : t \geq 0\}$ be a Wiener process. Let $\tau > 0$, $b > 0$ and $\tau + \delta b \geq 0$. Then

$$\begin{aligned}
(I.3.1) \quad P\{W(t) \leq \tau + \delta t : 0 \leq t \leq b\} \\
= \Phi [(\tau + \delta b) / \sqrt{b}] - e^{-2\tau\delta} \Phi [(-\tau + \delta b) / \sqrt{b}].
\end{aligned}$$

Proof. Applying Doob's transformation in (I.2.5), we obtain

$$\begin{aligned}
&P\{W(t) \leq \tau + \delta t : 0 \leq t \leq b\} \\
&= P\{(1+t) B(t/(1+t)) \leq \tau + \delta t : 0 \leq t \leq b\}
\end{aligned}$$

$$= P\{ B(s) \leq \tau(1-s) + \delta s : 0 \leq s \leq b' \},$$

where $B(\cdot)$ is Brownian bridge process and $b' = b/(1+b)$.

Now applying Theorem I.C, we get

$$P\{ W(t) \leq \tau + \delta t : 0 \leq t \leq b \}$$

$$= \lim_{n \rightarrow \infty} P\{ \alpha_n(s) \leq \tau(1-s) + \delta s : 0 \leq s \leq b' \}$$

$$= \lim_{n \rightarrow \infty} P\{ U_n(s) \geq [1 + (\tau - \delta)/\sqrt{n}]s - \tau/\sqrt{n} : 0 \leq s \leq b' \}.$$

If we apply Theorem I.D with $u = \tau - \delta$ and $v = \tau$, then we obtain the assertion of the theorem.

Remark I.3.1. For $\delta = 0$, we obtain (cf. (1.5.1) of M. Csörgö and Révész (1981))

$$(I.3.2) \quad P\{ W(t) \leq \tau : 0 \leq t \leq b \} = 2 \Phi(\tau/\sqrt{b}) - 1,$$

which is also an immediate result of the reflection principle in (I.2.3).

Having (I.3.1) and Doob's transformation in (I.2.4), it leads to the following corollary.

Corollary I.3.1. Let $\{ B(s) : 0 \leq s \leq 1 \}$ be a Brownian bridge process. Let $0 < b < 1$ and $c \geq 0$. Then

$$\begin{aligned}
 (I.3.3) \quad & P\{ B(s) \leq c : 0 \leq s \leq b \} \\
 & = \Phi \left[\frac{c}{\sqrt{b(1-b)}} \right] - e^{-2c^2} \Phi \left[\frac{-c(1-2b)}{\sqrt{b(1-b)}} \right].
 \end{aligned}$$

Proof. Apply Doob's transformation in (I.2.4), and then Theorem I.3.1 with $\tau = \delta = c$. Then we obtain

$$\begin{aligned}
 & P\{ B(s) \leq c : 0 \leq s \leq b \} \\
 & = P\{ (1-s)W(s/(1-s)) \leq c : 0 \leq s \leq b \} \\
 & = P\{ W(t) \leq c(1+t) : 0 \leq t \leq b/(1-b) \} \\
 & = \Phi \left[\frac{c}{\sqrt{b(1-b)}} \right] - e^{-2c^2} \Phi \left[\frac{-c(1-2b)}{\sqrt{b(1-b)}} \right].
 \end{aligned}$$

Remark I.3.2. For $b = 1$, formula (I.3.3) yields

$$(I.3.4) \quad P\{ B(s) \leq c : 0 \leq s \leq 1 \} = 1 - e^{-2c^2},$$

which is the limiting distribution function of the one-sided Kolmogorov-Smirnov statistic

$$\sqrt{n} \sup_{-\infty < x < \infty} (F_n(x) - F(x)),$$

where $F_n(\cdot)$ is the empirical distribution function of a continuous distribution function $F(\cdot)$ (cf. Doob, 1949).

Remark I.3.3. Koziol and Byar (1975) derived (L3.3) denoting it by $G_b(c)$ ((2.3) of Koziol and Byar (1975)). There is a misprint in their formula (2.3). Namely, instead of having $(T-T^2)^{-\frac{1}{2}}$ in the probability argument of (2.3), we should have $(T-T^2)^{-\frac{1}{2}d}$. Hall and Wellner (1980) have also derived a formula for the probability statement of (I.3.3) and denoted it by $G_b^+(c)$ (cf. (2.8) of Hall and Wellner (1980)). However, their (2.8) is not correct.

We will now extend Theorem I.3.1 to the general case of $a > 0$ which, of course, covers Theorem I.3.1.

Theorem I.3.2. Let $\{ W(t) : t \geq 0 \}$ be a Wiener process. Let $\tau \geq 0$, $a > 0$, $b > a$ and $\tau + \delta b \geq 0$. Then

$$(I.3.5) \quad P\{ W(t) \leq \tau + \delta t : a \leq t \leq b \}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\tau+\delta a)/\sqrt{a}} e^{-\frac{1}{2}x^2} \left[\Phi \left[(\tau+\delta b - \sqrt{ax})/\sqrt{b-a} \right] - e^{-\frac{2(\tau+\delta a - \sqrt{ax})\delta}{b-a}} \Phi \left[(-\tau - 2a\delta + b\delta + \sqrt{ax})/\sqrt{b-a} \right] \right] dx.$$

Proof. Apply the facts that, for a fixed a , $W(s+a)-W(a)$ and $W(a)$ are two independent random variables and $W(s+a)-W(a) = W(s)$ in distribution. Then

$$(I.3.6) \quad P\{ W(t) \leq \tau + \delta t : a \leq t \leq b \}$$

$$= P\{ W(t+a) \leq \tau + \delta(t+a) : 0 \leq t \leq b-a \}$$

$$\begin{aligned}
&= P\{ W(t+a) - W(a) \leq \tau + \delta(t+a) - W(a) : 0 \leq t \leq b-a \} \\
&= \int_{-\infty}^{\infty} P\{ W(t+a) - W(a) \leq \tau + \delta(t+a) - h : 0 \leq t \leq b-a \} dP\{ W(a) \leq h \} \\
&= \int_{-\infty}^{\infty} P\{ W(t) \leq \tau + \delta a - h + \delta t : 0 \leq t \leq b-a \} dP\{ W(a) \leq h \}.
\end{aligned}$$

However, using the normality of Wiener process, we have

$$(I.3.7) \quad dP\{ W(a) \leq h \} = d\Phi(h/\sqrt{a}) = \frac{1}{\sqrt{2\pi a}} e^{-h^2/2a} dh.$$

In addition, it follows from (I.3.1) that

$$\begin{aligned}
(I.3.8) \quad &P\{ W(t) \leq (\tau + \delta a - h) + \delta t : 0 \leq t \leq b-a \} \\
&= \Phi\left[(\tau + \delta b - h)/\sqrt{b-a} \right] - e^{-2(\tau + \delta a - h)\delta} \Phi\left[(-\tau - 2\delta a + \delta b + h)/\sqrt{b-a} \right],
\end{aligned}$$

if $\tau + \delta a \geq h$. By substituting (I.3.7) and (I.3.8) into (I.3.6), we obtain

$$\begin{aligned}
&P\{ W(t) \leq \tau + \delta t : a \leq t \leq b \} \\
&= \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\tau + \delta a} e^{-h^2/2a} \left[\Phi\left[(\tau + \delta b - h)/\sqrt{b-a} \right] \right. \\
&\quad \left. - e^{-2(\tau + \delta a - h)\delta} \Phi\left[(-\tau - 2\delta a + \delta b + h)/\sqrt{b-a} \right] \right] dh.
\end{aligned}$$

If we set $x = h/\sqrt{a}$, then we obtain the assertion of the theorem. $\square$

Remark I.3.4. As $a \rightarrow 0$ in (I.3.3), (I.3.3) becomes (I.3.1). That is, (I.3.1) is a special case of (I.3.3).

Remark I.3.5. For $\delta = 0$, we obtain

$$(I.3.9) \quad P\{W(t) \leq \tau : a \leq t \leq b\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tau/\sqrt{a}} e^{-\frac{1}{2}x^2} \left[2 \Phi \left[(\tau - \sqrt{a}x)/\sqrt{b-a} \right] - 1 \right] dx,$$

which is also discussed by Rényi(1953, (3.6)) and M. Csörgö(1967, (3.4)).

As an analogue of Corollary I.3.1 of Theorem I.3.1, we have the following corollary.

Corollary I.3.2. Let $\{B(s) : 0 \leq s \leq 1\}$ be a Brownian bridge process. Let $a \geq 0$, $a < b < 1$ and $c > 0$. Then

$$(I.3.10) \quad P\{B(s) \leq c : a \leq s \leq b\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c/(1-a)} e^{-\frac{1}{2}h^2} \left[\Phi \left[\frac{c\sqrt{1-a} - (1-b)\sqrt{a}h}{\sqrt{1-b}\sqrt{b-a}} \right] - e^{-2c[c - \sqrt{a(1-a)}h]/(1-a)} \Phi \left[\frac{c(2b-a-1) + (1-b)\sqrt{a(1-a)}h}{\sqrt{1-a}\sqrt{1-b}\sqrt{b-a}} \right] \right] dh.$$

Proof. Similar to the proof of Corollary I.3.1. Namely we apply Doob's transformation in (I.2.4) and Theorem I.3.2 with $\tau = \delta = c$. Then we derive

$$\begin{aligned}
 & P\{ B(s) \leq c : a \leq s \leq b \} \\
 &= P\{ (1-s)W(s/(1-s)) \leq c : a \leq s \leq b \} \\
 &= P\{ W(t) \leq c(1+t) : a/(1-a) \leq t \leq b/(1-b) \}. \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^q e^{-\frac{1}{2}h^2} \left[\Phi \left[(c+cb' - \sqrt{a'}h)/\sqrt{b'-a'} \right] \right. \\
 &\quad \left. - e^{-2(c^2+c^2a' - \sqrt{a'}ch)} \Phi \left[(-c-2ca'+cb'+\sqrt{a'}h)/\sqrt{b'-a'} \right] \right] dh,
 \end{aligned}$$

where $q = c(1+a')$, $a' = a/(1-a)$ and $b' = b/(1-b)$. If we apply algebraic manipulations, then we obtain the assertion of the corollary. $\square$

Remark I.3.6. Csáki((1981), (2.17)) obtained (I.3.10) as a limiting distribution of the uniform empirical process on a limited range. His formula is utilized by M. Csörgö(1983, (4.2.10)).

Remark I.3.7. For $a = 0$, (I.3.3) of Corollary I.3.1 is a special case of (I.3.10). On the other hand, when $b = 1$ and $a \neq 0$, by using the symmetricity property, we have

$$(I.3.11) \quad P\{ B(s) \leq c : a \leq s \leq 1 \}$$

$$= P\{ B(s) \leq c : 0 \leq s \leq 1-a \}$$

$$= \Phi \left[\frac{c}{\sqrt{a(1-a)}} \right] - e^{-2c^2} \Phi \left[\frac{c(1-2a)}{\sqrt{a(1-a)}} \right].$$

4. Formulae concerning $P\{ |W(t)| \leq \tau + \delta t : a \leq t \leq b \}$.

Anderson's formula in (I.2.9) of Theorem I.E immediately provides the following theorem which also appeared in Gillespie and Fisher (1979).

Theorem I.4.1. Let $\{ W(t) : t \geq 0 \}$ be a Wiener process. Let $\tau > 0$ and $\tau + \delta b \geq 0$. Then

$$(I.4.1) \quad P\{ |W(t)| \leq \tau + \delta t : 0 \leq t \leq b \}$$

$$= 1 - 2 \Phi \left(-u/\sqrt{b} \right) - 2 \sum_{k=1}^{\infty} (-1)^{k+1}$$

$$e^{-2k^2\tau\delta} \left[\Phi \left[(-2k\tau+u)/\sqrt{b} \right] - \Phi \left[(-2k\tau-u)/\sqrt{b} \right] \right],$$

where $u = \tau + \delta b$.

Proof. (I.4.1) is an immediate result of (I.2.9). $\square$

Remark I.4.1. For $\delta_1 = \delta_2 = 0$ in (I.2.9), Anderson's formula in Theorem I.E also implies that

$$(I.4.2) \quad P\{ \tau_2 \leq W(t) \leq \tau_1 : 0 \leq t \leq b \}$$

$$= 1 - 2 \sum_{k=1}^{\infty} (-1)^{k+1} \left[\Phi[(-k\tau' + \tau_1)/\sqrt{b}] + \Phi[(-k\tau' - \tau_2)/\sqrt{b}] \right],$$

where $\tau' = \tau_1 - \tau_2$.

However, Feller (1966) showed not only the assertion of (I.4.2) (cf. Feller (1966), (5.8) of Chapter X) but also another formula (cf. Feller (1966), (5.9) of Chapter X) for the probability in (I.4.2) which is

$$(I.4.3) \quad P\{ \tau_2 \leq W(t) \leq \tau_1 : 0 \leq t \leq b \}$$

$$= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \exp\left[\frac{-(2k+1)^2 \pi^2 b}{2(\tau_1 - \tau_2)^2} \right] \sin\left[\frac{(2k+1)\pi \tau_1}{\tau_1 - \tau_2} \right].$$

Feller (1966) noted that while the series in (I.4.2) converges quickly when b is small, that in (I.4.3) converges rapidly when b is large. We note also that M. Csörgö (1967, (2.5)) also derived (I.4.3).

Remark I.4.2. For $\tau_1 = -\tau_2 = \tau$ in (I.4.2) or $\delta = 0$ in (I.4.1), we have

$$(I.4.4) \quad P\{ |W(t)| \leq \tau : 0 \leq t \leq b \}$$

$$= 1 - 4 \sum_{k=1}^{\infty} (-1)^{k+1} \Phi\left[-(2k-1)\tau/\sqrt{b} \right]$$

$$= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp \left[\frac{-(2k+1)^2 \pi^2 b}{8\tau^2} \right].$$

From Theorem I.4.1, we have the following corollary for a Brownian bridge process.

Corollary I.4.1. Let $\{B(s) : 0 \leq s \leq 1\}$ be a Brownian bridge process. Let $0 < b < 1$ and $c > 0$. Then

$$(I.4.5) \quad P\{|B(s)| \leq c : 0 \leq s \leq b\}$$

$$= 1 - 2 \Phi\left(-c/\sqrt{b(1-b)}\right) - 2 \sum_{k=1}^{\infty} (-1)^{k+1} e^{-2k^2 c^2}$$

$$\left[\Phi\left(\frac{-c(2k-2kb-1)}{\sqrt{b(1-b)}}\right) - \Phi\left(\frac{-c(2k-2kb+1)}{\sqrt{b(1-b)}}\right) \right].$$

Proof. Similar to the proof of Corollary I.3.1. Apply Doob's transformation in (I.2.4), and then apply Theorem I.4.1. Thus we get

$$P\{|B(s)| \leq c : 0 \leq s \leq b\}$$

$$= P\{|W(t)| \leq c(1+t) : 0 \leq t \leq b/(1-b)\}.$$

$$= 1 - 2 \Phi\left(-\mu/\sqrt{b'}\right) - 2 \sum_{k=1}^{\infty} (-1)^{k+1}$$

$$e^{-2k^2 c^2} \left[\Phi\left[(-2kc+\mu)/\sqrt{b'}\right] - \Phi\left[(-2kc-\mu)/\sqrt{b'}\right] \right],$$

where $\mu = c(1+b')$ and $b' = b/(1-b)$, i.e. we obtain (I.4.5). $\square$

Remark I.4.3. For $b = 1$, as $b \rightarrow 1$ in (I.4.5), we have

$$(I.4.6) \quad P\{ |B(s)| \leq c : 0 \leq s \leq 1 \}$$

$$= 1 - 2 \sum_{k=1}^{\infty} (-1)^{k+1} e^{-2k^2 c^2},$$

which is the limiting distribution function of the two-sided Kolmogorov-Smirnov statistic

$$\sqrt{n} \sup_{-\infty < x < \infty} |F_n(x) - F(x)|,$$

where $F_n(\cdot)$ is the empirical distribution function of a continuous distribution function $F(\cdot)$ (cf. Doob (1949)).

Remark I.4.4. Hall and Wellner (1980, (2.9)) also derived (I.4.5) and denoted it by $G_b(c)$.

We will now extend Theorem I.E to the general case of $a > 0$.

Theorem I.4.2. Let $\{W(t) : t \geq 0\}$ be a Wiener process. Let $\tau_1 > 0$, $\tau_2 < 0$, $a > 0$, $\tau_2 + \delta_2 b \leq \tau_1 + \delta_1 b$. Then we have

$$(I.4.7) \quad P\{ \tau_2 + \delta_2 t \leq W(t) \leq \tau_1 + \delta_1 t : a \leq t \leq b \}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{q''/\sqrt{a}}^{q'/\sqrt{a}} \Gamma(q'' - \sqrt{a}h, \delta_2, q' - \sqrt{a}h, \delta_1, b-a) e^{-\frac{1}{2}h^2} dh,$$

where $q' = \tau_1 + \delta_1 a$, $q'' = \tau_2 + \delta_2 a$ and

$$(I.4.8) \quad \Gamma(u', v', u, v, w) = P\{ u' + v't \leq W(t) \leq u + vt : 0 \leq t \leq w \},$$

which has been discussed in (I.2.9) in Theorem I.E.

Proof. Similary to the proof of Theorem I.3.2, applying that, for fixed a , $W(s+a)-W(a)$ and $W(a)$ are two independent random variables and $W(s+a)-W(a) = W(s)$ in distribution, we get

$$\begin{aligned} & P\{ \tau_2 + \delta_2 t \leq W(t) \leq \tau_1 + \delta_1 t : a \leq t \leq b \} \\ &= P\{ \tau_2 + \delta_2(t+a) \leq W(t+a) \leq \tau_1 + \delta_1(t+a) : 0 \leq t \leq b-a \} \\ &= P\{ \tau_2 + \delta_2(t+a) - W(a) \leq W(t+a) - W(a) \leq \tau_1 + \delta_1(t+a) - W(a) : 0 \leq t \leq b-a \} \\ &= \int_{-\infty}^{\infty} P\{ \tau_2 + \delta_2(t+a) - h \leq W(t) \leq \tau_1 + \delta_1(t+a) - h : 0 \leq t \leq b-a \} dP\{W(a) \leq h\}. \end{aligned}$$

Now we use (I.3.5), apply Theorem I.E and observe that Theorem I.E is applicable only if $h \geq q''$ and $h \leq q'$, where $q' = \tau_1 + \delta_1 a$, $q'' = \tau_2 + \delta_2 a$.

Then

$$P\{ \tau_2 + \delta_2 t \leq W(t) \leq \tau_1 + \delta_1 t : a \leq t \leq b \}$$

$$= \frac{1}{\sqrt{2\pi a}} \int_{q''}^{q'} \Gamma(q'' - h, \delta_2, q' - h, \delta_1, b-a) e^{-h^2/2a} dh,$$

where $\Gamma(u', v', u, v, w) = P\{ u' + v't \leq W(t) \leq u + vt : 0 \leq t \leq w \}$.

If we set $x = h/\sqrt{a}$, then we obtain the assertion of the theorem.

Remark I.4.5. As $a \rightarrow 0$ in (I.4.7), (I.4.7) becomes (I.2.9).

Remark I.4.6. For $\delta_1 = \delta_2 = 0$ in (I.4.7), we obtain

$$(I.4.9) \quad P\{ \tau_2 \leq W(t) \leq \tau_1 : a \leq t \leq b \}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\tau_2/\sqrt{a}}^{\tau_1/\sqrt{a}} \Gamma'(\tau_2 - \sqrt{a}h, \tau_1 - \sqrt{a}h, b-a) e^{-\frac{1}{2}h^2} dh,$$

where $\Gamma'(u', u, w) = P\{ u' \leq W(t) \leq u : 0 \leq t \leq w \}$. $\Gamma'(\cdot)$ can be obtained by either (I.4.2) or (I.4.3).

Remark I.4.7. For $\tau_1 = -\tau_2 = \tau (> 0)$, $\delta_1 = -\delta_2 = \delta$ and $\tau + \delta b \geq 0$, we have

$$(I.4.10) \quad P\{ |W(t)| \leq \tau + \delta t : a \leq t \leq b \}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{q/\sqrt{a}} \Gamma(-q - \sqrt{a}h, \delta, q - \sqrt{a}h, \delta, b-a) e^{-\frac{1}{2}h^2} dh,$$

where $q = \tau + \delta a$. If $\delta = 0$ in (I.4.10), we obtain

$$(I.4.11) \quad P\{ |W(t)| \leq \tau : a \leq t \leq b \}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\tau/\sqrt{a}} \Gamma'(-\tau - \sqrt{a}h, \tau - \sqrt{a}h, b-a) e^{-\frac{1}{2}h^2} dh.$$

From Remark I.4.7, we have the following corollary for a Brownian bridge process.

Corollary I.4.2. Let $\{ B(s) : 0 \leq s \leq 1 \}$ be a Brownian bridge process. Let $0 \leq a < b < 1$ and $c > 0$. Then

$$(I.4.12) \quad P\{ |B(t)| \leq c : a \leq t \leq b \}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\mu/\sqrt{a}} \Gamma(-\mu - \sqrt{a'}h, -c, \mu - \sqrt{a'}h, c, b' - a') e^{-\frac{1}{2}h^2} dh,$$

where $\mu = c(1+a')$, $a' = a/(1-a)$, $b' = b/(1-b)$ and $\Gamma(\cdot)$ is defined in (I.4.8).

Proof. (I.4.12) is an immediate consequence of (I.4.10). $\square$

Remark I.4.8. For $b = 1$, if we apply the symmetricity of Brownian bridge process, and then apply (I.4.5) in Corollary I.4.1, we obtain a formula similar to that of (I.4.5).

Remark I.4.9. Anderson and Darling (1952, (5.9)) discuss the probability in Corollary I.4.2 and show that

$$(I.4.13) \quad P\{ |B(t)| \leq c : a \leq t \leq b \}$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k e^{-2c^2 k^2} M(2kc\sqrt{a'}, 2kc\sqrt{b'}, c\sqrt{a'}/a, c\sqrt{b'}/b, -\sqrt{a'b'}) ,$$

where $M(u, v, u', v', \rho)$ is the volume under the bivariate normal distribution surface with zero means, unit variances and correlation ρ which is above the rectangle with vertices $x = u \pm u'$ and $y = v \pm v'$.

5. Tables.

Using the computer programs WIENER PACK by Chung (1986) based upon (I.4.12), tables of the critical points τ of the distribution of $\sup_{a \leq s \leq b} |B(s)|$, for selected values of a , b and probability levels, are computed and presented in Tables I.1 - I.12.

The selected probability levels are:

$$P\left\{ \sup_{a \leq s \leq b} |B(s)| \leq \tau \right\} = 0.99, 0.975, 0.95, 0.9, 0.8, 0.7, 0.6, 0.5, \\ 0.4, 0.3, 0.2 \text{ and } 0.1.$$

1

►

[illegible]

Table I.12. Critical points τ of the distribution of $\sup_{a \leq s \leq b} |B(s)|^h$ for selected values of a and b , and

$$P\left\{ \sup_{a \leq s \leq b} |B(s)| < \tau \right\} = 0.1.$$

[illegible]

II. THE BASIC SETTING FOR THE RANDOM CENSORSHIP MODEL AND VARIOUS PRELIMINARIES.

1. Introduction.

Let $X_1^0, \dots, X_n^0$ be independent identically distributed(i.i.d.) random variables(rv) with continuous distribution function $F^0(x) = P\{X_1^0 \leq x\}$. Let $Y_1, \dots, Y_n$ be i.i.d. rv with distribution function H . Suppose that the two sequences $\{X_i^0\}$ and $\{Y_i\}$ are independent. In the random censorship model from the right, the X_i^0 may be censored on the right by the Y_i , so that one observes only the pairs (X_i, δ_i) , $i=1,2,\dots,n$, where X_i is the minimum of X_i^0 and Y_i , and δ_i is the indicator function of the event $X_i^0 \leq Y_i$, i.e.,

$$(II.1.1) \quad X_i = X_i^0 \wedge Y_i \quad \text{and} \quad \delta_i = \begin{cases} 1 & \text{if } X_i^0 \leq Y_i, \\ 0 & \text{if } X_i^0 > Y_i, \end{cases}$$

$i=1,2,\dots,n$. Thus the $\{X_i\}$ are i.i.d. rv with distribution function F given by

$$(II.1.2) \quad 1 - F(t) = (1 - F^0(t))(1 - H(t)), \quad -\infty < t < \infty,$$

and the subdistribution function of the uncensored observations is given by

$$(II.1.3) \quad \tilde{F}(t) := P\{X_i < t \text{ and } \delta_i = 1\} = \int_{-\infty}^t (1-H(s)) dF^0(s).$$

For any distribution function $L(\cdot)$, let

$$(II.1.4) \quad T_L := \inf\{t : L(t)=1\} \text{ and } t_L := \sup\{t : L(t)=0\}.$$

Then, by (II.1.2), $T_F = T_F^0 \wedge T_H$ and $t_F = t_F^0 \vee t_H$.

To estimate F^0 in the above model, the product-limit(PL) estimator (cf. Kaplan and Meier (1958)) F_n^0 has been widely used and is defined by

$$(II.1.5) \quad 1 - F_n^0(t) := \begin{cases} \prod_{1 \leq i \leq n: X_i < t} \left[\frac{n-R_i}{n-R_i+1} \right]^{\delta_i} & \text{if } t \leq X_{n:n} \\ 0 & \text{if } t > X_{n:n} \end{cases}$$

where $X_{n:n} = \text{Max}(X_1, \dots, X_n)$ and R_i is the rank of $(X_i, 1-\delta_i)$ in the lexicographic ordering of $\{(X_i, 1-\delta_i)\}$, $i=1, 2, \dots, n$.

2. Product-limit(PL) process:

F_n^0 has been generally accepted as a substitute for the empirical distribution function in the random censorship model from the right. An analogue of the empirical process is the PL-process defined as

$$(II.2.1) \quad \beta_n(t) := n^{1/2} (F_n^0(t) - F^0(t)), \quad -\infty < t < \infty.$$

Weak convergence of β_n to a Gaussian process over a finite interval was obtained by Breslow and Crowley (1974). Burke, S. Csörgö and Horváth (1981, 1986) established strong approximations of β_n in terms of appropriate Gaussian processes in a general setting over right closed infinite intervals, expanding to infinite on the right too. We quote their theorem here.

Theorem II.A. (Burke, S. Csörgö and Horváth (1981), (1986)). If the underlying probability space is rich enough, then one can define two-parameter Gaussian process $K(t,s)$ such that if $p < T_F$, then

$$(II.2.2) \quad \sup_{-\infty < t \leq p} | \beta_n(t) - n^{-\frac{1}{2}} K(t,n) | \stackrel{\text{a.s.}}{=} O [n^{-\frac{1}{2}} (\log n)^2],$$

where

$$(II.2.3) \quad \{ K(t,s) : -\infty < t < \infty, s \geq 0 \} \stackrel{D}{=} \{ (1-F^0(t))W(d(t),s) : -\infty < t < \infty, s \geq 0 \},$$

$$(II.2.4) \quad d(t) := \int_{-\infty}^t (1-F(s))^2 d\bar{F}(s),$$

$\bar{F}(t)$ is defined in (II.1.3), $\{ W(t,s) : t \geq 0, s \geq 0 \}$ is a two-parameter Wiener Process,

$$(II.2.5) \quad EW(t,s) = 0 \quad \text{and} \quad EW(t,s)W(t',s') = (t \wedge t')(s \wedge s'),$$

and $\stackrel{D}{=}$ denotes equality in distribution.

Remark II.2.1. We note that $\{ W(t,1) : t \geq 0 \}$ is a standard one-parameter Wiener process which is defined in Section 2 of Chapter I.

For simplicity of notation, we use the same W for both one- and two-parameter Wiener processes. Applying Lévy's modulus of continuity of one-parameter Wiener process, we have the following lemma, which will be used in the sequel.

Lemma II.A. If $\{W(t,s) : t \geq 0, s \geq 0\}$ is a two-parameter Wiener process, then

$$(II.2.6) \quad \lim_{h \rightarrow 0} P\left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq u \leq h} |W(t+u,1) - W(t,1)| > \epsilon \right\} = 0$$

for all $\epsilon > 0$ and $T > 0$.

Remark II.2.2. As a consequence of Theorem II.A, Burke et.al.(1981, 1986) and S. Csörgö and Horváth(1983) also show that the law of iterated logarithm holds for the PL-process via proving the law of iterated logarithm for the process K . We quote their result:

$$(II.2.7) \quad \sup_{-\infty < t \leq p} |\beta_n(t)| \stackrel{\text{a.s.}}{=} O\left[\left(\log \log n\right)^{\frac{1}{2}}\right].$$

In addition, we need the following lemma in our sequel.

Lemma II.B. (Burke, S. Csörgö and Horváth (1981), Lemma 6.2.).

Let

$$(II.2.8) \quad d_n(t) := \int_{-\infty}^t (1-F_n(s))^{-2} d\tilde{F}_n(s),$$

where

$$F_n(t) := n^{-1} \# \{ 1 \leq i \leq n : X_i < t \} \text{ and}$$

$$\tilde{F}_n(t) := n^{-1} \# \{ 1 \leq i \leq n : X_i < t \text{ and } \delta_i = 1 \}.$$

Then we have, for $p < T_{F_2}$

$$(II.2.9) \quad \sup_{-\infty < t \leq p} |d_n(t) - d(t)| \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{2}} (\log n)^{\frac{1}{2}}).$$

3. PL-quantile process.

Under the model of random censorship from the right, a parallel problem to estimating F^0 is that of estimating the quantile function Q of F^0 where Q is defined by

$$(II.3.1) \quad Q(y) := \inf\{t : F^0(t) \geq y\}, \quad 0 < y < 1.$$

The counterpart of the PL estimator F_n^0 of F^0 for Q is the PL-quantile function Q_n defined by

$$(II.3.2) \quad Q_n(y) := \inf\{t : F_n^0(t) \geq y\}, \quad 0 < y < 1.$$

An analogue of the normed quantile process (cf. M. Csörgö and Révész (1978)) of uncensored observation is the PL-normed quantile process ρ_n , which is defined as

$$(II.3.3) \quad \rho_n(y) := \sqrt{n} f^0(Q(y)) (Q(y) - Q_n(y)), \quad 0 < y < 1,$$

where f^0 is the derivative of F^0 and Q and Q_n are as in (II.3.1) and (II.3.2), respectively. For this process, Aly, M. Csorgo and Horváth(1985) proved the following strong approximation theorem, which will be used later on in the sequel.

Theorem II.B. (Aly, M. Csorgo and Horváth (1985), Theorem 4.4.).

Assume that the following conditions hold:

(i) F^0 is twice differentiable on (t_{F^0}, T_{F^0}) ,

(ii) $f^0(t) = d/dt F^0(t) > 0$ on (t_{F^0}, T_{F^0}) ,

(iii) for some $r > 0$ and $p^* \in [0,1]$, we have

$$(II.3.4) \quad \sup_{0 \leq t \leq p^*} t \frac{|f^{0'}(Q(t))|}{f^0(Q(t))^2} \leq r.$$

Then there exists a positive constant C such that, with

$\delta(n) = C n^{-1} \log \log n$, we have

$$(II.3.5) \quad \sup_{\delta(n) \leq t \leq p_0} \left| \rho_n(t) - n^{-\frac{1}{2}} K^*(t, n) \right| \stackrel{\text{a.s.}}{=} 0 \left[n^{-\frac{1}{2}} (\log n)^{\frac{1}{2}} (\log \log n)^{\frac{1}{2}} \right],$$

where

$$(II.3.6) \quad \{ K^*(t, s) : 0 \leq t \leq 1, s \geq 0 \} \stackrel{D}{=} \{ K(Q(t), s) : 0 \leq t \leq 1, s \geq 0 \},$$

and $K(., s)$ is defined in (II.2.3). If in addition to (i), (ii) and (iii), we have either

(iv, α) $0 < \lim_{t \downarrow t_{F0}} f^0(t) < \infty$, or

(iv, β) if $\lim_{t \downarrow t_{F0}} f^0(t) = 0$, then f^0 is non-decreasing on an interval to the right of t_{F0} ,

then

$$(II.3.7) \sup_{0 \leq t \leq p_0'} |\rho_n(t) - n^{-\frac{1}{2}} K^*(t, n)| \stackrel{\text{a.s.}}{=} O \left[n^{-\frac{1}{4}} (\log n)^{\frac{1}{2}} (\log \log n)^{\frac{1}{4}} \right],$$

where $p_0' < p^*$ and $Q(p_0') < T_F$.

Remark II.3.1. Under the conditions (i) and (ii) only, for any $\mu > 0$, we have

$$(II.3.8) \sup_{\mu \leq t \leq p_0'} |\rho_n(t) - n^{-\frac{1}{2}} K^*(t, n)| \stackrel{\text{a.s.}}{=} O \left[n^{-\frac{1}{4}} (\log n)^{\frac{1}{2}} (\log \log n)^{\frac{1}{4}} \right].$$

Remark II.3.2. From (II.3.5), (II.3.7) and (II.3.8), under the corresponding conditions of Theorem II.B, we have that (cf. Aly, M. Csörgö and Horváth (1985), Section 5)

$$(II.3.9) \sup_{\delta(n) \leq t \leq p_0'} |\rho_n(t)| \stackrel{\text{a.s.}}{=} O \left[(\log \log n)^{\frac{1}{2}} \right],$$

$$(II.3.10) \sup_{0 \leq t \leq p_0'} |\rho_n(t)| \stackrel{\text{a.s.}}{=} O \left[(\log \log n)^{\frac{1}{2}} \right],$$

$$(II.3.11) \quad \sup_{u \leq t \leq p_0} |\rho_n(t)| \stackrel{\text{a.s.}}{=} 0 [(\log \log n)^{-\frac{1}{2}}].$$

4. Uniform PL and uniform PL-quantile processes.

It will be a convenient device to define a new set of rv by setting

$$U_i^0 := F^0(X_i^0), \quad V_i := F^0(Y_i) \text{ and } U_i := F^0(X_i),$$

$i = 1, \dots, n$. Then the rv $\{U_i^0\}$ are independent and identically distributed according to the uniform distribution on $[0,1]$; the $\{V_i\}$ are iid with the distribution function $H(t) = H(Q(t))$ on $[0,1]$, and are independent of the $\{U_i^0\}$. The distribution function of $\{U_i\}$ will be denoted by F^* . Let F_n^{*0} be the PL-estimator based on $\{(U_i, \delta_i)\}$. Clearly then

$$(II.4.1) \quad F_n^{*0}(t) = F_n^0(Q(t)).$$

Let $e_n(t)$ denote the uniform PL-process

$$(II.4.2) \quad e_n(t) := \sqrt{n} (F_n^{*0}(t) - t), \quad 0 \leq t \leq 1.$$

Now define q_n , the inverse of F_n^{*0} , and the uniform PL-quantile process u_n by

$$(II.4.3) \quad q_n(y) := \inf\{t : F_n^{*0}(t) \geq y\}, \quad 0 < y < 1,$$

$$u_n(y) := \sqrt{n} (y - q_n(y)), \quad 0 < y < 1,$$

respectively. Concerning these auxiliary processes, Aly, M. Csorgo and Horváth (1985) proved the following theorem:

Theorem II.C. (Aly, M. Csorgo and Horváth (1985), Theorem 3.1).

On the probability space of Theorem II.A we have that if $p_0 < T_F^*$ ($F(Q(p_0)) < 1$ or $Q(p_0) < T_F$), then

$$(II.4.4) \quad \sup_{0 \leq t \leq p_0} |e_n(t) - n^{-\frac{1}{2}} K^*(t, n)| \stackrel{\text{a.s.}}{=} O[n^{-\frac{1}{2}} (\log n)^2],$$

$$(II.4.5) \quad \sup_{0 \leq t \leq p_0} |u_n(t) - n^{-\frac{1}{2}} K^*(t, n)| \stackrel{\text{a.s.}}{=} O[n^{-\frac{1}{2}} (\log n)^{\frac{1}{2}} (\log \log n)^{\frac{1}{4}}].$$

Remark II.4.1. As in Remark II.2.2, Theorem II.C implies the following laws of iterated logarithm:

$$(II.4.6) \quad \sup_{0 \leq t \leq p_0} |e_n(t)| \stackrel{\text{a.s.}}{=} O[(\log \log n)^{\frac{1}{2}}],$$

$$(II.4.7) \quad \sup_{0 \leq t \leq p_0} |u_n(t)| \stackrel{\text{a.s.}}{=} O[(\log \log n)^{\frac{1}{2}}].$$

Theorem II.D. (Aly, M. Csorgo and Horváth, 1985, Theorems 4.1 and 4.2). There exist two positive constants C_1 and C_2 such that with $\varepsilon(n) = C_2 n^{-1} \log n$, we have

$$(II.4.8) \quad \lim_{n \rightarrow \infty} \sup_{\varepsilon(n) \leq t \leq p^0} (t \log \log n)^{-\frac{1}{2}} |e_n(t)| \leq C_1,$$

and

$$(II.4.9) \quad \lim_{n \rightarrow \infty} \sup_{\varepsilon(n) \leq t \leq p^0} (t \log \log n)^{-\frac{1}{2}} |u_n(t)| \leq C_1.$$

5. Bootstrapped PL, uniform PL and uniform PL-quantile processes.

In our previous sections in this chapter, we quoted strong approximation theorems for the PL, PL-quantile, uniform PL and uniform PL-quantile processes to appropriate Wiener processes.

In Chapter V we would also like to study functionals of PL, PL-quantile, uniform PL and/or uniform PL-quantile processes. The limiting distributions of these functionals depend on the unknown underlying distribution and censoring distribution functions. Hence they cannot even be simulated directly without knowing the said distribution and quantile functions.

This type of problem can be solved by adapting the bootstrap method, proposed by Efron(1980, 1982). Weak and strong convergence of bootstrapped versions of PL, uniform PL and uniform PL-quantile

processes are discussed by Horváth and Yandell(1985) and we quote their results here.

Let us draw m independent identically distributed random vectors $\{(Z_j, \mu_j) ; j=1, 2, \dots, m\}$ from $\{(X_i, \delta_i) ; i=1, 2, \dots, n\}$, in which each (Z_j, μ_j) independently takes (X_i, δ_i) with probability $1/n$, for $j=1, 2, \dots, m$. In other words, $\{(Z_j, \mu_j) ; j=1, 2, \dots, m\}$ are an independent sample of size m drawn with replacement from $\{(X_i, \delta_i) ; i=1, 2, \dots, n\}$ (cf. Efron (1980)).

Let $F_{m,n}^{\circ}$ (cf. (II.1.5)) and $F_{m,n}^{*\circ}$ (cf. (II.4.1)) denote the bootstrapped PL estimators from $\{(Z_j, \mu_j)\}$ and $\{(F^{\circ}(Z_j), \mu_j)\}$, respectively. Then, clearly,

$$(II.5.1) \quad F_{m,n}^{*\circ}(t) = F_{m,n}^{\circ}(Q(t)).$$

Let $\beta_{m,n}(t)$ and $e_{m,n}(t)$ be the bootstrapped PL and uniform PL processes, respectively defined as

$$(II.5.2) \quad \begin{aligned} \beta_{m,n}(t) &:= \sqrt{m} (F_{m,n}^{\circ}(t) - F_n^{\circ}(t)), \\ e_{m,n}(t) &:= \sqrt{m} (F_{m,n}^{*\circ}(t) - F_n^{*\circ}(t)). \end{aligned}$$

Then, by (II.5.1), we immediately have

$$(II.5.3) \quad \beta_{m,n}(t) = e_{m,n}(F^{\circ}(t))$$

Define $Q_{m,n}$ and $q_{m,n}$, the inverses of $F_{m,n}^0$ and $F_{m,n}^{*0}(t)$ respectively by

$$(II.5.4) \quad Q_{m,n}(y) := \inf\{ t : F_{m,n}^0(t) \geq y \},$$

$$q_{m,n}(y) := \inf\{ t : F_{m,n}^{*0}(t) \geq y \}.$$

Then we have

$$(II.5.5) \quad q_{m,n}(y) = F^0(Q_{m,n}(y)).$$

Let us now define the bootstrapped uniform PL-quantile process $u_{m,n}$ by

$$(II.5.6) \quad u_{m,n}(y) := \sqrt{m} (q_n(y) - q_{m,n}(y)).$$

Then we have the following theorem:

Theorem II.E. (Horváth and Yandell (1985)). Assume that

$$0 < \liminf_{m,n \rightarrow \infty} m/n \leq \limsup_{m,n \rightarrow \infty} m/n < \infty,$$

$p_0 < T_F^*$ (or $Q(p_0) < T_F$). Then there exists a sequence of Wiener processes $\{ W_m'(t) : t \geq 0 \}$, $m=1, 2, \dots, \infty$ such that

$$(II.5.7) \quad \sup_{1/(m+1) \leq t \leq p_0} |e_{m,n}(t) - (1-t)W_m'(d(Q(t)))| = o_p(1),$$

$$(II.5.8) \quad \sup_{1/(m+1) \leq t \leq p_0} |u_{m,n}(t) - (1-t)W_m'(d(Q(t)))| = o_p(1).$$

Remark II.5.1. Horváth and Yandell (1985) also showed that $\{ W_m(t) : t \geq 0 \}$ are independent of $\{ (X_i, \delta_i) \}$ and $\{ n^{-1/2} K^*(t, n) : t \geq 0 \}$ in Theorem II.C.

Remark II.5.2. From (II.5.7) and (II.5.8), we also have

$$(II.5.9) \quad \sup_{0 \leq t \leq p_0} | F_{m,n}^*(t) - F_n^*(t) | = o_p(1) ,$$

$$(II.5.10) \quad \sup_{0 \leq y \leq p_0} | q_n(y) - q_{m,n}(y) | = o_p(1) .$$

III. CONFIDENCE BANDS FOR THE QUANTILE FUNCTION UNDER THE RANDOM CENSORSHIP MODEL FROM THE RIGHT.

1. Introduction.

When the samples are uncensored, the problem of constructing confidence bands for quantile functions was first studied by M. Csörgö and Révész (1984) via strong approximation methodology. Aly, M. Csörgö and Horváth (1985) extended the construction of the confidence band under the random censorship model from the right based upon a strong approximation theorem on the normed quantile process. In this chapter we generalize their approach by introducing a general class of bands which includes the Aly-Csörgö-Horváth band under the random censorship model.

2. Generalization of Aly-Csörgö-Horváth confidence band for quantile function.

As noted by Aly, M. Csörgö and Horváth (1985), using Theorem II.B directly, we can construct a confidence band for $Q(y)$ in terms of $Q_n(y) \pm (\cdot)$, but we must estimate the factor $f^0(Q(y))$ of $\rho_n(y)$ first from the censored data.

Aly, M. Csörgö and Horváth (1985) also proposed another method to construct a confidence band for $Q(y)$ in terms of $Q_n(y; \cdot)$ without estimating $f^0(Q(y))$. Their approach is an extension of the M. Csörgö and Révész confidence band (M. Csörgö and Révész (1984)) for $Q(y)$ for non-censored observations. We quote the Aly, M. Csörgö and Horváth result here:

Theorem III.A. Under the conditions (i), (ii) and (iii) of Theorem II.B, we have

$$\begin{aligned} \text{(III.2.1)} \quad \lim_{n \rightarrow \infty} P\{ Q_n(t - c_n(t)) \leq Q(t) \leq Q_n(t + c_n(t)) : \varepsilon(n) \leq t \leq p_0 \} \\ = P\{ |W(s)| \leq u : 0 \leq s \leq 1 \} , \end{aligned}$$

where p_0 is p_0' in Theorem II.B, $\sqrt{n} \varepsilon(n) \rightarrow \infty$ as $n \rightarrow \infty$, $u > 0$, $W(\cdot)$ is a Wiener process, and

$$\text{(III.2.2)} \quad c_n(t) := \frac{1-t}{\sqrt{n}} u \sqrt{d_n(Q_n(p_0))} .$$

Here we extend the above confidence band $\{ [Q_n(t - c_n(t)), Q_n(t + c_n(t))] : \varepsilon(n) \leq t \leq p_0 \}$ of Theorem III.A, into a confidence band like $\{ [Q_n(t - b_n(t)), Q_n(t + b_n(t))] : \varepsilon(n) \leq t \leq p_0 \}$, where $c_n(t)$ is a special case of $b_n(t)$.

Our main theorem is:

Theorem III.2.1. Under the conditions (i), (ii) and (iii) of Theorem II.B, we have

$$(III.2.3) \quad \lim_{n \rightarrow \infty} P\{ Q_n(t-b_n(t)) \leq Q(t) \leq Q_n(t+b_n(t)) : \varepsilon(n) \leq t \leq p_0 \} \\ = P\{ |W(s)| \leq \beta_1 s + \beta_2 : 0 \leq s \leq 1 \} ,$$

where p_0 is p_0' in Theorem II.B, $\sqrt{n} \varepsilon(n) \rightarrow \infty$ as $n \rightarrow \infty$, $\beta_1 > 0$, $\beta_1 + \beta_2 \geq 0$, $W(\cdot)$ is a Wiener process, and

$$(III.2.4) \quad b_n(t) := \frac{1-t}{\sqrt{n}} \left[\beta_1 \frac{d_n(Q_n(t))}{\sqrt{d_n(Q_n(p_0))}} + \beta_2 \sqrt{d_n(Q_n(p_0))} \right] .$$

To prove the theorem, we need the next two lemmas.

Lemma III.2.1. Under the conditions (i), (ii) and (iii) of Theorem II.B, we have

$$(III.2.5) \quad \left[\frac{t_1 \wedge t_2}{t_1 \vee t_2} \right]^r \leq \frac{f^0(Q(t_1))}{f^0(Q(t_2))} \leq \left[\frac{t_1 \vee t_2}{t_1 \wedge t_2} \right]^r$$

for any pair $t_1, t_2 \in [0,1]$ and with r as in the condition (iii).

Proof. Since

$$\frac{d}{dt} \log f^0(Q(t)) = \frac{f^{0'}(Q(t))}{[f^0(Q(t))]^2} ,$$

by the condition (iii), we have

$$\left| \frac{d}{dt} \log f^0(Q(t)) \right| = \frac{|f^{0'}(Q(t))|}{[f^0(Q(t))]^2} \leq r t^{-1} = r \frac{d}{dt} \log t.$$

Hence,

$$-r \frac{d}{dt} \log t \leq \frac{d}{dt} \log f^0(Q(t)) \leq r \frac{d}{dt} \log t,$$

which implies (III.2.5). $\square$

Lemma III.2.2. Under the condition (i), (ii) and (iii) of Theorem

II.B, we have

$$(i) \quad \sup_{\varepsilon(n) \leq t \leq p_0} \left| \frac{f^0(Q(t))}{f^0(Q(t+b_n(t)))} - 1 \right| = o_p(1),$$

$$(ii) \quad \sup_{\varepsilon(n) \leq t \leq p_0} \left| \frac{f^0(Q(t))}{f^0(Q(t-b_n(t)))} - 1 \right| = o_p(1),$$

$$(iii) \quad \sup_{\varepsilon(n) \leq t \leq p_0, t \leq x \leq b_n(t)+t} \left| \frac{f^0(Q(t))}{f^0(Q(x))} - 1 \right| = o_p(1),$$

$$(iv) \quad \sup_{\varepsilon(n) \leq t \leq p_0, t-b_n(t) \leq x \leq t} \left| \frac{f^0(Q(t))}{f^0(Q(x))} - 1 \right| = o_p(1),$$

where p_0 is p_0' in Theorem II.B, $\sqrt{n} \varepsilon(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $b_n(t)$ is defined in (III.2.4).

Proof. We will prove (i) only. The proofs of the other three statements are similar and we omit them here.

By Lemma III.2.1, we have, for every pair $t_1, t_2 \in (0,1)$ and r as in (iii) of Theorem II.B, that

$$\left[\frac{t_1 \wedge t_2}{t_1 \vee t_2} \right]^r \leq \frac{f^0(Q(t_1))}{f^0(Q(t_2))} \leq \left[\frac{t_1 \vee t_2}{t_1 \wedge t_2} \right]^r,$$

and we also have $b_n(t) \geq 0$. Hence

$$\begin{aligned} & \sup_{\varepsilon(n) \leq t \leq p_0} \left| \frac{f^0(Q(t))}{f^0(Q(t+b_n(t)))} - 1 \right| \\ & \leq \sup_{\varepsilon(n) \leq t \leq p_0} \left[\left(\frac{f^0(Q(t))}{f^0(Q(t+b_n(t)))} - 1 \right) \vee \left(1 - \frac{f^0(Q(t))}{f^0(Q(t+b_n(t)))} \right) \right] \\ & \leq 2 \sup_{\varepsilon(n) \leq t \leq p_0} \left[\left[\frac{t+b_n(t)}{t} \right]^r - 1 \right] \\ & \leq 2 \sup_{\varepsilon(n) \leq t \leq p_0} \left[\left[1 + \frac{b_n(t)}{t} \right]^r - 1 \right] \\ & \leq 2 \sup_{\varepsilon(n) \leq t \leq p_0} \left[\left[1 + \frac{b_n(t)}{\varepsilon(n)} \right]^r - 1 \right] \\ & \leq 2 \sup_{\varepsilon(n) \leq t \leq p_0} \left[\left[1 + \frac{1-t}{\sqrt{n}\varepsilon(n)} \left(\beta_1 \frac{d_n(Q_n(t))}{\sqrt{d_n(Q_n(p_0))}} + \beta_2 \sqrt{d_n(Q_n(p_0))} \right) \right]^r - 1 \right] \end{aligned}$$

$$\leq 2 \left[\left[1 + \frac{(\beta_1 + \beta_2)}{\sqrt{n} \epsilon(n)} \sqrt{d_n(Q_n(p_0))} \right]^r - 1 \right] \xrightarrow{P} 0 \text{ as } n \rightarrow \infty,$$

where the latter limit is by observing $Q_n(p_0) \xrightarrow{\text{a.s.}} Q(p_0)$ (cf. (II.3.9)),
 $d_n(Q(p_0)) \xrightarrow{\text{a.s.}} d(Q(p_0))$ (cf. (II.2.9)), and $\sqrt{n} \epsilon(n) \rightarrow \infty$ as $n \rightarrow \infty$.

This also completes the proof. $\square$

Now, using these two lemmas, we prove Theorem III.2.1.

Proof of Theorem III.2.1. We will prove (III.2.3) of Theorem III.2.1 for the one-sided case of $Q(y) \leq Q_n(t+b_n(t))$ only. The complete proof of our statement goes along the same lines. Consider

$$\begin{aligned} \text{(III.2.6)} \quad & P\{ Q(t) \leq Q_n(t+b_n(t)) : \epsilon(n) \leq t \leq p_0 \} \\ &= P\{ Q(t) - Q_n(t) \leq Q_n(t+b_n(t)) - Q_n(t) : \epsilon(n) \leq t \leq p_0 \} \\ &= P\{ \sqrt{n} f^0(Q(t)) (Q(t) - Q_n(t)) \leq \\ &\quad \sqrt{n} f^0(Q(t)) (Q_n(t+b_n(t)) - Q(t+b_n(t))) \\ &\quad + \sqrt{n} f^0(Q(t)) (Q(t+b_n(t)) - Q(t)) \\ &\quad + \sqrt{n} f^0(Q(t)) (Q(t) - Q_n(t)) : \epsilon(n) \leq t \leq p_0 \} \\ &= P\{ \rho_n(t) \leq \left[\left(\frac{f^0(Q(t))}{f^0(Q(t+b_n(t)))} - 1 \right) (-\rho_n(t+b_n(t))) \right] \right. \\ &\quad \left. + (\rho_n(t) - \rho_n(t+b_n(t))) \right. \\ &\quad \left. + \sqrt{n} f^0(Q(t)) (Q(t+b_n(t)) - Q(t)) : \epsilon(n) \leq t \leq p_0 \}. \end{aligned}$$

Applying one-term Taylor expansion on Q , we have

$$\begin{aligned}
 \text{(III.2.7)} \quad \sqrt{n} f^0(Q(t)) (Q(t+b_n(t)) - Q(t)) &= \frac{f^0(Q(t))}{f^0(Q(\delta_{t,n}))} \sqrt{n} b_n(t) \\
 &= \sqrt{n} b_n(t) \left[\frac{f^0(Q(t))}{f^0(Q(\delta_{t,n}))} - 1 \right] + \sqrt{n} b_n(t)
 \end{aligned}$$

where $t < \delta_{t,n} < t+b_n(t)$. Combining (III.2.6) and (III.2.7), we have

$$\begin{aligned}
 \text{(III.2.8)} \quad P\{ Q(t) \leq Q_n(t+b_n(t)) : \varepsilon(n) \leq t \leq p_0 \} \\
 = P\{ \rho_n(t) \leq \left[\left(\frac{f^0(Q(t))}{f^0(Q(t+b_n(t)))} - 1 \right) (- \rho_n(t+b_n(t))) \right] \right. \\
 \left. + (\rho_n(t) - \rho_n(t+b_n(t))) + \sqrt{n} b_n(t) \right. \\
 \left. + \sqrt{n} b_n(t) \left[\frac{f^0(Q(t))}{f^0(Q(\delta_{t,n}))} - 1 \right] : \varepsilon(n) \leq t \leq p_0 \} .
 \end{aligned}$$

First, we have

$$\begin{aligned}
 \text{(III.2.9)} \quad \sup_{\varepsilon(n) \leq t \leq p_0} \sqrt{n} |b_n(t)| &\leq (\beta_1 + \beta_2) \sqrt{d_n(Q_n(p_0))} \\
 &\xrightarrow{\text{a.s.}} (\beta_1 + \beta_2) \sqrt{d(Q(p_0))} \quad \text{as } n \rightarrow \infty,
 \end{aligned}$$

where the latter limit is by observing $Q_n(p_0) \xrightarrow{\text{a.s.}} Q(p_0)$ (cf. (II.3.9)),
 $d_n(Q(p_0)) \xrightarrow{\text{a.s.}} d(Q(p_0))$ (cf. (II.2.9)), and $\sqrt{n} \varepsilon(n) \rightarrow \infty$ as $n \rightarrow \infty$.

By Theorem II.B, we get, as $n \rightarrow \infty$

$$\text{(III.2.10)} \quad \sup_{\varepsilon(n) \leq t \leq p_0} |\rho_n(t+b_n(t)) - n^{-\frac{1}{2}} K^*(t+b_n(t), n)| \xrightarrow{\text{a.s.}} o(1).$$

Also, by the modulus of continuity for Wiener process in Lemma II.A and

$$(III.2.11) \quad \{ n^{-\frac{1}{2}} K^*(t, n) : 0 \leq t \leq 1 \} \stackrel{D}{=} \{ (1-t) W(d(Q(t), 1)) : 0 \leq t \leq 1 \}$$

for each $n = 1, 2, \dots$, we obtain

$$(III.2.12) \quad \sup_{\epsilon(n) \leq t \leq p_0} | \rho_n(t+b_n(t)) - \rho_n(t) |$$

$$\leq \sup_{\epsilon(n) \leq t \leq p_0} | \rho_n(t+b_n(t)) - n^{-\frac{1}{2}} K^*(t+b_n(t), n) |$$

$$+ \sup_{\epsilon(n) \leq t \leq p_0} | n^{-\frac{1}{2}} K^*(t+b_n(t), n) - n^{-\frac{1}{2}} K^*(t, n) |$$

$$+ \sup_{\epsilon(n) \leq t \leq p} | \rho_n(t) - n^{-\frac{1}{2}} K^*(t, n) | \xrightarrow{P} 0 \text{ as } n \rightarrow \infty.$$

Combining (III.2.10) and (III.2.11), we have, as $n \rightarrow \infty$

$$(III.2.13) \quad \sup_{\epsilon(n) \leq t \leq p_0} | \rho_n(t+b_n(t)) | \xrightarrow{\text{a.s.}} \text{a random variable.}$$

From (III.2.9), we have, for any $\epsilon > 0$, there exists $M(\epsilon) > 0$ such that

$$(III.2.14) \quad \lim_{n \rightarrow \infty} P \left\{ \sup_{\epsilon(n) \leq t \leq p_0} \sqrt{n} |b_n(t)| < M(\epsilon) \right\} \geq 1 - \epsilon.$$

Combining (III.2.13) and (i) of Lemma III.2.2, we have as $n \rightarrow \infty$

$$(III.2.15) \quad \sup_{\epsilon(n) \leq t \leq p_0} \left| \frac{f^0(Q(t))}{f^0(Q(t+b_n(t)))} - 1 \right| | \rho_n(t+b_n(t)) | \xrightarrow{P} 0.$$

Also combining (III.2.14) and (iii) of Lemma III.2.2, we obtain

$$(III.2.16) \quad \sup_{\varepsilon(n) \leq t \leq p_0} \left| \sqrt{n} b_n(t) \right| \left| \frac{f^0(Q(t))}{f^0(Q(\delta_{t,n}))} - 1 \right| \xrightarrow{P} 0, \text{ as } n \rightarrow \infty.$$

Finally, applying (III.2.12), (III.2.15) and (III.2.16) to (III.2.8), and using

$$\sup_{0 \leq t \leq \varepsilon(n)} |W(d(Q(t)), 1)| = o_p(1),$$

we have

$$\begin{aligned} (III.2.17) \quad & P\{Q(t) \leq Q_n(t+b_n(t)) : \varepsilon(n) \leq t \leq p_0\} \\ &= P\{\rho_n(t) \leq \sqrt{n} b_n(t) : \varepsilon(n) \leq t \leq p_0\} + o_p(1) \\ &= P\{n^{-1/2} K^*(t, n) \leq \sqrt{n} b_n(t) : \varepsilon(n) \leq t \leq p_0\} + o_p(1) \\ &= P\{(1-t) W(d(Q(t)), 1) \leq \sqrt{n} b_n(t) : \varepsilon(n) \leq t \leq p_0\} + o_p(1) \\ &= P\{(1-t) W(d(Q(t)), 1) \leq \sqrt{n} b_n(t) : 0 \leq t \leq p_0\} + o_p(1) \\ &= P\{W(d(Q(t))) \leq \beta_1 \frac{d(Q(t))}{\sqrt{d(Q(p_0))}} + \beta_2 \sqrt{d(Q(p_0))} : 0 \leq t \leq p_0\} + o_p(1) \\ &= P\{W\left(\frac{d(Q(t))}{d(Q(p_0))}\right) \leq \beta_1 \frac{d(Q(t))}{d(Q(p_0))} + \beta_2 : 0 \leq t \leq p_0\} + o_p(1) \\ &= P\{W(s) \leq \beta_1 s + \beta_2 : 0 \leq s \leq 1\} + o_p(1). \end{aligned}$$

To prove the other side of the inequality in (III.2.3), i.e.,

$$P\{ Q_n(t-b_n(t)) \leq Q(t) : \epsilon(n) \leq t \leq p_0 \}$$

$$= P\{ -\beta_1 s - \beta_2 \leq W(s) : 0 \leq s \leq 1 \} + o_p(1),$$

we make use of (ii) and (iv) of Lemma III.2.2, instead of (i) and (iii) as we have used in the proof. This also concludes the proof of Theorem III.2.1. ~~■~~

Remark III.2.1. When $\beta_1 = 0$ in (III.2.3), we get (III.2.1) of Theorem III.A.

Remark III.2.2. For convenience of notation we will denote $b_n(t)$ defined in (III.2.4) by $b_n(t, \beta_1, \beta_2)$ from here on.

Using Doob's transformation (cf. (I.2.5)) we may use Brownian bridge process instead of Wiener process in (III.2.3) as follows.

Corollary III.2.1. Under the conditions (i), (ii) and (iii) of Theorem II.B, we have

$$\begin{aligned} \text{(III.2.18)} \quad \lim_{n \rightarrow \infty} P\{ Q_n(t-b_n(t, \beta, \beta)) \leq Q(t) \leq Q_n(t+b_n(t, \beta, \beta)) : \epsilon(n) \leq t \leq p_0 \} \\ = P\{ |B(s)| \leq \beta : 0 \leq s \leq 1 \} \end{aligned}$$

where $B(\cdot)$ is a Brownian bridge process on $[0, 1]$.

Proof. By Doob's transformation, we have

$$P\{|W(t)| \leq \beta t + \beta : 0 \leq t \leq 1\}$$

$$= P\{|B(s)| \leq \beta : 0 \leq s \leq \frac{1}{2}\}.$$

Hence, (III.2.18) is proved. $\square$

From Theorem III.2.1 and Corollary III.2.1, we have the next corollary for one-sided confidence bands.

Corollary III.2.2. Under the conditions (i), (ii) and (iii) of Theorem II.B, we have

$$(III.2.19) \quad \lim_{n \rightarrow \infty} P\{Q(t) \leq Q_n(t + b_n(t, \beta_1, \beta_2)) : \epsilon(n) \leq t \leq p_0\}$$

$$= P\{W(s) \leq \beta_1 s + \beta_2 : 0 \leq s \leq 1\},$$

$$(III.2.20) \quad \lim_{n \rightarrow \infty} P\{Q(t) \leq Q_n(t + b_n(t, \beta, \beta)) : \epsilon(n) \leq t \leq p_0\}$$

$$= P\{B(s) \leq \beta : 0 \leq s \leq \frac{1}{2}\}.$$

Another important analogue of Theorem III.2.1 is the following theorem. It serves several applications in practical problems, which will be discussed in Section 4.

Theorem III.2.2. Under the conditions (i) and (ii) of Theorem II.B, we have, for $a > t_{p_0}$, $b > a$ and $b < T_{F-1}$

$$(III.2.21) \quad \lim_{n \rightarrow \infty} P\{ Q_n(t - b_n(t, \tau_1, \tau_2)) \leq Q(t) \leq Q_n(t + b_n(t, \tau_1, \tau_2)) : a \leq t \leq b \}$$

$$= P\left\{ |W(s)| \leq \tau_1 s + \tau_2 : \frac{d(Q(a))}{d(Q(b))} \leq s \leq 1 \right\},$$

where $b_n(\cdot)$ is defined in (III.2.4) (cf. Remark III.2.2).

Proof. The proof of (III.2.21) goes along similar lines as those of (III.2.3), resulting in

$$P\{ Q(t) \leq Q_n(t + b_n(t, \tau_1, \tau_2)) : a \leq t \leq b \}$$

$$= P\{ \rho_n(t) \leq \sqrt{n} b_n(t, \tau_1, \tau_2) : a \leq t \leq b \} + o_p(1)$$

$$= P\left\{ W\left(\frac{d(Q(t))}{d(Q(b))} \right) \leq \tau_1 \frac{d(Q(t))}{\sqrt{d(Q(b))}} + \tau_2 \sqrt{d(Q(b))} : a \leq t \leq b \right\} + o_p(1)$$

$$= P\left\{ |W(t)| \leq \tau_1 t + \tau_2 : \frac{d(Q(a))}{d(Q(b))} \leq t \leq 1 \right\} + o_p(1).$$

The proof for the other side of the inequality in (III.2.21) is similar, and the proof is now complete. $\square$

Remark III.2.3. We note here that condition (iii) of Theorem II.B regarding the tail behavior of the density function f^0 is no longer required for Theorem III.2.2. When $\tau(n)$ in Lemma III.2.1 is replaced by a $(> \tau_{p0})$, the statements in Lemma III.2.1 hold without condition (iii) of Theorem II.B.

Remark III.2.4. When the data are truncated on the left-side of distribution, (III.2.21) is still applicable, where a would be the known point of truncation.

Corollary III.2.3. Under the conditions (i) and (ii) of Theorem II.B, we have, for $a > t_{F0}$, $b > a$ and $b < T_{F1}$

$$(III.2.22) \quad \lim_{n \rightarrow \infty} P\{ Q_n(t - b_n(t, \tau, \tau)) \leq Q(t) \leq Q_n(t + b_n(t, \tau, \tau)) : a \leq t \leq b \}$$

$$= P\{ |B(s)| \leq \tau : \frac{d(Q(a))}{d(Q(a)) + d(Q(b))} \leq s \leq \frac{1}{2} \} ,$$

where $b_n(\cdot)$ is defined in (III.2.4) (cf. Remark III.2.2) and $B(\cdot)$ is a Brownian bridge process on $[0, 1]$.

Corollary III.2.4. Under the conditions (i) and (ii) of Theorem II.B, we have, for $a > t_{F0}$, $b > a$ and $b < T_{F1}$

$$(III.2.23) \quad \lim_{n \rightarrow \infty} P\{ Q(t) \leq Q_n(t + b_n(t, \tau', \tau')) : a \leq t \leq b \}$$

$$= P\{ B(s) \leq \tau' : \frac{d(Q(a))}{d(Q(a)) + d(Q(b))} \leq s \leq \frac{1}{2} \} .$$

Remark III.2.5. Contrary to (III.2.18) of Corollary III.2.1 where the range of Brownian bridge process is $[0, \frac{1}{2}]$, the range

$$(III.2.25) \quad \left[\frac{d(Q(a))}{d(Q(a)) + d(Q(b))} , \frac{1}{2} \right]$$

of (III.2.22) of Corollary III.2.3 depends upon the unknown functions $Q(\cdot)$ and $d(\cdot)$. Without knowing the range in (III.2.25), we cannot compute τ in (III.2.22), and consequently, the confidence band for $Q(\cdot)$ cannot be constructed by using (III.2.22) directly.

Hence, the next obvious question is the following: Can we first estimate τ in (III.2.22) by substituting $Q_n(\cdot)$ and $d_n(\cdot)$ for $Q(\cdot)$ and $d(\cdot)$ in it, and use $b_n(t, \hat{\tau}, \hat{\tau})$ where $\hat{\tau}$ is the estimator of τ , and then construct the confidence band for $Q(\cdot)$ by substituting $b_n(t, \hat{\tau}, \hat{\tau})$ for $b_n(t, \tau, \tau)$ in the first statement of (III.2.22)?

Answering this question leads to the next section.

We also note here that a similar remark can be made concerning the range of Wiener process in (III.2.21) and for the estimation of τ_1 and τ_2 there.

3. Continuity of $P\{ |B(s)| \leq c : a \leq s \leq b \}$ as a function of three parameters a, b and c .

Suppose we substitute $d_n(\cdot)$ and $Q_n(\cdot)$ for $d(\cdot)$ and $Q(\cdot)$ in (III.2.22) in order to estimate τ for a given level of significance. Then the probability

$$P\{ |B(s)| \leq \hat{\tau} : \frac{d_n(Q_n(a))}{d_n(Q_n(a)) + d_n(Q_n(b))} \leq s \leq \frac{1}{2} \}$$

cannot be computed without knowing how the random variable $\frac{d_n(Q_n(a))}{d_n(Q_n(a)) + d_n(Q_n(b))}$

and the Brownian bridge process $B(\cdot)$ interact. Hence the estimator $\hat{\tau}$ of τ cannot be computed directly.

Towards resolving this problem we define

$$p_2(a, b, c) := P\{ |B(s)| \leq c : a \leq s \leq b \},$$

a function of three parameters - a , b and c where $a \in [0, b]$, $b \in [a, 1]$ and $c \in (0, \infty)$. Define also

$$\varphi_2(a, b, p) := \sup \{ c \in (0, \infty) : p_2(a, b, c) \leq p \},$$

where $a \in [0, b]$, $b \in [a, 1]$ and $p \in (0, 1)$.

Suppose now that we can show that $\varphi_2(a, b, p)$ is continuous with respect to a for given b and p . If, in addition, we have also

$$\frac{d_n(Q_n(a))}{d_n(Q_n(a)) + d_n(Q_n(b))} \xrightarrow{\text{a.s.}} \frac{d(Q(a))}{d(Q(a)) + d(Q(b))}, \text{ as } n \rightarrow \infty,$$

then we immediately obtain that τ in (III.2.22) can be replaced by τ_n , where, for a given level of significance α ,

$$\tau_n := \varphi_2\left(\frac{d_n(Q_n(a))}{d_n(Q_n(a)) + d_n(Q_n(b))}, \frac{1}{2}, 1 - \alpha\right).$$

In order to prove the desired continuity of $\phi_2(\cdot)$, we need the following lemma and theorem.

Lemma III.A. (M. Csörgö and Révész (1981)). Let $\{ W(t): t \geq 0 \}$ and $\{ B(s) : 0 \leq s \leq 1 \}$ be Wiener process and Brownian bridge process, respectively. For any $T (< \infty)$, $S (\leq 1)$ and any $\epsilon > 0$, we have

$$(i) \quad \lim_{h \rightarrow 0} P \left\{ \sup_{0 \leq t \leq T-h} \sup_{0 \leq t' \leq h} | W(t+t') - W(t) | > \epsilon \right\} = 0 ,$$

$$(ii) \quad \lim_{h \rightarrow 0} P \left\{ \sup_{0 \leq s \leq S-h} \sup_{0 \leq s' \leq h} | B(s+s') - B(s) | > \epsilon \right\} = 0 .$$

Theorem III.3.1. Let $\{ B(s) : 0 \leq s \leq 1 \}$ be Brownian bridge process. Define three-parameter functions p_1 and p_2 such that

$$(III.3.1) \quad p_1(a, b, c) := P \{ B(s) \leq c : a \leq s \leq b \} ,$$

$$p_2(a, b, c) := P \{ |B(s)| \leq c : a \leq s \leq b \} ,$$

where $a \in [0, b]$, $b \in [a, 1]$ and $c \in (0, \infty)$. Then we have

(i) p_1 and p_2 are monotone non-decreasing continuous functions with respect to c on $(0, \infty)$ for fixed a and b ;

(ii) p_1 and p_2 are strictly monotone increasing uniformly continuous functions with respect to a on $[0, b]$ for fixed b and c ;

(iii) p_1 and p_2 are strictly monotone decreasing uniformly continuous functions with respect to b on $[a, 1]$ for fixed a and c .

Proof. We prove these statements for p_2 only. The proofs for p_1 are similar to those for p_2 . The monotonicity properties are obvious. Hence, we show continuity only.

Proof of (i). Case 1. Let $a = b = 0$ or $a = b = 1$. Then $p_2(0,0,c) = p_2(1,1,c) = 1$ for all $c \in (0, \infty)$, and hence p_2 is continuous.

Case 2. Let $a = b$, $a \neq 0$ and $a \neq 1$. Then

$$p_2(a,a,c) = P\{|B(a)| \leq c\} = 2\Phi\left(\frac{c}{\sqrt{a(1-a)}}\right) - 1,$$

where $\Phi(y)$ is defined in (I.2.8). Hence, p_2 is continuous on $a \in (0,1)$.

Since $p_2(a,a,c) \rightarrow 1$ as $a \rightarrow 0$ and $p_2(a,a,c) \rightarrow 1$ as $a \rightarrow 1$, p_2 is also continuous at 0 and 1.

Case 3. Let $a < b$ and β such that $a < \beta < b$. For that β and any $\varepsilon > 0$, there exists c' such that

$$\Phi\left(\frac{c+c'}{\sqrt{\beta(1-\beta)}}\right) - \Phi\left(\frac{c}{\sqrt{\beta(1-\beta)}}\right) < \varepsilon.$$

Then, for any c'' with $c < c'' < c+c'$, we have

$$\begin{aligned} p_2(a,b,c'') - p_2(a,b,c) &= P\{|B(s)| \leq c'' : a \leq s \leq b\} - P\{|B(s)| \leq c : a \leq s \leq b\} \\ &= P\{c \leq |B(s)| \leq c'' : a \leq s \leq b\} \end{aligned}$$

$$\leq P\{c \leq |B(\beta)| \leq c''\}$$

$$\leq \Phi\left(\frac{c+c'}{\sqrt{\beta(1-\beta)}}\right) - \Phi\left(\frac{c}{\sqrt{\beta(1-\beta)}}\right) < \varepsilon.$$

Hence p_2 is continuous and the proof of (i) is complete.

Proof of (ii). By the proof of (i) we have $c'(\varepsilon) > 0$ for any $\varepsilon > 0$ such that

$$p_2(a, b, c+c') - p_2(a, b, c) < \frac{1}{2}\varepsilon.$$

By the (ii) of Lemma III.B, for that c and c' , there exists $h > 0$ such that

$$P\left\{\sup_{0 \leq s \leq 1-h} \sup_{0 \leq s' \leq h} |B(s+s') - B(s)| > c'\right\} < \frac{1}{2}\varepsilon.$$

Case 1. Let $a < b$ and consider any a' with $a < a' < a+h \leq b$. Then

$$\begin{aligned} & p_2(a', b, c) - p_2(a, b, c) \\ &= P\{|B(s)| \leq c : a' \leq s \leq b\} - P\{|B(s)| \leq c : a \leq s \leq b\} \\ &= P\left\{\sup_{a' \leq s \leq b} |B(s)| \leq c, \sup_{0 \leq s \leq 1-h} \sup_{0 \leq s' \leq h} |B(s+s') - B(s)| \leq c'\right\} \\ &\quad + P\left\{\sup_{a' \leq s \leq b} |B(s)| \leq c, \sup_{0 \leq s \leq 1-h} \sup_{0 \leq s' \leq h} |B(s+s') - B(s)| > c'\right\} \\ &\quad - P\left\{\sup_{a \leq s \leq b} |B(s)| \leq c\right\}. \end{aligned}$$

$$\begin{aligned}
&< P\left\{ \sup_{a' \leq s \leq b} |B(s)| \leq c, \sup_{0 \leq s \leq 1-h} \sup_{0 \leq s' \leq h} |B(s+s') - B(s)| \leq c' \right\} \\
&\quad + \frac{1}{2}\epsilon - P\left\{ \sup_{a \leq s \leq b} |B(s)| \leq c \right\} \\
&\leq P\left\{ \sup_{(a'-h) \leq s \leq b} |B(s)| \leq c+c' \right\} + \frac{1}{2}\epsilon - P\left\{ \sup_{a \leq s \leq b} |B(s)| \leq c \right\} \\
&\leq P\left\{ \sup_{a \leq s \leq b} |B(s)| \leq c+c' \right\} - P\left\{ \sup_{a \leq s \leq b} |B(s)| \leq c \right\} + \frac{1}{2}\epsilon \\
&= p_2(a, b, c+c') - p_2(a, b, c) + \frac{1}{2}\epsilon < \epsilon.
\end{aligned}$$

Case 2. Let $a = b$ and consider any a' with $(b-h) < a' < b$. Then

$$\begin{aligned}
&p_2(b, b, c) - p_2(a', b, c) \\
&= P\{|B(b)| \leq c\} - P\{|B(s)| \leq c : a' \leq s \leq b\} \\
&\leq P\{|B(b)| \leq c\} - P\left\{ \sup_{(b-h) \leq s \leq b} |B(s)| \leq c \right\} \\
&= P\{|B(b)| \leq c, \sup_{0 \leq s' \leq h} |B(b-s') - B(b)| \leq c'\} \\
&\quad + P\{|B(b)| \leq c, \sup_{0 \leq s' \leq h} |B(b-s') - B(b)| > c'\} \\
&\quad - P\left\{ \sup_{(b-h) \leq s \leq b} |B(s)| \leq c \right\} \\
&< P\left\{ \sup_{(b-h) \leq s \leq b} |B(s)| \leq c+c' \right\} + \frac{1}{2}\epsilon - P\left\{ \sup_{(b-h) \leq s \leq b} |B(s)| \leq c \right\} \\
&= P\left\{ c \leq \sup_{(b-h) \leq s \leq b} |B(s)| \leq c+c' \right\} + \frac{1}{2}\epsilon
\end{aligned}$$

$$\leq P\{c \leq |B(b)| \leq c+c'\} + \frac{1}{2}c$$

$$= p_2(b,b,c+c') - p_2(b,b,c) + \frac{1}{2}c < \epsilon.$$

We proved that, for fixed b and c , p_2 is continuous with respect to a on $[0,b]$ and, of course, p_2 is bounded on $[0,b]$. Hence p_2 is uniformly continuous on $[0,b]$. $\square$

Proof of (iii). We omit the proof here, since it is similar to that of (ii).

Theorem III.3.2. Let $\{B(s) : 0 \leq s \leq 1\}$ be Brownian bridge process. Define three-parameter functions φ_1 and φ_2 such that

$$(III.3.2) \quad \varphi_1(a,b,p) := \sup\{c \in (0,\infty) : P\{B(s) \leq c : a \leq s \leq b\} \leq p\},$$

$$\varphi_2(a,b,p) := \sup\{c \in (0,\infty) : P\{|B(s)| \leq c : a \leq s \leq b\} \leq p\},$$

where $a \in (0,b]$, $b \in [a,1]$ and $p \in (0,1)$. Then we have

(i) φ_1 and φ_2 are monotone non-decreasing continuous functions with respect to p on $(0,1)$ for fixed a and b ;

(ii) φ_1 and φ_2 are strictly monotone decreasing continuous functions with respect to a on $(0,b]$ for fixed b and p ;

(iii) φ_1 and φ_2 are strictly monotone increasing continuous functions with respect to b on $[a,1]$ for fixed a and p .

Proof. We will prove these statements for ϕ_2 only. The monotonicity properties are obvious.

Proof of (i). Case 1. Let $a = b = a'$. Then, for any $p \in (0,1)$,

$$\begin{aligned}\phi_2(a', a', p) &= \sup \{ c \in (0, \infty) : P\{|B(a')| \leq c\} \leq p \} \\ &= \sup \{ c : 2 \Phi(c/\sqrt{a'(1-a')}) - 1 \leq p \} \\ &= \{ c : 2 \Phi(c/\sqrt{a'(1-a')}) - 1 = p \} \\ &= \sqrt{a'(1-a')} \Phi^{-1}(\tfrac{1}{2}(1+p)),\end{aligned}$$

where $\Phi(\cdot)$ is defined in (I.2.8) and is a strictly monotone continuous function. Hence the inverse function $\Phi^{-1}(\cdot)$ is also continuous.

Case 2. Let $a = a'$, $b = b'$ and $a' < b'$. Then, since we have that $p_2(a', b', c) = P\{|B(s)| \leq c : a' \leq s \leq b'\}$ is a strictly monotone continuous function in $c \in (0, \infty)$,

$$\begin{aligned}\phi_2(a', b', p) &= \sup \{ c : P\{|B(s)| \leq c : a' \leq s \leq b'\} \leq p \} \\ &= \{ c : p_2(a', b', c) = p \}\end{aligned}$$

for any $p \in (0,1)$. Hence ϕ_2 is the inverse function of p_2 and thus continuous.

Proof of (ii). For any fixed a and b , p_2 is monotone continuous in c and also for every fixed c and b , p_2 is monotone continuous in a . This implies that, for fixed $b = b'$ and $p = p'$, we have

$$\begin{aligned}\varphi_2(a, b', p') &= \sup \{ c : p_2(a, b', c) \leq p' \} \\ &= \{ c : p_2(a, b', c) = p' \},\end{aligned}$$

and hence φ_2 is continuous in a .

Proof of (iii). It is similar to that of (ii). $\square$

Remark III.3.1. We could restate Theorems III.3.1 and III.3.2 in terms of Wiener process and prove the resulting statements by using (i) of Lemma III.A, instead of (ii) of Lemma III.A, in the present proof of Theorems III.3.1 and III.3.2. We present these statements here without proofs. Let us define

$$\begin{aligned}(III.3.3) \quad w_1(c, a, b, p) &:= \sup \{ d\varepsilon(0, \infty) : P\{ |W(t)| \leq c+dt : a \leq t \leq b \} \leq p \}, \\ w_2(d, a, b, p) &:= \sup \{ c\varepsilon(0, \infty) : P\{ |W(t)| \leq c+dt : a \leq t \leq b \} \leq p \},\end{aligned}$$

$$\begin{aligned}(III.3.4) \quad w_3(c, a, b, p) &:= \sup \{ d\varepsilon(0, \infty) : P\{ W(t) \leq c+dt : a \leq t \leq b \} \leq p \}, \\ w_4(d, a, b, p) &:= \sup \{ c\varepsilon(0, \infty) : P\{ W(t) \leq c+dt : a \leq t \leq b \} \leq p \}.\end{aligned}$$

Then the functions defined are strictly monotone continuous functions with respect to their corresponding parameters.

4. Confidence bands for quantile function on $[a, b]$, $t_{F0} < a \leq b < T_F$.

We now state and prove our main theorem, answering the question asked at the end of Section 2 of this chapter.

Theorem III.4.1. Assume that (i) and (ii) of Theorem II.B. Let $a > t_{F0}$, $b > a$ and $b < T_F$. For a given level of significance α , we define

$$\tau_n := \varphi_2\left(\frac{d_n(Q_n(a))}{d_n(Q_n(a)) + d_n(Q_n(b))}, \frac{1}{2}, 1-\alpha \right),$$

where $\varphi_2(\cdot)$ is defined in (III.3.3). Then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} P\{ Q_m(t - b_m(t, \tau_n, \tau_n)) \leq Q(t) \leq Q_m(t + b_m(t, \tau_n, \tau_n)) : a \leq t \leq b \} \\ \text{(III.4.1)} \\ = 1 - \alpha, \end{aligned}$$

where $b_m(\cdot)$ is defined in (III.2.4) (cf. Remark III.2.2).

Proof. For fixed a and b , we have (cf. (II.3.9)) $Q_n(a) \xrightarrow{\text{a.s.}} Q(a)$ and $Q_n(b) \xrightarrow{\text{a.s.}} Q(b)$ as $n \rightarrow \infty$, and also $d_n(t) \xrightarrow{\text{a.s.}} d(t)$ as $n \rightarrow \infty$ for all $t \in [a, b]$ (cf. (II.2.9)).

In addition, we get $d_n(Q_n(a)) > 0$, since $a > t_{F0}$ implies $Q_n(a) > 0$. We also obtain $d_n(Q_n(b)) > 0$, since $b > a$. Hence

$$\frac{d_n(Q_n(a))}{d_n(Q_n(a)) + d_n(Q_n(b))} \xrightarrow{\text{a.s.}} \frac{d(Q(a))}{d(Q(a)) + d(Q(b))} \text{ as } n \rightarrow \infty.$$

For a given level of significance α , by Theorem III.3.2, $\phi_2(u, \frac{1}{2}, 1-\alpha)$ is continuous in u . Thus

$$\tau_n \xrightarrow{s.} \phi_2\left(\frac{d(Q(a))}{d(Q(a))+d(Q(b))}, \frac{1}{2}, 1-\alpha\right) \text{ as } n \rightarrow \infty.$$

Consequently

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} P\{Q_m(t-b_m(t, \tau_n, \tau_n)) \leq Q(t) \leq Q_m(t+b_m(t, \tau_n, \tau_n)) : a \leq t \leq b\}$$

$$= \lim_{n \rightarrow \infty} P\{|B(s)| \leq \tau_n : \frac{d(Q(a))}{d(Q(a))+d(Q(b))} \leq s \leq \frac{1}{2}\}$$

$$= P\{|B(s)| \leq \tau : \frac{d(Q(a))}{d(Q(a))+d(Q(b))} \leq s \leq \frac{1}{2}\},$$

$$\text{where } \tau := \phi_2\left(\frac{d(Q(a))}{d(Q(a))+d(Q(b))}, \frac{1}{2}, 1-\alpha\right).$$

This completes the proof. $\square$

Corollary III.4.1. Assume that (i) and (ii) of Theorem II.B. Let $a > t_{FO}$, $b > a$ and $b < T_F$. For a given level of significance α , we define

$$\tau_n := \phi_1\left(\frac{d_n(Q_n(a))}{d_n(Q_n(a))+d_n(Q_n(b))}, \frac{1}{2}, 1-\alpha\right),$$

where $\phi_1(\cdot)$ is defined in (III.3.3). Then we have

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} P\{ Q(t) \leq Q_m(t + b_m(t, \tau_n', \tau_n')) : a \leq t \leq b \} \\
 (III.4.2) \quad & = 1 - \alpha.
 \end{aligned}$$

Remark III.4.1. Opposed to τ in (III.2.22) and τ' in (III.2.23), τ_n in (III.4.1) and τ_n' in (III.4.2) do not depend on the unknown functions $d(\cdot)$ and $Q(\cdot)$. By using the formulae (cf.(I.3.10) and (I.4.12)) for $p_1(\cdot)$ and $p_2(\cdot)$ in (III.3.1) and observing that p_1 and p_2 are strictly monotone continuous functions as shown in Theorem III.3.1, τ_n and τ_n' can be easily evaluated (cf. Chung (1986)). Then using (III.4.1) and (III.4.2), we can construct two-sided and one-sided confidence bands for $Q(\cdot)$ on a given range $[a, b]$.

Remark III.4.2. Using Remark III.3.1, we can also prove Theorem III.4.1 and Corollary III.4.1 in terms of Wiener process instead of Brownian bridge process by following nearly identical arguments in the proofs in this section. We will again state the following without proof:

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} P\{ Q_m(t - b_m(t, \beta_1, \beta_2, n)) \leq Q(t) \leq Q_m(t + b_m(t, \beta_1, \beta_2, n)) : a \leq t \leq b \} \\
 (III.4.3) \quad & = 1 - \alpha,
 \end{aligned}$$

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} P\{ Q_m(t - b_m(t, \beta_1, n, \beta_2)) \leq Q(t) \leq Q_m(t + b_m(t, \beta_1, n, \beta_2)) : a \leq t \leq b \} \\
 (III.4.4) \quad & = 1 - \alpha,
 \end{aligned}$$

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} P\{ Q(t) \leq Q_m(t + b_m(t, \beta_1, \beta_2', n)) : a \leq t \leq b \} \\
 (III.4.5) \quad & = 1 - \alpha,
 \end{aligned}$$

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} P\{ Q(t) \leq Q_m(t + b_m(t, \beta'_{1,n}, \beta_2)) : a \leq t \leq b \} \\
 \text{(III.4.6)} \quad & = 1 - \alpha,
 \end{aligned}$$

where

$$\begin{aligned}
 \beta_{1,n} &:= w_1(\beta_2, d_n(Q_n(a))/d_n(Q_n(b)), 1, 1-\alpha) \\
 \beta_{2,n} &:= w_2(\beta_1, d_n(Q_n(a))/d_n(Q_n(b)), 1, 1-\alpha) \\
 \beta'_{1,n} &:= w_3(\beta_2, d_n(Q_n(a))/d_n(Q_n(b)), 1, 1-\alpha) \\
 \beta'_{2,n} &:= w_4(\beta_1, d_n(Q_n(a))/d_n(Q_n(b)), 1, 1-\alpha),
 \end{aligned}$$

and w_1, w_2, w_3 and w_4 are defined in (III.3.3) and (III.3.4).

5. Applications.

The methodology developed is applied to estimate the quantile function of the length of fractures in the granitic plutons near Lac du Bonnet, Manitoba, Canada.

The nuclear waste disposal program in Canada involves emplacement of a vault containing waste in a stable geological formation such as the granitic plutons in the Canadian Shield. The rock mass surrounding the vault containing the waste acts as a natural barrier between the waste and the biosphere. It is assumed, however, that the ground water system contaminated with radionuclides from the waste eventually migrates through the rock mass and reaches the biosphere. The fracture system in the host rock forms the main migration pathways.

Although the fractures are approximately linear planes in nature; we only observe linear lines on the surface of the rock mass. Studying the distribution and the quantile functions of the length of the fracture lines on the surface is the first step toward understanding the fracture system in the host rock.

The granitic pluton near Lac du Bonnet, Manitoba, where the Underground Research Laboratory is being built for the nuclear waste disposal program in Canada, is selected for this application. Stone et al.(1984) mapped the fractures on the surface of the granite. We chose a small area, shown as Figure III.1, from the map by Stone et al.(1984) for this study.

As shown in Figure III.1, about 67% of the granite pluton(shaded area) are covered by soil and vegetation, and thus we can only observe the fracture lines on the exposed areas(about 33%) of the granite. Consequently, most of the fractures are extended beyond the exposed areas. We have observed 1567 fractures. Of these, both ends are shown only for 256 fractures in the exposed areas whose lengths can be completely measured. The rest, namely 1311 fracture lines, are censored. The lengths of the longest un-censored and censored fracture lines are 0.991 and 2.361 (1 unit = 2.54 m), respectively. The length of the shortest fracture, which happens to be un-censored, is 0.01.

The PL-estimator F_{1567}^0 (cf. (II.1.5)) and the corresponding PL-quantile function Q_{1567} (cf. (II.3.2)), based on the above discussed

observations, are obtained and shown in Figure III.2 and III.3, respectively. Due to the heavy censoring, $F_{1567}^0(t)$ and $Q_{1567}(y)$ can not have any significance when $t > 0.9910$ and $y > F_n^0(0.991) = 0.3403$, respectively, although they are well defined on the whole line by definition.

To construct confidence bands for the quantile function by applying Theorem III.2.1, we set $\epsilon(n) = \log \log \log n / \sqrt{n}$, $p_0 = 0.3$ and $\alpha = 0.05$ for this example. We then obtain that $\epsilon(1567) = 0.0175$. We also get four sets of (β_1, β_2) for given $\alpha = 0.05$ such that $P\{|W(s)| \leq \beta_1 s + \beta_2 : 0 \leq s \leq 1\} = 1 - \alpha = 0.95$ by using Theorem I.E. The four sets are: $(0, 2.24141)$, $(0.5, 1.82445)$, $(1, 1.45132)$ and $(1.27306, 1.27306)$.

Using $b_n(t, \beta_1, \beta_2)$ defined in (III.2.4) of Theorem III.2.1, we construct the four confidence bands,

$$\{ [Q_{1567}(t - b_{1567}(t, \beta_1, \beta_2)), Q_{1567}(t + b_{1567}(t, \beta_1, \beta_2))] : \epsilon(1567) \leq t \leq p_0 \}$$

which are shown in Figures III.4, III.5, III.6 and III.7.

As mentioned in Remark III.2.1, the confidence band in Figure III.4 is the Aly-Csörgö-Horváth band in Theorem III.A.

For the second example, as applications of Theorem III.4.1 and (III.4.3) in Remark III.4.2, we construct 95% confidence bands for the

quantile function on a fixed interval $[a, b]$, where we set $a = 0.1$ and $b = 0.25$. Then, at $\alpha = 0.05$, we obtain that, since $d_{1567}(Q_{1567}(0.1)) = 0.1333$ and $d_{1567}(Q_{1567}(0.25)) = 0.6777$,

$$\begin{aligned}\tau_{1567} &= \inf\{c : P\{|B(t)| \leq c : (0.1333/0.8110) \leq t \leq 1\} = 0.95\} \\ &= 1.27297\end{aligned}$$

by using (I.4.12) of Corollary I.4.2.

Using Theorem III.4.1, we get the confidence band for the quantile function on $[0.1, 0.25]$

$$\{ [Q_{1567}(t - b_{1567}(t, \tau_{1567}, \tau_{1567}), Q_{1567}(t + b_{1567}(t, \tau_{1567}, \tau_{1567})) : 0.1 \leq t \leq 0.25] \}$$

which is shown in Figure III.8.

To apply (III.4.3) we also set $\beta_1 = 0, 0.5$ and 1 ; and for each β_1 , we compute $\beta_{2,1567} = w_2(\beta_1, 0.1333/0.6777, 1, 1-0.05)$ by using (III.3.3) and (I.4.7) (cf. Chung (1986)). The three sets of values for $(\beta_1, \beta_{2,1567})$ are $(0, 2.24141)$, $(0.5, 1.82443)$ and $(1, 1.45093)$. Applying these values, we obtain three confidence bands for $Q(t)$ on $[a, b]$ and they are shown in Figures III.9, III.10 and III.11. It should be noted that the four sets of values for Wiener process on $(0, 0.3]$ and those on $[0.1, 0.25]$ are very similar, since both $P\{|W(t)| \leq \beta_1 t + \beta_2 : 0 \leq t \leq 1\}$ and $P\{|W(t)| \leq \beta_1 t + \beta_2 : 0.2 \leq t \leq 1\}$ are approximately 0.95 for those (β_1, β_2) . Consequently, the widths of the confidence bands in

Figures III.8, III.9, III.10 and III.11 are quite similar to the corresponding confidence bands in Figures III.7, III.4, III.5 and III.6.

Figure III.1. Fracture patterns in a part of Stone et al. (1984) map of a granitic pluton near Lac du Bonnet, Manitoba, Canada. There are 1567 fractures. Of these, both ends are shown only for 256 fractures in the exposed areas whose lengths can be completely measured. The rest, namely 1311 fractures are censored.

Figure III.2. PL-estimator F_{1567}^0 (cf. (II.1.5)) of the distribution function F^0 of the length of fractures shown in Figure III.1.

Figure III.3. PL-quantile function Q_{1567} (cf. (II-3.2)) of the quantile function Q of the length of fractures shown in Figure III.1.

Figure III.4. PL-quantile function Q_{1567} and 95 % confidence band for Q on $[\epsilon(1567), 0.3]$, using $(\beta_1, \beta_2) = (0.0, 2.24141)$ in Theorem III.2.1.

Figure III.5. PL-quantile function Q_{1567} and 95 % confidence band for Q on $[e(1567), 0.3]$, using $(\beta_1, \beta_2) = (0.5, 1.82445)$ in Theorem III.2.1.

Figure III.6. PL-quantile function Q_{1567} and 95 % confidence band for Q on $[\varepsilon(1567), 0.3]$, using $(\hat{\beta}_1, \hat{\beta}_2) = (1.0, 1.45132)$ in Theorem III.2.1.

Figure III.7. PL-quantile function Q_{1567} and 95 % confidence band : r . . .
 on $[\varepsilon(1567), 0.3]$, using $(\beta_1, \beta_2) = (1.2^{*}306, 2.2^{*}306)$ in
 Theorem III.2.1.

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Figure III.8. PL-quantile function Q_{1567} and 95 % confidence band for Q on $[0.1, 0.25]$, using $\tau_{1567} = 1.27297$ in Theorem III.4.1.

Figure III.9. PL-quantile function Q_{1567} and 95 % confidence band for Q on $[0.1, 0.25]$, using $(\beta_1, \beta_{2,1567}) = (0.0, 2.24141)$ in (III.4.3) of Remark III.4.2.

Figure III.10. PL-quantile function Q_{1567} and 95 % confidence band for Q on $[0.1, 0.25]$, using $(\beta_1, \beta_{2,1567}) = (0.5, 1.82443)$ in (III.4.3) of Remark III.4.2.

Figure III.11. PL-quantile function Q_{1567} and 95 % confidence band for Q on $[0.1, 0.25]$, using $(\beta_1, \beta_{2,1567}) = (1.0, 1.45093)$ in (III.4.3) of Remark III.4.2.

IV. AN EXTENSION OF THE KOLMOGOROV-SMIRNOV CONFIDENCE BAND FOR TRUNCATED DATA

1. Introduction.

In this chapter a modified Kolomogorov-Smirnov procedure is proposed for constructing confidence bands for a distribution function from truncated data.

Let $X'_1, \dots, X'_n$ be independent random observations from a continuous distribution function F , and let F_n denote the empirical distribution function of the sample:

$$(IV.1.1) \quad F_n(x) = \begin{cases} 0 & X'_{1:n} > x, \\ k/n, & X'_{k:n} \leq x < X'_{k+1:n}, \quad k = 1, \dots, n-1, \\ 1 & X'_{n:n} \leq x, \end{cases}$$

where $X'_{i:n}$ is the i -th ordered statistic of $X'_1, \dots, X'_n$.

Suppose we want to construct a confidence band for F or we wish to test the null hypothesis:

$$(IV.1.2) \quad H_0 : F = F_0,$$

where F_0 is a completely specified distribution function.

A most commonly used methodology is the Kolmogorov-Smirnov(K.-S.) procedure based on the following limit theorems:

$$(IV.1.3) \quad \lim_{n \rightarrow \infty} P\left\{ \sqrt{n} \sup_{-\infty < x < \infty} |F_n(x) - F(x)| \leq c \right\} \\ = P\left\{ \sup_{0 \leq s \leq 1} |B(s)| \leq c \right\}, \text{ and}$$

$$(IV.1.4) \quad \lim_{n \rightarrow \infty} P\left\{ \sqrt{n} \sup_{-\infty < x < \infty} (F_n(x) - F(x)) \leq c' \right\} \\ = P\left\{ \sup_{0 \leq s \leq 1} B(s) < c' \right\},$$

where $\{B(s): 0 \leq s \leq 1\}$ is a Brownian bridge process(cf. Doob, 1949).

— Suppose that the random sample is doubly truncated at u and v , and instead of $X'_1, \dots, X'_n$, we have $X_1, \dots, X_n$ where

$$(IV.1.5) \quad X_i = \begin{cases} u, & u > X'_i, \\ X'_i, & u \leq X'_i \leq v, \\ v, & v < X'_i, \end{cases}$$

for $i = 1, \dots, n$. Then the empirical distribution $F_n(\cdot)$ in (IV.1.1) is no longer defined outside of $[u, v]$. We would like to construct confidence bands for F on $[u, v]$ or to test the hypothesis that

$$(IV.1.6) \quad H_0 : F(x) = F_0(x), \quad x \in [u, v].$$

Using Donsker's theorem(Donsker (1952)), or the Komlós, Major and Tusnády theorem(Theorem I.C), we get that

$$\begin{aligned}
 \text{(IV.1.7)} \quad \lim_{n \rightarrow \infty} P\{ \sqrt{n} \sup_{u \leq x \leq v} |F_n(x) - F(x)| \leq c \} \\
 = P\{ \sup_{F(u) \leq s \leq F(v)} |B(s)| \leq c \}, \text{ and}
 \end{aligned}$$

$$\begin{aligned}
 \text{(IV.1.8)} \quad \lim_{n \rightarrow \infty} P\{ \sqrt{n} \sup_{u \leq x \leq v} (F_n(x) - F(x)) < c' \} \\
 = P\{ \sup_{F(u) \leq s \leq F(v)} B(s) \leq c' \}.
 \end{aligned}$$

However, contrary to the non-truncated case in (IV.1.3) and (IV.1.4), c and c' in (IV.1.7) and (IV.1.8) depend upon the unknown distribution function F , and thus we can not immediately use Donsker's theorem to construct a confidence band for F .

Nevertheless (IV.1.7) and (IV.1.8) can be used for testing the null hypothesis in (IV.1.6) on $[u, v]$. Namely, we first define $c := c(\alpha, F(u), F(v))$ and $c' := c'(\alpha, F(u), F(v))$ for any $\alpha \in (0, 1)$ and any distribution function F by

$$\text{(IV.1.9)} \quad P\{ \sup_{F(u) \leq s \leq F(v)} |B(s)| \leq c \} = 1 - \alpha, \text{ and}$$

$$\text{(IV.1.10)} \quad P\{ \sup_{F(u) \leq s \leq F(v)} B(s) \leq c' \} = 1 - \alpha.$$

Then, for a given level of significance α , we compute $c(\alpha, F_0(u), F_0(v))$ and $c'(\alpha, F_0(u), F_0(v))$ from (IV.1.9) and (IV.1.10), and reject the hypothesis in (IV.1.6) if, for two-sided statistics,

$$\sqrt{n} \sup_{u \leq x \leq v} |F_n(x) - F_0(x)| \geq c(\alpha, F_0(u), F_0(v)),$$

and, for one-sided statistics,

$$\sqrt{n} \sup_{u \leq x \leq v} (F_n(x) - F_0(x)) \geq c'(\alpha, F_0(u), F_0(v)).$$

Unlike the K.-S. tests in the non-truncated case, the critical values c and c' depend on the null hypothesis.

For singly truncated data and censored data from the right, Barr and Davidson (1973), and Koziol and Byar (1975) proposed to use the special cases of (IV.1.7) and (IV.1.8) when $u = -\infty$ (i.e., $F(u) = 0$) for testing the hypothesis in (IV.1.6). Barr and Davidson (1973) suggested to use $c(\alpha, 0, F_0(v))$ and $c'(\alpha, 0, F_0(v))$ as the critical values, while Koziol and Byar (1975) suggested to use $c(\alpha, 0, w)$ and $c'(\alpha, 0, w)$ where w is slightly greater than both $F_0(v)$ and $F_n(v)$.

In addition, Gillespie and Fisher (1979), and Hall and Wellner (1980) used $c(\alpha, 0, F(w))$ in (IV.1.9), where $w < \inf\{x: F(x)=1\}$, to construct confidence bands for the product-limit survival curve in the random censorship model from the right. In particular, $c(\alpha, 0, \hat{F}_n(w))$ was suggested instead of $c(\alpha, 0, F(w))$ without a rigorous discussion of this substitution, where $\hat{F}_n$ is the PL estimator of F (cf. (II.1.5)).

In this chapter, we justify the use of $c(\alpha, F_n(u), F_n(v))$ for $c(\alpha, F(u), F(v))$ via proving that asymptotically the substitution would not make any difference. This will be the topic of the next section.

We conclude this section by remarking that formulae for the probability statements in (IV.1.9) and (IV.1.10) are derived in (I.4.12) and (I.3.10), respectively, (cf. Corollaries I.4.2 and I.3.2).

2. Modified K.-S. confidence band for distribution function.

As in Theorem III.3.2, let again

$$(IV.2.1) \quad \phi_1(a,b,p) := \sup\{ c : P\{ B(s) \leq c : a \leq s \leq b \} \leq p \}$$

and

$$(IV.2.2) \quad \phi_2(a,b,p) := \sup\{ c : P\{|B(s)| \leq c : a \leq s \leq b \} \leq p \},$$

where $a \in [0,1]$, $b \in [a,1]$ and $p \in (0,1)$. The latter theorem concludes that ϕ_1 and ϕ_2 are well-defined and continuous with respect to all three parameters a , b and p . By Glivenko-Cantelli theorem, for any u and v ,

$$F_n(u) \xrightarrow{\text{a.s.}} F(u) \text{ and } F_n(v) \xrightarrow{\text{a.s.}} F(v) \text{ as } n \rightarrow \infty.$$

Since ϕ_1 and ϕ_2 are continuous, we have, for any $\alpha \in (0,1)$,

$$(IV.2.3) \quad \phi_1(F_n(u), F_n(v), 1-\alpha) \xrightarrow{\text{a.s.}} \phi_1(F(u), F(v), 1-\alpha),$$

$$(IV.2.4) \quad \phi_2(F_n(u), F_n(v), 1-\alpha) \xrightarrow{\text{a.s.}} \phi_2(F(u), F(v), 1-\alpha),$$

as $n \rightarrow \infty$. Now (IV.2.3) and (IV.2.4) imply our main result here:

Theorem IV.2.1. Let $F_n(\cdot)$ be the empirical distribution function of a continuous distribution function $F(\cdot)$ on $[u, v]$ where $u > \sup\{t: F(t)=0\}$ and $v < \inf\{t: F(t)=1\}$ and $u < v$. Then, using ϕ_1 and ϕ_2 as defined in (IV.2.1) and (IV.2.2), we have, for any $\alpha \in (0, 1)$,

$$(IV.2.5) \quad \lim_{n \rightarrow \infty} P\left\{ \sqrt{n} \sup_{u \leq x \leq v} (F_n(x) - F(x)) \leq \phi_1(F_n(u), F_n(v), 1-\alpha) \right\} = 1 - \alpha, \text{ and}$$

$$(IV.2.6) \quad \lim_{n \rightarrow \infty} P\left\{ \sqrt{n} \sup_{u \leq x \leq v} |F_n(x) - F(x)| \leq \phi_2(F_n(u), F_n(v), 1-\alpha) \right\} = 1 - \alpha.$$

Remark IV.2.1. In order to construct a two-sided confidence band for F on $[u, v]$, where F_n are well-defined, for a given significance level α , $\phi_2(F_n(u), F_n(v), 1-\alpha)$ is first computed by using the formula discussed in Chapter I and the computer program by Chung (1986). Then the two-sided confidence band is constructed by

$$(IV.2.7) \quad [F_n(x) - n^{-\frac{1}{2}} \phi_2(F_n(u), F_n(v), 1-\alpha), \\ F_n(x) + n^{-\frac{1}{2}} \phi_2(F_n(u), F_n(v), 1-\alpha)]$$

for all $x \in [u, v]$.

Of course, when $u \leq \sup\{t: F(t)=0\}$ and $v \geq \inf\{t: F(t)=1\}$, then the confidence band in (IV.2.7) coincides with the K.-S. confidence band.

3. Applications.

The data to be considered here come from GSC Open file 88 (Goodfellow, 1981), which covers a geochemical reconnaissance survey of the Nahanni area, Yukon, Canada (Map - NTS 1051). There are 211 samples of stream waters in the north-eastern quarter of NTS 1051 and all 211 samples were analyzed in order to observe the amount of Mn(manganese) in stream waters.

When the amount of Mn in the samples is below the detection limit of 10 ppm(part per million), then it has been recorded as " $< 10\text{ppm}$ ". Among 211 samples, only 27 samples had Mn above the detection limit of 10ppm. The other 187 samples contained less than 10ppm of Mn. The 27 values above 10ppm are:

250	48	48	191	127	27	38	139	75	62	46	11	138	25
23	82	63	375	80	363	256	33	154	42	32	31	346.	

To apply (IV.2.6) in Theorem IV.2.1, we have $u = 10$, $v = \infty$ and thus $F_{211}(10) = 187/211 = 0.8863$ and $\phi_2(0.8863, 1, 0.95) = 0.724$ for $\alpha = 0.05$. The latter 0.724 is to be compared with $c = \phi_2(0, 1, 0.95) = 1.358$ of the K.-S. band in (IV.1.3). Figure IV.1 illustrates the truncated empirical distribution function on $[10, \infty)$ defined in (IV.1.5), the K.-S. confidence band in (IV.1.3) and the modified confidence band in Theorem IV.2.1. As expected, the modified band is certainly much narrower than the K.-S. band.

As a second example, a set of 100 pseudo-random numbers from a standard normal random variable with mean 0 and variance 1 was first generated using subroutine GGNML of IMSL Library (IMSL, 1984). Then, among the 100 observations, 65 values below 0.5 and 19 values above 1.0 are assumed to be truncated. Only 16 values between 0.5 and 1.0 are considered to have been observed. These 16 values are:

.547458	.562069	.593493	.615984	.620805	.669589	.688242
.726850	.753925	.781613	.809352	.886492	.894751	.946147
.997023	.998144					

Hence, we have $u = 0.5$, $v = 1.0$, $F_{100}(0.5) = 0.65$ and $F_{100}(1.0) = 0.81$. If we set $\alpha = 0.05$, then $\phi_2(F_{100}(0.5), F_{100}(1.0), 0.95) = \phi_2(0.65, 0.81, 0.95) = 1.140$, which is to be compared to $\phi_2(F(0.5), F(1.0), 0.95) = \phi_2(0.6915, 0.8413, 0.95) = 1.093$. This comparison implies that, in this example, the modified band in (IV.2.6) using 1.140 is wider than the theoretical band in (IV.1.7) using 1.093.

Using the empirical value 1.140 and Theorem IV.2.1, we now construct a 95% confidence band for F on $[0.5, 1.0]$ which is shown in Figure IV.2 together with the empirical distribution and the K.-S. confidence band.

Figure IV.1. Empirical distribution function of Mn(manganese) in stream waters in north-eastern quarter of map NTS 1051, Nahanni area, Yukon, Canada, with the 95% K.-S. confidence band, and the 95% modified confidence band of Theorem IV.2.1.

Figure IV.2. Empirical distribution function of 100 simulated values from $N(0,1)$ which are truncated at both sides of 0.5 and 1.0, with the 95% K.-S. confidence band, and the 95% modified confidence band of Theorem IV.2.1.

V. CONFIDENCE BANDS FOR PERCENTILE RESIDUAL LIFETIME UNDER THE RANDOM CENSORSHIP MODEL FROM THE RIGHT.

1. Introduction.

Let p be a fixed number with $0 < p < 1$ and let us define the $(1-p)$ percentile residual lifetime as:

$$(V.1.1) \quad R^{(p)}(t) := Q(1-p(1-F^0(t))) - t, \quad t > 0,$$

where F^0 is a continuous distribution with support $(0, b)$, where $b = \inf\{t > 0 : F^0(t) = 1\}$, and the corresponding quantile function

$$(V.1.2) \quad Q(x) := \inf\{t > 0 : F^0(t) \geq x\}, \quad 0 < x < 1,$$

$$Q(0) = 0 \text{ and } Q(1) = b.$$

The notion of the $(1-p)$ percentile residual lifetime was originally introduced by Haine and Singpurwalla (1974). Schmittlein and Morrison (1981) explain in detail the potential advantages of the median residual lifetime $R^{(1/2)}(t)$ over the more frequently used mean residual lifetime. In fact, without additional conditions, the mean residual lifetime cannot even be estimated under random censorship from the right. These two papers and the papers by Arnold and Brockett (1983), Joe and Proschan (1984a) and Gupta and Langford (1984) deal with

characterization theorems and provide results for the associated inversion problem. Joe and Proschan (1984b) use $R^{(p)}(t)$ to compare two lifetime distributions. The results of Arnold and Brockett (1983) and Joe and Proschan (1984a) imply that a single percentile lifetime function $R^{(p)}(t)$ does not in general determine the underlying lifetime distribution, but the knowledge of these functions for two or more values of p does determine the distribution. M. Csörgö and S. Csörgö (1986) developed the non-parametric large sample estimation theory of, and also constructed confidence bands for, the $(1-p)$ percentile residual lifetime based on non-censored data.

The aim of this chapter is to extend the Csörgö-Csörgö result to the random censorship model from the right, which is described in Chapter II.

We will first define the scaled PL $(1-p)$ percentile residual lifetime process $r_n^{(p)}(t)$ and show in Section 2 that the asymptotic distribution of $r_n^{(p)}(t)$ is normal for t and p fixed. Then we construct a confidence interval for the $(1-p)$ percentile residual lifetime, using the normality of the limiting distribution function, in Section 3.

In Section 4, we will view $r_n^{(p)}(t)$ as a stochastic process : (i) in t for fixed $p \in (0,1)$; (ii) in p for fixed $t \geq 0$; and (iii) in both t and p , and study strong approximations of these processes. Using the results obtained in Section 4, we construct confidence bands for the $(1-p)$ percentile residual lifetime in Section 5. However, in order to

practically construct these confidence bands, it is required that we should have the limiting distributions of the appropriate functionals of these processes which, unfortunately, also contain the unknown underlying distribution and quantile functions. This leads us to consider bootstrapping these asymptotic distributions.

In Section 6, we will define the bootstrapped PL $(1-p)$ percentile residual lifetime processes and discuss strong approximations of the processes to appropriate Gaussian processes. This, in turn, will finally lead to the bootstrapped confidence bands for the $(1-p)$ percentile residual lifetime in Section 7.

The methodology developed is then applied to the data set described in Section 6 in Chapter III.

2. PL $(1-p)$ percentile empirical residual lifetime - $R_n^{(p)}(t)$.

Under the random censorship model from the right, as an estimator of $R^{(p)}(t)$ in (V.1.1), it is natural to consider the PL $(1-p)$ percentile empirical residual lifetime at $t > 0$, defined as follows:

$$(V.2.1) \quad R_n^{(p)}(t) := Q_n(1-p(1-F_n^0(t))) - t, \quad 0 < p < 1,$$

where F_n^0 is the PL estimator of F^0 and Q_n is the PL-quantile function of Q . F_n^0 and Q_n are defined in (II.1.5) and (II.3.2), respectively.

The counterpart of the scaled empirical $(1-p)$ percentile residual lifetime process (cf. M. Csörgö and S. Csörgö, 1986), the scaled PL $(1-p)$ percentile residual lifetime process $r_n^{(p)}(t)$ is defined as

$$(V.2.2) \quad r_n^{(p)}(t) := \sqrt{n} f^0(Q(1-p(1-F^0(t)))) [R_n^{(p)}(t) - R^{(p)}(t)] ;$$

$$0 < p < 1, t > 0,$$

assuming that F^0 has a density function $f^0 = F^{0'}$ which is positive over the support $(0, T_{F^0})$ of F^0 .

In Section 4 we study the asymptotic distribution of $r_n^{(p)}(t)$ as a stochastic process in t , as that in p , as well as, in both t and p . However, we will follow the Csörgö-Csörgö approach, in which the pointwise asymptotic distribution of $r_n^{(p)}(t)$ is first studied, followed by the asymptotic distribution of $r_n^{(p)}(t)$ as one- and two-parameter stochastic processes. The pointwise asymptotic distribution of $r_n^{(p)}(t)$ may suffice in some applications. Hence we first prove a central limit theorem.

Theorem V.2.1. Let t and p be fixed numbers such that $t > 0$, $p \in (0,1)$ and $[t \vee Q(1-p(1-F^0(t)))] < T_F$. Assume that the density-quantile function $f^0(Q(.))$ is positive and continuous at $1-p(1-F^0(t))$. Then the asymptotic distribution of $r_n^{(p)}(t)$ is the normal with mean 0 and variance $p^2(1-F^0(t))^2 [d(Q(1-p(1-F^0(t)))) - d(t)]$, i.e.,

$$(V.2.3) \quad r_n^{(p)}(t) \xrightarrow{D} N(0, p^2(1-F^0(t))^2 [d(Q(1-p(1-F^0(t)))) - d(t)])$$

as $n \rightarrow \infty$, where $d(\cdot)$ is defined in (II.2.4) and $\xrightarrow{D}$ denotes convergence in distribution.

Proof. First, by the definition of q_n in (II.4.3) (cf. also (II.4.1)), we get

$$\begin{aligned} (V.2.4) \quad q_n(y) &= \inf\{t : F_n^{*0}(t) \geq y\} \\ &= \inf\{t : F_n^0(Q(t)) \geq y\} \\ &= \inf\{F^0(s) : F_n^0(s) \geq y\} \\ &= F^0(\inf\{s : F_n^0(s) \geq y\}) \\ &= F^0(Q_n(y)) \quad \text{for any } y \in (0,1). \end{aligned}$$

Applying (V.2.4) and a one-term Taylor expansion, we obtain

$$\begin{aligned} (V.2.5) \quad r_n^{(p)}(t) &= \sqrt{n} f^0(Q(1-p(1-F^0(t)))) [R_n^{(p)}(t) - R^{(p)}(t)] \\ &= \sqrt{n} f^0(Q(1-p(1-F^0(t)))) [Q_n(1-p(1-F_n^0(t))) - Q(1-p(1-F^0(t)))] \\ &= \sqrt{n} f^0(Q(1-p(1-F^0(t)))) [Q(F^0(Q_n(1-p(1-F_n^0(t))))) - Q(1-p(1-F^0(t)))] \\ &= \sqrt{n} f^0(Q(1-p(1-F^0(t)))) [Q(q_n(1-p(1-F_n^0(t)))) - Q(1-p(1-F^0(t)))] \\ &= \frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_{n,t,p}))} \sqrt{n} [q_n(1-p(1-F_n^0(t))) - (1-p(1-F^0(t)))] \\ &= \frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_{n,t,p}))} s_n^{(p)}(t), \end{aligned}$$

where $s_n^{(p)}(t) := \sqrt{n} [q_n(1-p(1-F_n^0(t))) - (1-p(1-F^0(t)))]$ and

$$(V.2.6) \quad q_n(1-p(1-F_n^0(t))) \wedge (1-p(1-F^0(t))) < \delta_{n,t,p} < q_n(1-p(1-F_n^0(t))) \vee (1-p(1-F^0(t))).$$

Now, since $t < T_F$ and $1-p(1-F^0(t)) < T_F^*$, $F_n^0(t) \xrightarrow{\text{a.s.}} F^0(t)$ (cf. (II.2.7))

and $q_n(1-p(1-F^0(t))) \xrightarrow{\text{a.s.}} 1-p(1-F^0(t))$ (cf. (II.4.7)) as $n \rightarrow \infty$,

the inequalities in (V.2.6) imply $\delta_{n,t,p} \xrightarrow{\text{a.s.}} 1-p(1-F^0(t))$ as $n \rightarrow \infty$.

Consequently, since $f^0(Q(\cdot))$ is assumed to be continuous at

$1-p(1-F^0(t))$, we get that

$$\frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_{n,t,p}))} \xrightarrow{\text{a.s.}} 1 \quad \text{as } n \rightarrow \infty.$$

Hence, by Slutsky's theorem, the asymptotic distribution of $r_n^{(p)}(t)$ is the same as that of $s_n^{(p)}(t)$, and concerning the latter we have

$$\begin{aligned} (V.2.7) \quad s_n^{(p)}(t) &= \sqrt{n} [q_n(1-p(1-F_n^0(t))) - (1-p(1-F^0(t)))] \\ &= \sqrt{n} [q_n(1-p(1-F_n^0(t))) - (1-p(1-F_n^0(t))) \\ &\quad + (1-p(1-F_n^0(t))) - (1-p(1-F^0(t)))] \\ &= -u_n(1-p(1-F_n^0(t))) + p \beta_n(t) \\ &= -u_n(1-p(1-F_n^0(t))) + p e_n(F^0(t)). \end{aligned}$$

Using (II.4.4) and (II.4.5) of Theorem II.C, there is a two-parameter Wiener process $K^*(t,s)$ defined in (II.3.6) and (II.2.3) such that

$$|s_n^{(p)}(t) + n^{-\frac{1}{2}} K^*(1-p(1-F^0(t)), n) - pn^{-\frac{1}{2}} K^*(F^0(t), n)| \xrightarrow{\text{a.s.}} 0 \text{ as } n \rightarrow \infty.$$

Hence, as $n \rightarrow \infty$,

$$\begin{aligned} s_n^{(p)}(t) &\xrightarrow{D} p K^*(F^0(t), 1) - K^*(1-p(1-F^0(t)), 1) \\ (V.2.8) \quad &= p (1-F^0(t)) W(d(t)) - p (1-F^0(t)) W(d(Q(1-p(1-F^0(t))))) \\ &= p (1-F^0(t)) [W(d(t)) - W(d(Q(1-p(1-F^0(t)))))]. \end{aligned}$$

In order to compute the variance of the latter normal random variable $W(d(t)) - W(d(Q(1-p(1-F^0(t)))))$, we first note that $t \leq Q(1-p(1-F^0(t)))$, since if we assume that $t > Q(1-p(1-F^0(t)))$, then we have $F^0(t) > 1-p(1-F^0(t))$ and subsequently $p > 1$, which contradicts $0 < p < 1$. We also have that $Q(\cdot)$ and $d(\cdot)$ are monotone increasing functions and so we have also that $d(t) \leq d(Q(1-p(1-F^0(t))))$. Hence the variance of $W(d(t)) - W(d(Q(1-p(1-F^0(t)))))$ is

$$(V.2.9) \quad d(t) - 2 d(t) + d(Q(1-p(1-F^0(t)))) = d(Q(1-p(1-F^0(t)))) - d(t).$$

Combining (V.2.8) and (V.2.9) with the normality of Wiener process for fixed time parameter, we obtain, as $n \rightarrow \infty$,

$$s_n^{(p)}(t) \xrightarrow{D} N(0, p^2(1-F^0(t))^2[d(Q(1-p(1-F^0(t)))) - d(t)]).$$

This also completes the proof. $\square$

Remark V.2.1. When there is no censoring, $d(t) = F^0(t)/(1-F^0(t))$ and the variance simply becomes $p(1-p)(1-F^0(t))$. Thus, in non-censored case, we have, as $n \rightarrow \infty$,

$$r_n^{(p)}(t) \xrightarrow{D} N(0, p(1-p)(1-F^0(t))),$$

which is the result of M. Csörgö and S. Csörgö (1986, Theorem 1).

3. Confidence interval for $R^{(p)}(t)$ for fixed p and t .

For a given $\alpha \in (0,1)$, let $\tau = \tau(\alpha)$ be the solution of the equation $\Phi(\tau) = 1 - \frac{1}{2}\alpha$ where $\Phi(\cdot)$ is defined in (I.2.8).

Theorem V.3.1. Let t and p satisfy the conditions of Theorem V.2.1. Let $f^0(Q(\cdot))$ be positive and continuous at $1-p(1-F^0(t))$. Let f_n^0 be any sequence of density estimators for f^0 which is uniformly consistent in a neighbourhood of $Q(1-p(1-F^0(t)))$. Then we have

$$(V.3.1) \quad \lim_{n \rightarrow \infty} P\{ R_n^{(p)}(t) - c_n(\tau;p,t) \leq R^{(p)}(t) \leq R_n^{(p)}(t) + c_n(\tau;p,t) \} = 1 - \alpha,$$

$$\text{where } c_n(\tau;p,t) := \frac{\tau p(1-F_n^0(t)) [d_n(Q_n(1-p(1-F_n^0(t)))) - d_n(t)]^{\frac{1}{2}}}{\sqrt{n} f_n^0(Q_n(1-p(1-F_n^0(t))))}.$$

Proof. The probability statement in (V.3.1) is equal to

$$P\left\{ \sqrt{n} \left| \frac{f_n^o(Q_n(1-p(1-F_n^o(t))))[R_n^{(p)}(t)-R^{(p)}(t)]}{p(1-F_n^o(t))[d_n(Q_n(1-p(1-F_n^o(t))))-d_n(t)]^{\frac{1}{2}}} \right| \leq \tau \right\}.$$

Now, since $t < T_F$ and $1-p(1-F^o(t)) < T_F^*$, $F_n^o(t) \xrightarrow{\text{a.s.}} F^o(t)$ (cf. (II.2.7)) and $q_n(1-p(1-F^o(t))) \xrightarrow{\text{a.s.}} 1-p(1-F^o(t))$ (cf. (II.4.7)), $d_n(t) \xrightarrow{\text{a.s.}} d(t)$ (cf. (II.2.9)) and $d_n(Q(1-p(1-F^o(t)))) \xrightarrow{\text{a.s.}} d(Q(1-p(1-F^o(t))))$ as $n \rightarrow \infty$.

Applying Theorem V.2.1, the assumption for f_n^o and Slutsky's Theorem, the latter probability sequence converges to the standard normal distribution. It completes the proof. $\square$

In order to construct confidence interval for $R^{(p)}(t)$ using this theorem, we have to choose a proper uniformly consistent estimator f_n^o . For information concerning density estimation under random censorship, we refer to the papers by Blum and Suşarla (1980), Burke (1983), Burke and Horváth (1984), Földes, Rejtő and Winter (1981), McNichols and Padgett (1981), and Yandell (1983). However, the confidence interval in the following theorem does not require the estimation of the density function f , and thus eliminates the latter problem. This approach is similar to that for quantile function under the random censorship discussed in Chapter III, and is an extension of M. Csörgö and Révész (1984) confidence band for quantile function in terms of non-censored observations (cf. Chapter 4 in M. Csörgö (1983)).

Theorem V.3.2. Let t and p satisfy the conditions of Theorem V.2.1. Let $f^o(Q(.))$ is positive and continuous at $1-p(1-F^o(t))$. Then

$$(V.3.2) \quad \lim_{n \rightarrow \infty} P\{ Q_n(1-p(1-F_n^0(t)) - n^{-\frac{1}{2}}\tau \sigma_n(p,t)) - t \leq R^{(p)}(t) \leq \\ Q_n(1-p(1-F_n^0(t)) + n^{-\frac{1}{2}}\tau \sigma_n(p,t)) - t \} = 1 - \alpha,$$

where $\sigma_n(p,t) := p(1-F_n^0(t))[d_n(Q_n(1-p(1-F_n^0(t))))-d_n(t)]^{\frac{1}{2}}$ and $\tau = \tau(\alpha)$ is as in Theorem V.3.1.

Proof. Let $a_n(\tau;p,t) := 1-p(1-F_n^0(t)) - n^{-\frac{1}{2}}\tau \sigma_n(p,t)$ and $b_n(\tau;p,t) := 1-p(1-F_n^0(t)) + n^{-\frac{1}{2}}\tau \sigma_n(p,t)$. Then the probability in (V.3.2) is equal to

$$(V.3.3) \quad P_n(\tau;p,t) := P\{ Q_n(a_n(\tau;p,t)) \leq Q(1-p(1-F^0(t))) \leq Q_n(b_n(\tau;p,t)) \}.$$

Consider the right side inequality in (V.3.3). Then

$$(V.3.4) \quad \{ Q(1-p(1-F^0(t))) \leq Q_n(b_n(\tau;p,t)) \} \\ = \{ Q_n(b_n(\tau;p,t)) - Q(b_n(\tau;p,t)) + Q(b_n(\tau;p,t)) - Q(1-p(1-F^0(t))) \geq 0 \}.$$

Applying one-term Taylor expansion for the second two terms in (V.3.4) and multiplying (V.3.4) by $\sqrt{n} f^0(Q(b_n(\tau;p,t)))$, we obtain that, in terms of the PL-normed quantile process ρ_n defined in (II.3.3), (V.3.4) is equivalent to the event

$$B_n(\tau;p,t) := \{ -\rho_n(b_n(\tau;p,t)) + \frac{f^0(Q(b_n(\tau;p,t)))}{f^0(Q(\delta_n(\tau;p,t)))} [p\beta_n(t) + \tau\sigma_n(p,t)] \geq 0 \},$$

where $b_n(\tau;p,t) \wedge (1-p(1-F^0(t))) < \delta_n(\tau;p,t) < b_n(\tau;p,t) \vee (1-p(1-F^0(t)))$.

Similarly, we obtain that the left side inequality in (V.3.3) is equivalent to the event

$$A_n(\tau; p, t) := \left\{ -\rho_n(a_n(\tau; p, t)) + \frac{f^0(Q(a_n(\tau; p, t)))}{f^0(Q(\eta_n(\tau; p, t)))} [p\beta_n(t) - \tau\sigma_n(p, t)] \leq 0 \right\},$$

where $a_n(\tau; p, t) \wedge (1-p(1-F^0(t))) < \eta_n(\tau; p, t) < a_n(\tau; p, t) \vee (1-p(1-F^0(t)))$.

Now, since $t < T_F$ and $1-p(1-F^0(t)) < T_F^*$, $F_n^0(t) \xrightarrow{\text{a.s.}} F^0(t)$ (cf. (II.2.7)) and $q_n(1-p(1-F^0(t))) \xrightarrow{\text{a.s.}} 1-p(1-F^0(t))$ (cf. (II.4.7)), $d_n(t) \xrightarrow{\text{a.s.}} d(t)$ (cf. (II.2.9)) and $d_n(Q(1-p(1-F^0(t)))) \xrightarrow{\text{a.s.}} d(Q(1-p(1-F^0(t))))$ as $n \rightarrow \infty$. Thus, $\sigma_n(p, t) \xrightarrow{\text{a.s.}} a$ constant. Consequently we have

$$a_n(\tau; p, t), b_n(\tau; p, t), \delta_n(\tau; p, t), \eta_n(\tau; p, t) \xrightarrow{\text{a.s.}} 1-p(1-F^0(t))$$

as $n \rightarrow \infty$. Hence, by the continuity of f^0 at $Q(1-p(1-F^0(t)))$,

$$\begin{aligned} \lim_{n \rightarrow \infty} P_n(\tau; p, t) &= \lim_{n \rightarrow \infty} P\{ a_n(\tau; p, t) \cap b_n(\tau; p, t) \} \\ &= \lim_{n \rightarrow \infty} P\{ \{ -\tau \sigma_n(p, t) \leq -\rho_n(b_n(\tau; p, t)) + p \beta_n(t) \} \\ &\quad \cap \{ -\rho_n(a_n(\tau; p, t)) + p \beta_n(t) \leq \tau \sigma_n(p, t) \} \} \\ &= \lim_{n \rightarrow \infty} P\{ -\tau \sigma(p, t) \leq -\rho_n(1-p(1-F^0(t))) + p e_n(F^0(t)) \leq \tau \sigma(p, t) \}, \end{aligned}$$

where $\sigma(p, t) := p(1-F^0(t)) \{ d(Q(1-p(1-F^0(t)))) - d(t) \}^{\frac{1}{2}}$. Since $1-p(1-F^0(t)) > 0$,

$$\begin{aligned} \rho_n(1-p(1-F^0(t))) &\xrightarrow{D} K^*(1-p(1-F^0(t)), 1) \\ &\xrightarrow{D} p(1-F^0(t))W(d(Q(1-p(1-F^0(t))))). \end{aligned}$$

i.e., $-\rho_n(b_n(\tau; p, t)) + p\beta_n(t)$ and $-\rho_n(a_n(\tau; p, t)) + p\beta_n(t)$ behave asymptotically like $s_n(p, t)$ which is defined in the proof of Theorem V.2.1. Hence, as in the proof of Theorem V.3.1, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} P_n(\tau; p, t) &= P\left\{ -\tau \leq \frac{p(1-F^0(t))[W(d(t)) - W(d(Q(1-p(1-F^0(t)))))]}{\sigma(p, t)} \leq \tau \right\} \\ &= 1 - \alpha, \end{aligned}$$

and the proof is now complete. $\square$

4. Strong approximations of PL (1-p) percentile residual lifetime process.

In Sections 2 and 3, we considered the asymptotic distribution of $r_n^{(p)}(t)$ for fixed p and t . In this section the following three types of problems will be discussed under the random censorship model:

- (i) for fixed $p \in (0, 1)$, joint estimation of (1-p) percentile residual lifetimes belonging to varying conditions of survival by time t ($t > 0$);

- (ii) for fixed $t > 0$, joint estimation of $(1-p)$ percentile residual lifetimes belonging to varying conditions of probability $p \in (0,1)$;
- (iii) joint estimation of $(1-p)$ percentile residual lifetimes belonging to varying conditions of both survival by time t ($t > 0$) and probability $p \in (0,1)$.

Problem (i) requires viewing $r_n^{(p)}(t)$ as a stochastic process in $t > 0$ for fixed $p \in (0,1)$, (ii) as a process in $p \in (0,1)$ for fixed t ($t > 0$), while (iii) necessitates viewing $r_n^{(p)}(t)$ as a two-parameter process in $(p,t) \in (0,1) \times (0,\infty)$.

To simplify our notation, we define the Gaussian process $G_n^{(p)}(t)$ by

$$(V.4.1) \quad G_n^{(p)}(t) := n^{-\frac{1}{2}} [K^*(1-p(1-F^0(t)), n) - p K^*(F^0(t), n)].$$

Then, $G_n^{(p)}(t)$ is like $G^{(p)}(t)$, defined as

$$(V.4.2) \quad G^{(p)}(t) := p(1-F^0(t)) [W(d(t)) - W(d(Q(1-p(1-F^0(t)))))]$$

in (V.2.8), for each $n = 1, 2, \dots$, i.e., we have

$$(V.4.3) \quad \{ G_n^{(p)}(t) : t \geq 0, 0 < p < 1 \} \stackrel{D}{=} \{ G^{(p)}(t) : t \geq 0, 0 < p < 1 \}$$

for each $n = 1, 2, \dots$.

Theorem V.4.1. For a fixed $p \in (0,1)$, let $T > 0$ and $[TVQ(1-p(1-F^0(t)))] < T_F$. Let F^0 be twice differentiable on $(0, T_F)$, and let f^0 be positive on $[Q(1-p), Q(1-p(1-F^0(T)))]$. Then we have, as $n \rightarrow \infty$,

$$(V.4.4) \quad \sup_{0 \leq t \leq T} |r_n^{(p)}(t) - G_n^{(p)}(t)| = \text{a.s. } O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}(\log \log n)^{\frac{1}{2}}).$$

Proof. By (V.2.5) in the proof of Theorem V.2.1 we have

$$(V.4.5) \quad r_n^{(p)}(t) = \frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_{n,t,p}))} s_n^{(p)}(t) \\ = \left[\frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_{n,t,p}))} - 1 \right] s_n^{(p)}(t) + s_n^{(p)}(t),$$

where $s_n^{(p)}(t) := \sqrt{n} [q_n(1-p(1-F_n^0(t))) - (1-p(1-F^0(t)))]$, and

$$(V.4.6) \quad q_n(1-p(1-F_n^0(t))) \wedge (1-p(1-F^0(t))) < \delta_{n,t,p} < q_n(1-p(1-F_n^0(t))) \vee (1-p(1-F^0(t))).$$

Using (II.2.7) and (II.4.7), we obtain

$$(V.4.7) \quad \sup_{0 \leq t \leq T} |F_n^0(t) - F^0(t)| = \text{a.s. } O(n^{-\frac{1}{2}}(\log \log n)^{\frac{1}{2}}), \\ \sup_{0 \leq y \leq Q(1-p(1-F^0(T)))} |y - q_n(y)| = \text{a.s. } O(n^{-\frac{1}{2}}(\log \log n)^{\frac{1}{2}}),$$

where $[T \vee Q(1-p(1-F^0(t)))] < T_F$. Hence,

$$\begin{aligned}
& \sup_{0 \leq t \leq T} | \delta_{n,t,p} - (1-p(1-F^0(t))) | \\
& \leq \sup_{0 \leq t \leq T} n^{-\frac{1}{2}} | s_n^{(p)}(t) | \\
& \leq \sup_{0 \leq t \leq T} | 1-p(1-F_n^0(t)) - q_n(1-p(1-F_n^0(t))) | \\
& \quad + \sup_{0 \leq t \leq T} p | F_n^0(t) - F^0(t) | \\
& \text{a.s.} \\
& = O(n^{-\frac{1}{2}} (\log \log n)^{\frac{1}{2}}).
\end{aligned}$$

Consequently, since $f^0(\cdot)$ is assumed to be differentiable and positive on $[Q(1-p), Q(1-p(1-F^0(T)))]$, we get that

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \left| \frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_{n,t,p}))} - 1 \right| \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{2}} (\log \log n)^{\frac{1}{2}}), \text{ and} \\
& \sup_{0 \leq t \leq T} | s_n^{(p)}(t) | \stackrel{\text{a.s.}}{=} O((\log \log n)^{\frac{1}{2}}).
\end{aligned}$$

(V.4.8)

As in (V.2.7), we have

$$s_n^{(p)}(t) = -u_n(1-p(1-F_n^0(t))) + p e_n(F^0(t)),$$

and applying (II.4.4) and (II.4.5) of Theorem II.C, we obtain

$$\sup_{0 \leq t \leq T} | s_n^{(p)}(t) - G_n^{(p)}(t) | \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{2}} (\log n)^{\frac{1}{2}} (\log \log n)^{\frac{1}{2}}).$$

(V.4.9)

We complete the proof by combining (V.4.9) with (V.4.5) and (V.4.8).

Theorem V.4.2. Let $t > 0$ be a fixed number with $t < T_F$. Let Γ be $\Gamma \in (0,1)$ and fixed so that $Q(1-\Gamma(1-F^0(t))) < T_F$. Let F^0 be twice differentiable on $(0, T_{F^0})$ and let f^0 be positive on $[t, Q(1-\Gamma(1-F^0(t)))]$. Then we have, as $n \rightarrow \infty$,

$$(V.4.10) \quad \sup_{\Gamma < p < 1} |r_n^{(p)}(t) - G_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}(\log \log n)^{\frac{1}{2}}).$$

Proof. As in the proof of Theorem V.4.1, from (V.4.6) and (V.4.7) we obtain

$$\begin{aligned} & \sup_{\Gamma < p < 1} | \delta_{n,t,p} - (1-p(1-F^0(t))) | \\ & \leq \sup_{\Gamma < p < 1} n^{-\frac{1}{2}} |s_n^{(p)}(t)| \\ & \leq \sup_{\Gamma < p < 1} |1-p(1-F_n^0(t)) - q_n(1-p(1-F_n^0(t)))| + |F_n^0(t) - F^0(t)| \\ & \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{2}}(\log \log n)^{\frac{1}{2}}). \end{aligned}$$

Consequently, since $f^0(\cdot)$ is assumed to be differentiable and positive on $[t, Q(1-\Gamma(1-F^0(t)))]$, we get that

$$(V.4.11) \quad \sup_{\Gamma < p < 1} \left| \frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_{n,t,p}))} - 1 \right| \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{2}}(\log \log n)^{\frac{1}{2}}), \text{ and}$$

$$\sup_{\Gamma < p < 1} |s_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O((\log \log n)^{\frac{1}{2}}).$$

On applying (II.4.4) and (II.4.5) of Theorem II.C to $s_n^{(p)}(t)$ as in (V.2.7), we obtain

$$(V.4.12) \quad \sup_{\Gamma < p < 1} |s_n^{(p)}(t) - G_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}(\log \log n)^{\frac{1}{2}}).$$

We complete the proof by combining (V.4.12) with (V.4.5) and (V.4.11). $\square$

Theorem V.4.3. Let $T > 0$ and $F \in (0,1)$ be so that $[TVQ(1-\Gamma(1-F^0(T)))] < T_F$. Let F^0 be twice differentiable on $(0, T_F)$ and let $\mu > 0$ be such that f^0 is positive on $[\mu, Q(1-\Gamma(1-F^0(T)))]$. Then we have, as $n \rightarrow \infty$,

$$(V.4.13) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} |r_n^{(p)}(t) - G_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}(\log \log n)^{\frac{1}{2}}).$$

Proof. As in the proof of Theorem V.4.1, from (V.4.6) and (V.4.7) we obtain

$$\begin{aligned} & \sup_{\mu \leq t \leq T, \Gamma < p < 1} |\delta_{n,t,p} - (1-p(1-F^0(t)))| \\ & \leq \sup_{\mu \leq t \leq T, \Gamma < p < 1} n^{-\frac{1}{2}} |s_n^{(p)}(t)| \\ & \leq \sup_{\mu \leq t \leq T, \Gamma < p < 1} |1-p(1-F_n^0(t)) - q_n(1-p(1-F_n^0(t)))| \\ & \quad + \sup_{\mu \leq t \leq T} |F_n^0(t) - F^0(t)| \\ & \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{2}}(\log \log n)^{\frac{1}{2}}). \end{aligned}$$

Consequently, since $f^0(\cdot)$ is assumed to be differentiable and positive on $[\mu, Q(1-\Gamma(1-F^0(T)))]$, we get that

$$(V.4.14) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} \left| \frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_{n,t,p}))} - 1 \right| \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{2}}(\log \log n)^{\frac{1}{2}}), \text{ and}$$

$$\sup_{\mu \leq t \leq T, \Gamma < p < 1} |s_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O((\log \log n)^{\frac{1}{2}}).$$

On applying (II.4.4) and (II.4.5) of Theorem II.C, we obtain

$$(V.4.15) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} |s_n^{(p)}(t) - G_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}(\log \log n)^{\frac{1}{2}}).$$

We complete the proof by combining (V.4.15) with (V.4.5) and (V.4.14). $\square$

In Theorem V.4.3 the range $[\mu, T]$ for t with any $\mu > 0$ should suffice for most applications. However, this range can be extended to $[0, T]$ under further assumptions on f^0 , which is the topic of the next theorem.

Theorem V.4.4. Let $T > 0$ and $\Gamma \in (0, 1)$ be so that $[T \vee Q(1-\Gamma(1-F^0(T)))] < T_F$. Let F^0 be twice differentiable on $(0, T_F)$ and let f^0 be positive on $(0, T_F)$. Let

$$(V.4.16) \quad \sup_{0 \leq t \leq p^*} t \frac{|f^{0'}(Q(t))|}{(f^0(Q(t)))^2} \leq r$$

for some $r > 0$ and $Q(p^*) > [T \vee Q(1 - \Gamma(1 - F^0(T)))]$. Then there exists a constant $C > 0$ such that, with $\varepsilon(n) = Cn^{-1} \log \log n$ and $\varphi(n) = Q(\varepsilon(n))$, we have, as $n \rightarrow \infty$,

$$(V.4.17) \quad \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} |r_n^{(p)}(t) - G_n^{(p)}(t)| = O(n^{-\frac{1}{4}} (\log n)^{\frac{1}{2}} (\log \log n)^{\frac{1}{4}}) \text{ a.s.}$$

If we assume further that

$$(V.4.18) \quad 0 < \lim_{t \downarrow 0} f^0(t) < \infty,$$

then we have, as $n \rightarrow \infty$,

$$(V.4.19) \quad \sup_{0 \leq t \leq T, \Gamma < p < 1} |r_n^{(p)}(t) - G_n^{(p)}(t)| = O(n^{-\frac{1}{4}} (\log n)^{\frac{1}{2}} (\log \log n)^{\frac{1}{4}}) \text{ a.s.}$$

Proof. As in (V.2.5),

$$(V.4.20) \quad r_n^{(p)}(t) = \sqrt{n} f^0(Q(1 - p(1 - F^0(t)))) \\ [Q(q_n(1 - p(1 - F_n^0(t)))) - Q(1 - p(1 - F^0(t)))]$$

By applying two-term Taylor expansion on Q and (V.2.7), we obtain

$$(V.4.21) \quad r_n^{(p)}(t) = s_n^{(p)}(t) \\ - \frac{1}{2} n^{-\frac{1}{2}} [s_n^{(p)}(t)]^2 \frac{f^0(Q(1 - p(1 - F^0(t))))}{f^0(Q(\delta_{n,t,p}))} \frac{f^{0'}(Q(\delta_{n,t,p}))}{(f^0(Q(\delta_{n,t,p})))^2},$$

where

$$(V.4.22) \quad s_n^{(p)}(t) = \sqrt{n} [q_n(1-p(1-F_n^0(t))) - (1-p(1-F^0(t)))] \\ = -u_n(1-p(1-F_n^0(t))) + p e_n(F^0(t)),$$

$$(V.4.23) \quad q_n(1-p(1-F_n^0(t))) \wedge (1-p(1-F^0(t))) < \delta_{n,t,p} \\ < q_n(1-p(1-F_n^0(t))) \vee (1-p(1-F^0(t))).$$

Now (V.4.23) and (V.4.22) imply that

$$(V.4.24) \quad |\delta_{n,t,p} - (1-p(1-F^0(t)))| \leq n^{-\frac{1}{2}} |s_n^{(p)}(t)|.$$

Applying (II.4.6) and (II.4.7) to (V.4.22), we obtain

$$(V.4.25) \quad \sup_{0 \leq t \leq T, \Gamma < p < 1} |s_n^{(p)}(t)| \stackrel{a.s.}{=} O[(\log \log n)^{\frac{1}{2}}].$$

Observing that $1-p(1-F^0(t)) > p F^0(t) > p^2 F^0(t)$, we get

$$(V.4.26) \quad \left| \frac{s_n^{(p)}(t)}{(1-p(1-F^0(t)))^{\frac{1}{2}} (\log \log n)^{\frac{1}{2}}} \right| \\ \leq \left| \frac{u_n(1-p(1-F_n^0(t)))}{(1-p(1-F_n^0(t)))^{\frac{1}{2}} (\log \log n)^{\frac{1}{2}}} \right| \left| \frac{1-p(1-F_n^0(t))}{1-p(1-F^0(t))} \right|^{\frac{1}{2}} \\ + \left| \frac{e_n(F^0(t))}{(F^0(t))^{\frac{1}{2}} (\log \log n)^{\frac{1}{2}}} \right|,$$

where

$$(V.4.27) \quad \frac{1-p(1-F_n^0(t))}{1-p(1-F^0(t))} = 1 + \frac{p e_n(F^0(t))}{n^{\frac{1}{2}} (1-p(1-F^0(t)))^{\frac{1}{2}} (1-p(1-F^0(t)))^{\frac{1}{2}}}$$

$$\leq 1 + \frac{e_n(F^0(t))}{(F^0(t))^{\frac{1}{2}}} \frac{1}{n^{\frac{1}{2}}} \frac{1}{(F^0(t))^{\frac{1}{2}}}$$

$$\leq 1 + \frac{e_n(F^0(t))}{C (F^0(t))^{\frac{1}{2}} (\log \log n)^{\frac{1}{2}}}$$

Applying (II.4.8) and (II.4.9) to (V.4.26) and (V.4.27), we obtain that there exists C' such that

$$(V.4.28) \quad \lim_{n \rightarrow \infty} \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} \left| \frac{s_n^{(p)}(t)}{(1-p(1-F^0(t)))^{\frac{1}{2}} (\log \log n)^{\frac{1}{2}}} \right| \leq C'.$$

Now, (V.4.21) implies that

$$(V.4.29) \quad |r_n^{(p)}(t) - s_n^{(p)}(t)| \leq \frac{1}{2} n^{-\frac{1}{2}} \left| \frac{s_n^{(p)}(t)}{(1-p(1-F^0(t)))^{\frac{1}{2}}} \right|^2 \left| \frac{1-p(1-F^0(t))}{\delta_{n,t,p}} \right|$$

$$\left| \frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_{n,t,p}))} \right| \delta_{n,t,p} \frac{f^0(Q(\delta_{n,t,p}))}{(f^0(Q(\delta_{n,t,p})))^2},$$

where the fourth term is bounded by r by (V.4.16) while both the second and third (cf. Lemma III.2.1) terms are bounded by

$$(V.4.30) \quad \frac{1-p(1-F^0(t)) + n^{-\frac{1}{2}} |s_n^{(p)}(t)|}{1-p(1-F^0(t)) - n^{-\frac{1}{2}} |s_n^{(p)}(t)|} \leq \frac{1-p(1-F^0(t)) + n^{-\frac{1}{2}} C''(\log \log n)^{\frac{1}{2}}}{1-p(1-F^0(t)) - n^{-\frac{1}{2}} C''(\log \log n)^{\frac{1}{2}}}$$

$$\leq 2 \text{ for any sufficiently large } n.$$

Combining (V.4.28), (V.4.29) and (V.4.30), we obtain

$$(V.4.31) \quad \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} |r_n^{(p)}(t) - s_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{4}} (\log \log n)^{\frac{1}{2}}).$$

Also applying (II.4.4) and (II.4.5), we have

$$(V.4.32) \quad \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} |s_n^{(p)}(t) - G_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O(n^{-\frac{1}{4}} (\log n)^{\frac{1}{2}} (\log \log n)^{\frac{1}{4}}),$$

and (V.4.17) is immediately derived from (V.4.31) and (V.4.32).

Under the assumption (V.4.18), we can replace $\varphi(n)$ in (V.4.28) and (V.4.31) with zero and hence (V.4.19) is also proven. $\square$

Remark V.4.1. As in Remark II.2.2, Theorems V.4.1, V.4.2, V.4.3 and V.4.4 imply the following laws of iterated logarithms under the corresponding conditions of Theorems V.4.1, V.4.2, V.4.3 and V.4.4:

$$(V.4.33) \quad \sup_{0 \leq t \leq T} |r_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O((\log \log n)^{\frac{1}{2}}) \text{ for each } 0 < p < 1;$$

$$(V.4.34) \quad \sup_{\Gamma < p < 1} |r_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O((\log \log n)^{\frac{1}{2}}) \text{ for each } t > 0;$$

$$(V.4.35) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} |r_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O((\log \log n)^{\frac{1}{2}});$$

$$(V.4.36) \quad \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} |r_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O((\log \log n)^{\frac{1}{2}});$$

$$(V.4.37) \quad \sup_{0 \leq t \leq T, \Gamma < p < 1} |r_n^{(p)}(t)| \stackrel{\text{a.s.}}{=} O((\log \log n)^{\frac{1}{2}}).$$

5. Confidence bands for $R^{(p)}(t)$.

In this section, we generalize Theorem V.3.2 to construct confidence bands for $R^{(p)}(t)$ in the light of Section 4.

First we define

$$(V.5.1) \quad a_n(\tau; p, t) := 1 - p(1 - F_n^0(t)) - n^{-\frac{1}{2}}\tau,$$

$$b_n(\tau; p, t) := 1 - p(1 - F_n^0(t)) + n^{-\frac{1}{2}}\tau,$$

where τ , for the time being, is any positive number. Consider the events

$$(V.5.2) \quad \begin{aligned} D_n'(\tau; p, t) &:= \{ Q_n(a_n(\tau; p, t)) - t \leq R^{(p)}(t) \} \\ &= \{ Q_n(a_n(\tau; p, t)) \leq Q(1 - p(1 - F^0(t))) \}, \text{ and} \end{aligned}$$

$$\begin{aligned} D_n(\tau; p, t) &:= \{ R^{(p)}(t) \leq Q_n(b_n(\tau; p, t)) - t \} \\ &= \{ Q(1 - p(1 - F^0(t))) \leq Q_n(b_n(\tau; p, t)) \}. \end{aligned}$$

Then we have the following asymptotic confidence bands for $R^{(p)}(t)$:

Theorem V.5.1. Under the conditions of Theorem V.4.1,

$$(V.5.3) \quad \lim_{n \rightarrow \infty} P\{ D_n'(\tau; p, t) \cap D_n(\tau; p, t) : 0 \leq t \leq T \} = 1 - \alpha,$$

where $\tau = \tau(\alpha; p)$ such that $P\{|G^{(p)}(t)| \leq \tau : 0 \leq t \leq T\} = 1 - \alpha$
(cf. (V.4.2)).

Proof. Subtracting $Q_n(1-p(1-F^0(t)))$ and multiplying by $\sqrt{n}f^0(Q(1-p(1-F^0(t))))$ on the event $D_n(\tau; p, t)$ in (V.5.2), we obtain

$$\begin{aligned}
 (V.5.4) \quad & P\{D_n(\tau; p, t) : 0 \leq t \leq T\} \\
 &= P\{Q(1-p(1-F^0(t))) \leq Q_n(b_n(\tau; p, t)) : 0 \leq t \leq T\} \\
 &= P\{\sqrt{n}f^0(Q(1-p(1-F^0(t)))) [Q(1-p(1-F^0(t))) - Q_n(1-p(1-F^0(t)))] \\
 &\quad \leq \sqrt{n}f^0(Q(1-p(1-F^0(t)))) [Q_n(b_n(\tau; p, t)) - Q(b_n(\tau; p, t)) \\
 &\quad + Q(b_n(\tau; p, t)) - Q(1-p(1-F^0(t))) \\
 &\quad + Q(1-p(1-F^0(t))) - Q_n(1-p(1-F^0(t)))] : 0 \leq t \leq T\}.
 \end{aligned}$$

Using the PL-normed quantile process ρ_n as defined in (II.3.3) and one-term Taylor expansion on Q , we have

$$\begin{aligned}
 (V.5.5) \quad & P\{D_n(\tau; p, t) : 0 \leq t \leq T\} \\
 &= P\{\rho_n(1-p(1-F^0(t))) \leq \rho_n(b_n(\tau; p, t)) - \rho_n(1-p(1-F^0(t))) \\
 &\quad + \left[\frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(b_n(\tau; p, t)))} - 1 \right] (-\rho_n(b_n(\tau; p, t))) \\
 &\quad + \left[\frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_n(\tau; p, t)))} - 1 \right] (\tau + p\beta_n(t)) \\
 &\quad + (\tau + p\beta_n(t)) : 0 \leq t \leq T\},
 \end{aligned}$$

where

$$(V.5.6) \quad b_n(\tau; p, t) \wedge (1-p(1-F^0(t))) < \delta_n(\tau; p, t) < b_n(\tau; p, t) \vee (1-p(1-F^0(t))).$$

By Remark II.3.1, we have, as $n \rightarrow \infty$,

$$(V.5.7) \quad \sup_{0 \leq t \leq T} | \rho_n(b_n(\tau; p, t)) - n^{-\frac{1}{2}} K^*(b_n(\tau; p, t), n) | \stackrel{\text{a.s.}}{=} o(1).$$

Since $\sup_{0 \leq t \leq T} | b_n(\tau; p, t) | \stackrel{\text{a.s.}}{\longrightarrow} 1-p(1-F^0(t))$, by the modulus of continuity for Wiener process of Lemma II.A and

$$(V.5.8) \quad \{ n^{-\frac{1}{2}} K^*(s, n) : 0 \leq s \leq 1 \} \stackrel{D}{=} \{ (1-s)W(d(Q(s)), 1) : 0 \leq s \leq 1 \}$$

for each $n = 1, 2, \dots$, we also have that, as $n \rightarrow \infty$,

$$(V.5.9) \quad \sup_{0 \leq t \leq T} | n^{-\frac{1}{2}} K^*(b_n(\tau; p, t), n) - n^{-\frac{1}{2}} K^*(1-p(1-F^0(t)), n) | = o_p(1).$$

By combining (V.5.7) and (V.5.9), we obtain

$$(V.5.10) \quad \sup_{0 \leq t \leq T} | \rho_n(b_n(\tau; p, t)) - \rho_n(1-p(1-F^0(t))) | = o_p(1),$$

and on combining (V.5.7) and (V.5.8), we get

$$(V.5.11) \quad \sup_{0 \leq t \leq T} | \rho_n(b_n(\tau; p, t)) | \stackrel{\text{a.s.}}{\longrightarrow} \text{a random variable.}$$

Now since f^0 is differentiable and positive on $[Q(1-p), Q(1-p(1-F^0(T)))]$, and, as $n \rightarrow \infty$, we have $\sup_{0 \leq t \leq T} | b_n(\tau; p, t) | \stackrel{\text{a.s.}}{\longrightarrow} 1-p(1-F^0(t))$, we get

$$(V.5.12) \quad \sup_{0 \leq t \leq T} \left| \frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(b_n(\tau; p, t)))} - 1 \right| \xrightarrow{\text{a.s.}} 0.$$

Thus we also have

$$(V.5.13) \quad \sup_{0 \leq t \leq T} \left| \frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_n(\tau; p, t)))} - 1 \right| \xrightarrow{\text{a.s.}} 0.$$

By (II.2.2) and (II.2.3)

$$(V.5.14) \quad \tau + p \sup_{0 \leq t \leq T} |\beta_n(t)| \xrightarrow{\text{a.s.}} \text{a random variable.}$$

Applying (V.5.11), (V.5.14), (V.5.12), (V.5.13) to (V.5.5), we finally obtain

$$(V.5.15) \quad P\{ D_n(\tau; p, t) : 0 \leq t \leq T \} \\ = P\{ \rho_n(1-p(1-F^0(t))) \leq \tau + p \beta_n(t) : 0 \leq t \leq T \}.$$

Similarly, we also derive

$$(V.5.16) \quad P\{ D_n'(\tau; p, t) : 0 \leq t \leq T \} \\ = P\{ \rho_n(1-p(1-F^0(t))) \geq -\tau + p \beta_n(t) : 0 \leq t \leq T \}.$$

Hence, (V.5.15) and (V.5.16) imply that

$$\lim_{n \rightarrow \infty} P\{ D_n'(\tau; p, t) \cap D_n(\tau; p, t) : 0 \leq t \leq T \} \\ = \lim_{n \rightarrow \infty} P\{ -\tau \leq \rho_n(1-p(1-F^0(t))) - p \beta_n(t) \leq \tau : 0 \leq t \leq T \}$$

$$= \lim_{n \rightarrow \infty} P\{ -\tau \leq G_n^{(p)}(t) \leq \tau : 0 \leq t \leq T \}$$

$$= P\{ |G^{(p)}(t)| \leq \tau : 0 \leq t \leq T \}$$

$$= 1 - \alpha.$$

Theorem V.5.2. Under the conditions of Theorem V.4.2,

$$(V.5.17) \quad \lim_{n \rightarrow \infty} P\{ D_n'(\tau'; p, t) \cap D_n(\tau'; p, t) : \Gamma < p < 1 \} = 1 - \alpha,$$

where $\tau' = \tau'(a; t)$ is such that $P\{ |G^{(p)}(t)| \leq \tau' : \Gamma < p < 1 \} = 1 - \alpha$
(cf. (V.4.2)).

Proof. As in (V.5.4), and (V.5.5), we have

$$\begin{aligned} (V.5.18) \quad & P\{ D_n'(\tau'; p, t) : \Gamma < p < 1 \} \\ &= P\{ \rho_n(1-p(1-F^0(t))) \leq \rho_n(b_n(\tau'; p, t)) - \rho_n(1-p(1-F^0(t))) \\ &\quad + \left[\frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(b_n(\tau'; p, t)))} - 1 \right] (\rho_n(b_n(\tau'; p, t))) \\ &\quad + \left[\frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_n(\tau'; p, t)))} - 1 \right] (\tau' + p \beta_n(t)) \\ &\quad + (\tau' + p \beta_n(t)) : \Gamma < p < 1 \} , \end{aligned}$$

where

$$\begin{aligned} (V.5.19) \quad & b_n(\tau'; p, t) \wedge (1-p(1-F^0(t))) < \delta_n(\tau'; p, t) \\ & < b_n(\tau'; p, t) \vee (1-p(1-F^0(t))). \end{aligned}$$

By Remark II.3.1, we have, as $n \rightarrow \infty$,

$$(V.5.20) \quad \sup_{\Gamma < p < 1} | \rho_n(b_n(\tau'; p, t)) - n^{-\frac{1}{2}} K^*(b_n(\tau'; p, t), n) | \xrightarrow{\text{a.s.}} o(1).$$

Since $\sup_{\Gamma < p < 1} |b_n(\tau'; p, t)| \xrightarrow{\text{a.s.}} 1 - p(1 - F^0(t))$, by the modulus of continuity for Wiener process of Lemma II.A and (V.5.8), we also have, as $n \rightarrow \infty$,

$$(V.5.21) \quad \sup_{\Gamma < p < 1} | n^{-\frac{1}{2}} K^*(b_n(\tau'; p, t), n) - n^{-\frac{1}{2}} K^*(1 - p(1 - F^0(t)), n) | = o_p(1).$$

Combining (V.5.20) and (V.5.21), we obtain

$$(V.5.22) \quad \sup_{\Gamma < p < 1} | \rho_n(b_n(\tau'; p, t)) - \rho_n(1 - p(1 - F^0(t))) | = o_p(1),$$

and combining (V.5.20) and (V.5.8), we get

$$(V.5.23) \quad \sup_{\Gamma < p < 1} | \rho_n(b_n(\tau'; p, t)) | \xrightarrow{\text{a.s.}} \text{a random variable.}$$

Now since f^0 is differentiable and positive on $[t, Q(1 - p(1 - F^0(t)))]$,

and as $n \rightarrow \infty$, we have $\sup_{\Gamma < p < 1} |b_n(\tau'; p, t)| \xrightarrow{\text{a.s.}} 1 - p(1 - F^0(t))$, we get

$$(V.5.24) \quad \sup_{\Gamma < p < 1} \left| \frac{f^0(Q(1 - p(1 - F^0(t))))}{f^0(Q(b_n(\tau'; p, t)))} - 1 \right| \xrightarrow{\text{a.s.}} 0.$$

Thus we also have

$$(V.5.25) \quad \sup_{\Gamma < p < 1} \left| \frac{f^0(Q(1 - p(1 - F^0(t))))}{f^0(Q(\delta_n(\tau'; p, t)))} - 1 \right| \xrightarrow{\text{a.s.}} 0.$$

By (II.2.2) and (II.2.3)

$$(V.5.26) \quad \tau' + \sup_{\Gamma < p < 1} p \mid \beta_n(t) \mid \xrightarrow{\text{a.s.}} \text{a random variable.}$$

Applying (V.5.22), (V.5.25), (V.5.23), (V.5.24) to (V.5.18), we finally obtain

$$(V.5.27) \quad P\{ D_n(\tau'; p, t) : \Gamma < p < 1 \} \\ = P\{ \rho_n(1-p(1-F^0(t))) \leq \tau' + p \beta_n(t) : \Gamma < p < 1 \}.$$

Similarly, we also derive

$$(V.5.28) \quad P\{ D_n'(\tau'; p, t) : \Gamma < p < 1 \} \\ = P\{ \rho_n(1-p(1-F^0(t))) \geq -\tau' + p \beta_n(t) : \Gamma < p < 1 \}.$$

Hence, (V.5.27) and (V.5.28) imply that

$$\begin{aligned} & \lim_{n \rightarrow \infty} P\{ D_n(\tau'; p, t) \cap D_n'(\tau'; p, t) : \Gamma < p < 1 \} \\ &= \lim_{n \rightarrow \infty} P\{ -\tau' \leq \rho_n(1-p(1-F^0(t))) - p \beta_n(t) \leq \tau' : \Gamma < p < 1 \} \\ &= \lim_{n \rightarrow \infty} P\{ -\tau' \leq G_n^{(p)}(t) \leq \tau' : \Gamma < p < 1 \} \\ &= P\{ \mid G^{(p)}(t) \mid \leq \tau' : \Gamma < p < 1 \} \\ &= 1 - \alpha. \end{aligned}$$

Theorem V.5.3. Under the conditions of Theorem V.4.3,

$$(V.5.29) \quad \lim_{n \rightarrow \infty} P\{ D_n'(\tau''; p, t) \cap D_n(\tau''; p, t) : \mu \leq t \leq T, \Gamma < p < 1 \} = 1 - \alpha,$$

where $\tau'' = \tau''(\alpha)$ such that $P\{ |G^{(p)}(t)| \leq \tau'' : \mu \leq t \leq T, \Gamma < p < 1 \} = 1 - \alpha$
(cf. (V.4.2)).

Proof. As in (V.5.4) and (V.5.5), we have

$$\begin{aligned} (V.5.30) \quad & P\{ D_n(\tau''; p, t) : \mu \leq t \leq T, \Gamma < p < 1 \} \\ &= P\{ \rho_n(1-p(1-F^0(t))) \leq \rho_n(b_n(\tau''; p, t)) - \rho_n(1-p(1-F^0(t))) \\ &\quad + \left[\frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(b_n(\tau''; p, t)))} - 1 \right] (- \rho_n(b_n(\tau''; p, t))) \\ &\quad + \left[\frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_n(\tau''; p, t)))} - 1 \right] (\tau'' + p \beta_n(t)) \\ &\quad + (\tau'' + p \beta_n(t)) : \mu \leq t \leq T, \Gamma < p < 1 \} , \end{aligned}$$

where

$$\begin{aligned} (V.5.31) \quad & b_n(\tau''; p, t) \wedge (1-p(1-F^0(t))) < \delta_n(\tau''; p, t) \\ & < b_n(\tau''; p, t) \vee (1-p(1-F^0(t))). \end{aligned}$$

By Remark II.3.1, we have, as $n \rightarrow \infty$,

$$(V.5.32) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} | \rho_n(b_n(\tau''; p, t)) - n^{-1/2} K^*(b_n(\tau''; p, t), n) | \stackrel{\text{a.s.}}{=} o(1).$$

Since $\sup_{\mu \leq t \leq T, \Gamma < p < 1} |b_n(\tau''; p, t)| \xrightarrow{\text{a.s.}} 1 - p(1 - F^0(t))$, by the modulus of continuity for Wiener process of Lemma II.A and (V.5.8), we also have, as $n \rightarrow \infty$,

$$(V.5.33) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} |n^{-\frac{1}{2}} K^*(b_n(\tau''; p, t), n) - n^{-\frac{1}{2}} K^*(1 - p(1 - F^0(t)), n)| = o_p(1).$$

Combining (V.5.32) and (V.5.33), we obtain

$$(V.5.34) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} |\rho_n(b_n(\tau''; p, t)) - \rho_n(1 - p(1 - F^0(t)))| = o_p(1),$$

and combining (V.5.32) and (V.5.8), we get

$$(V.5.35) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} |\rho_n(b_n(\tau''; p, t))| \xrightarrow{\text{a.s.}} \text{a random variable.}$$

Now since f^0 is differentiable and positive on $[\mu, Q(1 - \Gamma(1 - F^0(T)))]$, and, as $n \rightarrow \infty$, we have $\sup_{\Gamma < p < 1, \mu \leq t \leq T} |b_n(\tau''; p, t)| \xrightarrow{\text{a.s.}} 1 - p(1 - F^0(t))$, we have

$$(V.5.36) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} \left| \frac{f^0(Q(1 - p(1 - F^0(t))))}{f^0(Q(b_n(\tau''; p, t)))} - 1 \right| \xrightarrow{\text{a.s.}} 0,$$

and also

$$(V.5.37) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} \left| \frac{f^0(Q(1 - p(1 - F^0(t))))}{f^0(Q(\delta_n(\tau''; p, t)))} - 1 \right| \xrightarrow{\text{a.s.}} 0.$$

By (II.2.2) and (II.2.3)

$$(V.5.38) \quad \tau'' + \sup_{\mu \leq t \leq T, \Gamma < p < 1} p |\beta_n(t)| \xrightarrow{\text{a.s.}} \text{a random variable.}$$

Applying (V.5.34), (V.5.37), (V.5.35), (V.5.36) and (V.5.38) to (V.5.30), we finally obtain

$$(V.5.39) \quad P\{ D_n(\tau''; p, t) : \mu \leq t \leq T, \Gamma < p < 1 \} \\ = P\{ \rho_n(1-p(1-F^0(t))) \leq \tau'' + p \beta_n(t) : \mu \leq t \leq T, \Gamma < p < 1 \}.$$

Similarly, we also derive

$$(V.5.40) \quad P\{ D_n'(\tau''; p, t) : \mu \leq t \leq T, \Gamma < p < 1 \} \\ = P\{ \rho_n(1-p(1-F^0(t))) \geq -\tau'' + p \beta_n(t) : \mu \leq t \leq T, \Gamma < p < 1 \}.$$

Hence, (V.5.39) and (V.5.40) imply that

$$\begin{aligned} & \lim_{n \rightarrow \infty} P\{ D_n'(\tau''; p, t) \cap D_n(\tau''; p, t) : \mu \leq t \leq T, \Gamma < p < 1 \} \\ &= \lim_{n \rightarrow \infty} P\{ -\tau'' \leq \rho_n(1-p(1-F^0(t))) - p \beta_n(t) \leq \tau'' : \mu \leq t \leq T, \Gamma < p < 1 \} \\ &= \lim_{n \rightarrow \infty} P\{ -\tau'' \leq G_n^{(p)}(t) \leq \tau'' : \mu \leq t \leq T, \Gamma < p < 1 \} \\ &= P\{ |G^{(p)}(t)| \leq \tau'' : \mu \leq t \leq T, \Gamma < p < 1 \} \\ &= 1 - \alpha. \end{aligned}$$

Theorem V.5.4. Let $T > 0$, $\Gamma \in (0, 1)$ be so that $[T \vee Q(1-\Gamma(1-F^0(T)))] < T_F$. Let F^0 be twice differentiable on $(0, T_{F0})$ and let f^0 be positive on $(0, T_{F0})$. Assume also (V.4.16). Then there exists $C > 0$ such that, with $\epsilon(n) = Cn^{-1} \log \log n$ and $\varphi(n) = Q(\epsilon(n))$, we have

$$(V.5.41) \quad \lim_{n \rightarrow \infty} P\{ D_n'(n;p,t) \cap D_n(n;p,t) : \varphi(n) \leq t \leq T, \Gamma < p < 1 \} = 1 - \alpha,$$

where $n = n(\alpha)$ such that $P\{ |G^{(p)}(t)| \leq n : 0 \leq t \leq T, \Gamma < p < 1 \} = 1 - \alpha$

(cf. (V.4.2)).

Proof. As in (V.5.4) and (V.5.5), we have

$$\begin{aligned} (V.5.42) \quad & P\{ D_n(n;p,t) : \varphi(n) \leq t \leq T, \Gamma < p < 1 \} \\ &= P\{ \rho_n(1-p(1-F^0(t))) \leq \rho_n(b_n(n;p,t)) - \rho_n(1-p(1-F^0(t))) \\ &\quad + \left[\frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(b_n(n;p,t)))} - 1 \right] (- \rho_n(b_n(n;p,t))) \\ &\quad + \left[\frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_n(n;p,t)))} - 1 \right] (n + p \beta_n(t)) \\ &\quad + (n + p \beta_n(t)) : \varphi(n) \leq t \leq T, \Gamma < p < 1 \} , \end{aligned}$$

where

$$(V.5.43) \quad b_n(n;p,t) \wedge (1-p(1-F^0(t))) < \delta_n(n;p,t) < b_n(n;p,t) \vee (1-p(1-F^0(t))).$$

By (II.3.5) of Theorem II.B, we have, as $n \rightarrow \infty$,

$$(V.5.44) \quad \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} | \rho_n(b_n(n;p,t)) - n^{-1/2} K^*(b_n(n;p,t), n) | \stackrel{\text{a.s.}}{=} o(1).$$

Since $\sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} | b_n(n;p,t) | \xrightarrow{\text{a.s.}} 1-p(1-F^0(t))$, by the modulus of continuity for Wiener process of Lemma II.A and (V.5.8), we also have, as $n \rightarrow \infty$,

$$(V.5.45) \quad \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} | n^{-\frac{1}{2}} K^*(b_n(n; p, t), n) - n^{-\frac{1}{2}} K^*(1-p(1-F^0(t)), n) | = o_p(1).$$

Combining (V.5.44) and (V.5.45), we obtain

$$(V.5.46) \quad \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} | \rho_n(b_n(n; p, t)) - \rho_n(1-p(1-F^0(t))) | = o_p(1),$$

and combining (V.5.44) and (V.5.8), we get

$$(V.5.47) \quad \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} | \rho_n(b_n(n; p, t)) | \xrightarrow{\text{a.s.}} \text{a random variable.}$$

Since $\sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} | b_n(n; p, t) | \xrightarrow{\text{a.s.}} 1-p(1-F^0(t))$, we get

$$(V.5.48) \quad \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} \left| \frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(b_n(n; p, t)))} - 1 \right| = o_p(1)$$

by Lemma III.2.1, and thus also

$$(V.5.49) \quad \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} \left| \frac{f^0(Q(1-p(1-F^0(t))))}{f^0(Q(\delta_n(n; p, t)))} - 1 \right| = o_p(1).$$

By (II.2.2) and (II.2.3)

$$(V.5.50) \quad n + \sup_{\varphi(n) \leq t \leq T, \Gamma < p < 1} p | \beta_n(t) | \xrightarrow{\text{a.s.}} \text{a random variable.}$$

Applying (V.5.46), (V.5.49), (V.5.47), (V.5.48) and (V.5.50) to (V.5.42), we finally obtain

$$\begin{aligned}
 (V.5.51) \quad & P\{ D_n(n;p,t) : \varphi(n) \leq t \leq T, \Gamma < p < 1 \} \\
 & = P\{ \rho_n(1-p(1-F^0(t))) \leq n + p \beta_n(t) : \varphi(n) \leq t \leq T, \Gamma < p < 1 \}.
 \end{aligned}$$

Similarly, we also derive

$$\begin{aligned}
 (V.5.52) \quad & P\{ D_n'(n;p,t) : \varphi(n) \leq t \leq T, \Gamma < p < 1 \} \\
 & = P\{ \rho_n(1-p(1-F^0(t))) \geq -n + p \beta_n(t) : \varphi(n) \leq t \leq T, \Gamma < p < 1 \}
 \end{aligned}$$

Hence, (V.5.51) and (V.5.52) imply that

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} P\{ D_n'(n;p,t) \cap D_n(n;p,t) : \varphi(n) \leq t \leq T, \Gamma < p < 1 \} \\
 & = \lim_{n \rightarrow \infty} P\{ -n \leq \rho_n(1-p(1-F^0(t))) - p \beta_n(t) \leq n : \varphi(n) \leq t \leq T, \Gamma < p < 1 \} \\
 & = \lim_{n \rightarrow \infty} P\{ -n \leq G_n^{(p)}(t) \leq n : \varphi(n) \leq t \leq T, \Gamma < p < 1 \} \\
 & = P\{ |G^{(p)}(t)| \leq n : \varphi(n) \leq t \leq T, \Gamma < p < 1 \}
 \end{aligned}$$

On the other hand, combining (II.4.4), (II.4.5), (V.5.8) and (II.2.6) of Lemma II.A, we obtain

$$P\{ |G^{(p)}(t)| \leq n : 0 \leq t \leq \varphi(n), \Gamma < p < 1 \} = o_p(1).$$

Hence

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} P\{ D_n'(n;p,t) \cap D_n(n;p,t) : \varphi(n) \leq t \leq T, \Gamma < p < 1 \} \\
 & = P\{ |G^{(p)}(t)| \leq n : 0 \leq t \leq T, \Gamma < p < 1 \} \\
 & = 1 - \alpha.
 \end{aligned}$$

6. Bootstrapped PL (1-p) percentile residual lifetime process.

Although we have constructed confidence bands for $\bar{R}^{(p)}(t)$ in Theorems V.5.1, V.5.2, V.5.3 and V.5.4, it is impossible either to compute or to simulate directly τ , τ' , τ'' and n in (V.5.3), (V.5.17), (V.5.29) and (V.5.41), respectively, since these limiting distributions for $\sup |G^{(p)}(t)|$ contain not only Wiener processes but also the unknown underlying distribution and quantile functions.

Bootstrapping provides a procedure to estimate τ , τ' , τ'' and n . The bootstrapped estimations of these parameters will enable us to construct confidence bands for $R^{(p)}(t)$ in practice. For a bootstrapped version of the M. Csörgő and S. Csörgő (1986) paper in the non-censored case we refer to Barabás, M. Csörgő, Horváth and Yandell (1985). In this section we will first discuss approximations of the bootstrapped (1-p) percentile residual lifetime process.

Under the basic setting discussed in Section 5 of Chapter II, define the bootstrapped PL (1-p) percentile residual lifetime as

$$(V.6.1) \quad R_{m,n}^{(p)}(t) := Q_{m,n}(1-p(1-F_{m,n}^0(t))) - t,$$

and the bootstrapped version of $r_n^{(p)}(t)$ as

$$(V.6.2) \quad r_{m,n}^{(p)}(t) := \sqrt{m} (R_{m,n}^{(p)}(t) - R_n^{(p)}(t)).$$

We also define a sequence of stochastic processes $\{H_m^{(p)}(t)\}$ such that

$$(V.6.3) \quad H_m^{(p)}(t) := p(1-F^0(t)) \left[\frac{W_m'(d(t)) - W_m'(d(Q(1-p(1-F^0(t))))}{f^0(Q(1-p(1-F^0(t))))} \right],$$

where $\{W_m'(t) : t \geq 0\}$ is that of Theorem II.E.

The following three theorems provide strong approximations for $r_{m,n}^{(p)}(t)$ which will be used for construction of the bootstrapped confidence bands for $R^{(p)}(t)$.

Theorem V.6.1. In addition to the conditions in Theorem V.4.1, we assume that

$$(V.6.4) \quad 0 < \liminf_{m,n \rightarrow \infty} m/n \leq \limsup_{m,n \rightarrow \infty} m/n < \infty.$$

Then we have

$$(V.6.5) \quad \sup_{0 \leq t \leq T} |r_{m,n}^{(p)}(t) - H_m^{(p)}(t)| = o_p(1) \text{ as } m \text{ and } n \rightarrow \infty.$$

Proof. Using $Q_{m,n}(y) = Q(q_{m,n}(y))$ (cf. (II.5.5)) and $Q_n(y) = Q(q_n(y))$ (cf. (V.2.4)), we have

$$(V.6.6) \quad r_{m,n}^{(p)}(t) = \sqrt{m} (Q_{m,n}(1-p(1-F_{m,n}^0(t))) - Q_n(1-p(1-F_n^0(t))))$$

$$\begin{aligned}
&= \sqrt{m} (Q(q_{m,n}(1-p(1-F_{m,n}^0(t)))) - Q(q_n(1-p(1-F_{m,n}^0(t)))) \\
&\quad + \sqrt{m} (Q(q_n(1-p(1-F_{m,n}^0(t)))) - Q(1-p(1-F_{m,n}^0(t))) \\
&\quad + \sqrt{m} (Q(1-p(1-F_{m,n}^0(t))) - Q(1-p(1-F_n^0(t))) \\
&\quad + \sqrt{m} (Q(1-p(1-F_n^0(t))) - Q(q_n(1-p(1-F_n^0(t)))) .
\end{aligned}$$

Applying one-term Taylor expansion on Q , and then using (II.5.2), (II.5.3) and (II.5.6), we obtain

$$\begin{aligned}
(V.6.7) \quad r_{m,n}^{(p)}(t) &= (f^0(Q(\delta_a)))^{-1}(-u_{m,n}(1-p(1-F_{m,n}^0(t)))) \\
&\quad + \sqrt{m/n} (f^0(Q(\delta_b)))^{-1}(-u_n(1-p(1-F_{m,n}^0(t)))) \\
&\quad + (f^0(Q(\delta_c)))^{-1} p_{e_{m,n}}(F^0(t)) \\
&\quad + \sqrt{m/n} (f^0(Q(\delta_d)))^{-1}(u_n(1-p(1-F_n^0(t)))) \\
&=: a_{m,n}^{(p)}(t) + \sqrt{m/n} b_{m,n}^{(p)}(t) + c_{m,n}^{(p)}(t) + \sqrt{m/n} d_{m,n}^{(p)}(t),
\end{aligned}$$

where

$$\begin{aligned}
&q_{m,n}(1-p(1-F_{m,n}^0(t))) \wedge q_n(1-p(1-F_{m,n}^0(t))) < \delta_a \\
&\quad < q_{m,n}(1-p(1-F_{m,n}^0(t))) \vee q_n(1-p(1-F_{m,n}^0(t))), \\
&q_n(1-p(1-F_{m,n}^0(t))) \wedge (1-p(1-F_{m,n}^0(t))) < \delta_b \\
&\quad < q_n(1-p(1-F_{m,n}^0(t))) \vee (1-p(1-F_{m,n}^0(t))), \\
(V.6.8) \quad &(1-p(1-F_{m,n}^0(t))) \wedge (1-p(1-F_n^0(t))) < \delta_c \\
&\quad < (1-p(1-F_{m,n}^0(t))) \vee (1-p(1-F_n^0(t))), \\
&(1-p(1-F_n^0(t))) \wedge q_n(1-p(1-F_n^0(t))) < \delta_d \\
&\quad < (1-p(1-F_n^0(t))) \vee q_n(1-p(1-F_n^0(t))) .
\end{aligned}$$

We now want to show that δ_a , δ_b , δ_c and δ_d in (V.6.8) converge to $1-p(1-F^0(t))$ a.s. and uniformly in t for $0 \leq t \leq T$.

First we observe

$$(V.6.9) \quad \sup_{0 \leq t \leq T} | (1-p(1-F_{m,n}^0(t))) - (1-p(1-F^0(t))) |$$

$$\leq \sup_{0 \leq t \leq T} | F_{m,n}^0(t) - F_n^0(t) | + \sup_{0 \leq t \leq T} | F_n^0(t) - F^0(t) | \xrightarrow{P} 0,$$

as m and $n \rightarrow \infty$, by (II.5.9) and (II.2.7). Obviously, by (II.2.7), we get

$$(V.6.10) \quad \sup_{0 \leq t \leq T} | (1-p(1-F_n^0(t))) - (1-p(1-F^0(t))) | \stackrel{\text{a.s.}}{=} o(1).$$

Combining (V.6.9) and (V.6.10), we have

$$(V.6.11) \quad \sup_{0 \leq t \leq T} | \delta_c - (1-p(1-F^0(t))) | = o_P(1).$$

Secondly,

$$(V.6.12) \quad \sup_{0 \leq t \leq T} | q_n(1-p(1-F_n^0(t))) - (1-p(1-F^0(t))) |$$

$$\leq \sup_{0 \leq t \leq T} | q_n(1-p(1-F_n^0(t))) - (1-p(1-F_n^0(t))) |$$

$$+ \sup_{0 \leq t \leq T} | (1-p(1-F_n^0(t))) - (1-p(1-F^0(t))) | \stackrel{\text{a.s.}}{\longrightarrow} 0,$$

as $n \rightarrow \infty$, by (II.4.7). Together with (V.6.10), (V.6.12) implies

$$(V.6.13) \quad \sup_{0 \leq t \leq T} | \delta_d - (1-p(1-F^0(t))) | = o_p(1) .$$

Thirdly, we observe

$$(V.6.14) \quad \sup_{0 \leq t \leq T} | q_n(1-p(1-F_{m,n}^0(t))) - (1-p(1-F^0(t))) |$$

$$\leq \sup_{0 \leq t \leq T} | q_n(1-p(1-F_{m,n}^0(t))) - (1-p(1-F_{m,n}^0(t))) |$$

$$+ \sup_{0 \leq t \leq T} | (1-p(1-F_{m,n}^0(t))) - (1-p(1-F^0(t))) | \xrightarrow{P} 0$$

as m and $n \rightarrow \infty$, by (II.4.7) and (V.6.9). Combining (V.6.9) and (V.6.14), we obtain

$$(V.6.15) \quad \sup_{0 \leq t \leq T} | \delta_b - (1-p(1-F^0(t))) | = o_p(1) .$$

Lastly, we have

$$(V.6.16) \quad \sup_{0 \leq t \leq T} | q_{m,n}(1-p(1-F_{m,n}^0(t))) - (1-p(1-F^0(t))) |$$

$$\leq \sup_{0 \leq t \leq T} | q_{m,n}(1-p(1-F_{m,n}^0(t))) - q_n(1-p(1-F_{m,n}^0(t))) |$$

$$+ \sup_{0 \leq t \leq T} | q_n(1-p(1-F_{m,n}^0(t))) - (1-p(1-F^0(t))) | \xrightarrow{P} 0$$

as m and $n \rightarrow \infty$, by (II.5.10) and (V.6.14). Together with (V.6.14), (V.6.16) implies

$$(V.6.17) \quad \sup_{0 \leq t \leq T} | \delta_a - (1-p(1-F^0(t))) | = o_p(1).$$

Now by (V.6.9), (V.6.17), (II.5.9), Lemma II.A, and the continuity and positivity of $f^0(Q(.))$ on $[1-p, 1-p(1-F^0(T))]$, we obtain

$$\begin{aligned} (V.6.18) \quad & \sup_{0 \leq t \leq T} \left| a_{m,n}^{(p)}(t) + \frac{p(1-F^0(t))W_m'(d(Q(1-p(1-F^0(t))))}{f^0(Q(1-p(1-F^0(t))))} \right| \\ &= \sup_{0 \leq t \leq T} \left| - \frac{p(1-F_{m,n}^0(t))}{f^0(Q(\delta_a))} W_m'(d(Q(1-p(1-F_{m,n}^0(t)))) \right. \\ & \quad \left. + \frac{p(1-F^0(t))}{f^0(Q(1-p(1-F^0(t))))} W_m'(d(Q(1-p(1-F^0(t)))) \right| + o_p(1) \\ &= o_p(1). \end{aligned}$$

By (II.4.5), (V.6.9), (V.6.15), Lemma II.A, and the continuity and positivity of $f^0(Q(.))$ on $[1-p, 1-p(1-F^0(T))]$, we get

$$\begin{aligned} (V.6.19) \quad & \sup_{0 \leq t \leq T} \left| b_{m,n}^{(p)}(t) + \frac{p(1-F^0(t))W(d(Q(1-p(1-F^0(t))))}{f^0(Q(1-p(1-F^0(t))))} \right| \\ &= \sup_{0 \leq t \leq T} \left| - \frac{p(1-F_{m,n}^0(t))}{f^0(Q(\delta_b))} W(d(Q(1-p(1-F_{m,n}^0(t)))) \right. \\ & \quad \left. + \frac{p(1-F^0(t))}{f^0(Q(1-p(1-F^0(t))))} W(d(Q(1-p(1-F^0(t)))) \right| + o_p(1) \\ &= o_p(1). \end{aligned}$$

Again by (II.5.7), (V.6.11), Lemma II.A, and the continuity and positivity of $f^0(Q(\cdot))$ on $[1-p, 1-p(1-F^0(T))]$, we get

$$\begin{aligned}
 (V.6.20) \quad & \sup_{0 \leq t \leq T} \left| c_{m,n}^{(p)}(t) - \frac{p(1-F^0(t))}{f^0(Q(1-p(1-F^0(t))))} w_m'(d(t)) \right| \\
 &= \sup_{0 \leq t \leq T} \left| \frac{p(1-F^0(t))}{f^0(Q(\delta_c))} w_m'(d(t)) - \frac{p(1-F^0(t))}{f^0(Q(1-p(1-F^0(t))))} w_m'(d(t)) \right| \\
 &\quad + o_p(1) = o_p(1).
 \end{aligned}$$

Finally, by (II.4.5), (V.6.10), (V.6.13), Lemma II.A, and the continuity and positivity of $f^0(Q(\cdot))$ on $[1-p, 1-p(1-F^0(T))]$, we get

$$\begin{aligned}
 (V.6.21) \quad & \sup_{0 \leq t \leq T} \left| d_{m,n}^{(p)}(t) - \frac{p(1-F^0(t))w(d(Q(1-p(1-F^0(t)))))}{f^0(Q(1-p(1-F^0(t))))} \right| \\
 &= \sup_{0 \leq t \leq T} \left| \frac{p(1-F_n^0(t))}{f^0(Q(\delta_d))} w(d(Q(1-p(1-F_n^0(t)))) \right. \\
 &\quad \left. - \frac{p(1-F^0(t))}{f^0(Q(1-p(1-F^0(t))))} w(d(Q(1-p(1-F^0(t)))) \right| + o_p(1) \\
 &= o_p(1).
 \end{aligned}$$

Combining (V.6.7), (V.6.18), (V.6.19), (V.6.20), (V.6.21) and (V.6.3), and observing the condition in (V.6.4), we obtain the assertion in (V.6.5) of Theorem V.6.i. $\square$

Theorem V.6.2. In addition to the conditions in Theorem V.4.2, assume also (V.6.4). Then we have

$$(V.6.22) \quad \sup_{0 < p < 1} |r_{m,n}^{(p)}(t) - H_m^{(p)}(t)| = o_p(1) \text{ as } m \text{ and } n \rightarrow \infty.$$

Proof. Since the proof of this theorem is almost identical to that of Theorem V.6.1, we omit it here. $\square$

Theorem V.6.3. In addition to the conditions in Theorem V.4.3, assume also (V.6.4). Then we have

$$(V.6.23) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} |r_{m,n}^{(p)}(t) - H_m^{(p)}(t)| = o_p(1) \text{ as } m \text{ and } n \rightarrow \infty.$$

Proof. We now want to show that $\delta_a, \delta_b, \delta_c$ and δ_d in (V.6.8) converge to $1-p(1-F^0(t))$ a.s. and uniformly in t and p for $\mu \leq t \leq T$ and $\Gamma < p < 1$.

First we observe

$$(V.6.24) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} |(1-p(1-F_{m,n}^{(p)}(t))) - (1-p(1-F^0(t))))| \\ \leq \sup_{\mu \leq t \leq T, \Gamma < p < 1} |F_{m,n}^{(p)}(t) - F_n^{(p)}(t)| + \sup_{\mu \leq t \leq T, \Gamma < p < 1} |F_n^{(p)}(t) - F^0(t)| \xrightarrow{P} 0,$$

as m and $n \rightarrow \infty$, by (II.5.9) and (II.2.7). Obviously, by (II.2.7), we get

$$(V.6.25) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} | (1-p(1-F_n^0(t))) - (1-p(1-F^0(t))) | \stackrel{\text{a.s.}}{=} o(1).$$

Combining (V.6.24) and (V.6.25), we have

$$(V.6.26) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} | \delta_c - (1-p(1-F^0(t))) | = o_p(1).$$

Secondly, we have

$$\begin{aligned} (V.6.27) \quad & \sup_{\mu \leq t \leq T, \Gamma < p < 1} | q_n(1-p(1-F_n^0(t))) - (1-p(1-F^0(t))) | \\ & \leq \sup_{\mu \leq t \leq T, \Gamma < p < 1} | q_n(1-p(1-F_n^0(t))) - (1-p(1-F_n^0(t))) | \\ & \quad + \sup_{\mu \leq t \leq T, \Gamma < p < 1} | (1-p(1-F_n^0(t))) - (1-p(1-F^0(t))) | \stackrel{\text{a.s.}}{\longrightarrow} 0, \end{aligned}$$

as $n \rightarrow \infty$, by (II.4.7). Together with (V.6.25), (V.6.27) implies

$$(V.6.28) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} | \delta_d - (1-p(1-F^0(t))) | = o_p(1).$$

Thirdly, we observe

$$\begin{aligned} (V.6.29) \quad & \sup_{\mu \leq t \leq T, \Gamma < p < 1} | q_n(1-p(1-F_{m,n}^0(t))) - (1-p(1-F^0(t))) | \\ & \leq \sup_{\mu \leq t \leq T, \Gamma < p < 1} | q_n(1-p(1-F_{m,n}^0(t))) - (1-p(1-F_{m,n}^0(t))) | \\ & \quad + \sup_{\mu \leq t \leq T, \Gamma < p < 1} | (1-p(1-F_{m,n}^0(t))) - (1-p(1-F^0(t))) | \stackrel{P}{\longrightarrow} 0, \end{aligned}$$

as m and $n \rightarrow \infty$, by (II.4.7) and (V.6.9). Combining (V.6.24) and (V.6.29), we obtain

$$(V.6.30) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} | \delta_b - (1-p(1-F^0(t))) | = o_p(1).$$

Lastly, we have

$$(V.6.31) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} | q_{m,n}(1-p(1-F_{m,n}^0(t))) - (1-p(1-F^0(t))) |$$

$$\leq \sup_{\mu \leq t \leq T, \Gamma < p < 1} | q_{m,n}(1-p(1-F_{m,n}^0(t))) - q_n(1-p(1-F_{m,n}^0(t))) |$$

$$+ \sup_{\mu \leq t \leq T, \Gamma < p < 1} | q_n(1-p(1-F_{m,n}^0(t))) - (1-p(1-F^0(t))) | \xrightarrow{P} 0,$$

as m and $n \rightarrow \infty$, by (II.5.10) and (V.6.29). Together with (V.6.29), (V.6.31) implies

$$(V.6.32) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} | \delta_a - (1-p(1-F^0(t))) | = o_p(1).$$

Now by (V.6.24), (V.6.32), (II.5.9), Lemma II.A, and the continuity and positivity of $f^0(\cdot)$ on $[\mu, Q(1-\Gamma(1-F^0(T)))]$, we obtain

$$(V.6.33) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} \left| a_{m,n}^{(p)}(t) + \frac{p(1-F^0(t))W_m'(d(Q(1-p(1-F^0(t))))}{f^0(Q(1-p(1-F^0(t))))} \right|$$

$$\begin{aligned}
&= \sup_{\mu \leq t \leq T, \Gamma < p < 1} \left| - \frac{p(1-F_{m,n}^0(t))}{f^0(Q(\delta_a))} W_m'(d(Q(1-p(1-F_{m,n}^0(t)))) \right. \\
&\quad \left. + \frac{p(1-F^0(t))}{f^0(Q(1-p(1-F^0(t))))} W_m'(d(Q(1-p(1-F^0(t)))) \right| + o_p(1) \\
&= o_p(1).
\end{aligned}$$

Applying (II.4.5), (V.6.24), (V.6.30), Lemma II.A, and the continuity and positivity of $f^0(\cdot)$ on $[\mu, Q(1-\Gamma(1-F^0(T)))]$, we get

$$\begin{aligned}
(V.6.34) \quad &\sup_{\mu \leq t \leq T, \Gamma < p < 1} \left| b_{m,n}^{(p)}(t) + \frac{p(1-F^0(t))W(d(Q(1-p(1-F^0(t))))}{f^0(Q(1-p(1-F^0(t))))} \right| \\
&= \sup_{\mu \leq t \leq T, \Gamma < p < 1} \left| - \frac{p(1-F_{m,n}^0(t))}{f^0(Q(\delta_b))} W(d(Q(1-p(1-F_{m,n}^0(t)))) \right. \\
&\quad \left. + \frac{p(1-F^0(t))}{f^0(Q(1-p(1-F^0(t))))} W(d(Q(1-p(1-F^0(t)))) \right| + o_p(1) \\
&= o_p(1).
\end{aligned}$$

Again by (II.5.7), (V.6.24), Lemma II.A, and the continuity and positivity of $f^0(\cdot)$ on $[\mu, Q(1-\Gamma(1-F^0(T)))]$, we get

$$(V.6.35) \quad \sup_{\mu \leq t \leq T, \Gamma < p < 1} \left| c_{m,n}^{(p)}(t) - \frac{p(1-F^0(t))}{f^0(Q(1-p(1-F^0(t))))} W_m'(d(t)) \right|$$

$$\begin{aligned}
&= \sup_{\mu \leq t \leq T, \Gamma < p < 1} \left| \frac{p(1-F^0(t))}{f^0(Q(\delta_c))} w_m'(d(t)) \right. \\
&\quad \left. - \frac{p(1-F^0(t))}{f^0(Q(1-p(1-F^0(t))))} w_m'(d(t)) \right| + o_p(1) \\
&= o_p(1) .
\end{aligned}$$

Finally, by (II.4.5), (V.6.25), (V.6.28), Lemma II.A, and the continuity and positivity of $f^0(\cdot)$ on $[\mu, Q(1-\Gamma(1-F^0(T)))]$, we get

$$\begin{aligned}
(V.6.36) \quad &\sup_{\mu \leq t \leq T, \Gamma < p < 1} \left| d_{m,n}^{(p)}(t) - \frac{p(1-F^0(t))W(d(Q(1-p(1-F^0(t))))))}{f^0(Q(1-p(1-F^0(t))))} \right| \\
&= \sup_{\mu \leq t \leq T, \Gamma < p < 1} \left| \frac{p(1-F_n^0(t))}{f^0(Q(\delta_d))} W(d(Q(1-p(1-F_n^0(t)))))) \right. \\
&\quad \left. - \frac{p(1-F^0(t))}{f^0(Q(1-p(1-F^0(t))))} W(d(Q(1-p(1-F^0(t)))))) \right| + o_p(1) \\
&= o_p(1) .
\end{aligned}$$

Combining (V.6.7), (V.6.33), (V.6.34), (V.6.35), (V.6.36) and (V.6.3), and observing the condition in (V.6.4), we obtain the assertion in (V.6.23) of Theorem V.6.3. $\square$

7. Bootstrapped confidence bands for $R^{(p)}(t)$.

Let $r_{m,n}^{(i)(p)}(t)$ be the i -th generated bootstrapped process $r_{m,n}^{(p)}(t)$ for $i = 1, 2, \dots, N$. We also define

$$(V.7.1) \quad \begin{aligned} J_n^{(p)}(t) &:= G_n^{(p)}(t) / f^0(Q(1-p(1-F^0(t)))) , \\ J^{(p)}(t) &:= G^{(p)}(t) / f^0(Q(1-p(1-F^0(t)))) , \end{aligned}$$

where $G_n^{(p)}(t)$ and $G^{(p)}(t)$ are defined in (V.4.1) and (V.4.2), respectively. Then we immediately have, for each m and n ,

$$(V.7.2) \quad \begin{aligned} \{ H_m^{(p)}(t) : t \geq 0, 0 < p < 1 \} &\stackrel{D}{=} \{ J_n^{(p)}(t) : t \geq 0, 0 < p < 1 \} \\ &\stackrel{D}{=} \{ J^{(p)}(t) : t \geq 0, 0 < p < 1 \} , \end{aligned}$$

where $H_m^{(p)}(t)$ is defined in (V.6.3). We also have the following theorem.

Theorem V.7.1. Let

$$(i) \quad M^{(1)}(y) = M^{(1)}(y; p, T) := P\left\{ \sup_{0 \leq t \leq T} |J^{(p)}(t)| \leq y \right\} ;$$

$$(ii) \quad M^{(2)}(y) = M^{(2)}(y; t) := P\left\{ \sup_{0 < p < 1} |J^{(p)}(t)| \leq y \right\} ;$$

$$(iii) \quad M^{(3)}(y) = M^{(3)}(y; \mu, \Gamma, T) := P\left\{ \sup_{\mu \leq t \leq T, \Gamma < p < 1} |J^{(p)}(t)| \leq y \right\} .$$

Then $M^{(1)}(y)$, $M^{(2)}(y)$ and $M^{(3)}(y)$, under the corresponding conditions of Theorems V.4.1, V.4.2 and V.4.3, respectively, are strictly monotone increasing continuous functions in $y \in (0, \infty)$.

Proof. We prove only that $M^{(3)}(y)$ is continuous in y under the conditions of Theorem V.4.3.

The strict monotonicity of the function is obvious, since $J^{(p)}(t)$ is a Gaussian process and the standard normal density function is positive on $(-\infty, \infty)$. Due to the continuity and positivity of $f^0(\cdot)$ on $[\mu, Q(1 - \Gamma(1 - F^0(T)))]$, we only have to show that

$$P\left\{ \sup_{\mu \leq t \leq T, \Gamma < p < 1} |G^{(p)}(t)| \leq y \right\}$$

is continuous in y . Using Lemma 1.1.1 of Csörgö and Révész (1981), for any ϵ and $\delta > 0$, we can define the finite sequences $\Gamma = p_1 < \dots < p_k = 1$ and $\mu = t_1 < \dots < t_m = T$ such that,

$$P\left\{ \max_{1 \leq i \leq k-1} \max_{1 \leq j \leq m-1} \sup_{p_i \leq p \leq p_{i+1}} \sup_{t_j \leq t \leq t_{j+1}} |G^{(p_i)}(t_j) - G^{(p)}(t)| > \epsilon \right\} \leq \delta.$$

The random variables $\phi_{i,j} = G^{(p_i)}(t_j)$ have Gaussian joint distributions, $E\phi_{i,j} = 0$ and $E\phi_{i,j}^2 > 0$, $1 \leq i \leq k$, $1 \leq j \leq m$. By Theorem 1 of Tsirel'son (1975), we know that the distribution function of $\max(1 \leq i \leq k-1) \max(1 \leq j \leq m-1) |\phi_{i,j}|$ is continuous on $(0, \infty)$, and hence the proof is complete. $\square$

Observing that $M^{(1)}(y)$, $M^{(2)}(y)$ and $M^{(3)}(y)$ are strictly monotone increasing continuous functions, we define their inverse functions for a fixed $\alpha \in (0,1)$ as

$$c^{(1)}(\alpha) = c^{(1)}(\alpha; p, T) := \inf\{ y > 0 : M^{(1)}(y) = 1 - \alpha \} ;$$

$$(V.7.3) \quad c^{(2)}(\alpha) = c^{(2)}(\alpha; t) := \inf\{ y > 0 : M^{(2)}(y) = 1 - \alpha \} ;$$

$$c^{(3)}(\alpha) = c^{(3)}(\alpha; \mu, \Gamma, T) := \inf\{ y > 0 : M^{(3)}(y) = 1 - \alpha \} .$$

Then $c^{(1)}(\alpha)$, $c^{(2)}(\alpha)$ and $c^{(3)}(\alpha)$ are continuous functions in α .

Now we define

$$\begin{aligned} M_{N,m,n}^{(1)}(y) &= M_{N,m,n}^{(1)}(y; p, T) \\ &:= N^{-1} \#\{ 1 \leq i \leq N : \sup_{0 \leq t \leq T} |r_{m,n}^{(i)}(p)(t)| \leq y \} ; \end{aligned}$$

$$\begin{aligned} (V.7.4) \quad M_{N,m,n}^{(2)}(y) &= M_{N,m,n}^{(2)}(y; t) \\ &:= N^{-1} \#\{ 1 \leq i \leq N : \sup_{0 < p < 1} |r_{m,n}^{(i)}(p)(t)| \leq y \} ; \end{aligned}$$

$$\begin{aligned} M_{N,m,n}^{(3)}(y) &= M_{N,m,n}^{(3)}(y; \mu, \Gamma, T) \\ &:= N^{-1} \#\{ 1 \leq i \leq N : \sup_{\mu \leq t \leq T, \Gamma < p < 1} |r_{m,n}^{(i)}(p)(t)| \leq y \} . \end{aligned}$$

Theorem V.7.2. As N , n and $m \rightarrow \infty$, we have

$$(i) \quad \sup_{y > 0} |M_{N,m,n}^{(i)}(y) - M^{(i)}(y)| \stackrel{\text{a.s.}}{=} o(1) ;$$

$$(ii) \quad \sup_{y>0} | M_{N,m,n}^{(2)}(y) - M^{(2)}(y) | \stackrel{\text{a.s.}}{=} o(1) ;$$

$$(iii) \quad \sup_{y>0} | M_{N,m,n}^{(3)}(y) - M^{(3)}(y) | \stackrel{\text{a.s.}}{=} o(1) .$$

Proof. We prove (iii) only. By the Glivenko-Cantelli theorem we get, for n and m fixed,

$$(V.7.5) \quad \sup_{y>0} | M_{N,m,n}^{(3)}(y) - P \left\{ \sup_{\mu \leq t \leq T, \Gamma < p < 1} | r_{m,n}^{(p)}(t) | \leq y \right\} |$$

$$\stackrel{\text{a.s.}}{=} o(1) ,$$

as $N \rightarrow \infty$. Now applying (V.6.23) of Theorem V.6.3, (V.7.2), (iii) of Theorem V.7.1 and (V.7.5), we obtain the assertion in (iii). $\square$

We define empirical counterparts for $c^{(1)}(\alpha)$, $c^{(2)}(\alpha)$ and $c^{(3)}(\alpha)$ in terms of $M_{N,m,n}^{(1)}(y)$, $M_{N,m,n}^{(2)}(y)$ and $M_{N,m,n}^{(3)}(y)$ as

$$c_{N,m,n}^{(1)}(\alpha) = c_{N,m,n}^{(1)}(\alpha; p, T)$$

$$:= \inf \{ y > 0 : M_{N,m,n}^{(1)}(y) \geq 1-\alpha \} ;$$

$$(V.7.6) \quad c_{N,m,n}^{(2)}(\alpha) = c_{N,m,n}^{(2)}(\alpha; t)$$

$$:= \inf \{ y > 0 : M_{N,m,n}^{(2)}(y) \geq 1-\alpha \} ;$$

$$c_{N,m,n}^{(3)}(\alpha) = c_{N,m,n}^{(3)}(\alpha; \mu, \Gamma, T)$$

$$:= \inf \{ y > 0 : M_{N,m,n}^{(3)}(y) \geq 1-\alpha \} .$$

Now, we are in the position to construct our promised bootstrapped confidence bands for $R^{(p)}(t)$.

Theorem V.7.3. Under the assumptions in Theorem V.6.1, we have, for given $\alpha \in (0,1)$,

$$(V.7.7) \quad \lim_{N,n,m \rightarrow \infty} P\{ R_n^{(p)}(t) - n^{-\frac{1}{2}} c_{N,m,n}^{(1)}(\alpha) \leq R^{(p)}(t) \leq R_n^{(p)}(t) + n^{-\frac{1}{2}} c_{N,m,n}^{(1)}(\alpha) : 0 \leq t \leq T \} = 1 - \alpha.$$

Proof. We first prove that $c_{N,m,n}^{(1)}(\alpha)$ converges to $c^{(1)}(\alpha)$ a.s. for α fixed. For all y in some neighbourhood of $c^{(1)}(\alpha)$, we have (i) of Theorem V.7.2. Hence by Lemma of Horváth(1984) and the continuity of $M^{(1)}(y)$ proven in Theorem V.7.1, we have, as N, m and $n \rightarrow \infty$,

$$(V.7.8) \quad c_{N,m,n}^{(1)}(\alpha) \xrightarrow{\text{a.s.}} c^{(1)}(\alpha).$$

This implies that

$$\begin{aligned} (V.7.9) \quad & \lim_{N,n,m \rightarrow \infty} P\{ R_n^{(p)}(t) - n^{-\frac{1}{2}} c_{N,m,n}^{(1)}(\alpha) \leq R^{(p)}(t) \leq R_n^{(p)}(t) + n^{-\frac{1}{2}} c_{N,m,n}^{(1)}(\alpha) : 0 \leq t \leq T \} \\ & = \lim_{n \rightarrow \infty} P\{ -c^{(1)}(\alpha) \leq \frac{r_n^{(p)}(t)}{f^0(Q(1-p(1-F^0(t))))} \leq c^{(1)}(\alpha) : 0 \leq t \leq T \} \\ & = P\{ -c^{(1)}(\alpha) \leq J^{(p)}(t) \leq c^{(1)}(\alpha) : 0 \leq t \leq T \} = 1 - \alpha. \end{aligned}$$

Theorem V.7.4. Under the assumptions in Theorem V.6.2, we have,
for given $\alpha \in (0,1)$,

$$(V.7.10) \quad \lim_{N,n,m \rightarrow \infty} P\{ R_n^{(p)}(t) - n^{-\frac{1}{2}} c_{N,m,n}^{(2)}(\alpha) \leq R^{(p)}(t) \\ \leq R_n^{(p)}(t) + n^{-\frac{1}{2}} c_{N,m,n}^{(2)}(\alpha) : 0 < p < 1 \} \\ = 1 - \alpha .$$

Proof. Since the proof of this theorem is similar to that of Theorem V.7.3, we omit it. $\square$

Theorem V.7.5. Under the assumptions in Theorem V.6.3, we have,
for given $\alpha \in (0,1)$,

$$(V.7.11) \quad \lim_{N,n,m \rightarrow \infty} P\{ R_n^{(p)}(t) - n^{-\frac{1}{2}} c_{N,m,n}^{(3)}(\alpha) \leq R^{(p)}(t) \\ \leq R_n^{(p)}(t) + n^{-\frac{1}{2}} c_{N,m,n}^{(3)}(\alpha) : \mu \leq t \leq T, \Gamma < p < 1 \} \\ = 1 - \alpha .$$

Proof. An argument like that of Theorem V.7.3 gives

$$(V.7.12) \quad c_{N,m,n}^{(3)}(\alpha) \xrightarrow{\text{a.s.}} c^{(3)}(\alpha) .$$

This in turn implies

$$(V.7.13) \quad \lim_{N,n,m \rightarrow \infty} P\{ R_n^{(p)}(t) - n^{-\frac{1}{2}} c_{N,m,n}^{(3)}(\alpha) \leq R^{(p)}(t) \\ \leq R_n^{(p)}(t) + n^{-\frac{1}{2}} c_{N,m,n}^{(3)}(\alpha) : \mu \leq t \leq T, \Gamma < p < 1 \} \\ = \lim_{n \rightarrow \infty} P\{ -c^{(3)}(\alpha) \leq \frac{r_n^{(p)}(t)}{f^0(Q(1-p(1-F^0(t))))} \leq c^{(3)}(\alpha) : \mu \leq t \leq T, \Gamma < p < 1 \}$$

$$= P\{ -c^{(3)}(\alpha) \leq J^{(p)}(t) \leq c^{(3)}(\alpha) : \mu \leq t \leq T, \Gamma < p < 1 \}$$

$$= 1 - \alpha$$

8. Applications.

The methodology developed in the previous sections is now applied to the data set described in Section 5 of Chapter III. Figure V.1 illustrates the PL $(1-p)$ percentile residual lifetime $R_n^{(p)}(t)$ of the length of the fractures in Figure III.1 defined in (V.2.1). $R_n^{(p)}(t)$ is considered as two-parameter function of t and p in Figure V.1.

In Figure V.2, $R_{1567}^{(0.75)}(t)$ and $R_{1567}^{(0.90)}(t)$ are shown as functions of p and $R_{1567}^{(p)}(0.10)$ and $R_{1567}^{(p)}(0.25)$ are illustrated in Figure V.3. Because of heavy censoring, as we have noted in Chapter III, $R_n^{0.75}(t) = 2.371 - t$ for all $t \geq 0.228$ and $R_n^{0.90}(t) = 2.371 - t$ for all $t \geq 0.608$, where 2.371 is the maximum observed value among all 1567 data. Similarly, $R_{1567}^{(p)}(0.10) = 2.371 - 0.1 = 2.271$ and $R_{1567}^{(p)}(0.25) = 2.371 - 0.25 = 2.121$ for all p in $(0, 0.6748)$ and $(0, 0.7600)$, respectively.

As first application of Theorem V.3.2, we fix $t = 0.1$, $p = 0.9$ and $\alpha = 0.05$. Then $Q_{1567}(1 - 0.9(1 - F_{1567}^0(0.1))) = Q_{1567}(0.1201) = 0.228$, $R_{1567}^{0.9}(0.1) = 0.128$, $\tau(0.05) = 1.96$ and $\sigma_{1567}(0.9, 0.1) = 0.3389$. Thus, the 95% confidence interval for $R^{(0.9)}(0.1)$ is:

$$\begin{aligned}
 (V.8.1) \quad & [Q_{1567}(0.1201 - 1.96 \times 0.3389 / \sqrt{1576}) - 0.1, \\
 & \quad Q_{1567}(0.1201 + 1.96 \times 0.3389 / \sqrt{1576}) - 0.1] \\
 & = [Q_{1567}(0.1033) - 0.1, Q_{1567}(0.1369) - 0.1] \\
 & = [0.1112, 0.1660],
 \end{aligned}$$

which is shown as the short vertical segment at $t = 0.1$ in Figure V.5.

For the second application of Theorem V.3.2, let $t = 0.5$, $p = 0.9$ and $\alpha = 0.05$. Then $Q_{1567}(1 - 0.9(1 - F_{1567}^0(0.5))) = Q_{1567}(0.3173) = 0.933$, $R_{1567}^{0.9}(0.5) = 0.433$, $\tau(0.05) = 1.96$ and $\sigma_{1567}(0.9, 0.1) = 0.7885$. Thus, the 95% confidence interval for $R^{(0.9)}(0.5)$ is:

$$\begin{aligned}
 (V.8.2) \quad & [Q_{1567}(0.3173 - 1.96 \times 0.7885 / \sqrt{1576}) - 0.5, \\
 & \quad Q_{1567}(0.3173 + 1.96 \times 0.7885 / \sqrt{1576}) - 0.5] \\
 & = [Q_{1567}(0.2783) - 0.5, Q_{1567}(0.3563) - 0.5] \\
 & = [0.140, 1.871],
 \end{aligned}$$

which is also shown as the long vertical segment at $t = 0.5$ in Figure V.5.

Comparing the confidence interval of $R^{(0.9)}(0.1)$ in (V.8.1) with that of $R^{(0.9)}(0.5)$ in (V.8.2), the latter is much wider than the first one. This is not only because of the variance of the latter which is nearly twice of that of the first, but also the PL-quantile function is extremely sensitive near the largest non-censored observation as can be

seen in Figure III.3. That is, $Q_{1567}(0.34033) = 0.991$ (the largest non-censored value), but $Q_{1567}(0.34033+\epsilon) = 2.371$ (the largest observation) for any $\epsilon > 0$.

Suppose now that we wish to construct the 95% confidence band for $R^{(0.9)}(t)$ in $t \in (0, 0.3)$ shown in Figure V.2. Using Theorem V.7.3, we did as follows. Taking $m = 500$ and $N = 500$, $M_{N,m,n}^{(1)}(y)$ (cf. (V.7.4)) were computed from the original 1567 data and shown in Figure V.4. From $M_{500,500,1567}^{(1)}(y)$ shown in Figure V.4, we have also obtained $C_{500,500,1567}^{(1)}(0.05) = 8.140$ (cf. (V.7.6)). Hence, using (V.7.7), we have constructed the confidence band:

$$\begin{aligned} & [R_{1567}^{(0.9)}(t) - C_{500,500,1567}^{(1)}(0.05)/\sqrt{1567}, \\ & \quad R_{1567}^{(0.9)}(t) + C_{500,500,1567}^{(1)}(0.05)/\sqrt{1567}] : 0 \leq t \leq 0.3 \} \\ & = \{ [R_{1567}^{(0.9)}(t) - 0.2056, R_{1567}^{(0.9)}(t) + 0.2056 : 0 \leq t \leq 0.3 \} \end{aligned}$$

which is shown as band in Figure V.5.

Similarly, suppose that we wish to construct the 95% confidence band for $R^{(p)}(0.1)$ in $p \in (0.75, 1)$ shown in Figure V.3. Using Theorem V.7.4, we can do this as follows. Again, let $m = 500$ and $N = 500$. Then we obtain $M_{N,m,n}^{(2)}(y)$ (cf. (V.7.4)) as they are shown in Figure V.6. We note here that we have taken $(0.75, 1)$ for the range of p of $M_{N,m,n}^{(2)}(y)$, instead of $(0, 1)$ as defined in (V.7.4). From $M_{500,500,1567}^{(2)}(y)$ shown in Figure V.6, we have also computed

$C_{500,500,1567}^{(2)}(0.05) = 11.070$ (cf. (V.7.6)). Hence, using (W.7.8), we have constructed the confidence band:

$$\begin{aligned} & \{ [R_{1567}^{(p)}(0.1) - C_{500,500,1567}^{(2)}(0.05)/\sqrt{1567}, \\ & \quad R_{1567}^{(p)}(0.1) + C_{500,500,1567}^{(2)}(0.05)/\sqrt{1567}] : 0.75 \leq p \leq 1 \} \\ & = \{ [R_{1567}^{(p)}(0.1) - 0.2796, R_{1567}^{(p)}(0.1) + 0.2796 : 0.75 \leq p \leq 1 \} \end{aligned}$$

which is shown in Figure V.7.

Figure V.1. PL $(1-p)$ percentile empirical residual lifetime $R_{1567}^{(p)}(t)$, defined in (V.2.1), of the length of the fractures in Figure III.1.

Figure V.2. $PL_{(1-p)}$ percentile empirical residual lifetime $R_{1567}^{(0.75)}(t)$ and $R_{1567}^{(0.9)}(t)$ for $t \in (0, 2.5)$.

Figure V.3. PL $(1-p)$ percentile empirical residual lifetime

$R_{1567}^{(p)}(0.1)$ and $R_{1567}^{(p)}(0.25)$ for $p \in (0, 1)$.

Figure V.4. Simulations of bootstrapped distribution of $\sup\{ |r_{m,n}^{(0.9)}(t)| : 0 < t \leq 0.3 \}$ with $N = 500$ and $m = 500$ from the original $n = 1567$ data.

Figure V.5. PL (1-p) percentile empirical residual lifetime $R_{1567}^{(0.9)}(t)$ for $t \in (0, 2.5)$, with the 95% bootstrapped confidence band for $R^{(0.9)}(t)$ on $t \in (0, 0.3)$. The short vertical segment at $t = 0.1$ is the confidence interval of $R^{(0.9)}(0.1)$ and the long vertical segment at $t = 0.5$ is the confidence interval of $R^{(0.9)}(0.5)$.

Figure V.6. Simulations of bootstrapped distribution of $\sup\{ |r_{m,n}^{(p)}(0.1)| : 0.75 < p < 1 \}$ with $N = 500$ and $m = 500$ from the original $n = 1567$ data.

Figure V.7. PL $(1-p\%)$ percentile empirical residual lifetime $R_{1567}^{(p)}(0.1)$ for $p \in (0, 1)$ with the 95% bootstrapped confidence band for $R^{(p)}(0.1)$ on $p \in (0.75, 1)$.

References.

- Aly, E.-E. and Bleuer, S. (1983). Confidence bands for quantile-quantile plots. Tech. Report No.16, Carleton Univ., p.23-46.
- Aly, E.-E., Csörgö, M. and Horváth, L. (1985) Strong Approximations of the quantile process of the product-limit estimator. J. Mult. Analysis, V. 16, N. 2, p.185-210.
- Anderson, T.W. (1960). A modification of the sequential probability ratio test to reduce the sample size. Ann. Math. Stat., V. 31, p.165-197.
- Anderson, T.W. and Darling, D.A. (1952). Asymptotic theory of certain 'goodness of fit' criteria based on stochastic processes. Ann. Math. Stat., V. 23, p.193-212.
- Arnold, B. and Brockett P.L. (1983). When does the β th percentile residual life function determine the distribution? Operat. Res. V. 31, p.391-396.
- Barabas, B., Csörgö, M., Horváth, L. and Yandell, B.S. (1985). Bootstrapped confidence bands for percentile lifetime. Tech. Report No.71, Carleton Univ., p.26-39.
- Barr, D.R. and Davidson, T. (1973). A Kolmogorov-Smirnov test for censored samples. Technometrics V. 15, N. 4, p.737-757.

- Billingsley, P. (1968). Convergence of Probability Measures. John Wiley, 253p.
- Blum, J.R. and Susarla, V. (1980). Maximal deviation theory of density and failure rate function estimates based on censored data. In Multivariate Analysis-V (P.R. Krishnaiah, Ed.), p.213-222.
- Breslow, N. and Crowley, J. (1974). A large sample study of the life table and product limit estimates under random censorship. *Ann. Stat.*, V. 2, p.437-453.
- Burke, M.D. (1983). Approximations of some hazard rate estimators in a competing risks model. *Stochastic Process Appl.*, V. 14, p.157-174.
- Burke, M.D., Csörgö, S. and Horváth, L. (1981). Strong approximations of some biometric estimates under random censorship. *Z. Wahrsch. Verw. Gebiete* 56, p.87-112.
- Burke, M.D., Csörgö, S. and Horváth, L. (1986). An improved approximation rate for the product-limit process. Tech. Report No.79, Carleton Univ., p.26-35.
- Burke, M.D. and Horváth, L. (1982). Density and failure rate estimation in competing risks model. *Sankhya Ser. A*, V. 46, p.135-154.
- Chung, C.F. (1986). WIENER PACK - A subroutine package for computing probabilities associated with Wiener and Brownian bridge processes. *Geol. Surv. of Canada Paper* 85-xx, in Press.

- Csáki, E. (1981). Empirical distribution function. Selected Transl. in Math. Stat. & Prob., V. 15, p.229-317.
- Csörgő, M. (1967). A new proof of some results of Rényi and the asymptotic distribution of the range of his Kolomogorov-Smirnov type random variables. Can. J. Math. V. 19, p.550-558.
- Csorgo, M. (1983). Quantile Processes with Statistical Applications. SIAM CBMS-NSF No. 42, 156p.
- Csörgő, M. and Csörgő, S. (1986). Estimation of percentile residual life. Tech. Report No.79, Carleton Univ., p.36-53.
- Csörgő, M., Csörgő, S., Horváth, L. and Mason, D.M. (1986). An Asymptotic Theory for Empirical Reliability and Concentration Processes. Lecture Notes in Stat., Springer-Verlag, 161p.
- Csörgő, M. and Révész, P. (1978). Strong approximations of the quantile process. Ann. Stat., V. 4, p.882-894.
- Csörgő, M. and Révész, P. (1981). Strong Approximations in Probability and Statistics. Academic Press, 284p.
- Csörgő, M. and Révész, P. (1984). Two approaches to constructing simultaneous confidence bounds for quantiles. Prob. Math. Stat. (Warsaw/Wroclaw) 4, p.221-236.
- Csörgő, S. and Horváth L. (1981). On the Koziol-Green model for random censorship. Biometrika, V. 68, p.391-401.

- Csorgo, S. and Horváth L. (1983). The rate of strong uniform consistency for the product-limit estimator. *Z. Wahrsch. Verw. Gebiete* 62, p.411-426.
- Csorgo, S. and Horváth L. (1985). Confidence bands from censored samples. Tech. Report No. 44, Carleton Univ., p.39-76.
- Doksum, K.A. and Yandell, B.S. (1984). Tests for exponentiality. In Handbook of Statistics-IV (P.R. Krishnaiah and P.K. Sen, Eds.), p.579-611.
- Donsker, M. (1952). Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Stat.*, V.23, p.277-281.
- Doob, J.L. (1949). A heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Stat.*, V. 20, p.393-403.
- Efron, B. (1980). Censored data and the bootstrap. *J. Amer. Stat. Asso.*, V.76, p.312-319.
- Efron, B. (1982). The Jackknife, the Bootstrap and Other Resampling Plans. SIAM CBMS-NSF 38, 92p.
- Feller, W. (1966). An Introduction to Probability Theory and its Applications. V. II, John Wiley, 626p.
- Földes, A., Rejtő, L. and Winter, B.B. (1981). Strong consistency properties of nonparametric estimators for randomly censored data

II: Estimation of density and failure rate. Period. Math. Hungar.
V.12, p.15-29.

Gillespie, M.J. and Fisher, L. (1979). Confidence bands for the Kaplan-Meier survival curve estimate. Ann. Stat., V. 7, N. 4, p.920-924.

Goodfellow, W.D. (1981). Regional stream sediment and water geochemistry reconnaissance data, Yukon and Northwest Territories; G.S.C. Open file 868, NTS 1051.

Gupta, R.C. and Langford, E.S. (1984). On the determination of a distribution by its median residual life function: A functional equation. J. Appl. Prob., 21, p.120-128.

Haine, A.L. and Singpurwalla, N.A. (1974). Some contributions to the stochastic characterization of wear. In Reliability and Biometry: Statistical Analysis of Life Length (F. Prochan and R.J. Serfling Eds.), SIAM.

Hall, W.J. and Wellner J.A. (1980). Confidence bands for a survival curve from censored data. Biometrika V. 67, N.1, p.133-143.

Horváth, L. (1984). Strong approximations of extended renewal processes. Ann. Prob., V.12, p.1149-1166.

Horváth, L. and Yandell, B.S. (1985). Convergence rates for the bootstrapped product-limit process. Tech. Report No. 71, Carleton Univ., p.67-107.

IMSL (1984). IMSL Library - User's Manual V.1-4.

Joe, H. and Prochan, F. (1984a). Percentile residual life functions. Operat. Res., 32, p.668-678.

Joe, H. and Prochan, F. (1984b). Comparison of two life distributions on the basis of their percentile residual life functions. Can. J. Stat., V.12, p.91-97.

Kaplan, E.L. and Meier, P. (1958). Nonparametric estimation from incomplete observations. J. Amer. Stat. Asso., V.53, p.457-481.

Komlós, J., Major, P. and Tusnády, G. (1975). Weak convergence and embedding. In Limit Theorems of Probability Theory (P. Révész ed.), p.149-165.

Koziol, J. A. and Byar, D.P. (1975). Percentage points of the asymptotic distributions of one and two sample K-S statistics for truncated or censored data. Technometrics V. 17, p.507-510.

McNichols, D.T. and Padgett, W.J. (1981). Kernel density estimation under random censorship. Tech. Report N.74., Univ. of South Carolina.

Rényi, A. (1953). On the theory of order statistics, Acta Math. Acad. Sci. Hungary, V. 4, p.191-231.

Schmittlein, D.C. and Morrison, D.G. (1981). The median residual life time: A characterization theorem and an application. Operat. Res. 29, p.392-399.

Schumacher, M. (1984). Two-sample tests of Cramer-von Mises- and Kolmogorov-Smirnov-type for randomly censored data. Int. Stat. Review, V. 52, N. 3, p.263-281.

Stone, D., Kamineni, D.C. and Brown, A. (1984). Geology and fracture characteristics of the Underground Research Laboratory lease near Lac du Bonnet, Manitoba. T.R. 243, Atomic Energy of Canada Ltd Research Co..

Tsirel'son, V.S. (1975). The density of the distribution of the maximum of a Gaussian process. Theory Prob. Appl., V.20, p.847-856.

Yandell, B.S. (1983). Nonparametric inference for rates with censored survival data. Ann. Stat., V.11, p.1119-1135.

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