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**ON THE HOMOTOPY OF POLYGONS IN
GRAPH EMBEDDINGS**

by

Richard Brunet M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Ontario

July 1993

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ON THE HOMOTOPY OF POLYGONS IN
GRAPH EMBEDDINGS

submitted by
RICHARD BRUNET M.Sc.

in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

Chair, Department of Mathematics and Statistics

Thesis Supervisor

External Examiner

Carleton University

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ABSTRACT

This thesis concerns disjoint polygons in graph embeddings. Let Σ be a non-trivial surface. We define a loop in Σ to be a particular equivalence class of closed paths. We introduce the notion of complete partition of loops: a partition $(\mathcal{C}, \mathcal{E})$ of the set of loops of Σ is said to be complete if $\mathcal{C}$ contains all freely homotopic copies of the loops corresponding to the elements of some normal subgroup of the fundamental group of Σ . Let $\Psi : G \rightarrow G_\Psi \subset \Sigma$ be an embedding of a connected graph G onto Σ . Let $\rho(\Psi)$ be the representativity of the embedding.

It was previously known that if Σ is orientable, then G_Ψ has at least $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ disjoint freely homotopic noncontractible polygons. In Chapter II, we improve this to show that G_Ψ has at least $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ disjoint freely homotopic noncontractible polygons **whose corresponding loops are in $\mathcal{E}$, for any complete partition $(\mathcal{C}, \mathcal{E})$** (the previous case was when $\mathcal{E}$ is the set of noncontractible loops of Σ). Specifically, there are $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ **nonseparating** disjoint freely homotopic polygons.

In Chapter III, we improve slightly the known results to show that G_Ψ has at least $\lfloor \frac{\rho(\Psi)-1}{8} \rfloor - 1$ noncontractible separating disjoint freely homotopic polygons if the genus of the orientable surface Σ is at least 2. Bounds are also given for the case of nonorientable surfaces.

In Chapter IV, we apply our results to show that any 5-connected toroidal triangulation is Hamiltonian.

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LIST OF SYMBOLS

G	3
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$E(G)$	3
$\hat{K}$	8
D^a	10
Σ	14
$\mathbf{S}^n$	14
$\mathbf{P}^n$	14
$\overrightarrow{\Gamma}_\Sigma$	15
$\vec{\gamma}$	15
$[\vec{\gamma}]$	16
$\vec{\gamma}^{-1}$	16
$\pi(\Sigma, x)$	16
$\mathcal{P}(\Sigma)$	17
ψ	17
V_ψ	17
E_ψ	17
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G_ψ	17
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∂f	18
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$\sim_f$	19
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INTRODUCTION

In this thesis, we look for particular types of polygons (or cycles) in graphs embedded onto surfaces. Roughly speaking, an embedding of a graph is a "way of drawing" a graph on a given surface, without crossing the edges otherwise than at their ends. One is naturally interested in knowing how densely a graph is embedded in the surface: this measure is the representativity (or face-width) of the embedding, denoted by ρ (this concept will be defined in Section 1.8).

Much activity in Topological Graph Theory is about the issue of how representativity tells us about the structure of the given embedded graph. Nice results about these questions came from the work of Vitray and Robertson [RoV]. More particularly, one may ask how the representativity determines the presence of some kinds of polygons, their number and their homotopy type (that is to say, their general position on the surface). It is known that there are $\frac{\rho}{2}$ contractible disjoint polygons nested around any face of a given embedded graph (see for example [FHRRo]). Mohar, Richter, and the author [BMR] have proved that there are $\frac{\rho}{2}$ noncontractible disjoint homotopic polygons in an embedded graph (this can also be shown from a more general result of Schrijver [Sc2]). In Chapter II, new results are developed on disjoint noncontractible homotopic polygons by generalizing the notion of homotopy, and with the above mentioned results (in the noncontractible case) obtained as consequences. We will also show that there are $\frac{\rho}{2}$ **nonseparating** disjoint homotopic polygons in an embedded graph of an orientable surface (so far, there was no known nontrivial bound on the number of these polygons).

We may ask the same question about separating polygons (that is to say, a polygon that separates the surface in at least two pieces). Richter and Vitray

[RV1], and Zha and Zhao [ZZ], have given a lower bound on ρ to assure the existence of a separating polygon in the embedded graph. Mohar, Richter and the author [BMR], and Richter and Vitray [RV2], have shown that there are $\frac{g}{8}$ non-contractible homotopic disjoint polygons that are separating in a given embedded graph. In Chapter III, this same bound is obtained, with some refinements on the homotopy type of these polygons.

These structures have important implications: in certain cases, we can use our construction to find a separating polygon that bounds a planar spanning subgraph of the original embedded graph. As an application, we will show that a 5-connected triangulation of the torus has a polygon that contains all the vertices of the graph, and moreover, this polygon is contractible (Chapter IV).

To pursue this search, some concepts from Topology are required, but some are also developed in Chapter I. We will refine the definition of homotopy (free homotopy), and we will also define some classes of paths, called loops, so that these new concepts will be independent of basepoints. Another new notion, complete partitions of loops, will be the basis of this framework: their properties are exactly those needed in order to obtain the results. We will then introduce the new concept of free homotopy modulo a complete set of loops, which is essential to this thesis.

The origin of this work lies in the ideas developed by Richter and Vitray [RV1], especially the notion of minimal O-circuit with respect to a face chain in the graph. This concept is refined, and by the same way extend the definition of representativity, in order to somehow "measure" sets of closed curves of a surface with the help of the embedded graph. It will then be possible to indicate the homotopy type of any given closed curve.

Using representativity is one way of obtaining information on the general structure of embedded graphs. Other methods, such as those used by Negami [N], have led to interesting results.

CHAPTER I

FOUNDATIONS

1.1. Introduction.

In this chapter, we will first present the basic notions and results from Graph Theory, Topology and Algebraic Topology that will be needed in the following chapters. In Section 1.6, we will extend the notion of closed paths to the notion of loops, which is an equivalence class of closed paths independent of the basepoint of its paths. We will then introduce the concept of complete partition of loops, an extremely important notion for the rest of this work. We will also generalize the representativity to complete sets of loops in order to “measure” these sets. In Section 1.7, we will see that there is a close relation between our theory of complete sets of loops and the fundamental group of a surface, and we will define crossing partitions, an essential concept for Chapter II. In Section 1.8, we will introduce the basic structure of embedded graph. We will close this chapter by developing a theory of curves.

1.2. Graph Theoretical tools.

We introduce now some concepts from Graph Theory that we will need through this work. The reader can refer to [Tu1], [Tu2],[BM] for more.

We assume that the reader has some knowledge of the basic concepts of Graph Theory. To begin with, a *graph* G is an ordered triple $(V(G), E(G), \phi_G)$ consisting of a nonempty set $V(G)$ of *vertices*, a set $E(G)$, disjoint from $V(G)$, of *edges*, and an *incidence function* ϕ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G , called the *ends* of this edge. If an edge

has distinct ends, we call such an edge a *link*. For simplicity of notation, we will denote a graph G by $G = (V, E)$.

The graphs we deal with in this work are *finite*, that is to say, $V(G)$ and $E(G)$ are both finite sets. A graph is *simple* if each edge has distinct ends and no two edges have the same ends. The *degree* of a vertex v of G is the number of edges of G incident with v (if an edge has the same vertex for its ends, we say that this edge is a *loop-edge* and it counts for two). An n -*bond* of G is a subgraph of G formed by two vertices and n edges whose endpoints are these two vertices.

If $G = (V, E)$ and $e \in E$, we denote by $G - \{e\}$ the graph $(V, E - \{e\})$. If $v \in V$, we denote by $G - \{v\}$ the graph $(V - \{v\}, E - \{e \in E \mid e \text{ is incident with } v\})$. If $G = (V, E)$ and $G' = (V', E')$, we denote by $G \cup G'$ the graph $(V \cup V', E \cup E')$.

Definition 1.2.1. A walk in a graph G is a finite nonnull sequence $W = (v_0, e_1, v_1, \dots, e_k, v_k)$ whose terms are alternately vertices and edges, and such that the ends of e_i are v_{i-1} and v_i , for $i = 1, \dots, k$. The vertices v_0 and v_k are respectively the origin and the terminus of W . The number k is the length of W . If all the e_i and v_i are distinct, W is a *path*. The subgraph A of G whose vertices are the vertices of W and whose edges are the edges of W is an *arc* of G .

Definition 1.2.2. A cycle is a walk $W = (v_0, e_1, v_1, \dots, e_k, v_k)$ such that

- (1) the edges $e_1, e_2, \dots, e_k$ of W are distinct;
- (2) the vertices $v_1, \dots, v_k$ are distinct; and
- (3) $v_0 = v_k$.

The subgraph C whose vertices are the vertices of W and whose edges are the edges of W is a *polygon* of G . W is a *Hamilton cycle* (or *Hamiltonian cycle*) if W is a cycle and all the vertices of G are vertices of W (we also say that G is *Hamiltonian*). A *Hamilton polygon* (or *Hamiltonian polygon*) is the polygon of a *Hamiltonian cycle*.

This concept of path allows us to describe the structure of a graph.

Definition 1.2.3. A graph is *connected* if, for any vertices $v, v' \in V$, there is a path in G with v as origin and v' as terminus. Otherwise, G is *disconnected*. A maximal connected subgraph of G is a *component* of G .

Definition 1.2.4. Let G be a connected graph. Let n be a positive integer. An *n -separation* of G is a pair (H, K) of edge-disjoint subgraphs of G such that

- (i) $H \cup K = G$;
- (ii) $|V(H) \cap V(K)| = n$; and
- (iii) $|E(H)| \geq n, |E(K)| \geq n$.

The set $V(H) \cap V(K)$ is an *n -cut* of G . If $n = 1$, the unique vertex of $|V(H) \cap V(K)|$ is a *cut vertex* of G .

Definition 1.2.5. Let G be a connected graph that is not the complete graph K_n for any n . Let m a positive integer. Then G is *m -connected* if G has no *k -separation* for $k < m$. The complete graph K_n is $(n - 1)$ -connected for all n .

Remark. It is important to notice that if G is 2-connected, it contains no loop-edge, and if it is 3-connected, G contains also no 2-bond.

Definition 1.2.6. Let G be a graph and n a positive integer. A pair (H, K) of subgraphs of G that satisfies (i), (ii), (iii) of Definition 1.2.4 and

- (iv) H and K have each at least one vertex not belonging to the other
- is a *vertical n -separation* of G . The graph G is *vertically m -connected* if it has no vertical *n -separation* for $n < m$.

Vertical connectivity is the notion of connectivity most often found in the literature. It is the type of connectivity we will use in Chapter IV only. Figure I shows an example of a graph that has a 3-separation but that contains no vertical 3-separation.

Figure I

Remark. Notice that if a graph G is m -connected for some $m \geq 0$, then G is vertically m -connected. If $m \leq 3$, a simple graph is m -connected if and only if it is vertically m -connected.

Definition 1.2.7. A block of a graph G is a maximal 2-connected subgraph of G , that is to say, it is a 2-connected subgraph of G not properly contained in another 2-connected subgraph of G .

Proposition 1.2.1. Let G be a graph. Each edge of G belongs to exactly one block of G , and each vertex of G belongs to at least one block of G . The vertices that belong to more than one block are the cut vertices of G .

Proof. See [Tu2] pp. 60-62. $\square$

The blocks of a graph G are some kind of "bricks" of G : we can decompose G uniquely into its blocks, and these are linked by the cut vertices of G .

Definition 1.2.8. Let H be a subgraph of G . A vertex of attachment of H in G is a vertex that is incident with at least one edge that is not in H . The subgraph H is detached in G if it has no vertex of attachment in G . Let J be another subgraph of G . Then H is J -detached in G if all its vertices of attachment are in J .

Definition 1.2.9. A bridge B of G is a subgraph of G that satisfies the following conditions:

- (i) B is not a subgraph of G ;
- (ii) B is J -detached in G ; and
- (iii) no proper subgraph of B satisfies both (i) and (ii).

We can also say that a bridge B of J in G is a subgraph such that either B is an edge not in J together with its ends which are in J (we then say that B is *degenerate*), or $B - V(J)$ is a component K of $G - V(J)$ and B is K , together with the edges joining K to J with their ends (we then say that B is *nondegenerate*).

1.3. Cleavage units.

As we have seen in the previous section, we can decompose a connected graph into blocks in a unique way. In the present section, we describe a decomposition of a 2-connected graph into parts called cleavage units. These are not exactly subgraphs of the original graph: they consist of subgraphs to which we add some new edges, called virtual edges. The removal of these virtual edges and the assembling of the remaining subgraphs give the original graph. We will see that the decomposition of a 2-connected graph into cleavage units is unique. The reader can learn much more by consulting the excellent textbook [Tu1], from which this overview of the theory was made.

Definition 1.3.1. *Let K be a subgraph of G . The complement of K in G is the subgraph $\bar{K}$ of G such that*

- (1) $E(\bar{K}) = E(G) \setminus E(K)$;
- (2) $V(\bar{K}) = (V(G) \setminus V(K)) \cup \{x \in V(K) : x \text{ is an endpoint of an edge of } E(\bar{K})\}$.

Definition 1.3.2. *Let G be a 2-connected graph. Let x, y be vertices of G , and let K be a bridge of $\{x, y\}$. Then K is excisable if*

- (1) $|E(K)| \geq 2$ and $|E(\bar{K})| \geq 2$; and
- (2) either K or $\bar{K}$ is 2-connected.

K is an excisable $\{x, y\}$ -component of G and $\{x, y\}$ is an excising pair of hinges.

Remark. *If K is an excisable $\{x, y\}$ -component, then $(\bar{K}, K)$ is a 2-separation.*

We say that $(\bar{K}, K)$ is an excisable 2-separation of G .

Theorem 1.3.1. *Let G be a 2-connected graph. Then G has no excisable 2-separation if and only if G is either:*

- (1) *a 3-connected graph;*
- (2) *a polygon with at least four edges; or*
- (3) *a bond with at least four edges.*

Proof. See [Tu1] p 119. $\square$

Definition 1.3.3. *Let $\{x, y\}$ be an excising pair of hinges of a 2-connected graph G , and let K be an excisable $\{x, y\}$ -component of G . Let D_1 and D_2 be the graphs derived from K and $\bar{K}$ respectively by adjoining a new edge e with ends x and y to both K and $\bar{K}$. The edge e is a virtual edge of G associated with K . The graphs D_1 and D_2 are formed by cleaving G along e , and D_1 and D_2 are the cleavage graphs of G at e .*

Definition 1.3.4. *Let $\{x, y\}$ be an excising pair of hinges of a 2-connected graph G . Let K be a corresponding excisable component of G . Let e be the virtual edge of G associated with K , and let D_1, D_2 be the cleavage graphs of G at e corresponding to K and $\bar{K}$ respectively. Let $S(D_1)$ denote the class of subgraphs of D_1 , and let $S(G, K)$ denote the class of subgraphs L of G such that either $L \subseteq K$ or $\bar{K} \subseteq L$. The D_1 -mapping f_{D_1} is a mapping of $S(D_1)$ into $S(G, K)$ defined by*

- (a) *If $N \subseteq K$, then $f_{D_1}(N) = N$;*
- (b) *if $N \subseteq D_1$ and $e \in E(N)$, then $f_{D_1}(N) = (N - \{e\}) \cup \bar{K}$.*

Proposition 1.3.2. *Let D be a cleavage graph of G at e . Let x and y be distinct vertices of D , and let X be an excisable $\{x, y\}$ -component of D that is not $D - \{e\}$ or $(\{x, y\}, e)$. Then $f_D(X)$ is an excisable $\{x, y\}$ -component of G if and only if X is excisable in D .*

Proof. See [Tu1] p 120. $\square$

Definition 1.3.3. Let G^a be the graph obtained from G by adjoining the virtual edges associated with its excisable components. Then G^a is the *augmented graph* of G .

Remark. The virtual edges of G are of course real edges of G^a , and one can show that G^a is 2-connected.

Figure I shows an example of the augmented graph G^a of a graph G . The virtual edges are represented by dotted lines.

Proposition 1.3.3. Let e be a virtual edge of G , and let D be a cleavage graph of G at e . Then we can construct an augmented graph D^a of D so that the following condition is satisfied: if e' is a virtual edge of D associated with an excisable component X of D , then e' is also a virtual edge of G associated with the excisable component $f_{D_1}(X)$ of G .

Proof. See [Tu1] p 121. $\square$

Proposition 1.3.4. Let e be a virtual edge of G , let D_1 and D_2 be two cleavage graphs of G at e . Let their augmented graphs $(D_1)^a$ and $(D_2)^a$ be constructed from G^a by the method of Proposition 1.3.3. Then $(D_1)^a$ and $(D_2)^a$ are subgraphs of G^a having the following properties:

- (a) $(D_1)^a \cap (D_2)^a = (\{x, y\}, e)$, where x and y are the ends of e ;
- (b) $(D_1)^a \cup (D_2)^a = G^a$.

Proof. See [Tu1] p 122. $\square$

Definition 1.3.6. Let G, e, D_1 and D_2 be as in Proposition 1.3.4. Given the augmented graph G^a of G , the subgraphs $(D_1)^a$ and $(D_2)^a$ of G^a are the *wings* of e in G^a . If e' is a virtual edge of D_1 , we can likewise define the wings of e' in the augmented graph $(D_1)^a$ of D_1 .

Proposition 1.3.5. *Let e be a virtual edge of G . Let D be a cleavage graph of G at e . Let e' be a virtual edge of D , let its wings in G^a be W_1 and W_2 . Then its wings in D^a are $W_1 \cap D^a$ and $W_2 \cap D^a$. Moreover, either W_1 or W_2 is a subgraph of D^a .*

Proof. See [Tu1] p 124. $\square$

Definition 1.3.7. *Let G be a 2-connected graph, and let G^a be a given augmented graph of G . Let e be a virtual edge of G with wings W_1 and W_2 in G^a . A subgraph L of G^a is cut by e if neither $W_1 \cap L$ nor $W_2 \cap L$ is a subgraph of $(\{x, y\}, e)$ (here x and y are the ends of e).*

Proposition 1.3.6. *Let e be a virtual edge of G , and let the cleavage graphs of G at e be D_1 and D_2 . Let the corresponding wings of e in G^a be W_1 and W_2 respectively. Let L be any subgraph of W_1 . Then L is cut in D_1 by a virtual edge e' of D_1 if and only if it is cut by e' in G^a . Moreover, L is not cut in G^a by any virtual edge of D_2 .*

Proof. See [Tu1] p 124. $\square$

With all these tools, we can now define the cleavage units of G .

Definition 1.3.8. *Let $Q(G)$ denote the set of all subgraphs L of G^a that are cut in G^a by no virtual edge of G . The maximal members of $Q(G)$ are the cleavage units of G .*

One can prove that the cleavage units of G are nonnull and that their union is G^a .

Figure II

Figure III

Proposition 1.3.7. *Let e be a virtual edge of G , and let the cleavage graphs of G at e be D_1 and D_2 . Then a subgraph of G^a is a cleavage unit if and only if it is a cleavage unit of D_1 or D_2 .*

Proof. See [Tu1]p 125. $\square$

Proposition 1.3.8. *Let G be a 2-connected graph such that $|E(G)| \geq 3$. Then each cleavage unit of G is either:*

- (1) *a 3-connected graph with at least three edges; or*
- (2) *a polygon with at least four edges; or*
- (3) *a bond with at least four edges.*

Each edge of G belongs to one and only one cleavage unit, and each virtual edge of G to exactly two.

Proof. See [Tu1] p 125. $\square$

As an example, Figure III shows the cleavage units of the graph of Figure II.

1.4 Topological tools.

We introduce, in this section, the concepts from Topology and Algebraic Topology that will be needed through the present work. For more on Topology, the reader can refer to [Mu] or [Ma].

We assume that the reader is familiar with the basic concepts of Topology. We remind that a topological space Σ is *t-connected* (for *topologically connected*) if the only subsets that are both open and closed in Σ are the empty set and Σ itself. A subset A of Σ is *t-connected* if the subspace A is t-connected. Given Σ , we define an equivalence relation by setting $x \simeq y$ if there is a t-connected subset of Σ that contains both x and y . The equivalence classes are called the *t-components* (or topological components) of Σ .

In the rest of this work, when we consider a subset of a topological space, its topology will be the subspace topology.

We now introduce some definitions particularly relevant to this work.

Definition 1.4.1. A surface is a compact, t -connected Hausdorff space Σ , such that every point of Σ has a neighbourhood homeomorphic to $\mathbb{R}^2$.

Definition 1.4.2. Let Σ be a surface. Suppose we give the same orientation (for example, a clockwise rotation around a point) to each point of Σ . Now suppose that, for all $x \in \Sigma$, all the points contained in any neighbourhood N of x have the same orientation. Then Σ is *orientable*. Otherwise, Σ is *nonorientable*.

Remark. We can also say that Σ is nonorientable if and only if it contains a Möbius strip.

An example of a nonorientable surface is given by the *projective plane* $\mathbb{P}^1$ (the quotient space of the closed unit disc $\bar{\Delta}$ obtained by identifying every pair of diametrically opposite points of the boundary). An example of orientable surface is the torus (the surface of a doughnut).

Definition 1.4.3. Given two surfaces Σ_1 and Σ_2 , the *connected sum* of Σ_1 and Σ_2 is the surface obtained by removing the interiors Δ_1 and Δ_2 of closed discs $\Delta_1, \bar{\Delta}_2$ in Σ_1 and Σ_2 respectively, and identifying their borders (the new surface is $(\Sigma_1 \setminus \Delta_1) \cup (\Sigma_2 \setminus \Delta_2)$ with the appropriate identifications of their borders).

Theorem 1.4.1. Any surface is either homeomorphic to the 2-sphere S_2 , or to a connected sum of n -tori S^n , or to a connected sum of n projective planes $\mathbb{P}^n$.

Proof. See [Ma] p 10 $\square$

We also denote the 2-sphere S_2 by S^0 .

Definition 1.4.4. Let Σ be a surface. If Σ is homeomorphic to S_2 , then Σ is a *trivial surface*. Otherwise, Σ is *nontrivial*.

Definition 1.4.5. Let Σ be a surface. If Σ is homeomorphic to S^n for some $n \geq 0$, then $g(\Sigma) = n$ is the *orientable genus* of Σ . If Σ is homeomorphic to P^n , then $\tilde{g}(\Sigma) = n$ is the *nonorientable genus* of Σ .

Definition 1.4.6. Let Σ be a surface.

- (1) A *path* in Σ is a continuous map $\tilde{\gamma} : I = [0, 1] \longrightarrow \Sigma$. $\tilde{\gamma}(0)$ is the *origin* of $\tilde{\gamma}$, and $\tilde{\gamma}(1)$ is the *terminus* of $\tilde{\gamma}$.
- (2) A path is *simple* if $\tilde{\gamma}$ is a bijection on $[0, 1]$, except possibly at the points $0, 1$, where we allow $\tilde{\gamma}(0) = \tilde{\gamma}(1)$.
- (3) A path is *closed* if $\tilde{\gamma}(0) = \tilde{\gamma}(1)$. The point $\tilde{\gamma}(0) = \tilde{\gamma}(1)$ is the *basepoint* of $\tilde{\gamma}$. We represent the set of closed paths in Σ by $\overline{\Gamma}_\Sigma$.

Definition 1.4.7. Let $\tilde{\gamma}$ be a path in Σ . If $\tilde{\gamma}(I)$ has a neighbourhood in Σ that is orientable, then $\tilde{\gamma}$ is an *orientation-preserving path*. Otherwise, $\tilde{\gamma}$ is an *orientation-reversing path*.

We now come to the definition of homotopy of paths, which leads to the important notion of fundamental group of a surface.

Definition 1.4.8. Two paths $\tilde{\gamma}, \tilde{\gamma}'$ in a surface Σ are *homotopic* if they have the same origin and the same terminus, and if there is a continuous map $h : I \times I \longrightarrow \Sigma$ such that

$$\begin{cases} h(0, t) = \tilde{\gamma}(t), \\ h(1, t) = \tilde{\gamma}'(t), \\ h(s, 0) = \tilde{\gamma}(0) = \tilde{\gamma}'(0), \\ h(s, 1) = \tilde{\gamma}(1) = \tilde{\gamma}'(1) \end{cases}$$

for all $(s, t) \in I \times I$. Then h is a *homotopy* between $\tilde{\gamma}$ and $\tilde{\gamma}'$. If $\tilde{\gamma}$ and $\tilde{\gamma}'$ are homotopic, we write $\tilde{\gamma} \sim \tilde{\gamma}'$.

Definition 1.4.9. If $\tilde{\gamma}$ is a path in Σ from x_0 to x_1 , and if $\tilde{\gamma}'$ is a path in Σ from x_1 to x_2 , the composition $\tilde{\gamma} \circ \tilde{\gamma}'$ of $\tilde{\gamma}$ and $\tilde{\gamma}'$ is the path $\tilde{\gamma}''$ defined by

$$\tilde{\gamma}''(t) = \begin{cases} \tilde{\gamma}(2t), & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \tilde{\gamma}'(2t - 1), & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

We can define the composition of their homotopy classes by

$$[\tilde{\gamma}] \circ [\tilde{\gamma}'] = [\tilde{\gamma} \circ \tilde{\gamma}'].$$

The following facts are elementary.

Proposition 1.4.2. The composition of homotopy classes is a well-defined operation with the following properties:

- (a) If $[\tilde{\gamma}_0] \circ ([\tilde{\gamma}_1] \circ [\tilde{\gamma}_2])$ is defined, then so is $([\tilde{\gamma}_0] \circ [\tilde{\gamma}_1]) \circ [\tilde{\gamma}_2]$, and they are equal;
- (b) Given $x \in \Sigma$, let $\tilde{x}$ denote the constant path $\tilde{x} : I \rightarrow \Sigma$ carrying all of I to the point x . If $\tilde{\gamma}$ is a path from x_0 to x_1 , then

$$[\tilde{\gamma}] \circ [\tilde{x}_1] = [\tilde{\gamma}] \quad \text{and} \quad [\tilde{x}_0] \circ [\tilde{\gamma}] = [\tilde{\gamma}];$$

- (c) Given the path $\tilde{\gamma}$ in Σ from x_0 to x_1 , let $\tilde{\gamma}^{-1}$ be the path defined by $\tilde{\gamma}^{-1}(t) = \tilde{\gamma}(1 - t)$, for $t \in I$. Then

$$[\tilde{\gamma}] \circ [\tilde{\gamma}^{-1}] = [\tilde{x}_0] \quad \text{and} \quad [\tilde{\gamma}^{-1}] \circ [\tilde{\gamma}] = [\tilde{x}_1].$$

Proof. See [Mu] p 322. $\square$

The path $\tilde{\gamma}^{-1}$ is the *inverse* of $\tilde{\gamma}$. The last Proposition shows that the set of homotopy classes is a groupoid under $\circ$. However, in the case of closed paths with the same basepoint, we obtain a group.

Definition 1.4.10. Let Σ be a space and x a point of Σ . The set of homotopy classes of closed paths based at x , with the operation $\circ$, is the *fundamental group* of Σ relative to the base point x . It is denoted by $\pi(\Sigma, x)$.

One can show that the fundamental group of a surface is independent of the choice of the basepoint.

1.5 Embeddings of graphs.

This section is devoted to the introduction of the basic concepts and results about embeddings of graphs.

Intuitively, an embedding of a graph G onto a surface Σ is "a way of drawing" G (with vertices and edges) on Σ such that the lines representing the edges never intersect otherwise than at their endpoints, that is, at their incident vertices.

We use $]a, b[$ to denote the open interval $\{x : a < x < b\}$. The closure of a subset of a topological space is denoted $\bar{A}$.

Definition 1.5.1. Let $G = (V, E)$ be a graph, Σ a surface and $\mathcal{P}(\Sigma)$ the set of all subsets of Σ . An embedding of G onto Σ is a function $\Psi : V \cup E \longrightarrow \Sigma \cup \mathcal{P}(\Sigma)$ such that

- (1) $\Psi(V) \subset \Sigma$ and $\Psi(E) \subset \mathcal{P}(\Sigma)$;
- (2) if $e \in E$, then $\Psi(e)$ is homeomorphic to $]0, 1[$;
- (3) if $v, v' \in V$ with $v \neq v'$, then $\Psi(v) \neq \Psi(v')$;
- (4) if $e, e' \in E$ with $e \neq e'$, then $\Psi(e) \cap \Psi(e') = \emptyset$;
- (5) $\Psi(V) \cap \Psi(e) = \emptyset$, for all $e \in E$;
- (6) if $e \in E$ has endpoints $v, w \in V$, then $\overline{\Psi(e)} = \{\Psi(v), \Psi(w)\} \cup \Psi(e)$.

A face of Ψ is a t -component of $\Sigma \setminus (\Psi(V) \cup \bigcup_{e \in E} \Psi(e))$. Let $G_\Psi = (\Psi(V), \Psi(E))$. Then $V_\Psi = \Psi(V)$ is the set of vertices of G_Ψ , $E_\Psi = \Psi(E)$ is the set of edges of G_Ψ and $F_\Psi = \Sigma \setminus (\Psi(V) \cup \bigcup_{e \in E} \Psi(e))$ is the set of faces of G_Ψ . We use the notation $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$.

It thus follows from this that G_Ψ has the structure of a graph. We will therefore refer to G_Ψ as a graph. The definition also implies that the surface Σ is partitioned into vertices, edges and faces of G_Ψ . We shall impose a topological structure on G_Ψ by using the induced topology from the surface. We slightly abuse the word embedding to denote this.

Definition 1.5.2. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a surface Σ . The trace of Ψ is the topological space defined by $G_\Psi(\Sigma) = \Psi(V) \cup \bigcup_{e \in E} \Psi(e)$.

We now describe a type of embedding, which will be considered for the rest of this work unless stated otherwise.

Definition 1.5.3. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a surface Σ . If all the faces of G_Ψ are homeomorphic to open discs, the embedding Ψ is an open 2-cell embedding.

The following relation, called Euler's formula, determines for open 2-cell embeddings the surface Σ on which G is embedded up to homeomorphism.

Theorem 1.5.1. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an open 2-cell embedding of a graph G onto a surface Σ . Let $\chi(\Psi) = |V_\Psi| - |E_\Psi| + |F_\Psi|$. Then

$$\chi(\Psi) = \begin{cases} 2 - 2g & \text{if } \Sigma \simeq \mathbf{S}^g \text{ for } g \geq 0, \\ 2 - h & \text{if } \Sigma \simeq \mathbf{P}^h \text{ for } h \geq 1. \end{cases}$$

Proof. See Gross and Tucker[GT] pp.121-123. $\square$

It follows from Theorem 1.5.1 that, given an open 2-cell embedding Ψ , $\chi(\Psi)$ depends only on the surface Σ .

Definition 1.5.4. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph $G = (V, E)$ onto a surface Σ . Let $f \in F_\Psi$. The boundary graph of f is the subgraph of G_Ψ whose vertices and edges are respectively elements and subsets of ∂f . We use the notation $bd(f)$ for representing the boundary of f . The sequence of vertices and edges obtained in a traversal of $bd(f)$ is the boundary walk of f . More generally, let $\tilde{\gamma}$ be a path and suppose that there exists $E' \subseteq E$, $V' \subseteq V$ such that $\gamma = V' \cup E'$. The skeleton γ of $\tilde{\gamma}$ is the subgraph (V', E') .

1.6 Loop-equivalence of paths.

In this section, we will consider some classes of closed paths, independent of the basepoints of their paths, called loops. We will also introduce complete sets of loops, an essential concept for the rest of this thesis.

We now extend the notion of homotopy between two paths.

Definition 1.6.1. *Let Σ be a topological space. Two closed paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ in Σ are freely homotopic if there exists a continuous map $h : [0, 1] \times [0, 1] \longrightarrow \Sigma$ such that*

$$(1) \quad \begin{cases} h(0, t) = \tilde{\gamma}_0(t) \text{ for } 0 \leq t \leq 1, \\ h(1, t) = \tilde{\gamma}_1(t) \text{ for } 0 \leq t \leq 1, \\ h(s, 0) = h(s, 1) \text{ for } 0 \leq s \leq 1. \end{cases}$$

We use the notation $\tilde{\gamma}_0 \sim_f \tilde{\gamma}_1$ to indicate that the closed paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are freely homotopic. The continuous map h is a free homotopy between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$.

Proposition 1.6.1. $\sim_f$ is an equivalence relation on $\overrightarrow{\Gamma}_\Sigma$.

Proof. Reflexivity is clear. We first prove symmetry. Suppose that $\tilde{\gamma}_0 \sim_f \tilde{\gamma}_1$, for some $\tilde{\gamma}_0, \tilde{\gamma}_1 \in \overrightarrow{\Gamma}_\Sigma$, and let f be the free homotopy between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$. Let $g : [0, 1] \times [0, 1] \longrightarrow \Sigma$ be defined by $g(s, t) = f(1 - s, t)$, for $(s, t) \in [0, 1] \times [0, 1]$. It is easy to see that g is continuous and satisfies (1) with $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ interchanged. Hence, $\tilde{\gamma}_1 \sim_f \tilde{\gamma}_0$, and the relation is symmetric.

We next prove the transitivity of the relation. Suppose that $\tilde{\gamma}_0 \sim_f \tilde{\gamma}_1$ and $\tilde{\gamma}_1 \sim_f \tilde{\gamma}_2$, for some $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2 \in \overrightarrow{\Gamma}_\Sigma$. Let f be the free homotopy between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$, and let g be the free homotopy between $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. Then it is easy to verify that the map $h : [0, 1] \times [0, 1] \longrightarrow \Sigma$ defined by

$$h(s, t) = \begin{cases} f(2s, t) \text{ for } 0 \leq s \leq \frac{1}{2}, \\ g(2s - 1, t) \text{ for } \frac{1}{2} \leq s \leq 1, \end{cases}$$

is continuous and satisfies

$$\begin{cases} h(0, t) = \tilde{\gamma}_0(t) \text{ for } 0 \leq t \leq 1, \\ h(1, t) = \tilde{\gamma}_2(t) \text{ for } 0 \leq t \leq 1, \\ h(s, 0) = h(s, 1) \text{ for } 0 \leq s \leq 1, \end{cases}$$

hence $\tilde{\gamma}_0 \sim_f \tilde{\gamma}_1$ and the relation is transitive. $\square$

Definition 1.6.2. Two closed paths $\tilde{\gamma}_0(t)$ and $\tilde{\gamma}_1(t)$ are *orientation-equivalent* if

- (1) $\tilde{\gamma}_0([0, 1]) = \tilde{\gamma}_1([0, 1])$, and
- (2) $\tilde{\gamma}_0 \sim_f \tilde{\gamma}_1$ in the topological space $\tilde{\gamma}_0([0, 1])$.

We use the notation $\tilde{\gamma}_0 \sim_o \tilde{\gamma}_1$ to indicate that the two closed paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are orientation-equivalent.

Remark. It is easy to see that $\sim_o$ is an equivalence relation. We call the equivalence classes under $\sim_o$ *orientation classes*.

Lemma 1.6.2. Let $\tilde{\gamma}_0, \tilde{\gamma}_1 \in \vec{\Gamma}_\Sigma$, and suppose that $\tilde{\gamma}_0 \sim_o \tilde{\gamma}_1$. Then $\tilde{\gamma}_0^{-1} \sim_o \tilde{\gamma}_1^{-1}$.

Proof. Let h be a free homotopy between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$. Then $g(s, t) = h(s, 1 - t)$ is a free homotopy between $\tilde{\gamma}_0^{-1}$ and $\tilde{\gamma}_1^{-1}$. $\square$

Proposition 1.6.3. Let $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \vec{\Gamma}_\Sigma$, and suppose that $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) = x$. Then

$$\tilde{\gamma}_1 \circ \tilde{\gamma}_2 \sim_o \tilde{\gamma}_2 \circ \tilde{\gamma}_1.$$

Proof. Condition (1) of Definition 1.6.2 is satisfied since $\tilde{\gamma}_1 \circ \tilde{\gamma}_2([0, 1]) = \tilde{\gamma}_2 \circ \tilde{\gamma}_1([0, 1])$.

Now the function $h : I \times I \rightarrow \Sigma$ defined by

$$h(s, t) = \begin{cases} (\tilde{\gamma}_1 \circ \tilde{\gamma}_2)(t + \frac{s}{2}) \text{ for } 0 \leq t \leq 1 - \frac{s}{2}, \\ (\tilde{\gamma}_1 \circ \tilde{\gamma}_2)(t + \frac{s}{2} - 1) \text{ for } 1 - \frac{s}{2} \leq t \leq 1, \end{cases}$$

is a free homotopy in $\tilde{\gamma}_1 \circ \tilde{\gamma}_2([0, 1])$ between $\tilde{\gamma}_1 \circ \tilde{\gamma}_2$ and $\tilde{\gamma}_2 \circ \tilde{\gamma}_1$. Therefore condition (2) of Definition 1.6.3 is satisfied. $\square$

Definition 1.6.3. Two closed paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are loop-equivalent if either

- (1) $\tilde{\gamma}_0 \sim_o \tilde{\gamma}_1$, or
- (2) $\tilde{\gamma}_0 \sim_o \tilde{\gamma}_1^{-1}$.

We use the notation $\tilde{\gamma}_0 \sim_l \tilde{\gamma}_1$ to indicate that the two closed paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are loop-equivalent.

Proposition 1.6.4. Let $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \vec{\Gamma}_\Sigma$, and suppose that $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) = x$. Then

$$\tilde{\gamma}_1 \circ \tilde{\gamma}_2 \sim_l \tilde{\gamma}_2 \circ \tilde{\gamma}_1.$$

Proof. This follows from Proposition 1.6.3 and Definition 1.6.3. $\square$

Remark. Proposition 1.6.4 implies that composition of closed paths is commutative with respect to loop equivalence. This will be helpful in the beginning of the next section.

Lemma 1.6.5. Let $\tilde{\gamma}_0, \tilde{\gamma}_1 \in \vec{\Gamma}_\Sigma$. Then the following assertions are equivalent:

- (1) $\tilde{\gamma}_0 \sim_l \tilde{\gamma}_1$;
- (2) $\tilde{\gamma}_0 \sim_l \tilde{\gamma}_1^{-1}$;
- (3) $\tilde{\gamma}_0^{-1} \sim_l \tilde{\gamma}_1$;
- (4) $\tilde{\gamma}_0^{-1} \sim_l \tilde{\gamma}_1^{-1}$.

Proof. This follows from Definition 1.6.3 and Lemma 1.6.2. $\square$

Proposition 1.6.6. $\sim_l$ is an equivalence relation over $\vec{\Gamma}_\Sigma$.

Proof. Reflexivity is clear. We prove first the symmetry of the relation. Suppose that $\tilde{\gamma}_0 \sim_l \tilde{\gamma}_1$, for some $\tilde{\gamma}_0, \tilde{\gamma}_1 \in \vec{\Gamma}_\Sigma$. Suppose first that $\tilde{\gamma}_0 \sim_o \tilde{\gamma}_1$. Then by the remark after Definition 1.6.2, we have that $\tilde{\gamma}_1 \sim_o \tilde{\gamma}_0$, and hence $\tilde{\gamma}_1 \sim_l \tilde{\gamma}_0$. If $\tilde{\gamma}_0 \sim_o \tilde{\gamma}_1^{-1}$, then $\tilde{\gamma}_1^{-1} \sim_o \tilde{\gamma}_0$, and so $\tilde{\gamma}_1 \sim_o \tilde{\gamma}_0^{-1}$ by Lemma 1.6.2. Hence $\tilde{\gamma}_1 \sim_l \tilde{\gamma}_0$, and the symmetry is proved.

We now prove the transitivity of $\sim_l$. Let $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2 \in \vec{\Gamma}_\Sigma$, and suppose that $\tilde{\gamma}_0 \sim_l \tilde{\gamma}_1$ and $\tilde{\gamma}_1 \sim_l \tilde{\gamma}_2$. Suppose first that $\tilde{\gamma}_0 \sim_o \tilde{\gamma}_1$ and that $\tilde{\gamma}_1 \sim_o \tilde{\gamma}_2$. By the remark following Definition 1.6.3, we have that $\tilde{\gamma}_0 \sim_o \tilde{\gamma}_2$, and so $\tilde{\gamma}_0 \sim_l \tilde{\gamma}_2$. The other 3 cases are easily dealt with using Lemma 1.6.2 and Lemma 1.6.5. $\square$

Definition 1.6.4. A loop in Σ is an equivalence class with respect to the loop-equivalence. Given a closed path $\tilde{\gamma}$, we represent its loop by $[\tilde{\gamma}]_l$. If $[\tilde{\gamma}]_l$ is a loop, a closed path of $[\tilde{\gamma}]_l$ is an element $\tilde{\gamma}'$ of the class $[\tilde{\gamma}]_l$. We represent the set of loops of Σ by $\overrightarrow{\Gamma}_\Sigma$.

This concept will allow us a lot of flexibility when dealing with paths in the next chapters.

Definition 1.6.5. A closed path $\tilde{\gamma}$ is contractible if it is freely homotopic to a constant path. Otherwise, $\tilde{\gamma}$ is noncontractible.

Definition 1.6.6. A loop $[\tilde{\gamma}]_l$ is contractible if it has a contractible closed path. Otherwise, the loop is said to be noncontractible.

If a closed path of a loop $[\tilde{\gamma}]_l$ is contractible, then, by transitivity of $\sim_l$, any other closed path of $[\tilde{\gamma}]_l$ is also contractible. Therefore, Definition 1.6.6 is equivalent to saying that a loop is contractible if any closed path of this loop is contractible.

Definition 1.6.7. Let $[\tilde{\gamma}]_l$ be a loop of Σ . Given a closed path $\tilde{\gamma}$ of $[\tilde{\gamma}]_l$, let $[\tilde{\gamma}]_o$ be the set of closed paths of $[\tilde{\gamma}]_l$ that are orientation-equivalent to $\tilde{\gamma}$. The set $[\tilde{\gamma}]_o$ is an orientation class of the loop $[\tilde{\gamma}]_l$.

Any non-constant loop has exactly two orientation classes: that is, given a closed path $\tilde{\gamma}$ of the loop, these orientation classes are respectively $\{\tilde{\gamma}' : \tilde{\gamma}' \in [\tilde{\gamma}]_l, \tilde{\gamma}' \sim_o \tilde{\gamma}\}$ and $\{\tilde{\gamma}' : \tilde{\gamma}' \in [\tilde{\gamma}]_l, \tilde{\gamma}' \sim_o \tilde{\gamma}^{-1}\}$.

Definition 1.6.8. If $[\tilde{\gamma}]_l$ is a loop, the circuit γ corresponding to the loop $[\tilde{\gamma}]_l$ is the image $\tilde{\gamma}'([0, 1])$ of any closed path $\tilde{\gamma}'$ in the loop $[\tilde{\gamma}]_l$ (clearly, if $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are any closed paths of the loop $[\tilde{\gamma}]_l$, then $\tilde{\gamma}_0([0, 1]) = \tilde{\gamma}_1([0, 1])$). The set of circuits in Σ is denoted by Γ_Σ .

A circuit can correspond to infinitely many loops. For example, if $\tilde{\gamma}$ is a closed path in Σ , then the circuit corresponding to $[\tilde{\gamma}]_l$ corresponds also to the loop $[\tilde{\gamma}^m]_l$, for $m \in \mathbb{Z} \setminus \{0\}$.

Definition 1.6.9. An *O-loop* is a loop $[\tilde{\gamma}]_l$ that has at least one closed path which is a simple closed path. We denote the set of *O-loops* in Σ by $\overrightarrow{\Omega}_\Sigma$.

Definition 1.6.10. An *O-circuit* is a circuit γ that corresponds to at least one *O-loop*. We denote the set of *O-circuits* of Σ by Ω_Σ .

Definition 1.6.11. Two loops $[\tilde{\gamma}_0]_l$ and $[\tilde{\gamma}_1]_l$ are *freely homotopic* if they have two freely homotopic closed paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$. We use the notation $[\tilde{\gamma}_0]_l \sim_f [\tilde{\gamma}_1]_l$.

Proposition 1.6.7. $\sim_f$ is an equivalence relation over $\overrightarrow{\Gamma}_\Sigma$.

Proof. The proof is similar to the proof of Proposition 1.6.6. $\square$

Proposition 1.6.8. There is a bijection between Ω_Σ and $\overrightarrow{\Omega}_\Sigma$.

Proof. Let $f : \overrightarrow{\Omega}_\Sigma \rightarrow \Omega_\Sigma$ be the function that assigns to each *O-loop* its corresponding *O-circuit*. Since for each loop of Σ there corresponds a unique circuit, the function f is well defined.

Clearly, f is surjective. We now prove that f is also injective. Let γ be an *O-circuit* of Σ , and let $[\tilde{\gamma}_0]_l, [\tilde{\gamma}_1]_l$ be two *O-loops* corresponding to γ with $[\tilde{\gamma}_0]_l \neq [\tilde{\gamma}_1]_l$. Without loss of generality, we can suppose that $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are simple closed paths of $[\tilde{\gamma}_0]_l$ and $[\tilde{\gamma}_1]_l$ and have the same basepoint x_0 . Clearly, γ is homeomorphic to the unit circle S^1 . Therefore, $\pi(S^1, (1, 0))$ is isomorphic to $\pi(\gamma, x_0)$. This isomorphism induces an isomorphism of $\overrightarrow{\Gamma}_\gamma$ and $\overrightarrow{\Gamma}_{S^1}$. There is only one element of $\overrightarrow{\Gamma}_{S^1}$ that contains simple closed paths, namely, $[e^{2\pi i t}]_l$ where $t \in [0, 1]$. Therefore, the corresponding loop in γ under the isomorphism must contain both $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$. It follows that $[\tilde{\gamma}_0]_l = [\tilde{\gamma}_1]_l$. $\square$

Definition 1.6.12. Two *O-circuits* γ_0 and γ_1 are *freely homotopic* if their corresponding *O-loops* $[\tilde{\gamma}_0]_l$ and $[\tilde{\gamma}_1]_l$ are freely homotopic. We use the notation $\gamma_0 \sim_f \gamma_1$ to indicate that the two *O-circuits* γ_0 and γ_1 are freely homotopic. More generally, two circuits γ_0 and γ_1 are *freely homotopic* if they have two respective closed paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ that are freely homotopic.

Proposition 1.6.9. $\sim_f$ is an equivalence relation over $\overrightarrow{\Omega}_\Sigma$ and Ω_Σ .

Proof. Clear from Proposition 1.6.6 and Proposition 1.6.8. $\square$

Definition 1.6.13. Let γ_0 and γ_1 be two O -circuits in Σ with corresponding O -loops $[\tilde{\gamma}_0]_l$ and $[\tilde{\gamma}_1]_l$. Then γ_0 is n -freely homotopic to γ_1 if $[\tilde{\gamma}_0]_l \sim_f [\tilde{\gamma}_1]_l$. We use the notation $\gamma_0 \sim_f^n \gamma_1$ to indicate that the O -circuit γ_0 is n -freely homotopic to the O -circuit γ_1 . The case where $n = 1, -1$ corresponds to Definition 1.6.12.

Notation. Let $\mathcal{A}$ be a set of loops in Σ . We denote the set of closed paths of elements of $\mathcal{A}$ by $\tilde{\Gamma}_{\mathcal{A}}$, the set of circuits corresponding to elements of $\mathcal{A}$ by $\Gamma_{\mathcal{A}}$, the set of O -loops of $\mathcal{A}$ by $\overrightarrow{\Omega}_{\mathcal{A}}$ and the set of O -circuits corresponding to elements of $\mathcal{A}$ by $\Omega_{\mathcal{A}}$.

Proposition 1.6.10. Let $\tilde{\gamma} \in \overrightarrow{\Gamma}_\Sigma$, and let x be any point of Σ . Then there exists $\tilde{\gamma}' \in \overrightarrow{\Gamma}_\Sigma$ such that $\tilde{\gamma}'$ has basepoint x and $\tilde{\gamma}' \sim_f \tilde{\gamma}$.

Proof. Since Σ is a path-connected topological space, there is a path $\tilde{\sigma} : [0, 1] \rightarrow \Sigma$ such that $\tilde{\sigma}(0) = x$, $\tilde{\sigma}(1) = x_0$, where x_0 is the base point of $\tilde{\gamma}$. Let $\tilde{\gamma}' = \tilde{\sigma} \circ \tilde{\gamma} \circ \tilde{\sigma}^{-1}$. By the remark after Proposition 1.6.4, a closed path of $[\tilde{\gamma}']_l$ is also given by $\tilde{\gamma} \circ \tilde{\sigma}^{-1} \circ \tilde{\sigma}$ (the sum is at the base point x_0), which is homotopic to $\tilde{\gamma}$, with respect to the base point x_0 . By transitivity of $\sim_f$ over $\overrightarrow{\Gamma}_\Sigma$, it follows that $\tilde{\gamma}' \sim_f \tilde{\gamma}$. $\square$

We now come to our main topic of this section. These ideas were motivated by the following observation: in Figure IV, $\tilde{\gamma}_0 \circ \tilde{\gamma}_1$ is contractible while in Figure V, $\tilde{\gamma}_0 \circ \tilde{\gamma}_1$ is noncontractible. Note that the circuit formed by each $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ is the same in both figures. On the other hand, the sum of two contractible paths is always contractible, even if we take the inverse of either paths.

Figure IV

Figure V

Definition 1.6.14. A set of loops $\mathcal{C}$ in Σ is a complete set of loops if:

- (1) $\mathcal{C}$ is closed under free homotopy; and
- (2) all $[\tilde{\gamma}_0]_I, [\tilde{\gamma}_1]_I \in \mathcal{C}$ satisfy the following property: for any loops $[\tilde{\gamma}_0']_I$ and $[\tilde{\gamma}_1']_I$ that are freely homotopic to $[\tilde{\gamma}_0]_I$ and $[\tilde{\gamma}_1]_I$ respectively, and for any closed paths $\tilde{\gamma}_0'', \tilde{\gamma}_1''$ of $[\tilde{\gamma}_0']_I$ and $[\tilde{\gamma}_1']_I$ respectively that share the same base point x_0 , $\tilde{\gamma}_0'' \circ \tilde{\gamma}_1'' \in \tilde{\Gamma}_{\mathcal{C}}$, where $\circ$ is the composition law of closed paths with base point x_0 .

Proposition 1.6.11. Let $\mathcal{C}$ be a set of loops of Σ , and $\mathcal{E} = \overleftarrow{\Gamma}_{\Sigma} \setminus \mathcal{C}$. Then $\mathcal{C}$ is complete if and only if the elements of $\mathcal{E}$ satisfy the following conditions:

- (3) $\mathcal{E}$ is closed under free homotopy;
- (4) let $[\tilde{\gamma}_0]_I, [\tilde{\gamma}_1]_I \in \overleftarrow{\Gamma}_{\Sigma}$ such that for some closed path $\tilde{\gamma}_0, \tilde{\gamma}_1$ of $[\tilde{\gamma}_0]_I, [\tilde{\gamma}_1]_I$ with common base point $x_0 = \tilde{\gamma}_0(0) = \tilde{\gamma}_1(0)$, we have $\tilde{\gamma}_0 \circ \tilde{\gamma}_1 \in \tilde{\Gamma}_{\mathcal{E}}$, where $\circ$ is the composition law of closed paths with base point x_0 . Then at least one of $[\tilde{\gamma}_0]_I$ and $[\tilde{\gamma}_1]_I$ is an element of $\mathcal{E}$.

Proof. ($\implies$) (3) is clear, since otherwise transitivity of free homotopy equivalence would imply that some element of $\mathcal{E}$ is in $\mathcal{C}$.

Now suppose that $[\tilde{\gamma}_0]_I, [\tilde{\gamma}_1]_I \in \overleftarrow{\Gamma}_{\Sigma}$, and some closed paths $\tilde{\gamma}_0, \tilde{\gamma}_1$ of γ_0, γ_1 respectively, gives $\tilde{\gamma}_0 \circ \tilde{\gamma}_1 \in \tilde{\Gamma}_{\mathcal{E}}$. Suppose that both $[\tilde{\gamma}_0]_I$ and $[\tilde{\gamma}_1]_I$ are in $\mathcal{C}$. Then, by (2), we would have that $\tilde{\gamma}_0 \circ \tilde{\gamma}_1 \in \tilde{\Gamma}_{\mathcal{C}}$, a contradiction. Therefore, we must have at least one of $[\tilde{\gamma}_0]_I$ and $[\tilde{\gamma}_1]_I$ in $\mathcal{E}$, and we are done.

($\impliedby$) The proof is similar. $\square$

Proposition 1.6.12. *Conditions (3) and (4) are equivalent to Condition (3) and the following condition:*

- (5) *Let $[\tilde{\gamma}]_I$ be an element of $\mathcal{E}$ and $\tilde{\sigma}$ a path in Σ such that $\tilde{\sigma}(0), \tilde{\sigma}(1) \in \gamma$. Let $\tilde{\gamma}' \in [\tilde{\gamma}]_I$ such that $\tilde{\gamma}'(0) = \tilde{\sigma}(1)$. Let t_1 be a number in $[0, 1[$ such that $\tilde{\gamma}'(t_1) = \tilde{\sigma}(0)$. Let $\tilde{\gamma}''$ be the path induced by $\tilde{\gamma}'|_{[0, t_1]}$ and let $\tilde{\gamma}'''$ be the path induced by $\tilde{\gamma}'|_{[t_1, 1]}$. Let $\tilde{\gamma}_0 = \tilde{\gamma}'' \circ \tilde{\sigma}$, and $\tilde{\gamma}_1 = \tilde{\sigma}^{-1} \circ \tilde{\gamma}'''$. Then at least one of $[\tilde{\gamma}_0]_I, [\tilde{\gamma}_1]_I$ is an element of $\mathcal{E}$.*

Proof. ((3) and (4) $\implies$ (3) and (5)) With the hypothesis of Conditions (3) and (5), suppose that neither $[\tilde{\gamma}_0]_I$ nor $[\tilde{\gamma}_1]_I$ is in $\mathcal{E}$, that is, $[\tilde{\gamma}_0]_I$ and $[\tilde{\gamma}_1]_I$ are in $\mathcal{C}$. Then $\tilde{\gamma}_0 \circ \tilde{\gamma}_1 \in \bar{\Gamma}_{\mathcal{C}}$, but since $\tilde{\gamma} \sim_f \tilde{\gamma}_0 \circ \tilde{\gamma}_1$, we obtain that $[\tilde{\gamma}]_I \in \mathcal{C}$, a contradiction with the hypothesis. Hence, there must be at least one of $[\tilde{\gamma}_0]_I$ and $[\tilde{\gamma}_1]_I$ that is in $\mathcal{E}$.

((3) and (5) $\implies$ (3) and (4)) With the hypothesis of Conditions (3) and (4), suppose that neither $[\tilde{\gamma}_0]_I$ nor $[\tilde{\gamma}_1]_I$ is in $\mathcal{E}$. We apply condition (5) on $[\tilde{\gamma}]_I = [\tilde{\gamma}_0 \circ \tilde{\gamma}_1]_I \in \mathcal{E}$. We thus have a contradiction, and so at least one of $[\tilde{\gamma}_0]_I$ and $[\tilde{\gamma}_1]_I$ must be in $\mathcal{E}$. $\square$

Definition 1.6.15. *Let $\tilde{\gamma}$ be a path in Σ . The segment corresponding to $\tilde{\gamma}$ is the image $\tilde{\gamma}([0, 1])$.*

Property (5) is what we will call the *3 paths property* or *TPP*: let γ be an element of $\Gamma_{\mathcal{E}}$, and σ be a segment with both endpoints in γ (we allow σ to be a single point). There are two curves γ'' and γ''' in γ that have the same endpoints as σ and such that $\gamma = \gamma'' \cup \gamma'''$. Then at least one of $\gamma'' \cup \sigma$ and $\gamma''' \cup \sigma$ is an element of $\Gamma_{\mathcal{E}}$. In part, this is motivated by Thomassen's work on three path property (see [Th2]).

The next Definition introduces one of the most important concept of this work. We will see, in the next section, that it corresponds to normality of a subgroup of the fundamental group of the given surface.

Definition 1.6.16. A complete partition of loops of Σ (or a complete partition of $\overrightarrow{\Gamma}_\Sigma$) is a partition $(\mathcal{C}, \mathcal{E})$ of the set $\overrightarrow{\Gamma}_\Sigma$ such that $\mathcal{C}$ is a complete set of loops in Σ and $\mathcal{E} = \overrightarrow{\Gamma}_\Sigma \setminus \mathcal{C}$.

Remark. The reader can verify that $(\overrightarrow{\Gamma}_\Sigma, \emptyset)$ and $(\emptyset, \overrightarrow{\Gamma}_\Sigma)$ are complete partitions. These are called *trivial complete partitions*. All through the rest of this work, we consider only complete partitions that are not trivial.

Later, we will need to “measure” O-loops, O-circuits and complete sets of loops. We therefore introduce the following notion.

Definition 1.6.17. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a nontrivial surface Σ .

- (1) Let $\mathcal{A}$ be a set of loops of Σ that is closed under free homotopy. We define the $\mathcal{A}$ -representativity (or representativity with respect to $\mathcal{A}$), denoted by $\rho_{\mathcal{A}}(\Psi)$, by

$$\rho_{\mathcal{A}}(\Psi) = \min \{ |\gamma \cap G_\Psi(\Sigma)| : \gamma \in \Omega_{\mathcal{A}} \}.$$

If $\Omega_{\mathcal{A}} = \emptyset$, then we set $\rho_{\mathcal{A}}(\Psi) = \infty$.

- (2) If Υ is a set of O-circuits in Γ_Σ that is closed under free homotopy, then we define the Υ -representativity (or representativity with respect to Υ), denoted by $\rho_\Upsilon(\Psi)$, by

$$\rho_\Upsilon(\Psi) = \min \{ |\gamma \cap G_\Psi(\Sigma)| : \gamma \in \Upsilon \}.$$

- (3) If γ is an O-circuit of Σ , and in (2), we let $\Upsilon = \{ \gamma' : \gamma' \in \overrightarrow{\Omega}_\Sigma, \text{ and } \gamma' \sim_f \gamma \}$, the length of γ , denoted by $\ell(\gamma)$, is defined by

$$\ell(\gamma) = \rho_\Upsilon(\Psi).$$

Figure VI shows an embedding of K_5 in the torus (we identify the top and bottom arrow, and the right and left arrow). The O-circuit that corresponds to the polygon $v_1, v_1v_3, v_3, v_3v_5, v_5, v_5v_1$ has length 1: the O-circuit γ is a freely homotopic copy of this polygon through the face f_1 that intersects the graph only at v_4 .

Definition 1.6.18. Let $\gamma \in \Gamma_\Sigma$.

- (1) Let $p \in \gamma$. The multiplicity of p is given by $\text{mult}(p) = \inf \{ |\{t \in [0, 1[: \tilde{\gamma}_0(t) = p\}| : \tilde{\gamma}_0 \text{ is a closed path of } \gamma \}$.
- (2) Let $S_\gamma = \{p \in \gamma \mid \text{mult}(p) \geq 2\}$. The total multiplicity of γ is given by $\text{mult}(\gamma) = \sum_{p \in S_\gamma} \text{mult}(p)$.

Proposition 1.6.13. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a nontrivial surface Σ . Let $\mathcal{A}$ be a set of loops of $\overleftarrow{\Gamma}_\Sigma$ that is closed under free homotopy, and let $[\tilde{\gamma}]_l \in \mathcal{A}$. Then there is a loop $[\tilde{\gamma}']_l \in \mathcal{A}$ such that $[\tilde{\gamma}']_l \sim_f [\tilde{\gamma}]_l$ and $|\gamma' \cap G_\Psi(\Sigma)|$ is finite. Furthermore, the total multiplicity of γ' is finite.

Proof. This comes from the fact that $\pi(\Sigma, x_0)$ is finitely generated by a set of simple closed paths that meet $G_\Psi(\Sigma)$ finitely often, and therefore γ is freely homotopic to a finite product of such elements. $\square$

Theorem 1.6.14. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a nontrivial surface Σ . Let $(\mathcal{C}, \mathcal{E})$ be a complete partition of $\overleftarrow{\Gamma}_\Sigma$. Then there is an O-loop $[\tilde{\gamma}]_l \in \mathcal{E}$ such that $|\gamma \cap G_\Psi| = \rho_{\mathcal{E}}(\Psi)$ and $\gamma \cap G_\Psi(\Sigma) \subseteq V_\Psi$.

Proof. By an argument similar to the one in the next paragraph, one can show that $\overleftarrow{\Omega}_{\mathcal{E}} \neq \emptyset$. Let $\gamma' \in \overleftarrow{\Omega}_{\mathcal{E}}$ be such that $\rho_{\mathcal{E}}(\Psi) = |\gamma' \cap G_\Psi(\Sigma)|$. Suppose that there are some points in $\gamma' \cap G_\Psi(\Sigma)$ that are not in V_Ψ , that is, they belong to some edges of G_Ψ . Then we can move homotopically these parts of γ' along those edges so that the resulting circuit will intersect one of the endpoints of the corresponding edges. We then obtain a freely homotopic copy γ'' of γ' such that $\gamma'' \cap G_\Psi(\Sigma) \subseteq V_\Psi$. Notice that since $|\gamma' \cap G_\Psi(\Sigma)|$ is finite and γ' is an O-circuit, the total multiplicity of γ'' is also finite.

Suppose now that γ'' is not an O-circuit, that is, there is a point p that is a point of multiplicity of γ'' . Then p splits γ'' into two circuits γ_1 and γ_2 (we can actually choose any two defined by p). Since $[\gamma'']_l \in \mathcal{E}$ by homotopy with γ' , one of $[\gamma_1]_l$ and $[\gamma_2]_l$ belongs to $\mathcal{E}$ by the TPP, and its circuit contains one less multiplicity point than γ'' . If there is another point of multiplicity, we continue the process until we obtain an O-loop $[\gamma]_l \in \mathcal{E}$. It is clear that $\gamma \cap G_\Psi(\Sigma) \subseteq V_\Psi$, and that $|\gamma \cap G_\Psi(\Sigma)| \leq |\gamma' \cap G_\Psi(\Sigma)|$. $\square$

It follows from the last Proposition that we can always give a measure of a complete partition using a graph in a natural way. The following concepts will also play an important role in these matters.

Definition 1.6.19. Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a surface Σ . A face chain Λ in G_Ψ is a sequence $v_0, f_1, v_1, f_2, v_2, \dots, f_\ell, v_\ell$ such that

- (1) $v_i \in V_\Psi, f_j \in F_\Psi$, for $i = 0, \dots, \ell$ and $j = 1, \dots, \ell$;
- (2) $v_i \in \bar{f}_i \cap \bar{f}_{i+1}$, for $i = 1, \dots, \ell - 1$;
- (3) $v_0 \in \bar{f}_0, v_\ell \in \bar{f}_\ell$.

Definition 1.6.20. With the notations of Definition 1.6.19, the inner vertices of Λ are the vertices v_i for $i = 1, \dots, \ell - 1$ and the inner faces of Λ are the faces f_i for $i = 2, \dots, \ell - 1$.

Definition 1.6.21. A face chain $\Lambda = v_0, f_1, v_1, \dots, f_\ell, v_\ell$ is a face path if $f_1, \dots, f_\ell$ are all distinct and $v_0, v_1, \dots, v_\ell$ are all distinct.

Definition 1.6.22. A face chain $\Lambda = v_0, f_1, v_1, f_2, v_2, \dots, f_\ell, v_\ell$ is a face cycle if $f_1, \dots, f_\ell$ are all distinct and $v_0, v_1, \dots, v_\ell$ are all distinct except for $v_0 = v_\ell$.

Figure VI

Definition 1.6.23. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma$ be an embedding of a graph G onto a surface Σ . Let γ be an O-circuit in Σ . Let $\Lambda = v_0, f_1, v_1, f_2, v_2, \dots, f_\ell, v_\ell$ be a closed face chain in G_Ψ . The O-circuit γ is *face-represented* by Λ if there is an O-circuit γ' such that

- (1) $\gamma \sim_f \gamma'$;
- (2) $\gamma' \subseteq \bigcup_{i=1}^{\ell} \bar{f}_i$;
- (3) for $i = 1, \dots, \ell$, $v_i \in \gamma'$;
- (4) for $i = 1, \dots, \ell$, $f_i \cap \gamma' \neq \emptyset$.

For example, the O-circuit γ in Figure VI is face-represented by the face cycle v_4, f_1, v_4 .

1.7 Free homotopy modulo $\mathcal{C}$.

Let Σ be a nontrivial surface. In this section, we first show that there is a bijection between the set of complete sets of loops of Σ and the set of normal subgroups of $\pi(\Sigma, x)$. Then we introduce some complete partitions of loops of Σ : the *fundamental partition* $(\mathcal{C}_0, \mathcal{E}_0)$ of $\overleftarrow{\Gamma}_\Sigma$, the *separating partition* $(\mathcal{C}_s, \mathcal{E}_s)$ of $\overleftarrow{\Gamma}_\Sigma$ and the *orientation-preserving partition* $(\mathcal{C}_p, \mathcal{E}_p)$. These complete partitions will be of interest in the next chapters. We will also introduce the concept of *free homotopy modulo $\mathcal{C}$* (for a given complete set of loops $\mathcal{C}$), which is an extension of free homotopy. We end up this section with the other very important notion of crossing partition, which has exactly the required properties for our main results.

Let $\mathcal{C}$ be a complete set of loops, and let x be any point of Σ . Let $\overrightarrow{\Gamma}_\mathcal{C}^x$ be the set of $\tilde{\gamma} \in \overrightarrow{\Gamma}_\mathcal{C}$ such that $\tilde{\gamma}(0) = x$. Let $\sim$ denote the closed path homotopy equivalence with basepoint x . For a closed path $\tilde{\gamma}$ with $\tilde{\gamma}(0) = x$, we denote by $[\tilde{\gamma}]$ the homotopy class of $\tilde{\gamma}$ with respect to x . Let $\pi_\mathcal{C}^x \subseteq \pi(\Sigma, x)$ be the set of $[\tilde{\gamma}]$ such that $\tilde{\gamma} \in \overrightarrow{\Gamma}_\mathcal{C}^x$.

Proposition 1.7.1. *Let C be a complete set of loops and x a point of Σ . Then π_C^Σ is a normal subgroup of $\pi(\Sigma, x)$.*

Proof. Since C is nonempty and closed under free homotopy, π_C^Σ is also nonempty.

Let $[\tilde{\gamma}], [\tilde{\gamma}'] \in \pi_C^\Sigma$. We first prove that π_C^Σ is a subgroup of $\pi(\Sigma, x)$. This will be achieved by showing that $[\tilde{\gamma}]^{-1} \circ [\tilde{\gamma}'] \in \pi_C^\Sigma$. The reader will notice that $[\tilde{\gamma}]^{-1} = [\tilde{\gamma}^{-1}]$. So, since $[\tilde{\gamma}^{-1}] \circ [\tilde{\gamma}'] = [\tilde{\gamma}^{-1} \circ \tilde{\gamma}']$, we will be done if we prove that $\tilde{\gamma}^{-1} \circ \tilde{\gamma}' \in \overrightarrow{\Gamma}_C^\Sigma$. By Lemma 1.6.2, $\tilde{\gamma}^{-1} \sim_I \tilde{\gamma}$, and so $\tilde{\gamma}^{-1} \in \overrightarrow{\Gamma}_C^\Sigma$. Therefore, by Definition 1.6.14, we must have $\tilde{\gamma}^{-1} \circ \tilde{\gamma}' \in \overrightarrow{\Gamma}_C^\Sigma$, and so $[\tilde{\gamma}]^{-1} \circ [\tilde{\gamma}'] \in \pi_C^\Sigma$.

We now prove that π_C^Σ is normal in $\pi(\Sigma, x)$. Let $[\tilde{\gamma}] \in \pi(\Sigma, x)$, and $[\tilde{\gamma}'] \in \pi_C^\Sigma$. We have to prove that $[\tilde{\gamma}] \circ [\tilde{\gamma}'] \circ [\tilde{\gamma}]^{-1} \in \pi_C^\Sigma$. As in the last paragraph, it suffices to prove that $\tilde{\gamma} \circ \tilde{\gamma}' \circ \tilde{\gamma}^{-1} \in \overrightarrow{\Gamma}_C^\Sigma$. By the remark after Proposition 1.6.3, $(\tilde{\gamma} \circ \tilde{\gamma}') \circ \tilde{\gamma}^{-1} \sim_0 \tilde{\gamma}^{-1} \circ (\tilde{\gamma} \circ \tilde{\gamma}')$. But $\tilde{\gamma}^{-1} \circ (\tilde{\gamma} \circ \tilde{\gamma}') \sim (\tilde{\gamma}^{-1} \circ \tilde{\gamma}) \circ \tilde{\gamma}' \sim \tilde{\gamma}'$, and so $\tilde{\gamma} \circ \tilde{\gamma}' \circ \tilde{\gamma}^{-1} \in \overrightarrow{\Gamma}_C^\Sigma$. Hence π_C^Σ is a normal subgroup of $\pi(\Sigma, x)$. $\square$

We will prove that the converse is also true. Let H be a normal subgroup of $\pi(\Sigma, x)$. Let $\overrightarrow{\Gamma}_H^\Sigma$ denote the set of all closed paths $\tilde{\gamma}$ such that $[\tilde{\gamma}] \in H$. Let $\overleftarrow{\Gamma}_H$ denote the set of all closed paths $\tilde{\gamma}$ such that there is a closed path $\tilde{\gamma}' \in \overrightarrow{\Gamma}_H^\Sigma$ and $\tilde{\gamma} \sim_f \tilde{\gamma}'$.

If $\tilde{\gamma} \in \overrightarrow{\Gamma}_H^\Sigma$ and $\tilde{\gamma}'$ is a closed path such that $\tilde{\gamma}' \sim_I \tilde{\gamma}$, then it is a routine exercise to show that $\tilde{\gamma}' \in \overrightarrow{\Gamma}_H^\Sigma$. Therefore, $\overrightarrow{\Gamma}_H^\Sigma$ partitions into loops. Let $\overleftarrow{\Gamma}_H = \{[\tilde{\gamma}]_I \mid \tilde{\gamma} \in \overrightarrow{\Gamma}_H^\Sigma\}$.

Proposition 1.7.2. *If H is a normal subgroup of $\pi(\Sigma, x)$, then $\overleftarrow{\Gamma}_H$ is a complete set of loops.*

Proof. Let $[\tilde{\gamma}]_I \in \overleftarrow{\Gamma}_H$, and $[\tilde{\gamma}']_I \sim_f [\tilde{\gamma}]_I$. Then there is some $\tilde{\gamma}_0 \in H$ such that $\tilde{\gamma} \sim_f \tilde{\gamma}_0$. By hypothesis, we either have $\tilde{\gamma}' \sim_f \tilde{\gamma}$ or $\tilde{\gamma}' \sim_f \tilde{\gamma}^{-1}$. Without loss of

generality, suppose that $\tilde{\gamma}' \sim_f \tilde{\gamma}$. Then, by transitivity of $\sim_f$, $\tilde{\gamma}' \sim_f \tilde{\gamma}_0$, and so $\tilde{\gamma}' \in \overrightarrow{\Gamma}_H$. Therefore, $[\tilde{\gamma}']_l \in \overrightarrow{\Gamma}_H$, and so $\overrightarrow{\Gamma}_H$ is closed under free homotopy.

Now let $[\tilde{\gamma}_1]_l, [\tilde{\gamma}_2]_l \in \overrightarrow{\Gamma}_H$, and let $\tilde{\gamma}_1'$ and $\tilde{\gamma}_2'$ be any two closed paths having a common basepoint, say y , such that $[\tilde{\gamma}_1']_l \sim_f [\tilde{\gamma}_1]_l$ and $[\tilde{\gamma}_2']_l \sim_f [\tilde{\gamma}_2]_l$. We have to prove that $\tilde{\gamma}_1' \circ \tilde{\gamma}_2' \in \overrightarrow{\Gamma}_H$, where $\circ$ is the composition of paths with respect to the basepoint y .

Let $\tilde{\gamma}_i'' \in \overrightarrow{\Gamma}_H^x$ be such that $\tilde{\gamma}_i' \sim_f \tilde{\gamma}_i''$, for $i = 1, 2$. Let $h_i : I \times I \rightarrow \Sigma$ be a free homotopy such that $h_i(t \times \{0\}) = \tilde{\gamma}_i'(t)$ and $h_i(t \times \{1\}) = \tilde{\gamma}_i''(t)$, for $t \in I$. Define $\tilde{\alpha}_i : I \rightarrow \Sigma$ by $\tilde{\alpha}_i(t) = h_i((0, t))$. It follows that $\tilde{\alpha}_i(0) = \tilde{\gamma}_i'(0) = y$ and $\tilde{\alpha}_i(1) = x$.

The closed curve $\tilde{\delta}_i = \tilde{\alpha}_i^{-1} \circ \tilde{\gamma}_i' \circ \tilde{\alpha}_i$ is freely homotopic to $\tilde{\gamma}_i'$ and has basepoint x . Furthermore, $\tilde{\delta}_i \in [\tilde{\gamma}_i'']$.

Since H is normal, we have that $(\tilde{\alpha}_1^{-1} \circ \tilde{\alpha}_2)^{-1} \circ \tilde{\delta}_1 \circ (\tilde{\alpha}_1^{-1} \circ \tilde{\alpha}_2)$ is in $\overrightarrow{\Gamma}_H^x$. This implies that $\tilde{\alpha}_2^{-1} \circ \tilde{\gamma}_1' \circ \tilde{\alpha}_2$ is in $\overrightarrow{\Gamma}_H^x$. Therefore, $[\tilde{\alpha}_2^{-1} \circ \tilde{\gamma}_1' \circ \tilde{\alpha}_2] \circ [\tilde{\alpha}_2^{-1} \circ \tilde{\gamma}_2' \circ \tilde{\alpha}_2] \in H$, so that $\tilde{\alpha}_2^{-1} \circ \tilde{\gamma}_1' \circ \tilde{\gamma}_2' \circ \tilde{\alpha}_2 \in \overrightarrow{\Gamma}_H^x$. But this last path is freely homotopic to $\tilde{\gamma}_1' \circ \tilde{\gamma}_2'$, so $\tilde{\gamma}_1' \circ \tilde{\gamma}_2' \in \overrightarrow{\Gamma}_H$. $\square$

We now show that the operations of Propositions 1.7.1 and 1.7.2 are actually inverses.

Theorem 1.7.3. *There is a bijection between the set of complete sets of loops of Σ and the set of normal subgroups of $\pi(\Sigma, x)$ given by the relations*

$$H = \pi_{\overrightarrow{\Gamma}_H}^x \text{ and } \overrightarrow{\Gamma}_{\pi_x^H} = C$$

Proof. Let H be a normal subgroup of $\pi(\Sigma, x)$.

$$\begin{aligned}
[\tilde{\gamma}] \in \pi_{\vec{\Gamma}_H}^x &\iff \tilde{\gamma} \in \vec{\Gamma}_{\vec{\Gamma}_H}^x \\
&\iff \tilde{\gamma}(0) = x \text{ and } [\tilde{\gamma}]_t \in \vec{\Gamma}_H \\
&\iff \tilde{\gamma} \in \vec{\Gamma}_H \text{ and } \tilde{\gamma}(0) = x \\
&\iff \tilde{\gamma} \sim_f \tilde{\gamma}' \text{ where } [\tilde{\gamma}'] \in H \\
&\iff [\tilde{\gamma}] \in H.
\end{aligned}$$

The proof of the second relation is similar. $\square$

Let $\tilde{x}$ be the constant path at x , so that $[\tilde{x}]$ is the identity of $\pi(\Sigma, x)$. It is easy to see that the subgroup $\langle [\tilde{x}] \rangle$ of $\pi(\Sigma, x)$ generated by $[\tilde{x}]$ is normal. This brings us to the following definition.

Definition 1.7.1. Let $C_0 = \vec{\Gamma}_{\langle [\tilde{x}] \rangle}$. The complete partition $(C_0, \mathcal{E}_0)$ of $\vec{\Gamma}_\Sigma$, where C_0 is the set of all contractible loops of Σ , is the fundamental partition of Σ .

Remarks. Notice that the set $\mathcal{E}_0$ is the set of all noncontractible loops of Σ . Also, we will see in Proposition 1.7.7 why we call this partition fundamental.

For an orientable surface, we now consider the subgroup H_s of the commutators of $\pi(\Sigma, x)$. It is known that H_s is the kernel of the natural map from $\pi(\Sigma, x)$ to the first homology group (see [GH] p 63). It follows that H_s is normal. Notice that the O-circuits γ of Σ such that $\Sigma \setminus \gamma$ has two t -components are trivial in the first homology group.

Definition 1.7.2. Let Σ be an orientable surface, and let $C_s = \overrightarrow{\Gamma}_{H_s}$, that is, C_s is the smallest complete set of loops that contains all the O-loops $[\tilde{\gamma}]_l$ such that either $[\tilde{\gamma}]_l$ is contractible or $\Sigma \setminus \tilde{\gamma}'([0, 1])$ is composed of more than one t-component, for all $\tilde{\gamma}' \sim_l \tilde{\gamma}$. The complete partition $(C_s, \mathcal{E}_s)$ of $\overrightarrow{\Gamma}_\Sigma$ is the separating partition of Σ .

Now suppose that Σ is a nonorientable surface. Then the subgroup H_p of $\pi(\Sigma, x)$ that consists of orientation-preserving classes is a subgroup of index 2 in $\pi(\Sigma, x)$. Therefore, H_p is a normal subgroup of $\pi(\Sigma, x)$.

Definition 1.7.3. Let $C_p = \overrightarrow{\Gamma}_{H_p}$, that is, C_p is the set of all loops $[\tilde{\gamma}]_l$ such that $\tilde{\gamma}$ is orientation-preserving. The complete partition $(C_p, \mathcal{E}_p)$ is the orientation-preserving partition of Σ .

We will have to deal with situations where a self-intersecting circuit contains circuits that are contractibles. We will see that these contractible circuits can be "deleted" from the main circuit without changing the homotopy. But before, we need to develop a way of amalgamating loops and circuits respectively.

We now define a sum over the set of loops.

Definition 1.7.4. Let $[\tilde{\gamma}_0]_l, [\tilde{\gamma}_1]_l$ be two loops in Σ , and let $\tilde{\gamma}'_0$ and $\tilde{\gamma}'_1$ be two closed paths of $[\tilde{\gamma}_0]_l$ and $[\tilde{\gamma}_1]_l$ respectively. Suppose that $\tilde{\gamma}'_0$ and $\tilde{\gamma}'_1$ share the same basepoint x_0 . The sum of $[\tilde{\gamma}_0]_l$ and $[\tilde{\gamma}_1]_l$ with respect to $\tilde{\gamma}'_0$ and $\tilde{\gamma}'_1$, denoted $[\tilde{\gamma}_0]_l \circ [\tilde{\gamma}_1]_l$, is the loop of $\tilde{\gamma}'_0 \circ \tilde{\gamma}'_1$, where $\circ$ is the composition of closed paths with respect to the basepoint x_0 .

Remark. The reader might think that, in the last definition, it would have been sufficient to define the sum of two loops with respect to their orientation classes rather than two respective paths. However, it is not true that if $\tilde{\gamma}_1 \sim_o \tilde{\gamma}_2$ and $\tilde{\gamma} \in \overrightarrow{\Gamma}_\Sigma$ with $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) = \tilde{\gamma}(0) = x$, then $\tilde{\gamma} \circ \tilde{\gamma}_1 \sim_o \tilde{\gamma} \circ \tilde{\gamma}_2$. This can be readily seen if x is a point of multiplicity greater than one for $\tilde{\gamma}_1$.

Proposition 1.7.4. Let $(C, \mathcal{E})$ be a complete partition of $\overrightarrow{\Gamma}_\Sigma$. Let $[\tilde{\gamma}_0]_l, [\tilde{\gamma}_1]_l \in C$, and let $\tilde{\gamma}_0'$ and $\tilde{\gamma}_1'$ be two respective closed paths, and suppose that they have a common basepoint x_0 . Then $[\tilde{\gamma}_0]_l \circ [\tilde{\gamma}_1]_l$ with respect to $\tilde{\gamma}_0'$ and $\tilde{\gamma}_1'$ is an element of C .

Proof. This follows from Definition 1.6.14. $\square$

The following proposition shows that, given a complete partition $(C, \mathcal{E})$, the loops of C play the role of identity with respect to the sum of loops. This will be an important tool in the proof of many results in chapters 2 and 3.

Proposition 1.7.5. Let $(C, \mathcal{E})$ be a complete partition of $\overrightarrow{\Gamma}_\Sigma$. Let $[\tilde{\gamma}_0]_l \in \mathcal{E}$ and $[\tilde{\gamma}_1]_l \in C$, and let $\tilde{\gamma}_0', \tilde{\gamma}_1'$ be closed paths of $[\tilde{\gamma}_0]_l$ and $[\tilde{\gamma}_1]_l$ respectively. Suppose that $\tilde{\gamma}_0'$ and $\tilde{\gamma}_1'$ have a common basepoint x_0 . Then $[\tilde{\gamma}_0]_l \circ [\tilde{\gamma}_1]_l$, with respect to $\tilde{\gamma}_0'$ and $\tilde{\gamma}_1'$, is an element of $\mathcal{E}$.

Proof. Suppose that $[\tilde{\gamma}_0]_l \circ [\tilde{\gamma}_1]_l$, with respect to $\tilde{\gamma}_0'$ and $\tilde{\gamma}_1'$, is an element of C . Then $[\tilde{\gamma}_0' \circ \tilde{\gamma}_1']_l \in C$. Therefore, $[\tilde{\gamma}_0]_l = [\tilde{\gamma}_0' \circ \tilde{\gamma}_1']_l \circ [\tilde{\gamma}_1]_l$, with respect to $\tilde{\gamma}_0' \circ \tilde{\gamma}_1'$ and $(\tilde{\gamma}_1')^{-1}$, is an element of C by Definition 1.6.14. But this is a contradiction. $\square$

Definition 1.7.5. Suppose that $[\tilde{\gamma}]_l$ is a loop of Σ , and that $\tilde{\gamma} = \tilde{\gamma}_0 \circ \tilde{\gamma}_1$ for some paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$. Then the loops $[\tilde{\gamma}_0]_l$ and $[\tilde{\gamma}_1]_l$ are subloops of $[\tilde{\gamma}]_l$.

Definition 1.7.6. Let $(C, \mathcal{E})$ be a complete partition of $\overrightarrow{\Gamma}_\Sigma$. Let $[\tilde{\sigma}]_l, [\tilde{\sigma}']_l \in \overrightarrow{\Gamma}_\Sigma$, and $n \in \mathbb{Z}$. Then $[\tilde{\sigma}]_l$ is n -freely homotopic modulo C to $[\tilde{\sigma}']_l$ if there are loops $[\tilde{\gamma}_1]_l, [\tilde{\gamma}_2]_l, \dots, [\tilde{\gamma}_k]_l \in C$, closed paths $\tilde{\gamma}_1', \tilde{\gamma}_2', \dots, \tilde{\gamma}_k', \tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_{k+1}$ and points $x_1, \dots, x_k$ such that, for $i = 1, \dots, k$;

- (1) $\tilde{\gamma}_i'$ is a closed path of $[\tilde{\gamma}_i]_l$;
- (2) x_i is a common basepoint of $\tilde{\gamma}_i'$ and $\tilde{\sigma}_i$;
- (3) x_k is the basepoint of $\tilde{\sigma}_{k+1}$;
- (4) $\tilde{\sigma}_1$ is a closed path of $[\tilde{\sigma}]_l$;
- (5) $\tilde{\sigma}_{i+1}$ is a closed path of $[\tilde{\sigma}_i]_l \circ [\tilde{\gamma}_i]_l$ with respect to $\tilde{\sigma}_i$ and $\tilde{\gamma}_i'$ at x_i ; and
- (6) $[\tilde{\sigma}_{k+1}]_l \sim^n [\tilde{\sigma}']_l$.

We use the notation $[\vec{\sigma}]_I \sim_C^n [\vec{\sigma}']_I$ to indicate that $[\vec{\sigma}]_I$ and $[\vec{\sigma}']_I$ are n -freely homotopic modulo C .

In the case where $n = 1$, we say that $[\vec{\sigma}]_I$ is freely homotopic to $[\vec{\sigma}']_I$ modulo C . We use the notation $[\vec{\sigma}]_I \sim_C [\vec{\sigma}']_I$ to indicate that $[\vec{\sigma}]_I$ and $[\vec{\sigma}']_I$ are freely homotopic modulo C .

Figure VII shows the example of an orientation-preserving closed path $\vec{\sigma}$ that is 2-freely homotopic to an orientation-reversing closed path $\vec{\sigma}'$. Therefore, $[\vec{\sigma}]_I \sim_f^2 [\vec{\sigma}']_I$. In Figure VIII, the reader can see that the loop $[\vec{\sigma}]_I$ is freely homotopic modulo C , to the 0-loop $[\vec{\sigma}']_I$.

Remark. It follows from the last definition that, if $[\vec{\sigma}]_I$ and $[\vec{\sigma}']_I$ are two loops and $[\vec{\sigma}]_I \sim_f [\vec{\sigma}']_I$, then $[\vec{\sigma}]_I \sim_C [\vec{\sigma}']_I$ for any complete set of loops C . Notice also that, in the last definition, we consider k loops in C instead of 1 because we will have to deal with situations where there will be more than one loop of C "attached" at different points to the given loop.

Definition 1.7.7. Let $(C, \mathcal{E})$ be a complete partition of $\overleftarrow{\Gamma}_\Sigma$. Let σ_0 and σ_1 be two 0-circuits in Σ with corresponding 0-loops $[\vec{\sigma}_0]_I$ and $[\vec{\sigma}_1]_I$, and let $n \in \mathbb{Z}$. Then σ_0 and σ_1 are n -freely homotopic modulo C if $[\vec{\sigma}_0]_I \sim_C^n [\vec{\sigma}_1]_I$. We use the notation $\sigma_0 \sim_C^n \sigma_1$ to indicate that σ_0 and σ_1 are n -freely homotopic modulo C .

Proposition 1.7.6. Let $(C_0, \mathcal{E}_0)$ be the fundamental partition of $\overleftarrow{\Gamma}_\Sigma$. Let $[\vec{\sigma}_0]_I, [\vec{\sigma}_1]_I$ be some loops of $\overleftarrow{\Gamma}_\Sigma$. Then $[\vec{\sigma}_0]_I \sim_f [\vec{\sigma}_1]_I$ iff $[\vec{\sigma}_0]_I \sim_{C_0} [\vec{\sigma}_1]_I$.

Proof. ($\Rightarrow$) It is a trivial case of Definition 1.7.6 where the set of loops to add is empty.

($\Leftarrow$) Suppose that the loops $[\vec{\sigma}_0]_I$ and $[\vec{\sigma}_1]_I$ are freely homotopic modulo C_0 . Then there are loops $[\vec{\gamma}_1]_I, [\vec{\gamma}_2]_I, \dots, [\vec{\gamma}_k]_I \in C_0$, closed paths $\vec{\gamma}'_1, \vec{\gamma}'_2, \dots, \vec{\gamma}'_k, \vec{\sigma}_1, \vec{\sigma}_2, \dots, \vec{\sigma}_{k+1}$ and points $x_1, \dots, x_k$ such that, for $i = 1, \dots, k$:

Figure VII

Figure VIII

- (1) $\tilde{\gamma}'_i$ is a closed path of $[\tilde{\gamma}_i]_I$;
- (2) x_i is a common basepoint of $\tilde{\gamma}'_i$ and $\tilde{\sigma}_i$;
- (3) x_k is the basepoint of $\tilde{\sigma}_{k+1}$;
- (4) $\tilde{\sigma}_1$ is a closed path of $[\tilde{\sigma}_0]_I$;
- (5) $\tilde{\sigma}_{i+1}$ is a closed path of $[\tilde{\sigma}_i]_I \circ [\tilde{\gamma}_i]_I$ with respect to $\tilde{\sigma}_i$ and $\tilde{\gamma}'_i$ at x_i ; and
- (6) $[\tilde{\sigma}_{k+1}]_I \sim_f [\tilde{\sigma}_1]_I$.

There exists a free homotopy over $[\tilde{\sigma}]_I \circ [\tilde{\gamma}_1]_I \circ \cdots \circ [\tilde{\gamma}_k]_I$ that brings all the $[\tilde{\gamma}_i]_I$ for $i = 1, \dots, k$ to constant circuits respectively. The result follows. $\square$

Proposition 1.7.7. *Let $\mathcal{C}$ be a complete set of loops of Σ . Then $\mathcal{C}_0 \subseteq \mathcal{C}$.*

Proof. Let $[\tilde{\gamma}]_I \in \mathcal{C}$. Let $\tilde{\gamma}$ be a closed path of $[\tilde{\gamma}]_I$ with basepoint x . Now $[\tilde{\gamma}]_I \circ [\tilde{\gamma}]_I$ at x with respect to $\tilde{\gamma}$ and $\tilde{\gamma}^{-1}$ is a contractible loop. By Definition 1.6.14, this sum is a loop of $\mathcal{C}$. It follows that there is a contractible loop in $\mathcal{C}$. Since $\mathcal{C}_0$ is the set of the free homotopic copies of any contractible loop, we obtain that $\mathcal{C}_0 \subseteq \mathcal{C}$. $\square$

Proposition 1.7.7 justifies the name "fundamental partition" given to $(\mathcal{C}_0, \mathcal{E}_0)$.

The next three definitions extend the definition of the length of O-circuits according to the preceding concepts.

Definition 1.7.8. *Let $(\mathcal{C}, \mathcal{E})$ be a complete partition of the loops of Σ . Let $[\tilde{\gamma}]_I$ be a loop of Σ . Then $[\tilde{\gamma}]_{\mathcal{C}}$ is the set*

$$\{[\tilde{\gamma}']_I : [\tilde{\gamma}']_I \sim_{\mathcal{C}} [\tilde{\gamma}]_I\}.$$

Now we generalize some results of Section 1.6. This will show the importance of the TPP.

Definition 1.7.9. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a nontrivial surface Σ . Let $(\mathcal{C}, \mathcal{E})$ be a complete partition of $\overleftarrow{\Gamma}_\Sigma$. Let $\gamma \in \Omega_{\mathcal{E}}$, and suppose there exists $\gamma' \in \Omega_{\mathcal{E}}$ such that $\gamma' \sim_{\mathcal{C}} \gamma$ and $\gamma' \cap G_\Psi(\Sigma) \subset V_\Psi$. Then γ is *face-represented modulo $\mathcal{C}$* by γ' .

Definition 1.7.10. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a nontrivial surface Σ . Let $(\mathcal{C}, \mathcal{E})$ be a complete partition of $\overleftarrow{\Gamma}_\Sigma$. Let $[\tilde{\gamma}]_l \in \mathcal{E}$, and let γ be its corresponding O -circuit. Suppose that γ is face-represented modulo $\mathcal{C}$. The length of γ modulo $\mathcal{C}$, denoted $\ell_\gamma^{\mathcal{C}}(\Psi)$, is

$$\ell_\gamma^{\mathcal{C}}(\Psi) = \min \{ |\gamma' \cap V_\Psi| : \gamma' \in [\tilde{\gamma}]_{\mathcal{C}} \cap \mathcal{O}_{\mathcal{C}}, \gamma' \cap G_\Psi \subseteq V_\Psi \}.$$

Proposition 1.7.8. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a nontrivial surface Σ . Let $(\mathcal{C}, \mathcal{E})$ be a complete partition of $\overleftarrow{\Gamma}_\Sigma$. Let γ be an O -circuit of $\Omega_{\mathcal{E}}$, and suppose that γ is face-represented modulo $\mathcal{C}$ with length modulo $\mathcal{C}$ $\ell_\gamma^{\mathcal{C}}(\Psi)$. Then $\rho_{\mathcal{E}}(\Psi) \leq \ell_\gamma^{\mathcal{C}}(\Psi)$.

Proof. Since γ is face-represented modulo $\mathcal{C}$, there is an O -circuit γ' such that $\gamma \sim_{\mathcal{C}} \gamma'$ and $\ell_\gamma^{\mathcal{C}}(\Psi) = |\gamma' \cap V_\Psi|$. By Proposition 1.7.5, $[\tilde{\gamma}]_l \in \mathcal{E}$. Therefore, $\rho(\Psi) \leq |\gamma' \cap V_\Psi| = \ell_\gamma^{\mathcal{C}}(\Psi)$. $\square$

The following result is central to the proofs of Theorem 2.2.2 and Theorem 3.2.1.

Proposition 1.7.9. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a nontrivial surface Σ . Let $(\mathcal{C}_1, \mathcal{E}_1), (\mathcal{C}_2, \mathcal{E}_2)$ be two complete partitions of $\overleftarrow{\Gamma}_\Sigma$ with $\mathcal{E}_2 \subseteq \mathcal{E}_1$. Let γ be an O -circuit of $\Omega_{\mathcal{E}_2}$ that is face-represented modulo $\mathcal{C}_1$ by a face cycle $\Lambda = v_0, f_1, v_1, \dots, f_\ell, v_\ell$ of length $\ell = \ell_\gamma^{\mathcal{C}_1}(\Psi)$. Let $\Lambda' = x_0, g_1, x_1, \dots, g_{k'}, x_{k'}$ (it might happen that Λ' is the vertex x_0 only) be a face chain such that

- (1) $x_0, x_k \in \bigcup_{i=1}^{\ell} \bar{f}_i$;
- (2) $(\bigcup_{i=2}^{k'} \bar{g}_i) \cap \bigcup_{i=1}^{\ell} \bar{f}_i = \emptyset$.

(If $\Lambda' = x_0$, we suppose that $x_0 \in \bar{f}_i \cap \bar{f}_j$, for some i, j with $i \neq j$). If either

(1)

$$k' \leq \left\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - \ell_{\gamma}^{c_1} - 1}{2} \right\rfloor - 1 \text{ or}$$

(2) if both end faces of Λ' are incident with two of the v_i and

$$k' \leq \left\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - \ell_{\gamma}^{c_1} - 1}{2} \right\rfloor,$$

then the closed face chain formed by Λ' and the shortest part of Λ between the ends of Λ' face-represents an O -circuit that belongs to Ω_{c_1} .

Proof. 1) Without loss of generality, suppose that $\{x_0, x_{k'}\} \cap \{v_1, v_2, \dots, v_l\} = \emptyset$ (the case where one of $x_0, x_{k'}$ is in $v_1, \dots, v_l$ is similar). Let i and j be the indices such that $x_0 \in \bar{f}_i$ and $x_{k'} \in \bar{f}_j$ respectively. Without loss of generality, suppose that the shortest sub-face chain of Λ between x_0 and $x_{k'}$ is $\Lambda_1 = x_0, f_i, v_{i+1}, f_{i+1}, \dots, f_j, x_{k'}$, and that the longest sub-face chain of Λ between x_0 and $x_{k'}$ is $\Lambda_2 = x_0, f_i, v_{i-1}, f_{i-1}, \dots, f_{j+1}, v_j, f_j, x_{k'}$ (see Figure IX). Of course, it might happen that Λ_1 and Λ_2 have the same length.

Let $\Lambda_1 \cup \Lambda' = x_0, f_i, v_i, \dots, f_j, x_{k'}, g_{k'}, x_{k'-1}, \dots, x_1, g_1, x_0$, and $\Lambda_2 \cup \Lambda' = x_0, f_i, x_{i-1}, f_{i-1}, \dots, x_j, f_j, x_{k'}, g_{k'}, x_{k'-1}, \dots, g_1, x_0$.

Since Λ face-represents an element of $\mathcal{E}_2$, the TPP implies that at least one of $\Lambda_1 \cup \Lambda'$ and $\Lambda_2 \cup \Lambda'$ face-represents a loop of $\mathcal{E}_2$.

We have the following inequalities:

$$\begin{aligned} \ell(\Lambda_1 \cup \Lambda') &\leq \left\lfloor \frac{l}{2} \right\rfloor + 1 + \left\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - l - 1}{2} \right\rfloor - 1 \\ &\leq \left\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - 1}{2} \right\rfloor \\ &\leq \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - 1}{2} \\ &< \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi)}{2} \end{aligned}$$

Figure IX

and

$$\begin{aligned}
 \ell(\Lambda_1 \cup \Lambda') + \ell(\Lambda_2 \cup \Lambda') &\leq 2k' + l + 2 \\
 &\leq 2\left(\left\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - l - 1}{2} \right\rfloor - 1\right) + l + 2 \\
 &\leq \rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - l - 1 + l \\
 &= \rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - 1,
 \end{aligned}$$

The first inequality implies that $\ell(\Lambda_1 \cup \Lambda') < \rho_{\mathcal{E}_2}(\Psi)$, so $\Lambda_1 \cup \Lambda'$ face-represents an O-circuit in $\Omega_{\mathcal{C}_2}$. Therefore, $\Lambda_2 \cup \Lambda'$ face-represents an O-circuit in $\Omega_{\mathcal{E}_2}$. Thus, $\ell(\Lambda_2 \cup \Lambda') \geq \rho_{\mathcal{E}_2}(\Psi)$, and it follows from the second inequality that $\ell(\Lambda_1 \cup \Lambda') \leq \rho_{\mathcal{E}_1}(\Psi) - 1$, and therefore the O-circuit face-represented by $\Lambda_1 \cup \Lambda'$ is in $\Omega_{\mathcal{C}_1}$.

Notice that the case where exactly one of $x_0, x_{k'}$ is one of $v_1, \dots, v_l$ is proved in a similar way.

2) Let i and j be the indices such that $x_0 = v_i$ and $x_{k'} = v_j$ respectively. Without loss of generality, suppose that the shortest sub-face chain of Λ between x_0 and $x_{k'}$ is $\Lambda_1 = x_0, f_{i+1}, v_{i+1}, \dots, f_j, x_{k'}$, and that the longest sub-face chain of Λ between x_0 and $x_{k'}$ is $\Lambda_2 = x_0, f_i, v_{i-1}, f_{i-1}, \dots, v_{j+1}, f_{j+1}, x_{k'}$. Of course, it might happen that Λ_1 and Λ_2 both have the same length.

Since Λ face-represents an element of $\mathcal{E}_2$, the TPP implies that at least one of $\Lambda_1 \cup \Lambda'$ and $\Lambda_2 \cup \Lambda'$ face-represents a loop of $\mathcal{E}_2$.

We have the following inequalities:

$$\begin{aligned}
 \ell(\Lambda_1 \cup \Lambda') &\leq \left\lfloor \frac{l}{2} \right\rfloor + \left\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - l - 1}{2} \right\rfloor \\
 &\leq \left\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - 1}{2} \right\rfloor \\
 &\leq \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - 1}{2} \\
 &< \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi)}{2}
 \end{aligned}$$

and

$$\begin{aligned}
 \ell(\Lambda_1 \cup \Lambda') + \ell(\Lambda_2 \cup \Lambda') &\leq 2k' + l \\
 &\leq 2\left(\left\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - l - 1}{2} \right\rfloor\right) + l \\
 &\leq \rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - l - 1 + l \\
 &= \rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - 1,
 \end{aligned}$$

In the first inequality, it follows that $\ell(\Lambda_1 \cup \Lambda') < \rho_{\mathcal{E}_2}(\Psi)$, so $\Lambda_1 \cup \Lambda'$ face-represents an O-circuit in $\Omega_{\mathcal{C}_2}$. Hence, $\Lambda_2 \cup \Lambda'$ face-represents an O-circuit in $\Omega_{\mathcal{E}_2}$. Therefore, $\ell(\Lambda_2 \cup \Lambda') \geq \rho_{\mathcal{E}_2}(\Psi)$, and it follows from the second inequality that $\ell(\Lambda_1 \cup \Lambda') \leq \rho_{\mathcal{E}_1}(\Psi) - 1$. Thus the O-circuit face-represented by $\Lambda_1 \cup \Lambda'$ is in $\Omega_{\mathcal{C}_1}$.

□

Definition 1.7.11. Let Σ be a nontrivial surface, and let $\gamma_0, \gamma_1 \in \Omega_\Sigma$. Suppose that there is a point $p \in \gamma_0 \cap \gamma_1$, a closed neighbourhood $\bar{\Delta}$ of p and a homeomorphism $f : \bar{\Delta} \longrightarrow \{(x, y) : x^2 + y^2 \leq 1\}$ such that S^1 meets $f(\gamma_0 \cup \gamma_1)$ in exactly four points, two from each γ_i and these points alternate in S^1 being in $f(\gamma_0)$ and $f(\gamma_1)$. Then we say that γ_0 and γ_1 cross transversely at p .

The next concept puts in evidence exactly the characteristics we require from some complete partitions to obtain our main results in the next chapters.

Definition 1.7.12. Let Σ be a nontrivial surface. A complete partition $(\mathcal{C}, \mathcal{E})$ of $\overrightarrow{\Gamma}_\Sigma$ is a crossing partition if, for any $\gamma_0 \in \Omega_{\mathcal{C}}$ and for any $\gamma_1 \in \Gamma_\Sigma$ that crosses γ_0 transversely, $|\gamma_0 \cap \gamma_1| \geq 2$.

We can now show a profound characteristic common to both $(\mathcal{C}_0, \mathcal{E}_0)$ and $(\mathcal{C}_s, \mathcal{E}_s)$.

Proposition 1.7.10. $(C_0, \mathcal{E}_0)$ is a crossing partition.

Proof. Let $\gamma_0 \in \Omega_{C_0}$ and $\gamma_1 \in \Gamma_\Sigma$ such that γ_0 and γ_1 cross transversely at a point p . By Corollary S.1 of [ARV], γ_0 bounds a disc Δ . Since γ_1 is t-connected, there is another point $p' \in \gamma_0 \cap \gamma_1$, since otherwise γ_1 would be included in $\bar{\Delta}$, a contradiction with the hypothesis. $\square$

Proposition 1.7.11. $(C_s, \mathcal{E}_s)$ is a crossing partition.

Proof. Let $\gamma_0 \in \Omega_{C_s}$ and $\gamma_1 \in \Gamma_\Sigma$ such that γ_0 and γ_1 cross transversely at a point p . The proof is somehow similar to the proof of Proposition 1.7.10: for this, we replace one of the t-components of $\Sigma \setminus \gamma_0$ by a disc. $\square$

The following Proposition will be very useful when dealing with nonorientable loops.

Proposition 1.7.12. Let Σ be a nontrivial surface, and let $(C, \mathcal{E})$ be a crossing partition of $\overleftarrow{\Gamma}_\Sigma$. Then all orientation-reversing O-loops of Σ are contained in $\mathcal{E}$.

Proof. Let $[\tilde{\gamma}]_t$ be an orientation-reversing O-loop. Let $\tilde{\gamma}$ be a simple closed path of $[\tilde{\gamma}]_t$. We can enclose γ inside a Möbius band whose boundary is an O-circuit γ' that is 2-freely homotopic to γ . Let σ be a curve from one point of γ' to another point of γ' and that crosses transversely γ only once. Then $\gamma' \cup \sigma$ contains an O-circuit α with $\sigma \subseteq \alpha$, and α intersects γ only once. Therefore, $\gamma \notin C$. $\square$

1.8 Representativity and types of embeddings.

Definition 1.8.1. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a nontrivial surface Σ . Let $\Omega_{\mathcal{E}_0}$ be the set of noncontractible O-circuits of Σ . The representativity of Ψ , denoted $\rho(\Psi)$, is defined by

$$\rho(\Psi) = \min\{|G_\Psi(\Sigma) \cap \gamma| : \gamma \in \Omega_{\mathcal{E}_0}\}.$$

For example, the representativity of the embedding of K_5 in the torus illustrated in Figure VI is 1.

Remark. Notice that Definition 1.8.1 is a special case of Definition 1.6.17 (2) with $\Upsilon = \Omega_{C_0}$.

Proposition 1.8.1. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a nontrivial surface Σ . Then there is a noncontractible 0-circuit γ of Σ such that $|\gamma \cap G_\Psi(\Sigma)| = \rho(\Psi)$ and $\gamma \cap G_\Psi(\Sigma) \subseteq V_\Psi(\Sigma)$.

Proof. This is an application of Theorem 1.6.14 with $(\mathcal{C}, \mathcal{E}) = (C_0, \mathcal{E}_0)$. $\square$

Definition 1.8.2. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a surface Σ .

- (1) If the faces of G_Ψ are homeomorphic to the unit disc, then Ψ is an open 2-cell embedding (see Figure VI):
- (2) if the closure of the faces of G_Ψ are homeomorphic to the closed unit disc, then Ψ is a closed 2-cell embedding (see Figure X):
- (3) if Ψ is a closed 2-cell embedding and the subgraph of G_Ψ bounding the faces incident with any vertex is a wheel with at least three spokes and a possibly subdivided rim, then Ψ is a wheel-neighbourhood embedding (see Figure XI).

The following propositions give necessary and sufficient conditions for an embedding to be an open 2-cell embedding, a closed 2-cell embedding or a wheel-neighbourhood embedding depending on the representativity of the embedding and the properties of the graph. These propositions are due to Negami [N] and Vitray [V1].

Figure X

Figure XI

Lemma 1.8.2. *Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a nontrivial surface Σ . Then $\rho(\Psi) = 0$ if and only if there is a face of G_Ψ that contains a noncontractible O-circuit.*

Proof. ($\Rightarrow$) Suppose that $\rho(\Psi) = 0$. Then, by Proposition 1.8.1, there is a noncontractible O-circuit γ of Γ_Σ such that $|\gamma \cap G_\Psi(\Sigma)| = \rho(\Psi) = 0$. Hence, γ is in a t-component of $\Sigma \setminus G_\Psi(\Sigma)$. It follows that γ is contained in a face of G_Ψ .

($\Leftarrow$) Suppose that a face f of G_Ψ contains a noncontractible circuit γ . Since $f \cap G_\Psi(\Sigma) = \emptyset$, it follows that $|\gamma \cap G_\Psi(\Sigma)| = 0$, and so $\rho(\Psi) = 0$. $\square$

Proposition 1.8.3. *An embedding $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$, where Σ is a nontrivial surface, is an open 2-cell embedding if and only if G_Ψ is connected and $\rho(\Psi) \geq 1$.*

Proof. ($\Rightarrow$) Suppose that Ψ is an open 2-cell embedding. Then, since the faces of G_Ψ are homeomorphic to open discs, they cannot contain a noncontractible O-circuit, and therefore Lemma 1.8.2 implies that $\rho(\Psi) \geq 1$. Furthermore, G_Ψ must be connected for otherwise one face, say f , would be bounded by two disjoint subgraphs of G_Ψ , implying that the boundary of f is not connected, and so this face would not be homeomorphic to a disc.

($\Leftarrow$) Suppose that Ψ is not an open 2-cell embedding. Then there exists a face f of G_Ψ that is not homeomorphic to a disc, and therefore f contains O-circuits that are not contractible in f . If there is one O-circuit in f that is both noncontractible in f and noncontractible in Σ , then Lemma 1.8.2 implies that $\rho(\Psi) = 0$, a contradiction. Suppose then that all the noncontractible O-circuits in f are contractible in Σ . Let γ be such an O-circuit. Then γ bounds a disc Δ . Since $\Delta \not\subseteq f$, Δ contains a component K of G_Ψ . Since $\Sigma \setminus \Delta$ is not a disc, it contains a noncontractible O-circuit. Since $\rho(\Psi) > 0$, such an O-circuit meets part of G . Therefore, there is a component L of G_Ψ in $\Sigma \setminus \Delta$, a contradiction with the connectivity of G_Ψ . $\square$

Proposition 1.8.4. *An embedding $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$, where Σ is a nontrivial surface, is a closed 2-cell embedding if and only if $\rho(\Psi) \geq 2$ and G_Ψ is 2-connected.*

Proof. By Proposition 1.8.3, we can assume that Ψ is an open 2-cell embedding with $\rho(\Psi) \geq 1$ and G_Ψ is connected.

($\Rightarrow$) Suppose that Ψ is a closed 2-cell embedding. Let γ be a noncontractible O-circuit such that $|\gamma \cap G_\Psi(\Sigma)| = \rho(\Psi)$ and $\gamma \cap G_\Psi(\Sigma) \subseteq V(G_\Psi)$. For each face f , $\bar{f}$ is a closed disc, so that $\gamma \not\subseteq \bar{f}$. Since there is a face f such that $\gamma \cap f \neq \emptyset$, we have, for that face f , that $\gamma \cap \text{bd}(f)$ contains at least two points. Therefore, $\gamma \cap G_\Psi(\Sigma)$ has at least two points, and so $\rho(\Psi) \geq 2$.

If G_Ψ has a cut vertex v , then v is a repeated vertex of the boundary walk of some face of G_Ψ , a contradiction. Hence G_Ψ is 2-connected.

($\Leftarrow$) Suppose that Ψ is not a closed 2-cell embedding. Then there is a face of G_Ψ whose boundary walk has a repeated vertex v .

If there is an O-circuit $\gamma \subseteq \bar{f}$ through v that is noncontractible in Σ , then $\rho(\Psi) = 1$, a contradiction with the hypothesis. So suppose that all the O-circuits in $\bar{f}$ through v are contractible in Σ .

Since v is a repeated vertex of the boundary walk of f , there are two closed subwalks W_1 and W_2 that are traversals of the subgraphs J_1 and J_2 of $\text{bd}(f)$ respectively such that $J_1 \cup J_2 = \text{bd}(f)$. Let γ be an O-circuit in $\bar{f}$ such that $\gamma \cap \bar{f} = \{v\}$ and such that γ encloses either J_1 or J_2 inside an open disc Δ . Then γ separates $J_1 - \{v\}$ from $J_2 - \{v\}$, and so v is a cut vertex of G_Ψ , a contradiction with the connectivity of G_Ψ . $\square$

Proposition 1.8.5. *Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a non-trivial surface Σ . Let v be a vertex of G , and let $\Psi' = \Psi|_{G-\{v\}}$. Then $\rho(\Psi') \geq \rho(\Psi) - 1$.*

Proof. If $\rho(\Psi) \leq 1$, the result is trivial. So suppose that $\rho(\Psi) \geq 2$.

Consider $G_{\Psi'} = G_\Psi - \{v\}$. Let γ be an O-circuit of Σ such that $|\gamma \cap G_{\Psi'}(\Sigma)| = \rho(\Psi')$ and $\gamma \cap G_{\Psi'}(\Sigma) \subseteq V_{\Psi'}$.

If $|\gamma \cap G_\Psi(\Sigma)| = \rho(\Psi)$ and $\gamma \cap G_\Psi(\Sigma) \subseteq V_\Psi$, then $\rho(\Psi') = \rho(\Psi)$. Otherwise, γ intersects v and/or some edges incident with v . Let f' be the face of $G_{\Psi'}$ that contains v and its deleted incident edges in G_Ψ . Then, fixing the ends of γ in $\partial f'$, we can displace $\gamma \cap f'$ so that it intersects only v and it is contained in two faces of G_Ψ . Let γ' be the new O-circuit obtained. Since the two faces are homeomorphic to a disc, γ' is noncontractible. We also have that $|\gamma' \cap G_\Psi(\Sigma)| = |\gamma \cap G_{\Psi'}(\Sigma)| + 1$ and $\gamma' \cap G_\Psi(\Sigma) \subseteq V_\Psi$. It follows that $\rho(\Psi') \geq \rho(\Psi) - 1$. $\square$

Proposition 1.8.6. *An embedding $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$, where Σ is a nontrivial surface, is a wheel-neighbourhood embedding if and only if $\rho(\Psi) \geq 3$ and G_Ψ is 3-connected.*

Proof. By Proposition 1.8.4, we can assume that Ψ is a closed 2-cell embedding with $\rho(\Psi) \geq 2$ and that G_Ψ is 2-connected.

($\Rightarrow$) Suppose that Ψ is a wheel-neighbourhood embedding. There is a noncontractible O-circuit γ in Σ such that $|\gamma \cap G_\Psi(\Sigma)| = \rho(\Psi)$ and $\gamma \cap G_\Psi(\Sigma) \subseteq V_\Psi$. Let $v \in V_\Psi \cap \gamma$. Then as γ crosses v , it crosses at least two distinct vertices of the rim of the wheel-neighbourhood of v . It follows that $\rho(\Psi) \geq 3$.

If we delete any vertex from G_Ψ , we obtain a closed 2-cell embedding, and by Proposition 1.8.4, the graph $G_\Psi - \{v\}$ is 2-connected. It follows that G_Ψ is 3-connected.

($\Leftarrow$) Now suppose that $\rho(\Psi) \geq 3$ and that G_Ψ is 3-connected. Let $v \in V_\Psi$, and remove this vertex from G_Ψ . We obtain a 2-connected graph and the representativity of the induced embedding is at least 2, by Proposition 1.8.5. By Proposition 1.8.4, this new embedding is a closed 2-cell embedding. Therefore, the subgraph of G_Ψ that bounds the faces incident with v is a wheel. Since v was chosen arbitrarily, Ψ is thus a wheel-neighbourhood embedding. $\square$

Corollary 1.8.7. *Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a simple 3-connected graph G onto a nontrivial compact surface Σ . Suppose that $\rho(\Psi) \geq 3$. Let $\Lambda = v_0, f_1, v_1, \dots, f_l, v_l$ be a face-chain in G_Ψ . Then, for each i , $\bar{f}_i \cap \bar{f}_{i+1}$ is either the vertex v_i or $\{v_i, v_i v, v\}$, for some vertex v .*

Proof. Since $\rho(\Psi) \geq 3$ and G_Ψ is 3-connected, Ψ is a wheel-neighbourhood embedding with each wheel-neighbourhood containing at least three spokes.

Let N_i be the wheel-neighbourhood of v_i . The intersection of the closure of any two faces of N_i is either v_i or $\{v_i, v_i v, v\}$, where v is a vertex of the rim of N_i and $v_i v$ is a spoke of N_i . Since f_i and f_{i+1} are two faces of N_i , the result follows. $\square$

Definition 1.8.3. *Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding. A component, block or cleavage unit is contractible if all its polygons are contractible as O -circuits. It is noncontractible if it contains at least one noncontractible polygon.*

Proposition 1.8.8. *Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a connected graph G onto a nontrivial surface Σ . Then all the polygons of G_Ψ are contractible if and only if there exists a closed disc $\bar{\Delta}$ such that $G_\Psi(\Sigma) \subset \bar{\Delta}$.*

Proof. ($\implies$) By Theorem 4 of Anderson, Richter and Vitray [ARV], if H is a 2-connected graph embedded onto a surface Σ by an embedding Ψ , then either H_Ψ has a noncontractible polygon or there is a polygon that bounds a disc Δ_H containing H_Ψ . So let $B_1, \dots, B_k$ be the blocks of G_Ψ . Since by hypothesis there is no noncontractible polygon in G_Ψ , it follows that none of the blocks B_i contains a noncontractible polygon, and therefore each B_i has a contractible polygon P_i that bounds a disc Δ_i containing B_i , for $i = 1, \dots, k$. For each cut vertex v of G , there is an open disc Δ' such that $\bar{\Delta}'$ contains for only parts of G_Ψ , v and some segment from the incident edges to v . Since the B_i are blocks, the union of the $\bar{\Delta}_i$ for $i = 1, \dots, k$, together with the $\bar{\Delta}'$ is a disc $\bar{\Delta}$ that contains G_Ψ .

($\impliedby$) Suppose that there exists a closed disc $\bar{\Delta} \subset \Sigma$ such that $G_\Psi \subset \bar{\Delta}$. Then every polygon of G_Ψ is contained in that closed disc and is thus contractible. $\square$

Proposition 1.8.9. *Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a non-trivial surface Σ . Suppose that $\rho(\Psi) \geq 1$. Then G_Ψ contains a noncontractible polygon.*

Proof. Suppose that all the polygons of G_Ψ are contractible. Then, by Proposition 1.8.8, there exists a closed disc $\bar{\Delta} \subset \Sigma$ such that $G_\Psi \subset \bar{\Delta}$. However, $\Sigma \setminus \bar{\Delta}$ contains noncontractible O-circuits and so $\rho(\Psi) = 0$, a contradiction with the hypothesis. $\square$

Proposition 1.8.10. *If $\rho(\Psi) \geq 1$, then G_Ψ has a unique essential component, say L , and $\rho(\Psi|_L) = \rho(\Psi)$.*

Proof. Since $\rho(\Psi) \geq 1$, Proposition 1.8.9 implies that G_Ψ contains a noncontractible polygon C . Let L be the component of G_Ψ that contains C .

Suppose that there is another component K of G_Ψ that contains a noncontractible polygon. Now K is in a face f of L . Then an O-circuit in f near any one of the components (there might be more than one) of $\text{bd}(f)$ is noncontractible (for otherwise L or K would be contained in a disc, a contradiction). Therefore, $\rho(\Psi) = 0$, a contradiction with the hypothesis. $\square$

Proposition 1.8.11. *If $\rho(\Psi) \geq 2$, then G_Ψ has a unique noncontractible block, say B , and $\rho(\Psi|_B) = \rho(\Psi)$.*

Proof. Since $\rho(\Psi) \geq 2$, Proposition 1.8.8 implies that G_Ψ contains a noncontractible polygon C . Let B be the block of G_Ψ that contains C .

Suppose that there is another block B' that contains a noncontractible polygon C . By Proposition 1.8.10, we can assume that B and B' are in the same component. Then B' is part of a bridge L of B that has a unique vertex of attachment in B , say v . Now, there is a face f of $\Psi|_B$ that contains B' , and so there is an O-circuit γ that is contained in f and that crosses B' only at the vertex v . Since

$\rho(\Psi) \geq 2$, γ must be contractible. But then either B or B' is contained in a closed disc $\bar{\Delta}$, a contradiction with the hypotheses. It follows that B is the unique noncontractible block of G_Ψ .

Now let γ be an O-circuit that meets B at exactly $\rho(\Psi|_B)$ points. Since the other 2-components of G_Ψ are contractible, we can choose γ to avoid them. Therefore, $\gamma \cap G_\Psi(\Sigma) \subset B$, and so $\rho(\Psi|_B) \geq \rho(\Psi)$. It follows that $\rho(\Psi|_B) = \rho(\Psi)$. $\square$

Proposition 1.8.12. *If $\rho(\Psi) \geq 3$, then G_Ψ has a unique noncontractible cleavage unit, say K^a . Moreover, $\rho(\Psi|_{K^a}) = \rho(\Psi)$.*

Proof. We give a brief outline of the proof, we refer the reader to [RoV] p 308 for more details.

Since $\rho(\Psi) \geq 2$, we know by Proposition 1.8.11 that there is a unique noncontractible block B .

We iterate the following process: given a good excisable 2-separation (M_1, N_1) of $H_1 = B$, we adjoin a virtual edge e_1 between its corresponding pair of hinges $\{x_1, y_1\}$ to obtain two 2-connected subgraphs M_1^+ and N_1^+ of G_Ψ .

We have to be extremely careful about the position of e_1 on the surface Σ . The key to the proof is to notice that one of M_1 and N_1 , say N_1 , can be enclosed in a disc Δ_1 , and the other one, M_1 , is enclosed in a nontrivial bordered surface Σ_1 such that $\Sigma = \Sigma_1 \cup \Delta_1$. Now we choose to put e_1 inside Δ_1 between the endpoints x_1 and y_1 . We continue this process until there remains no good excisable 2-separation, and we then end up with a unique noncontractible cleavage unit $K^a = H_k$ topologically contained in $G_\Psi(\Sigma)$.

We now prove that $\rho(\Psi|_{K^a}) = \rho(\Psi)$. Let γ be an O-circuit of Σ that meets K^a at exactly $\rho(\Psi|_{K^a})$ vertices. Since K^a is topologically contained in B_1 , γ can avoid B except at vertices of K^a . All other blocks, which are contractible, can be avoided as in the proof of Proposition 1.8.11, except at vertices of K^a . Therefore, $\rho(\Psi|_{K^a}) \geq \rho(\Psi)$. However, since K^a is topologically contained in G_Ψ , we have that $\rho(\Psi) \geq \rho(\Psi|_{K^a})$. Hence, $\rho(\Psi|_{K^a}) = \rho(\Psi)$. $\square$

Remark. Since $3 \leq \rho(\Psi) = \rho(\Psi|_{K^*})$, K^* cannot be a bond or a polygon. Therefore, the unique noncontractible cleavage unit K^* of G_Ψ is 3-connected.

Proposition 1.8.13. Let $(\mathcal{C}, \mathcal{E})$ be a complete partition of $\overrightarrow{\Gamma}_\Sigma$, and suppose that $\rho(\Psi) \geq 3$. Let K^* be the unique noncontractible cleavage unit of G_Ψ . Then $\rho_{\mathcal{E}}(\Psi|_{K^*}) = \rho_{\mathcal{E}}(\Psi)$.

Proof. Let G_Ψ , B , K^* be as in the proof of Proposition 1.8.12. Let γ be an O-circuit of $\Omega_{\mathcal{E}}$ that meets K^* at exactly $\rho(\Psi|_{K^*})$ vertices. Since K^* is topologically contained in B , γ can avoid B except at vertices of K^* . All other blocks, which are contractible, can be avoided as in the proof of Proposition 1.8.11 (remember that $C_0 \in \mathcal{C}$ by Proposition 1.7.7), except at vertices of K^* . The rest of the proof is similar to the end of the proof of Proposition 1.8.12. $\square$

1.9 String-equivalence of paths.

In Sections 6 and 7, we defined some equivalence relations over closed paths and defined a sum over their equivalence classes. In this section, we will consider similar concepts on paths in general.

In this section, we let Σ be a surface and denote the set of paths in Σ by $\overrightarrow{\Theta}_\Sigma$.

Definition 1.9.1. Two paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are *direction-equivalent* if

- (1) $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ have the same endpoints;
- (2) $\tilde{\gamma}_0([0, 1]) = \tilde{\gamma}_1([0, 1])$; and
- (3) $\tilde{\gamma}_0 \sim \tilde{\gamma}_1$ in the topological space $\tilde{\gamma}_0([0, 1])$.

We use the notation $\tilde{\gamma}_0 \sim_d \tilde{\gamma}_1$ to indicate that the two paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are direction-equivalent.

Remarks. 1) It is easy to see that $\sim_d$ is an equivalence relation. We call the equivalence classes under $\sim_d$ *direction classes*.

2) Since the elements of a given direction class have the same origin and terminus, say x and y , we call x the *origin* of the direction class and y the *terminus* of the direction class.

3) Since $\tilde{\gamma}_0([0, 1])$ is a topological subspace of Σ with respect to the topology of Σ , Condition 3) implies that $\tilde{\gamma}_0 \sim \tilde{\gamma}_1$ in Σ .

Lemma 1.9.1. Let $\tilde{\gamma}_0, \tilde{\gamma}_1 \in \overrightarrow{\Theta}_\Sigma$, and suppose that $\tilde{\gamma}_0 \sim_d \tilde{\gamma}_1$. Then $\tilde{\gamma}_0^{-1} \sim_d \tilde{\gamma}_1^{-1}$.

Proof. Similar to the proof of Lemma 1.6.2. $\square$

Definition 1.9.2. Two paths $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are *string-equivalent* if either

- (1) $\tilde{\gamma}_0 \sim_d \tilde{\gamma}_1$, or
- (2) $\tilde{\gamma}_0 \sim_d \tilde{\gamma}_1^{-1}$.

We use the notation $\tilde{\gamma}_0 \sim_s \tilde{\gamma}_1$ to indicate that $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are string-equivalent.

Lemma 1.9.2. Let $\tilde{\gamma}_0, \tilde{\gamma}_1 \in \overrightarrow{\Theta}_\Sigma$. Then the following assertions are equivalent:

- (1) $\tilde{\gamma}_0 \sim_s \tilde{\gamma}_1$;
- (2) $\tilde{\gamma}_0 \sim_s \tilde{\gamma}_1^{-1}$;
- (3) $\tilde{\gamma}_0^{-1} \sim_s \tilde{\gamma}_1$;
- (4) $\tilde{\gamma}_0^{-1} \sim_s \tilde{\gamma}_1^{-1}$.

Proof. Follows from Definition 1.9.2 and Lemma 1.9.1. $\square$

Proposition 1.9.3. $\sim_s$ is an equivalence relation over $\overrightarrow{\Theta}_\Sigma$.

Proof. Reflexivity is clear. We first prove the symmetry of the relation. Suppose that $\tilde{\gamma}_0 \sim_s \tilde{\gamma}_1$, for some $\tilde{\gamma}_0, \tilde{\gamma}_1 \in \overrightarrow{\Theta}_\Sigma$. Then either $\tilde{\gamma}_0 \sim_d \tilde{\gamma}_1$ or $\tilde{\gamma}_0 \sim_d \tilde{\gamma}_1^{-1}$. Suppose first that $\tilde{\gamma}_0 \sim_d \tilde{\gamma}_1$. Then, by reflexivity of $\sim_d$ (see the remark after Definition 1.9.1), we have that $\tilde{\gamma}_1 \sim_d \tilde{\gamma}_0$ and so $\tilde{\gamma}_1 \sim_s \tilde{\gamma}_0$. If $\tilde{\gamma}_0 \sim_d \tilde{\gamma}_1^{-1}$, then $\tilde{\gamma}_0^{-1} \sim_d \tilde{\gamma}_1$ by Lemma 1.9.1, and $\tilde{\gamma}_1 \sim_d \tilde{\gamma}_0^{-1}$ by reflexivity of $\sim_d$. Hence, $\tilde{\gamma}_1 \sim_s \tilde{\gamma}_0$.

We now prove the transitivity of $\sim_s$. Let $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2 \in \overrightarrow{\Theta}_\Sigma$, and suppose that $\tilde{\gamma}_0 \sim_s \tilde{\gamma}_1$ and $\tilde{\gamma}_1 \sim_s \tilde{\gamma}_2$. Suppose first that $\tilde{\gamma}_0 \sim_d \tilde{\gamma}_1$ and that $\tilde{\gamma}_1 \sim_d \tilde{\gamma}_2$. By transitivity of $\sim_d$, we have that $\tilde{\gamma}_0 \sim_d \tilde{\gamma}_2$. The other three cases are dealt with using Lemma 1.9.1 and Lemma 1.9.2. $\square$

Definition 1.9.3. A string in Σ is an equivalence class with respect to string-equivalence. Given a path $\tilde{\gamma}$, we represent its string by $[\tilde{\gamma}]_s$. If $[\tilde{\gamma}]_s$ is a string, a path of $[\tilde{\gamma}]_s$ is an element of the class $[\tilde{\gamma}]_s$. We represent the set of strings of Σ by $\overrightarrow{\Theta}_\Sigma$.

Remark. Notice that in the case where a path $\tilde{\gamma}$ is closed, $[\tilde{\gamma}]_s \neq [\tilde{\gamma}]_l$. However, $[\tilde{\gamma}]_s \subseteq [\tilde{\gamma}]_l$.

Definition 1.9.4. Let $[\tilde{\gamma}]_s$ be a string of Σ . Given a path $\tilde{\gamma}'$ of $[\tilde{\gamma}]_s$, let $[\tilde{\gamma}']_d$ be the set of paths of $[\tilde{\gamma}]_s$ that are direction-equivalent to $\tilde{\gamma}'$. The set $[\tilde{\gamma}']_d$ is a direction class of the string $[\tilde{\gamma}]_s$.

Remarks. 1) Let $[\tilde{\gamma}]_s$ be a string and $\tilde{\gamma}'$ be a path of $[\tilde{\gamma}]_s$. Let $[\tilde{\gamma}']_d$ be the direction class of $[\tilde{\gamma}]_s$ containing $\tilde{\gamma}'$. Then $[\tilde{\gamma}']_d$ is the equivalence class of $\tilde{\gamma}'$ under $\sim_d$. Therefore, we will denote the direction class of a path $\tilde{\gamma}$ by $[\tilde{\gamma}]_d$.

2) Any nonconstant string has exactly two direction classes: given a path $\tilde{\gamma}'$ of the string $[\tilde{\gamma}]_s$, these direction classes are respectively $\{\tilde{\gamma}'' : \tilde{\gamma}'' \in [\tilde{\gamma}]_s, \tilde{\gamma}'' \sim_d \tilde{\gamma}'\}$ and $\{\tilde{\gamma}'' : \tilde{\gamma}'' \in [\tilde{\gamma}]_s, \tilde{\gamma}'' \sim_d \tilde{\gamma}'^{-1}\}$.

Definition 1.9.5. If $[\tilde{\gamma}]_s$ is a string of Σ , the curve corresponding to the string $[\tilde{\gamma}]_s$ is the image $\tilde{\gamma}'([0, 1])$ of any path $\tilde{\gamma}'$ of $[\tilde{\gamma}]_s$ (clearly, if $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are any paths of the string $[\tilde{\gamma}]_s$, then $\tilde{\gamma}_0([0, 1]) = \tilde{\gamma}_1([0, 1])$). The set of curves in Σ is denoted by Θ_Σ .

Definition 1.9.6. An I-string is a string that has at least one simple path.

Definition 1.9.7. A segment is a curve γ that corresponds to at least one I-string.

Definition 1.9.8. Two strings $[\tilde{\gamma}_0]_s$ and $[\tilde{\gamma}_1]_s$ are homotopic if they have two respective paths that are path-homotopic. We use the notation $[\tilde{\gamma}_0]_s \sim [\tilde{\gamma}_1]_s$ to indicate that $[\tilde{\gamma}_0]_s$ and $[\tilde{\gamma}_1]_s$ are homotopic.

Proposition 1.9.4. $\sim$ is an equivalence relation over $\vec{\Theta}_\Sigma$.

Proof. The proof is similar to the proof of Proposition 1.9.3 $\square$

Proposition 1.9.5. Let $\tilde{\gamma}_0, \tilde{\gamma}_0''$ and $\tilde{\gamma}_1, \tilde{\gamma}_1''$ be paths in Σ such that $\tilde{\gamma}_0 \sim_d \tilde{\gamma}_0''$ and $\tilde{\gamma}_1 \sim_d \tilde{\gamma}_1''$. Suppose that there is a point $x_1 \in \Sigma$ such that $\tilde{\gamma}_0'(1) = \tilde{\gamma}_0''(1) = x_1$ and $\tilde{\gamma}_1'(0) = \tilde{\gamma}_1''(0) = x_1$. Then $\tilde{\gamma}_0' \circ \tilde{\gamma}_0'' \sim_d \tilde{\gamma}_1' \circ \tilde{\gamma}_1''$, where $\circ$ is the composition of paths at x_1 .

Proof. The paths $\tilde{\gamma}_0 = \tilde{\gamma}_0' \circ \tilde{\gamma}_0''$ and $\tilde{\gamma}_1 = \tilde{\gamma}_1' \circ \tilde{\gamma}_1''$ clearly satisfy the conditions of Definition 1.9.1. $\square$

Definition 1.9.9. Let $[\tilde{\gamma}_0]_d$ and $[\tilde{\gamma}_1]_d$ be two direction classes of Σ . Suppose that the terminus x_1 of $[\tilde{\gamma}_0]_d$ is the origin of $[\tilde{\gamma}_1]_d$. The sum of $[\tilde{\gamma}_0]_d$ and $[\tilde{\gamma}_1]_d$ at x_1 , denoted $[\tilde{\gamma}_0]_d \circ [\tilde{\gamma}_1]_d$, is the direction class of $\tilde{\gamma}_0' \circ \tilde{\gamma}_1'$ with respect to the base point x_1 , where $\tilde{\gamma}_0' \in [\tilde{\gamma}_0]_d$, $\tilde{\gamma}_1' \in [\tilde{\gamma}_1]_d$.

Remark. By Proposition 1.9.5, the sum of two direction classes at a point x_1 is well-defined.

We now define a sum over the set of strings using the former concept.

Definition 1.9.10. Let $[\tilde{\gamma}_0]_s$ and $[\tilde{\gamma}_1]_s$ be two strings in Σ . Let $[\tilde{\gamma}_0]_d$ and $[\tilde{\gamma}_1]_d$ be two direction classes of $[\tilde{\gamma}_0]_s$ and $[\tilde{\gamma}_1]_s$, respectively. Suppose that the terminus x_1 of $[\tilde{\gamma}_0]_d$ is the origin of $[\tilde{\gamma}_1]_d$. The sum of $[\tilde{\gamma}_0]_s$ and $[\tilde{\gamma}_1]_s$ at x_1 with respect to $[\tilde{\gamma}_0]_d$ and $[\tilde{\gamma}_1]_d$, denoted $[\tilde{\gamma}_0]_s \circ [\tilde{\gamma}_1]_s$, is the string of $[\tilde{\gamma}_0]_d \circ [\tilde{\gamma}_1]_d$.

Definition 1.9.11. Let $[\tilde{\sigma}]_s$ be a string and $[\tilde{\gamma}]_l$ be a loop, and suppose that σ and γ have a common point x_1 . Let $[\tilde{\sigma}]_d$ be a direction class of $[\tilde{\sigma}]_s$, and let $[\tilde{\sigma}]_d = [\tilde{\sigma}']_d \circ [\tilde{\sigma}'']_d$ at x_1 for some paths $\tilde{\sigma}' \in [\tilde{\sigma}']_d$, $\tilde{\sigma}'' \in [\tilde{\sigma}'']_d$ with $\tilde{\sigma}'(1) = \tilde{\sigma}''(0) = x_1$. Let $\tilde{\gamma}'$ be a path of $[\tilde{\gamma}]_l$ such that $\tilde{\gamma}'(0) = \tilde{\gamma}'(1) = x_1$. Then the sum of $[\tilde{\sigma}]_s$ and $[\tilde{\gamma}]_l$ at x_1 with respect to $\tilde{\sigma}'$ and $\tilde{\gamma}'$ is the string of the path $\tilde{\sigma}' \circ \tilde{\gamma}' \circ \tilde{\sigma}''$.

Proposition 1.9.6. Let $(C_0, \mathcal{E}_0)$ be the fundamental partition of $\overleftarrow{\Gamma}_\Sigma$. Let $[\sigma]_s$ be a string and $[\gamma]_l$ a loop in C_0 , and suppose that σ and γ have a common point x_1 . Let $[\sigma]_d$ be a direction class of $[\sigma]_s$ such that $[\sigma]_d = [\sigma']_d \circ [\sigma'']_d$ at x_1 for some paths $\sigma' \in [\sigma']_d$, $\sigma'' \in [\sigma'']_d$ with $\sigma'(1) = \sigma''(0) = x_1$. Let $\tilde{\gamma}'$ be a path of $[\gamma]_l$ such that $\tilde{\gamma}'(0) = \tilde{\gamma}'(1) = x_1$. Then $[\sigma]_s \circ [\gamma]_l$ at x_1 with respect to σ' and $\tilde{\gamma}'$ is homotopic to $[\sigma]_s$.

Proof. This comes from the fact that, since $\tilde{\gamma}' \sim [e]$, $\sigma' \circ \tilde{\gamma}' \circ \sigma'' \sim \sigma' \circ \sigma''$. $\square$

Definition 1.9.12. Let $(C, \mathcal{E})$ be a complete partition of $\overleftarrow{\Gamma}_\Sigma$. Let $n \in \mathbb{Z}$, and let $[\sigma]_s, [\sigma']_s$ be two strings of Σ that have the same endpoints. Then $[\sigma]_s$ is homotopic modulo C to $[\sigma']_s$ if there are loops $[\tilde{\gamma}_1]_l, [\tilde{\gamma}_2]_l, \dots, [\tilde{\gamma}_k]_l \in C$, closed paths $\tilde{\gamma}'_1, \tilde{\gamma}'_2, \dots, \tilde{\gamma}'_k$, paths $\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_{k+1}$ and points $x_1, \dots, x_k$ such that, for $i = 1, \dots, k$;

- (1) $\tilde{\gamma}'_i$ is a closed path of $[\tilde{\gamma}_i]_l$;
- (2) x_i is the basepoint of $\tilde{\gamma}'_i$ and a point of σ_i ;
- (3) x_k is a point of σ_{k+1} ;
- (4) $\tilde{\sigma}_1$ is a path of $[\sigma]_s$;
- (5) $\tilde{\sigma}_{i+1}$ is a path of $[\sigma]_s \circ [\tilde{\gamma}_i]_l$ with respect to $\tilde{\sigma}_i$ and $\tilde{\gamma}'_i$ at x_i ; and
- (6) $[\tilde{\sigma}_{k+1}]_s \sim [\sigma']_s$.

We use the notation $[\sigma]_s \sim_C [\sigma']_s$ to indicate that $[\sigma]_s$ and $[\sigma']_s$ are freely homotopic modulo C .

Definition 1.9.13. Let $(C, \mathcal{E})$ be a complete partition of $\overleftarrow{\Gamma}_\Sigma$. Let σ_0 and σ_1 be two segments in Σ , with corresponding strings $[\sigma_0]_s$ and $[\sigma_1]_s$, and suppose that these two strings have the same endpoints. Then σ_0 and σ_1 are homotopic modulo C if $[\sigma_0]_s \sim_C [\sigma_1]_s$. We use the notation $\sigma_0 \sim_C \sigma_1$ to indicate that σ_0 and σ_1 are homotopic modulo C .

Proposition 1.9.7. Let $(C_0, \mathcal{E}_0)$ be the fundamental partition of $\overleftarrow{\Gamma}_\Sigma$. Let $[\sigma]_s, [\sigma']_s$ be two strings of Σ with the same endpoints. Then $[\sigma]_s \sim [\sigma']_s$ iff $[\sigma]_s \sim_{C_0} [\sigma']_s$.

Proof. The proof is similar to the proof of Proposition 1.7.6. $\square$

CHAPTER II

C-PARALLELS

2.1 Introduction.

The notions developed in Chapter I will now be used to describe some aspects of the structure of embedded graphs.

Theorems about the structure of embedded graphs are of great interest, in particular those about the number and homotopy type of disjoint polygons. Schrijver [Sch1] showed that an embedding of a graph in the torus with representativity ρ has at least $\lfloor \frac{3}{4}\rho \rfloor$ pairwise disjoint noncontractible polygons. Zha and Zhao [ZZ] proved that there are at least two such polygons for an embedding of representativity at least 5 in a surface. Schrijver characterizes embeddings with disjoint polygons $P_1, \dots, P_k$ having prescribed homotopy types [Sc2]. His result implies a version of the result of Mohar, Richter and the author [BMR] that there are at least $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ disjoint freely homotopic and noncontractible polygons in an embedded graph G_Ψ . Again, this was proved differently by Richter and Vitray [RV2]. In this chapter, a new proof of this result is given among other results.

We will be interested, in the present chapter, in disjoint polygons of an embedded graph which are freely homotopic in a more general way (free homotopy modulo $\mathcal{C}$). In this generality, this has never been done before, and has led to new results that we will now expose. In particular, we will show that, given any complete partition $(\mathcal{C}, \mathcal{E})$, G_Ψ has a set of $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ disjoint freely homotopic and noncontractible polygons that belong to $\Omega_{\mathcal{E}}$.

2.2 A theorem on $\mathcal{C}$ -parallels.

Definition 2.2.1. Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a surface Σ . Let $(\mathcal{C}, \mathcal{E})$ be a complete partition of $\overleftarrow{\Gamma}_\Sigma$. A set of $\mathcal{C}$ -parallels is a set of pairwise disjoint polygons of G_Ψ , all freely homotopic modulo $\mathcal{C}$.

Definition 2.2.2. Let $\mathcal{A}$ be a set of loops of Σ . Then, for $n \in \mathbb{Z}$, we define

$$\mathcal{A}^n = \{[\tilde{\gamma}]_l \in \overleftarrow{\Gamma}_\Sigma : [\tilde{\gamma}]_l \sim_f^n [\tilde{\gamma}']_l \text{ for some } [\tilde{\gamma}']_l \in \mathcal{A}\}.$$

The following theorem and its proof are the remainder of this section.

Theorem 2.2.2. Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a 3-connected graph G onto a nontrivial surface Σ . Suppose that $\rho(\Psi) \geq 3$. Let $(\mathcal{C}_1, \mathcal{E}_1)$ and $(\mathcal{C}_2, \mathcal{E}_2)$ be two complete partitions of $\overleftarrow{\Gamma}_\Sigma$ such that $(\mathcal{C}_1, \mathcal{E}_1)$ is a crossing partition and $\mathcal{E}_2 \subseteq \mathcal{E}_1$. Let γ be a face-represented O-circuit modulo $\mathcal{C}_1$ in $\Omega_{\mathcal{E}_2}$ of length $\ell_\gamma^{\mathcal{C}_1}(\Psi)$ in G_Ψ .

- (1) If γ is an orientation-preserving O-circuit, then G_Ψ has a set of $\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - \ell_\gamma^{\mathcal{C}_1}(\Psi) - 1}{2} \rfloor$ noncontractible $\mathcal{C}_1$ -parallels, all freely homotopic modulo $\mathcal{C}_1$ to γ , and belonging to $\Omega_{\mathcal{E}_2}$.
- (2) If γ is an orientation-reversing O-circuit, then G_Ψ has a set of $\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - \ell_\gamma^{\mathcal{C}_1}(\Psi) - 1}{4} \rfloor$ $\mathcal{C}_1$ -parallels, all 2-freely homotopic modulo $\mathcal{C}_1$ to γ , and belonging to $\Omega_{\mathcal{E}_2}$.

Proof. Since γ is face-represented modulo $\mathcal{C}_1$ of length $\ell_\gamma^{\mathcal{C}_1}(\Psi)$, there is a face cycle $\Lambda = v_0, f_1, v_1, f_2, v_2, \dots, f_l, v_l$ that face-represents an O-circuit $\alpha \sim_{\mathcal{C}_1} \gamma$, with $l = \ell_\gamma^{\mathcal{C}_1}(\Psi)$. Without loss of generality, we can assume that $\alpha \subseteq \bigcup_{i=1}^l \bar{f}_i$ with $\{v_1, \dots, v_l\} = \alpha \cap \bigcup_{i=1}^l \text{bd}(f_i)$.

(1) Suppose that γ is orientation-preserving.

Let $m = \lfloor \frac{\rho_{\ell_1}(\Psi) + \rho_{\ell_2}(\Psi) - \ell_{\gamma^1}^{\ell_1}(\Psi) - 1}{2} \rfloor$, and assume that $m \geq 1$. We will construct a set $\{C_1, C_2, \dots, C_m\}$ of C_1 -parallels, all freely homotopic modulo C_1 to α . We begin by constructing C_1 and C_2 , and we will then construct the other ones by induction. Since γ is orientation-preserving and α is freely homotopic modulo C_1 to γ , it follows from Proposition 1.7.12 and Proposition 1.7.5 that α is orientation-preserving. Thus, the face cycle A determines a right-hand side and a left-hand side for α . The C_i with odd indices will be constructed on the right-hand side of α , while the C_i of even indices will be on the left-hand side of α . The proof is actually divided into the following steps:

I. Construction of C_1 .

A) Construction of a subgraph J_1 of G_Ψ .

B) $C_1 = J_1$ is a polygon such that $C_1 \sim_{C_1} \gamma$.

II. Construction of C_2 .

A) Construction of a subgraph J_2 of G_Ψ .

B) J_2 contains a polygon C_2 such that $C_2 \sim_{C_1} \gamma$.

C) $V(C_1) \cap V(C_2) = \emptyset$.

III. Construction of C_t .

A) Construction of a subgraph J_t of G_Ψ .

B) J_t contains a polygon C_t such that $C_t \sim_{C_1} \gamma$.

C) $V(C_t) \cap V(C_{t-2}) = \emptyset$.

D) $V(C_t) \cap V(C_j) = \emptyset$, for $j = 1, \dots, t-3$.

E) $V(C_t) \cap V(C_{t-1}) = \emptyset$.

I) Construction of C_1 .

A) Construction of a subgraph J_1 of G_Ψ .

Since G is 3-connected and $\rho(\Psi) \geq 3$, Ψ is a wheel-neighbourhood embedding by Proposition 1.8.6. For $i = 1, \dots, l$, let $\tilde{\alpha}_i$ be a traversal of $\tilde{f}_i \cap \alpha$, with $\tilde{\alpha}_i(0) = v_{i-1}$ and $\tilde{\alpha}_i(1) = v_i$. For $i = 1, \dots, l$, let $f_{i,R}$ be the t-component of $f_i \setminus \alpha_i$ on the right-hand side of α , and let $f_{i,L}$ be the t-component of $f_i \setminus \alpha_i$ on the left-hand side of α . By Corollary 1.8.7, we can choose the v_i so that $\tilde{f}_{i,R} \cap \tilde{f}_{i+1,R}$ contains no other vertex than v_i , for $i = 1, \dots, l$.

For $i = 1, \dots, l$, let N_i be the wheel-neighbourhood of v_i in G_Ψ . Notice that $N_i \setminus \alpha$ is formed of two t-components. Let $N_{i,L}$ be the graph of $(N_i \setminus \alpha) \cup \{v_{i-1}, v_i, v_{i+1}\}$ on the left-hand side of α , for $i = 1, \dots, l$.

Since Ψ is a wheel-neighbourhood embedding, the closure of each face of G_Ψ is homeomorphic to a closed disc. Therefore, the boundary of each face of G_Ψ is a polygon.

We can assume, without loss of generality, that $\tilde{\alpha} = \tilde{\alpha}_1 \circ \tilde{\alpha}_2 \circ \dots \circ \tilde{\alpha}_l$. Let $\tilde{\eta}_i$ be a path that is a traversal of $(\partial(f_{i,R}) \setminus \tilde{\alpha}_i) \cap [0, 1]$ with $\tilde{\eta}_i(0) = v_{i-1}$ and $\tilde{\eta}_i(1) = v_i$, for $i = 1, \dots, l$. Let $\tilde{\sigma}_1 = \tilde{\eta}_1 \circ \tilde{\eta}_2 \circ \dots \circ \tilde{\eta}_l$. Since $\tilde{\alpha}_i$ and $\tilde{\eta}_i$ together bound $f_{i,R}$ for $i = 1, \dots, l$, we have that $\tilde{\alpha} \sim_f \tilde{\sigma}_1$. Let J_1 be the skeleton of the circuit σ_1 .

B) $C_1 = J_1$ is a polygon such that $C_1 \sim_{C_1} \gamma$.

Suppose, on the contrary, that J_1 contains a vertex v of degree greater than 2 in J_1 . Let i be the smallest index such that v is a vertex of $\tilde{f}_i \setminus \{v_i\}$, and let j be the largest integer such that v is a vertex of $\tilde{f}_j \setminus \{v_{j-1}\}$. By Corollary 1.8.7, i and j cannot be consecutive in the cyclic ordering of $(1, \dots, l)$.

Let $\Lambda' = v$. By Proposition 1.7.9, the closed face chain formed by Λ' and the shortest part of Λ between the two occurrences of v in f_i and f_j , respectively, face-represents an O-circuit that belongs to Ω_{C_1} . Without loss of generality, suppose

that $f_i, v_i, f_{i+1}, \dots, f_j$ is the shortest part of Λ between the two occurrences of v in f_i and f_j . Let β' be an O-circuit through this closed face chain. Then β' is an element of Ω_{C_1} .

Let β'' be an O-circuit through v, f_j (if $v \neq v_j$), $v_j, f_{j+1}, v_{j+1}, \dots, v_{i-1}, f_i$ (if $v \neq v_{i-1}$), v . Then $\beta'' \sim_{C_1} \alpha$, and $|\beta'' \cap V_\Phi| < |\alpha \cap V_\Phi|$, which is a contradiction with the hypothesis on the minimality of the length of α .

Therefore, all the vertices of σ_1 , and therefore all the points of σ_1 have multiplicity 2. Hence, σ_1 is an O-circuit. Therefore, the subgraph C_1 , whose circuit is σ_1 , is a polygon. Since $\vec{\sigma}_1 \sim_f \vec{\alpha}$, it follows that C_1 is freely homotopic as an O-circuit to α , and, therefore, $C_1 \sim_{C_1} \alpha$.

II) Construction of C_2 .

A) Construction of a subgraph J_2 of G_Φ .

Now, assuming that $m \geq 2$, we construct C_2 . The subgraphs $N_{i,L} - \{v_{i-1}, v_i, v_{i+1}\}$ are nonnull graphs, since otherwise there would be a shortest face chain face-representing γ modulo C_1 , a contradiction with the hypothesis. We orient the arcs $N_{i,L} - \{v_i\}$ in a direction that goes from v_{i-1} to v_{i+1} , for $i = 1, \dots, l$. Respecting this ordering, let x_i be the first vertex of $N_{i,L} - \{v_{i-1}, v_i, v_{i+1}\}$ that is a neighbour of v_i , and let y_i be the last vertex of $N_{i,L} - \{v_{i-1}, v_i, v_{i+1}\}$. Notice that $x_i = y_{i-1}$, where the indices are modulo l , for $i = 1, \dots, l$. Let $\vec{\eta}_i$ be a simple path that is a traversal of the part of $N_{i,L} - \{v_{i-1}, v_i, v_{i+1}\}$ from $x_i = \vec{\eta}_i(0)$ to $y_i = \vec{\eta}_i(1)$, for $i = 1, \dots, l$. Let $\vec{\sigma}_2 = \vec{\eta}_1 \circ \vec{\eta}_2 \circ \dots \circ \vec{\eta}_l$. We have that $\vec{\sigma}_2$ and $\vec{\alpha}$ are freely homotopic through the faces of $N_{i,L}$, for $i = 1, \dots, l$. Let J_2 be the subgraph of G_Φ that is the skeleton of σ_2 .

B) J_2 contains a polygon C_2 such that $C_2 \sim_{C_1} \gamma$.

We will "extract" from J_2 a polygon C_2 that is freely homotopic modulo C_1 as an O-circuit to α . Suppose that J_2 contains a vertex v of degree greater than 2 in J_2 . Let s_1 and s_2 be respectively the smallest and largest numbers in $]0, 1]$ such that $\bar{\sigma}_2(s_1) = \bar{\sigma}_2(s_2) = v$. Let $\bar{\sigma}'_2$ be the path induced by $\bar{\sigma}_2|_{[s_1, s_2]}$, and let $\bar{\sigma}''_2$ be the path induced by $\bar{\sigma}_2|_{[s_2, 1] \cup [0, s_1]}$. Then $[\bar{\sigma}_2]_l$ is the sum of $[\bar{\sigma}'_2]_l$ and $[\bar{\sigma}''_2]_l$ at v with respect to the closed paths $\bar{\sigma}'_2$ and $\bar{\sigma}''_2$.

Now let i be the smallest index such that $\bar{\sigma}_2(s_1) \in \eta_i$, and let j be the greatest index such that $\bar{\sigma}_2(s_2) \in \eta_j$. Let f be a face of $(F(N_{i,L}) \setminus \{f_{i,L}, f_{i+1,L}\}) \cup \{f_i, f_{i+1}\}$ such that $v \in V(\text{bd}(f))$ and such that $\text{bd}(f)$ shares at least one edge with η_i . Let f' be a face of $(F(N_{j,R}) \setminus \{f_{j,R}, f_{j+1,R}\}) \cup \{f_j, f_{j+1}\}$ such that $v \in V(\text{bd}(f'))$ and such that $\text{bd}(f')$ shares at least one edge with η_j .

Let Λ' be the face path whose traversal is given by v_i, f, v, f', v_j . By Proposition 1.7.9, the closed face chain formed by Λ' and the shortest part of Λ between the ends of Λ' face-represents an O-circuit that belongs to Ω_{C_1} . Without loss of generality, suppose that the face chain $v_i, f_{i+1}, v_{i+1}, \dots, v_{j-1}, f_j, v_j$ is the shortest part of Λ between v_i and v_j .

Let β' be an O-circuit such that $\beta' \subset \bar{f} \cup \bar{f}' \cup (\bigcup_{k=i+1}^j \bar{f}_k)$ and $\beta' \cap (\text{bd}(f) \cup \text{bd}(f') \cup (\bigcup_{k=i+1}^j \text{bd}(f_k))) \subseteq \{v, v_i, v_{i+1}, \dots, v_{j-1}, v_j\}$ (notice that the indices run in the cyclic ordering $(1, \dots, l)$). Then β' is an element of C_1 . Let β'' be an O-circuit such that $\beta'' \subset \bar{f} \cup \bar{f}' \cup (\bigcup_{k=j+1}^i \bar{f}_k)$ and $\beta'' \cap (\text{bd}(f) \cup \text{bd}(f') \cup (\bigcup_{k=j+1}^i \text{bd}(f_k))) \subseteq \{v, v_j, v_{j+1}, \dots, v_{i-1}, v_i\}$. Then, by the TPP, β'' is an element of $\mathcal{E}_2$. Also, $\beta'' \sim_{C_1} \alpha$ (see Figure XII).

Let $\tilde{\beta}'$ and $\tilde{\beta}''$ be simple closed paths of β' and β'' , respectively. Now, since $\bar{\sigma}_2$ is constructed from the boundaries of the faces $N_{i,L}$, we have that each one of $\bar{\sigma}'_2$, $\bar{\sigma}''_2$ is freely homotopic to one of $\tilde{\beta}', \tilde{\beta}''$.

Figure XII

Without loss of generality, suppose that $\vec{\sigma}'_2 \sim_f \vec{\beta}'$. Then $\vec{\sigma}''_2 \sim_f \vec{\beta}''$, and so $\vec{\sigma}'_2 \in \vec{\Omega}_{C_1}$ and $\vec{\sigma}''_2 \in \vec{\Omega}_{E_2}$. We discard $\vec{\sigma}'_2$, and we repeat the whole process with $\vec{\sigma}''_2$ instead of $\vec{\sigma}_2$: we scan for some vertex of degree greater than 2 in the skeleton of σ''_2 , and we eliminate some part of it that belongs to Ω_{C_1} .

We end up with a unique sub-O-circuit $\sigma''_2 \in \Omega_{E_2}$ of σ_2 . Let C_2 be the skeleton of σ''_2 . From the former process, $[\vec{\sigma}_2]_l$ is the sum of $[\vec{\sigma}''_2]_l$ with some loops of C_1 (the ones we have discarded). It follows that $[\vec{\sigma}''_2]_l \sim_{C_1} [\vec{\sigma}_2]_l$, and so C_2 is freely homotopic modulo C_1 as an O-circuit to α .

C) $V(C_2) \cap V(C_1) = \emptyset$.

Suppose that there is a vertex $v \in V(C_2) \cap V(C_1)$. Then there exists i, j such that $v \in V(N_i) \cap V(N_j)$, and a face f of $(F(N_{i,L}) \setminus \{f_{i,L}, f_{i+1,L}\}) \cup \{f_i, f_{i+1}\}$ incident with v_i and v . Let $\vec{\sigma}_2$ be a path between v_i and v_j through v such that $\vec{\sigma}_2 \subseteq f \cup f_j \cup \{v_i, v, v_j\}$ and $\sigma_2 \cap \alpha \subseteq \{v_i, v_j\}$. Now $\alpha \setminus \{v_i, v_j\}$ is composed of two segments β_0 and β_1 both having v_i and v_j as endpoints. Without loss of generality, assume that $\beta_0 \subseteq \bigcup_{r=i}^j \bar{f}_r$ and that $\beta_1 \subseteq \bigcup_{r=j}^i \bar{f}_r$. Suppose also that $|\beta_0 \cap V_\Psi| \leq |\beta_1 \cap V_\Psi|$. Then, applying Proposition 1.7.9 with $t = 1$, we establish that $\beta_0 \cup \sigma_2$ is an element of Ω_{C_1} . We can move homotopically $\beta_0 \cup \sigma_2$ such that the new O-circuit obtained intersects α only once. Since $\beta_0 \cup \sigma_2 \in \Omega_{C_1}$ and (C_1, E_1) is a crossing partition, we have a contradiction. Therefore, $C_2 \cap C_1 = \emptyset$.

III. Construction of C_t .

A) Construction of a subgraph J_t of G_Ψ .

We now describe the other polygons C_t , for $\lfloor \frac{\rho_{E_1}(\Psi) + \rho_{E_2}(\Psi) - \ell(\gamma) - 1}{2} \rfloor \geq t \geq 3$. We proceed by induction. Suppose that we have constructed $t - 1$ noncontractible polygons $C_1, \dots, C_{t-1}$, all freely homotopic modulo C_1 to α , for $t \geq 3$. Let

$w_0, e_1, w_1, e_2, w_2, \dots, e_k, w_k$ be a traversal of C_{t-2} (where $w_0 = w_k$) with the same orientation as $\vec{\alpha}$.

Let N_i be the wheel neighbourhood of w_i , for $i = 1, \dots, k$. Now $N_i \setminus C_{t-2}$ is formed of two t-components. Since γ is orientation-preserving (by Proposition 1.7.12 and Proposition 1.7.5), we can define a right-hand side and a left-hand side to C_{t-2} . Let $N_{i,L}$ (respectively $N_{i,R}$) be the component of $N_i \setminus C_{t-2}$ on the left-hand side (respectively the right-hand side) of C_{t-2} , including the vertices w_{i-1}, w_i, w_{i+1} and the edges e_i, e_{i+1} . Let $N_L = \bigcup_{i=1}^k N_{i,L}$, and $N_R = \bigcup_{i=1}^k N_{i,R}$.

The reader will notice that the polygons C_j we construct are subgraphs of the union of the wheel-neighbourhoods of the vertices of the preceding polygon C_{j-2} . Therefore, given C_j with $j \leq t-1$ and $v \in V(C_j)$, there is a face incident to v and a vertex v' of C_{j-2} . Repeat the same process with v' and C_{j-4} , and continue in this way until we reach one of the vertices of $X = \{v_1, \dots, v_l\} \cup (\bigcup_{i=1}^l V(\text{bd}(f_i)))$. We thus obtain a short face path A' from v to a vertex of X with inner vertices in $V(C_{j-2}), V(C_{j-4}), \dots, V(C_r)$, where r is 1 if j is odd and 2 if j is even. Furthermore, A' has length at most $\lfloor \frac{j}{2} \rfloor$.

Next, we construct C_t . Contrary to the case C_2 , it might happen that some subgraphs $N_{i,L} - \{w_{i-1}, w_i, w_{i+1}\}$ or $N_{i,R} - \{w_{i-1}, w_i, w_{i+1}\}$ are null (but not all of those on the right-hand side or on the left-hand side, since then we would have $\rho(\Psi) = 0$). In that case, we only keep those that are nonnull, and the rest of the argument still holds. So, without loss of generality and for the sake of notation, we assume that $N_{i,L} - \{w_{i-1}, w_i, w_{i+1}\}$ and $N_{i,R} - \{w_{i-1}, w_i, w_{i+1}\}$ are nonnull graphs for $i = 1, \dots, k$. We orient the arcs $N_{i,L} - \{w_i\}$ and $N_{i,R} - \{w_i\}$ in a direction that goes from w_{i-1} to w_{i+1} , for $i = 1, \dots, k$. Respecting this ordering, let x_i be the first vertex of $N_{i,L} - \{w_{i-1}, w_i, w_{i+1}\}$ that is a neighbour of w_i , and let y_i be the last vertex of $N_{i,L} - \{w_{i-1}, w_i, w_{i+1}\}$ if t is even. If t is odd, let x_i

be the first vertex of $N_{i,R} - \{w_{i-1}, w_i, w_{i+1}\}$ that is a neighbour of w_i , and let y_i be the last vertex of $N_{i,R} - \{w_{i-1}, w_i, w_{i+1}\}$.

If t is even, let $\tilde{\eta}_i$ be a simple path that is a traversal of the part of $N_{i,L} - \{w_{i-1}, w_i, w_{i+1}\}$ from $x_i = \tilde{\eta}_i(0)$ to $y_i = \tilde{\eta}_i(1)$, for $i = 1, \dots, k$. Notice that $x_i = y_{i-1}$, where the indices are modulo k , for $i = 1, \dots, k$. If t is odd, let $\tilde{\eta}_i$ be a path that is a traversal of the part of $N_{i,R} - \{w_{i-1}, w_i, w_{i+1}\}$ from $x_i = \tilde{\eta}_i(0)$ to $y_i = \tilde{\eta}_i(1)$, for $i = 1, \dots, k$. This allows us to properly define $\tilde{\sigma}_t = \tilde{\eta}_1 \circ \tilde{\eta}_2 \circ \dots \circ \tilde{\eta}_k$. We have that $\tilde{\sigma}_t$ and $\tilde{\sigma}_{t-2}$ are freely homotopic through the faces of N_L if t is even, and N_R if t is odd.

Let J_t be the subgraph of G_Ψ that is the skeleton of σ_t . As in the case of J_2 , we will extract a polygon C_t from J_t by removing subcircuits that belong to Ω_{C_1} , and it will result that C_t , as an O-circuit, is freely homotopic modulo C_1 to α .

B) J_t contains a polygon C_t such that $C_t \sim_{C_1} \gamma$.

Suppose that J_t contains a vertex v of degree greater than 2 in J_t . Let s_1 and s_2 be, respectively, the smallest and largest numbers in $[0, 1]$ such that $\tilde{\sigma}_t(s_1) = \tilde{\sigma}_t(s_2) = v$. Let $\tilde{\sigma}'_t$ be the path induced by $\tilde{\sigma}_t|_{[s_1, s_2]}$, and let $\tilde{\sigma}''_t$ be the path induced by $\tilde{\sigma}_t|_{[s_2, 1] \cup [0, s_1]}$. Then $[\tilde{\sigma}_t]_l$ is the sum of $[\tilde{\sigma}'_t]_l$ and $[\tilde{\sigma}''_t]_l$ at v with respect to closed paths $\tilde{\sigma}'_t$ and $\tilde{\sigma}''_t$, respectively. Let i be the smallest index such that $\tilde{\sigma}_t(s_1) \in \eta_i$, and let j be the largest index such that $\tilde{\sigma}_t(s_2) \in \eta_j$. Let f be a face of $N_{i,L}$ such that $v \in V(\text{bd}(f))$, and such that $\text{bd}(f)$ shares at least one edge with η_i . Correspondingly, let f' be a face of $N_{j,R}$ such that $v \in V(\text{bd}(f'))$, and such that $\text{bd}(f')$ shares at least one edge with η_j .

As we have seen before, there are two face paths Λ_0 and Λ_1 , through f and w_i , and f' and w_j , respectively. Both face paths link v to a vertex of X and have their inner vertices in $V(C_{t-2}), V(C_{t-4}), \dots, V(C_p)$, where $p = 1$ if t is odd and $p = 2$ if t is even. Each of Λ_0 and Λ_1 has length at most $\lfloor \frac{t}{2} \rfloor$. Let $\Lambda_0 = z_0, g_1, z_1, \dots, g_r, z_r$

where $z_0 = v$, $g_1 = f$, $z_1 = w_i$ and $z_r \in \bigcup_{n=1}^t \bar{f}_n$. Let $\Lambda_1 = z'_0, g'_1, z'_1, \dots, g'_s, z'_s$ where $z'_0 = v$, $g'_1 = f'$, $z'_1 = v_j$ and $z'_s \in \bigcup_{n=1}^t \bar{f}_n$ (see Figure XIII).

Suppose first that there is no common inner vertex u to both face paths Λ_0 and Λ_1 . Let $\Lambda' = z_r, g_r, z_{r-1}, \dots, z_1, g_1, z_0, g'_1, z'_1, \dots, g'_s, z'_s$. By Proposition 1.7.9, the closed face chain formed by Λ' and the shortest part of Λ between z_r and z_s face-represents an O-circuit that belongs to Ω_{C_1} . Suppose that $z_r \in V(\bar{f}_{i'} \setminus \{v_{i'}\})$, and that $z_s \in V(\bar{f}_{j'} \setminus \{v_{j'}\})$. Suppose, without loss of generality, that $\Lambda'' = z_r, f_{i'}, v_{i'}, f_{i'+1}, \dots, f_{j'}, z_s$ is the shortest part of Λ between z_r and z_s .

Our objective is to prove that $[\bar{\sigma}'_t]_t$ is an element of C_1 . For this, we will need some "intermediate" loops.

Let $\bar{\beta}$ be a path from v through Λ_0 , Λ'' , Λ_1 and back to v , such that $\beta \subset (\bigcup_{n=1}^r \bar{g}_n) \cup (\bigcup_{n=1}^s \bar{g}'_n) \cup (\bigcup_{n=i'}^{j'} \bar{f}_n)$ and such that $\beta \cap G_\Psi(\Sigma) = \{z_1, z_2, \dots, z_r, z'_1, z'_2, \dots, z'_s, v_{i'+1}, v_{i'+2}, \dots, v_{j'-1}\}$. Therefore, $[\bar{\beta}]_t$ is an element of C_1 . Let $\bar{\beta}_n$ be the path from v which coincides with $\bar{\beta}$ down to the vertex $z_{\frac{t-n}{2}} \in V(C_n)$, that then traverses the part of C_n parallel to Λ'' to $z'_{\frac{t-n}{2}}$ and then coincides with $\bar{\beta}$ back to v in Λ_1 . Without loss of generality, we can assume that $\bar{\beta}_n(\frac{1}{4}) = z_{\frac{t-n}{2}}$, and $\bar{\beta}_n(\frac{3}{4}) = z'_{\frac{t-n}{2}}$.

Now let $3 \leq n \leq t-1$ such that $n \equiv t \pmod{2}$. Let $t_1, t_2 \in [0, 1]$ such that $\bar{\beta}_{n-2}(t_1) = z_{\frac{t-n}{2}}$ and $\bar{\beta}_{n-2}(t_2) = z'_{\frac{t-n}{2}}$. Let $\bar{\beta}'_n = \bar{\beta}_n|_{[\frac{1}{4}, \frac{3}{4}]}$ and $\beta'_{n-2} = \bar{\beta}_{n-2}|_{[t_1, t_2]}$. Let $\bar{\beta}''_n$ be a traversal of the part of σ_n between the endpoints of $\bar{\beta}'_n$ in the same direction as $\bar{\sigma}_n$ (see part III A)) with the same endpoints as $\bar{\beta}'_n$.

Since β''_n and β'_{n-2} bound faces of wheel-neighbourhoods of vertices of C_{n-2} , we have that $[\bar{\beta}''_n]_s \sim [\bar{\beta}'_{n-2}]_s$. We will be done if we can show that $[\bar{\beta}''_n]_s \sim_{C_1} [\bar{\beta}'_n]_s$. We have seen that the subloops of $[\bar{\sigma}_n]_t$ (besides the principal loop $[\bar{\sigma}''_n]_t$) are elements of C_1 . This is also the case for $[\bar{\beta}''_n]_t$, unless there is an interior point of $\beta''_n \setminus \{z_{\frac{t-n}{2}}, z'_{\frac{t-n}{2}}\}$ that is also a point of $C_n \setminus \beta'_n$. But then there would be an O-curve in Ω_{C_1} , attached to this point, that intersects β'_{n-2} only once, a contradiction with the fact that $(C_1, \mathcal{E}_1)$ is a crossing partition. Therefore, $[\bar{\beta}''_n]_s \sim_{C_1} [\bar{\beta}'_n]_s$, and by transitivity, we obtain that $[\bar{\beta}'_n]_s \sim_{C_1} [\bar{\beta}'_{n-2}]_s$. Therefore, $[\bar{\beta}_{n-2}]_t \sim_{C_1} \bar{\beta}_n$ for $3 \leq n \leq t-1$ such that $n \equiv t \pmod{2}$.

Figure XIII

Figure XIV

On the other hand, $[\vec{\sigma}_t']_t$ is freely homotopic modulo $\mathcal{C}_1$ to $[\vec{\beta}_{t-2}]_t$ through the faces of $N_{i,L}, N_{i+1,L}, \dots, N_{j,L}$. Since $[\vec{\beta}]_t \in \mathcal{C}_1$, we obtain by transitivity that $[\vec{\sigma}_t']_t \in \mathcal{C}_1$. This now implies, by the TPP, that $[\vec{\sigma}_t'']_t \in \mathcal{E}_2$.

Now suppose that there is a common inner vertex u to both face paths Λ_0 and Λ_1 (see Figure XIV). Let i', j' be the smallest indices such that $u = x_{i'} = x_{j'}$. Now, let $\Lambda' = x_0, g_1, x_1, \dots, g_{i'}, x_{i'}, g'_{j'}, x'_{j'-1}, \dots, g'_1, x'_0$. Then Λ' is a face cycle of length at most $2\lfloor \frac{t-1}{2} \rfloor \leq t-1$, and it face-represents an O-circuit. Suppose that Λ' face-represents an O-circuit of $\Omega_{\mathcal{E}_1}$. We have that

$$\rho_{\mathcal{E}_1}(\Psi) \leq t-1 = \left\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - \ell_{\gamma}^{\mathcal{C}_1}(\Psi) - 1}{2} \right\rfloor - 1.$$

Since $\rho_{\mathcal{E}_2}(\Psi) \leq \ell_{\gamma}^{\mathcal{C}_1}(\Psi)$, this last inequality implies that $\rho_{\mathcal{E}_1}(\Psi)$ is negative, a contradiction. Therefore, Λ' face-represents an O-circuit that belongs to $\Omega_{\mathcal{C}_1}$. Using an argument similar to the case where Λ_0 and Λ_1 share no common inner vertex, we can prove that this O-circuit has an O-loop that is freely homotopic modulo $\mathcal{C}_1$ to $[\vec{\sigma}_t']_t$.

We discard σ_t' , and we repeat the whole process with σ_t'' instead of σ_t : we scan for some vertex of degree greater than 2 in the skeleton of σ_t'' , and we eliminate some part of it that belongs to $\Omega_{\mathcal{C}_1}$.

We end up, as in the case of J_2 , with a unique subcircuit $\sigma_t'' \in \Omega_{\mathcal{E}_2}$ of σ_t . Let C_t be the skeleton of σ_t'' . From the former process, $[\vec{\sigma}_t]_t$ is the sum of $[\vec{\sigma}_t'']_t$ with some loops of $\mathcal{C}_1$ (the ones we have discarded). Therefore, $[\vec{\sigma}_t'']_t \sim_f [\vec{\sigma}_t]_t$ modulo $\mathcal{C}_1$, and so C_t is freely homotopic modulo $\mathcal{C}_1$, as an O-circuit, to α .

$$C) V(C_t) \cap V(C_{t-2}) = \emptyset.$$

Notice that since C_t is a subgraph of the symmetric difference of the boundary graphs of the faces of N_L if t is even or of N_R if t is odd, and this symmetric difference has σ_t in one component and the vertices $w_1, \dots, w_k$ in another component. Therefore, σ_t'' contains none of the vertices $w_1, \dots, w_k$, and so $C_t \cap C_{t-2} = \emptyset$.

D) $V(C_t) \cap V(C_j) = \emptyset$, for $j = 1, \dots, t-3$.

Let $j < t$ be the largest integer such that $V(C_t)$ and $V(C_j)$ share a common vertex. From C), $j \neq t-2$ and the case $j = t-1$ is given in E). Thus, suppose that $j \leq t-3$. Let $v \in V(C_t) \cap V(C_j)$. If $j \equiv t \pmod{2}$, then there are two face paths from each occurrence of v in C_t and C_j to Λ of length at most $\lfloor \frac{t-1}{2} \rfloor$. Using an argument similar to the one used in the proof of B), we can show that these face paths contribute to form a closed face chain that face-represents an O-circuit that belongs to Ω_{C_1} that crosses C_{t-2} only once, a contradiction with the fact that $(C_1, \mathcal{E}_1)$ is a crossing partition. If $j \not\equiv t \pmod{2}$, then an argument similar to the one used in E) (next paragraph) shows that there is a closed face chain that face-represents an O-circuit of Ω_{C_1} that crosses α only once, and this is again a contradiction. Therefore, $V(C_t) \cap V(C_j) = \emptyset$, for $j = 1, \dots, t-3$.

E) $V(C_t) \cap V(C_{t-1}) = \emptyset$.

Suppose, on the contrary, that there is a vertex $v \in V(C_t) \cap V(C_{t-1})$. Without loss of generality, suppose that t is even. Then there are two face paths from v to Λ : one, say $\Lambda_0 = z_0, g_1, z_1, \dots, g_r, z_r$, with $z_0 = v$, $z_1 \in V(C_{t-2})$, $z_2 \in V(C_{t-4})$, $\dots$, $z_r \in V(\text{bd}(f_s))$ for some i , and another face path $\Lambda_1 = z'_0, g'_1, z'_1, \dots, g'_s, z'_s$ where $z'_0 = v$, $z'_1 \in V(C_{t-1})$, $z'_2 \in V(C_{t-3})$, $\dots$, $z'_s \in V(C_1)$. The concatenation of these two face paths gives a face path Λ' of length at most $\lfloor \frac{t}{2} \rfloor + \lfloor \frac{t-1}{2} \rfloor = t-1$. By Proposition 1.7.9, the closed face cycle formed by Λ' and the shortest part of Λ between the ends of Λ' face-represents an O-circuit that belongs to Ω_{C_1} . However, it is easy to see that this face cycle also face-represents an O-circuit that crosses α transversely only once, a contradiction with the hypothesis that $(C_1, \mathcal{E}_1)$ is a crossing partition. It follows that $V(C_t) \cap V(C_{t-1}) = \emptyset$.

We thus have constructed a set $\{C_1, C_2, \dots, C_m\}$ of C_1 -parallels, all freely homotopic modulo C_1 to α , and belonging to $\Omega_{\mathcal{E}_2}$.

(2) Suppose now that γ is orientation-reversing. Let $m = \lfloor \frac{\rho_{\mathcal{C}_1}(\Psi) + \rho_{\mathcal{C}_2}(\Psi) - \ell_{\gamma}^{\mathcal{C}_1}(\Psi) - 1}{4} \rfloor$.

We will construct a set $\{C_1, C_2, \dots, C_m\}$ of disjoint polygons, all 2-freely homotopic modulo $\mathcal{C}_1$ to α . They will actually be constructed on each side of α : more precisely, each of these polygons will bound a nonorientable surface that contains α . We begin by constructing C_1 , and we will then construct the other ones by induction. The proof will be split into the following parts:

I. Construction of C_1 .

A) Construction of a subgraph J_1 of G_Ψ .

B) $C_1 = J_1$ is a polygon such that $C_1 \sim_{\mathcal{C}_1}^2 \gamma$.

II. Construction of C_t .

A) Construction of a subgraph J_t of G_Ψ .

B) J_t contains a polygon C_t such that $C_t \sim_{\mathcal{C}_1}^2 \gamma$.

C) $V(C_t) \cap V(C_{t-1}) = \emptyset$.

D) $V(C_t) \cap V(C_j) = \emptyset$, for $j = 1, \dots, t-2$.

I. Construction of C_1 .

A) Construction of a subgraph J_1 of G_Ψ .

Since G_Ψ is 3-connected and $\rho(\Psi) \geq 3$, Ψ is a wheel-neighbourhood embedding by Proposition 1.8.6. For $i = 1, \dots, l$, let $\tilde{\alpha}_i$ be a traversal of $\bar{f}_i \cap \alpha$, with $\tilde{\alpha}_i(0) = v_{i-1}$ and $\tilde{\alpha}_i(1) = v_i$. With respect to this traversal, let f'_i be the t-component of $f_i \setminus \alpha$, on the left-hand side of α , for $i = 1, \dots, l$. Let f'_{i+l} be the other t-component of $f_i \setminus \alpha$, for $i = 1, \dots, l$.

For $i = 1, \dots, l$, let N_i be the wheel-neighbourhood of v_i in G_Ψ . Notice that $N_i \setminus \alpha$ is formed of two t-components. Let N'_i be the graph of $(N_i \setminus \alpha) \cup \{v_{i-1}, v_i, v_{i+1}\}$ that contains the face f'_i , and let N'_{i+l} be the graph of $(N_i \setminus \alpha) \cup$

$\{v_{i-1}, v_i, v_{i+1}\}$ that contains f'_{i+l} , for $i = 1, \dots, l$. Let $N' = \bigcup_{i=1}^{2l} N'_i$. Also, to simplify the proof, we let $u_i = v_i$, and $u_{l+i} = v_i$, for $i = 1, \dots, l$.

Since Ψ is a wheel-neighbourhood embedding, the closure of each face of G_Ψ is homeomorphic to a closed disc. Therefore, the boundary of each face of G'_Ψ is a polygon.

We can assume, without loss of generality, that $\bar{\alpha} = \bar{\alpha}_1 \circ \bar{\alpha}_2 \circ \dots \circ \bar{\alpha}_l$.

We now construct C_1 . The subgraphs $N'_i - \{u_{i-1}, u_i, u_{i+1}\}$ are nonnull graphs, since otherwise there would be a shortest face chain face-representing γ modulo C_1 . We orient the arcs $N'_i - \{u_i\}$ in a direction that goes from u_{i-1} to u_{i+1} , for $i = 1, \dots, 2l$. Respecting this ordering, let x_i be the first vertex of $N'_i - \{u_{i-1}, u_i, u_{i+1}\}$ that is a neighbour of u_i , and let y_i be the last vertex of $N'_i - \{u_{i-1}, u_i, u_{i+1}\}$. Notice that $x_i = y_{i-1}$, for $i = 1, \dots, 2l$. Let $\bar{\eta}_i$ be a simple path that is a traversal of the part of $N'_i - \{u_{i-1}, u_i, u_{i+1}\}$ from $x_i = \bar{\eta}_i(0)$ to $y_i = \bar{\eta}_i(1)$, for $i = 1, \dots, l$. Let $\bar{\sigma}_1 = \bar{\eta}_1 \circ \bar{\eta}_2 \circ \dots \circ \bar{\eta}_{2l}$. We have that $\bar{\sigma}_1$ is 2-freely homotopic to $\bar{\alpha}$ through the faces of N'_i , for $i = 1, \dots, 2l$. Let J_1 be the subgraph of G_Ψ that is the skeleton of σ_1 .

B) J_1 contains a polygon C_1 such that $C_1 \sim_{C_1}^2 \gamma$.

We will "extract" from J_1 a polygon C_1 that is 2-freely homotopic modulo C_1 as an O-circuit to α . Suppose that J_1 contains a vertex v of degree greater than 2 in J_1 . Let s_1 and s_2 be respectively the smallest and largest numbers in $[0, 1]$ such that $\bar{\sigma}_1(s_1) = \bar{\sigma}_1(s_2) = v$.

Let $\bar{\sigma}'_1$ be the path induced by $\bar{\sigma}_1|_{[s_1, s_2]}$, and let $\bar{\sigma}''_1$ be the path induced by $\bar{\sigma}_1|_{[s_2, 1] \cup [0, s_1]}$. Then $[\bar{\sigma}_1]_l$ is the sum of $[\bar{\sigma}'_1]_l$ and $[\bar{\sigma}''_1]_l$ at v with respect to the orientation of $\bar{\sigma}'_1$ and $\bar{\sigma}''_1$. Now let r be the smallest index such that $\bar{\sigma}_1(s_1) \in \eta_r$, and let s be the greatest index such that $\bar{\sigma}_1(s_2) \in \eta_s$. Let i, j be the residues modulo l of r and s , respectively. Let f be a face of $(F(N'_r) \setminus \{f'_r, f'_{r+1}\}) \cup \{f_i, f_{i+1}\}$

such that $v \in V(\text{bd}(f))$ and such that $\text{bd}(f)$ shares at least one edge with η_r . Let f' be a face of $(F(N'_s) \setminus \{f'_s, f'_{s+1}\}) \cup \{f_j, f_{j+1}\}$ such that $v \in V(\text{bd}(f'))$ and such that $\text{bd}(f')$ shares at least one edge with η_s . By Proposition 1.7.9, the closed face chain formed by $\Lambda' = v_i, f, v, f', v_j$ and the shortest part of Λ between v_i and v_j face-represents an O-circuit that belongs to $\Omega_{\mathcal{C}_1}$.

Without loss of generality, suppose that the face chain $v_i, f_{i+1}, v_{i+1}, \dots, v_{j-1}, f_j, v_j$ is the shortest part of Λ between v_i and v_j . Let β' be an O-circuit such that $\beta' \subset \bar{f} \cup \bar{f}' \cup (\bigcup_{k=i+1}^j \bar{f}_k)$ and $\beta' \cap (\text{bd}(f) \cup \text{bd}(f') \cup (\bigcup_{k=i+1}^j \text{bd}(f_k))) \subseteq \{v, v_i, v_{i+1}, \dots, v_{j-1}, v_j\}$. Then β' is an element of $\Omega_{\mathcal{C}_1}$. Let β'' be an O-circuit such that $\beta'' \subset \bar{f} \cup \bar{f}' \cup (\bigcup_{k=j+1}^i \bar{f}_k)$ and $\beta'' \cap (\text{bd}(f) \cup \text{bd}(f') \cup (\bigcup_{k=j+1}^i \text{bd}(f_k))) \subseteq \{v, v_j, v_{j+1}, \dots, v_{i-1}, v_i\}$. Then, by the TPP, β'' is an element of $\Omega_{\mathcal{C}_2}$. Also, $\beta'' \sim_{\mathcal{C}_1}^2 \alpha$.

Let $\vec{\beta}'$ and $\vec{\beta}''$ be two closed paths of β' and β'' respectively. Now since $\vec{\sigma}_1$ is constructed from the boundaries of the faces of N'_i , we have that exactly one of $\vec{\sigma}'_1, \vec{\sigma}''_1$ is freely homotopic to one of $\vec{\beta}', \vec{\beta}''$, and the other one is 2-freely homotopic to the remaining one. Without loss of generality, suppose that $\vec{\sigma}'_1 \sim_f \vec{\beta}'$. Then $\vec{\sigma}''_1 \sim_{\mathcal{C}_1}^2 \vec{\beta}''$, and so $\vec{\sigma}'_1 \in \vec{\Omega}_{\mathcal{C}_1}$, and $\vec{\sigma}''_1 \in \vec{\Omega}_{\mathcal{C}_2}$. We discard $\vec{\sigma}'_1$, and we repeat the whole process with $\vec{\sigma}''_1$ instead of $\vec{\sigma}_1$: we scan for some vertex of degree greater than 2 in the skeleton of σ''_1 , and we eliminate some part of it that belongs to $\Omega_{\mathcal{C}_1}$.

We end up with a unique sub-O-circuit $\sigma''_1 \in \Omega_{\mathcal{C}_2}$ of σ_1 . Let C_1 be the skeleton of σ''_1 . From the former process, $[\vec{\sigma}_1]_I$ is the sum of $[\vec{\sigma}''_1]_I$ with some loops of $\mathcal{C}_1$ (the ones we have discarded). It follows that $[\vec{\sigma}''_1]_I \sim_{\mathcal{C}_1} [\vec{\sigma}_1]_I$, and so C_1 is 2-freely homotopic modulo $\mathcal{C}_1$ as an O-circuit to α .

II. Construction of C_t .

A) Construction of a subgraph J_t of G_Ψ .

We now describe the other polygons C_t , for $\lfloor \frac{\rho\mathcal{L}_1(\Psi) + \rho\mathcal{L}_2(\Psi) - \mathcal{L}_3^{\mathcal{C}_1}(\Psi) - 1}{4} \rfloor \geq t \geq$

2. We proceed by induction. Suppose that we have constructed $t - 1$ non-contractible polygons $C_1, \dots, C_{t-1}$, all 2-freely homotopic modulo C_1 to α , for $t \geq 2$. Let $w_0, e_1, w_1, \dots, w_{k-1}, e_k, w_k$ (with $w_k = w_0$) be a traversal of C_{t-1} with the same orientation as $\vec{\alpha}$. Let N_i be the wheel-neighbourhood of w_i , for $i = 1, \dots, k$. Now C_{t-1} bounds a nonorientable bordered surface containing the polygons $C_1, \dots, C_{t-2}$. Let N'_i be the t -component of $N_i \setminus C_{t-2}$ that is not contained in this surface, together with the vertices w_{i-1}, w_i, w_{i+1} and the edges e'_i, e'_{i+1} . Let $N' = \bigcup_{i=1}^k N'_i$.

As in 1), we construct the polygons C_j on the wheel-neighbourhood of the vertices of the preceding polygon C_{j-1} . Therefore, given C_j with $j \leq t - 1$ and $v \in V(C_j)$, there is a face incident with v and a vertex $v' \in V(C_{j-2})$. Repeating the same process with v' and C_{j-2} , and continuing this way until we reach one of the vertices of $X = \{v_1, \dots, v_l\} \cup (\bigcup_{i=1}^l V(\text{bd}(f_i)))$. We thus obtain a short face path A' from v to a vertex of X with inner vertices in $V(C_{j-1}), V(C_{j-2}), \dots, V(C_1)$. Furthermore, A' has length at most j .

We now construct the polygon C_t . It might happen that some subgraphs $N'_i - \{w_{i-1}, w_i, w_{i+1}\}$ are null (but not all of them since then we would have $\rho(\Psi) = 0$). In that case, we only keep those that are nonnull, and the rest of the argument still holds. So, without loss of generality, and for the sake of notation, we assume that the N'_i are nonnull graphs, for $i = 1, \dots, k$. We orient the arcs $N'_i - \{w_i\}$ in a direction that goes from w_{i-1} to w_{i+1} , for $i = 1, \dots, k$. With respect to this ordering, let x_i be the first vertex of $N'_i - \{w_{i-1}, w_i, w_{i+1}\}$ that is a neighbour of w_i , and let y_i be the last vertex of $N'_i - \{w_{i-1}, w_i, w_{i+1}\}$. Notice that $x_i =$

y_{i-1} , for $i = 1, 2, \dots, k$. Let $\bar{\eta}_i$ be a simple path that is a traversal of the part of $N'_i - \{w_{i-1}, w_i, w_{i+1}\}$ from $x_i = \bar{\eta}_i(0)$ to $y_i = \bar{\eta}_i(1)$, for $i = 1, \dots, k$. Let $\bar{\sigma}_t = \bar{\eta}_1 \circ \bar{\eta}_2 \circ \dots \circ \bar{\eta}_k$. We have that $\bar{\sigma}_t$ and $\bar{\sigma}_{t-1}$ are freely homotopic through the faces of N' .

Let J_t be the subgraph of G_Ψ that is the skeleton of σ_t . We will extract a polygon C_t from J_t by removing subcircuits that belong to Ω_{C_1} , and it will result that C_t , as an O-circuit, is freely homotopic modulo C_1 to α .

B) J_t contains a polygon C_t such that $C_t \sim_{C_1}^2 \gamma$.

Suppose that J_t contains a vertex v of degree greater than 2 in J_t . Let s_1 and s_2 be, respectively, the smallest and largest numbers in $[0, 1]$ such that $\bar{\sigma}_t(s_1) = \bar{\sigma}_t(s_2) = v$. Let $\bar{\sigma}'_t$ be the path induced by $\bar{\sigma}_t|_{[s_1, s_2]}$, and let $\bar{\sigma}''_t$ be the path induced by $\bar{\sigma}_t|_{[s_2, 1] \cup [0, s_1]}$. Then $[\bar{\sigma}_t]_t$ is the sum of $[\bar{\sigma}'_t]_t$ and $[\bar{\sigma}''_t]_t$ at v with respect to the orientations $\bar{\sigma}'_t$ and $\bar{\sigma}''_t$ respectively. Let i be the smallest index such that $\bar{\sigma}_t(s_1) \in \eta_i$, and let j be the largest index such that $\bar{\sigma}_t(s_2) \in \eta_j$. Let f be a face of N'_i such that $v \in V(\text{bd}(f))$, and such that $\text{bd}(f)$ shares at least one edge with η_i . Correspondingly, let f' be a face of N'_j such that $v \in V(\text{bd}(f'))$, and such that $\text{bd}(f')$ shares at least one edge with η_j .

As we have seen before, there are two face paths Λ_0 and Λ_1 , through f and w_i , and f' and w_j respectively. Both face paths link v to a vertex of X and have their inner vertices in $V(C_{t-2}), V(C_{t-4}), \dots, V(C_1)$. Each of Λ_0 and Λ_1 has length at most t . Let $\Lambda_0 = z_0, g_1, z_1, \dots, g_r, z_r$ where $z_0 = v$, $g_1 = f$, $z_1 = w_i$ and $z_r \in \bigcup_{n=1}^t \bar{f}_n$. Let $\Lambda_1 = z'_0, g'_1, z'_1, \dots, g'_s, z'_s$ where $z'_0 = v$, $g'_1 = f'$, $z'_1 = w_j$ and $z'_s \in \bigcup_{n=1}^t \bar{f}_n$.

Suppose first that there is no common inner vertex u to both face paths Λ_0 and Λ_1 . Let $\Lambda' = z_r, g_r, z_{r-1}, \dots, z_1, g_1, z_0, g'_1, z'_1, \dots, g'_s, z'_s$. By Proposition 1.7.9, the the closed face chain formed by Λ' and the shortest part of Λ between z_r and z_s

face-represents an O-circuit that belongs to Ω_{C_1} . Suppose that $z_r \in V(\bar{f}_{j'} \setminus \{v_{j'}\})$, and that $z_s \in V(\bar{f}_{j'} \setminus \{v_{j'-1}\})$. Suppose that $\Lambda'' = z_r, f_{j'}, v_{j'}, f_{j'+1}, \dots, f_{j'}, z_s$ is the shortest part of Λ between z_r and z_s .

Our objective is to prove that $[\bar{\sigma}'_t]_I$ is an element of C_1 . For this, we will need some "intermediate" loops.

Let $\tilde{\beta}$ be a path from v through Λ_0 , Λ'' , Λ_1 and back to v , such that $\beta \subset (\bigcup_{n=1}^r \bar{g}_n) \cup (\bigcup_{n=1}^s \bar{g}'_n) \cup (\bigcup_{n=t'}^{t''} \bar{f}_n)$ and such that $\beta \cap G_\Psi(\Sigma) = \{z_1, z_2, \dots, z_r, z'_1, z'_2, \dots, z'_s, v_{j'+1}, v_{j'+2}, \dots, v_{j'-1}\}$. Therefore, $[\tilde{\beta}]_I$ is an element of C_1 . Let $\tilde{\beta}_n$ be the path from v which coincides with $\tilde{\beta}$ down to the vertex $z_{t-n} \in V(C_n)$, then traverses the part of C_n parallel to Λ'' to z'_{t-n} and then coincides with $\tilde{\beta}$ back to v in Λ_1 . Without loss of generality, we can assume that $\tilde{\beta}_n(\frac{1}{4}) = z_{t-n}$, and $\tilde{\beta}_n(\frac{3}{4}) = z'_{t-n}$.

Now let $2 \leq n \leq t-1$. Let $t_1, t_2 \in [0, 1]$ such that $\tilde{\beta}_{n-1}(t_1) = z_{t-n}$ and $\tilde{\beta}_{n-1}(t_2) = z'_{t-n}$. Let $\tilde{\beta}'_n = \tilde{\beta}_n|_{[\frac{1}{4}, \frac{3}{4}]}$ and $\beta'_{n-1} = \tilde{\beta}_{n-1}|_{[t_1, t_2]}$. Let $\tilde{\beta}''_n$ be a traversal of the part of σ_n between the endpoints of $\tilde{\beta}'_n$ in the same direction as $\tilde{\sigma}_n$ (see part III A)) with the same endpoints as $\tilde{\beta}'_n$.

Since β''_n and β'_{n-1} bound faces of wheel-neighbourhoods of vertices of C_{n-1} , we have that $[\tilde{\beta}''_n]_s \sim [\tilde{\beta}'_{n-1}]_s$. We will be done if we can show that $[\tilde{\beta}''_n]_s \sim_{C_1} [\tilde{\beta}'_n]_s$. We have seen that the subloops of $[\tilde{\sigma}_n]_I$ (besides the principal loop $[\tilde{\sigma}''_n]_I$) are elements of C_1 . This is also the case for $[\tilde{\beta}''_n]_I$, unless there is an interior point of $\beta''_n \setminus \{z_{t-n}, z'_{t-n}\}$ that is also a point of $C_n \setminus \beta'_n$. But then there would be an O-curve in Ω_{C_1} , attached to this point, that intersects β'_{n-1} only once, a contradiction with the fact that $(C_1, \mathcal{E}_1)$ is a crossing partition. Therefore, $[\tilde{\beta}''_n]_s \sim_{C_1} [\tilde{\beta}'_n]_s$, and by transitivity, we obtain that $[\tilde{\beta}'_n]_s \sim_{C_1} [\tilde{\beta}'_{n-1}]_s$. Therefore, $[\tilde{\beta}_{n-1}]_I \sim_{C_1} \tilde{\beta}_n$ for $2 \leq n \leq t-1$.

On the other hand, $[\vec{\sigma}_i']_l$ is freely homotopic modulo C_1 to $[\vec{\beta}_{i-1}]_l$ through the faces of $N_{i,L}, N_{i+1,L}, \dots, N_{j,L}$. Since $[\vec{\beta}]_l \in C_1$, we obtain by transitivity that $[\vec{\sigma}_i']_l \in C_1$. This now implies that $[\vec{\sigma}_i'']_l \sim_{C_1}^2 [\vec{\alpha}]_l$.

Now suppose that there is a common inner vertex u to both face paths Λ_0 and Λ_1 . Let i', j' be the smallest indices such that $u = x_{i'} = x'_{j'}$. Let $\Lambda' = x_0, g_1, x_1, \dots, g_{i'}, x_{i'}, g'_{j'}, x'_{j'-1}, \dots, g'_1, x'_0$. Then Λ' is a face cycle of length at most $2t$, and it face-represents an O-circuit. Suppose that Λ' face-represents an O-circuit of $\Omega_{\mathcal{E}_1}$. We have that

$$\begin{aligned} \rho_{\mathcal{E}_1}(\Psi) \leq 2t &\leq 2 \left\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - \ell_{\gamma}^{\mathcal{C}_1}(\Psi) - 1}{4} \right\rfloor \\ &\leq \left\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - \ell_{\gamma}^{\mathcal{C}_1}(\Psi) - 1}{2} \right\rfloor. \end{aligned}$$

Since $\rho_{\mathcal{E}_2}(\Psi) \leq \ell_{\gamma}^{\mathcal{C}_1}(\Psi)$, this last inequality implies that $\rho_{\mathcal{E}_1}(\Psi) \leq 0$, which is a contradiction since we assume $\rho(\Psi) \geq 3$. Therefore, Λ' face-represents an O-circuit that belongs to Ω_{C_1} . Using an argument similar to the case where Λ_0 and Λ_1 share no common inner vertex, we can prove that this O-circuit has an O-loop that is freely homotopic modulo C_1 to $[\vec{\sigma}_i']_l$.

We discard σ_i' , and we repeat the whole process with σ_i'' instead of σ_i : we scan for some vertex of degree greater than 2 in the skeleton of σ_i'' , and we eliminate some part of it that belongs to Ω_{C_1} .

We end up, as in the case of J_2 , with a unique subcircuit $\sigma_i'' \in \Omega_{\mathcal{E}_2}$ of σ_i . Let C_i be the skeleton of σ_i'' . From the former process, $[\vec{\sigma}_i]_l$ is the sum of $[\vec{\sigma}_i'']_l$ with some loops of C_1 (the ones we have discarded). Therefore, $[\vec{\sigma}_i'']_l \sim_f [\vec{\sigma}_i]_l$ modulo C_1 , and so C_i is 2-freely homotopic modulo C_1 , as an O-circuit, to α .

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C) $C_t \cap C_{t-1} = \emptyset$.

Notice that since C_t is a subgraph of the symmetric difference of the boundary graphs of the faces of N_L , and this symmetric difference has σ_t in one component and the vertices $w_1, \dots, w_k$ in another component. Therefore, σ_t'' contains none of the vertices $w_1, \dots, w_k$, and so $C_t \cap C_{t-1} = \emptyset$.

D) $V(C_t) \cap V(C_j) = \emptyset$, for $j = 1, \dots, t-2$.

Let $j < t$ be the largest integer such that $V(C_t)$ and $V(C_j)$ share a common vertex. From C), we must have that $j \neq t-1$, so suppose that $j \leq t-2$, and let $v \in V(C_t) \cap V(C_j)$. Then there are two face paths from each occurrence of v in C_t and C_j to A of length at most t . Using an argument similar to the one used in the proof of C), we can show that these face paths contribute to form a closed face chain that face-represents an O-circuit of Ω_{C_1} and that crosses C_{t-1} only once, a contradiction with the fact that $(C_1, \mathcal{E}_1)$ is a crossing partition. Therefore, $V(C_t) \cap V(C_j) = \emptyset$, for $j = 1, \dots, t-2$.

We thus have constructed a set $\{C_1, C_2, \dots, C_m\}$ of C_1 -parallels, all 2-freely homotopic modulo C_1 to α , and belonging to $\Omega_{\mathcal{E}_1}$. $\square$

2.3 Some consequences of Theorem 2.2.2.

Theorem 2.2.2 has many implications, depending on the choice of complete partitions or γ we make. In this section, we consider what we think to be interesting situations where we can apply Theorem 2.2.2. First, we look at the implications of this theorem in the case where Σ is an orientable surface, and later we will consider the nonorientable case.

We now consider a special case of C -parallels when $C = C_0$.

Definition 2.3.1. Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a graph G onto a surface Σ . A set of parallels is a set of pairwise disjoint, freely homotopic polygons of G_Ψ .

Remark. Of course, any polygon C of G_Ψ is a parallel, in the sense that the elements of $\{C\}$ are trivially freely homotopic and disjoint.

The next result is probably the most "natural" one. It is an improvement over the result of [BMR]. It will be used in the next chapter to construct sets of separating O-circuits.

Proposition 2.3.1. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a 3-connected graph G onto a nontrivial surface Σ . Suppose that $\rho(\Psi) \geq 3$. Let $(C, \mathcal{E})$ be a complete partition of $\overleftarrow{\Gamma}_\Sigma$, and let γ be an orientation-preserving face-represented O-circuit in $\Omega_\mathcal{E}$ of length $\ell_\gamma(\Psi)$ in G_Ψ . Then G_Ψ has a set of $\lfloor \frac{\rho(\Psi) + \rho_\mathcal{E}(\Psi) - \ell_\gamma(\Psi) - 1}{2} \rfloor$ noncontractible parallels, all freely homotopic to γ , and belonging to $\Omega_\mathcal{E}$.

Proof. By Proposition 1.7.10, $(C_0, \mathcal{E}_0)$ is a crossing partition and, by Proposition 1.7.7, $\mathcal{E} \subseteq \mathcal{E}_0$. We then apply Theorem 2.2.2. $\square$

Given $(C_1, \mathcal{E}_1), (C_2, \mathcal{E}_2)$, we obtain a maximum number of C_1 -parallels from Theorem 2.2.2 if the length of the O-circuit γ is $\rho_{\mathcal{E}_2}(\Psi)$, as we can see in the next Proposition.

Proposition 2.3.2. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a connected graph G onto a nontrivial orientable surface Σ . Suppose that $\rho(\Psi) \geq 3$. Let $(C_1, \mathcal{E}_1)$ and $(C_2, \mathcal{E}_2)$ be two complete partitions of $\overleftarrow{\Gamma}_\Sigma$ such that $(C_1, \mathcal{E}_1)$ is a crossing partition and $\mathcal{E}_2 \subseteq \mathcal{E}_1$. Then G_Ψ has a set of $\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) - 1}{2} \rfloor$ noncontractible C_1 -parallels belonging to $\Omega_{\mathcal{E}_2}$.

Proof. By Proposition 1.8.12 and Proposition 1.8.13, we can assume that G_Ψ is 3-connected. By Theorem 1.6.12, there is a face-represented O-circuit of $\Omega_{\mathcal{E}_1}$ such that $\ell_\gamma^{C_1}(\Psi) = \rho_{\mathcal{E}_1}(\Psi)$. Then we apply Theorem 2.2.2. $\square$

Remarks. 1) Notice that the number of C_1 -parallels in Proposition 2.3.2 is independent of any γ and of $(C_2, \mathcal{E}_2)$.

2) There is another way to look at Proposition 2.3.2. By Theorem 1.7.3, there is a bijection between the set of complete sets of loops of Σ and the set of normal subgroups of $\pi(\Sigma, x)$ for a given $x \in \Sigma$. Therefore, for any two normal subgroups N_1 and N_2 of $\pi(\Sigma, x)$ such that $N_1 \leq N_2$, the conclusion of Proposition 2.3.2 follows with $C_i = \overrightarrow{\Gamma}_{N_i}$ and $\mathcal{E}_i = \overrightarrow{\Gamma}_{\Sigma} \setminus \overrightarrow{\Gamma}_{N_i}$, for $i = 1, 2$.

3) The bound of the conclusion of Proposition 2.3.2 is better than the corresponding bound of Theorem 2.2.2. However, in Proposition 2.3.2, Σ is orientable and we have no control on the homotopy type of the C_1 -parallels besides the fact that they belong to $\Omega_{\mathcal{E}_2}$.

Now suppose that, in Proposition 2.3.2, we take $(C_1, \mathcal{E}_1)$ to be the fundamental partition $(C_0, \mathcal{E}_0)$. Then we obtain the following interesting result.

Theorem 2.3.3. Let $\Psi : G \longrightarrow G_{\Psi} \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a connected graph G onto a nontrivial orientable surface Σ . Let $(C, \mathcal{E})$ be a complete partition of Σ . Then G_{Ψ} has a set of $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ noncontractible parallels belonging to $\Omega_{\mathcal{E}}$.

Proof. Take $(C_1, \mathcal{E}_1) = (C_0, \mathcal{E}_0)$ in Proposition 2.3.2. $\square$

Remark. Theorem 2.3.3 says that given any complete partition $(C, \mathcal{E})$ of an orientable surface Σ , G_{Ψ} contains $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ noncontractible disjoint polygons of $\Omega_{\mathcal{E}}$ all freely homotopic.

Theorem 2.3.4. Let $\Psi : G \longrightarrow G_{\Psi} \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a connected graph G onto a nontrivial orientable surface Σ . Then G_{Ψ} has a set of $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ nonseparating parallels.

Proof. Take $(C, \mathcal{E}) = (C_0, \mathcal{E}_0)$ in Theorem 2.3.3. $\square$

Remark. We will prove in Chapter 3 that G_Ψ has a set of $\lfloor \frac{\rho(\Psi)-1}{8} \rfloor$ separating parallels.

The following Proposition is actually a more general form of this result:

Proposition 2.3.5. Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a 3-connected graph G onto a nontrivial orientable surface Σ with $\rho(\Psi) \geq 3$. Let γ be a nonseparating O-circuit that is face-represented with length $\ell_\gamma(\Psi)$ in G_Ψ . Then G_Ψ has a set of $\lfloor \frac{\rho(\Psi) + \rho_{\mathcal{E}_\gamma}(\Psi) - \ell_\gamma(\Psi) - 1}{2} \rfloor$ nonseparating parallels, all freely homotopic to γ .

Proof. Simply take $(\mathcal{C}, \mathcal{E}) = (\mathcal{C}_s, \mathcal{E}_s)$ in Proposition 2.3.1. $\square$

We now consider the nonorientable cases. We actually have the nonorientable versions of Proposition 2.3.1, 2.3.2, and Theorem 2.3.3.

Proposition 2.3.6. Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a 3-connected graph G onto a nontrivial surface Σ . Suppose that $\rho(\Psi) \geq 3$. Let $(\mathcal{C}, \mathcal{E})$ be a complete partition of $\overrightarrow{\Gamma}_\Sigma$, and let γ be an orientation-reversing face-represented O-circuit in $\Omega_{\mathcal{E}}$ of length $\ell_\gamma(\Psi)$ in G_Ψ . Then G_Ψ has a set of $\lfloor \frac{\rho(\Psi) + \rho_{\mathcal{E}}(\Psi) - \ell_\gamma(\Psi) - 1}{4} \rfloor$ orientation-preserving parallels, all 2-freely homotopic to γ , and belonging to $\Omega_{\mathcal{E}}$.

Proof. The proof is similar to the proof of Proposition 2.3.1. $\square$

Proposition 2.3.7. Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a connected graph G onto a nontrivial surface Σ . Suppose that $\rho(\Psi) \geq 3$. Let $(\mathcal{C}_1, \mathcal{E}_1)$ and $(\mathcal{C}_2, \mathcal{E}_2)$ be two complete partitions of $\overrightarrow{\Gamma}_\Sigma$ such that $(\mathcal{C}_1, \mathcal{E}_1)$ is a crossing partition and $\mathcal{E}_2 \subseteq \mathcal{E}_\Psi \cap \mathcal{E}_1$. Then G_Ψ has a set of $\lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) - 1}{4} \rfloor$ orientation-preserving $\mathcal{C}_1$ -parallels belonging to $\Omega_{\mathcal{E}_1}$.

Proof. Similar to the proof of Proposition 2.3.2. $\square$

Theorem 2.3.8. *Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a connected graph G onto a nontrivial surface Σ . Let $(\mathcal{C}, \mathcal{E})$ be a complete partition of Σ such that $\mathcal{E} \subseteq \mathcal{E}_p$. Then G_Ψ has a set of $\lfloor \frac{\rho(\Psi)-1}{4} \rfloor$ separating parallels belonging to $\Omega_{\mathcal{E}}$.*

Proof. Similar to the proof of Theorem 2.3.3. Notice that if γ is a nonorientable O-curve, then $\gamma^2 = \gamma \circ \gamma$, with respect to a closed path of γ that makes $\gamma \circ \gamma$ noncontractible, bounds a Möbius strip and is therefore a separating O-circuit. $\square$

Remark. We point out that if the nonorientable genus of Σ in Proposition 2.3.6 and Theorem 2.3.8 is at least 2, then the parallels obtained are also noncontractible: since each of them bound a Möbius strip, the "capping" of the rest of the surface with a closed disc gives a nontrivial surface, and therefore the parallels do not bound a disc.

CHAPTER III

SEPARATING PARALLELS

3.1 Introduction.

We now consider the question of the existence of noncontractible separating parallels in an embedded graph. Richter and Vitray [RV1] have shown the existence of noncontractible separating polygons in an embedding Ψ with $\rho(\Psi) \geq 11$ when the surface is orientable with genus at least 2. Zha and Zhao [ZZ] came to the same conclusion when $\rho(\Psi) \geq 7$ for a surface (orientable or nonorientable) of genus at least 2. Mohar, Richter and the author [BMR] then reduced the requirement to $\rho(\Psi) = 6$ in the case of an orientable surface of genus at least 2, and to $\rho(\Psi) = 5$ in the case of a nonorientable surface of genus at least 2. They also showed that if Σ is an orientable surface of genus at least 2 and if γ is an O-circuit of length $\ell_\gamma(\Psi)$ in G_Ψ , then G_Ψ has at least $\lfloor \frac{\rho(\Psi) - \lfloor \frac{\ell_\gamma(\Psi)}{2} \rfloor - 1}{4} \rfloor - 1$ noncontractible separating parallels that "isolate" γ (we will define this concept in Section 2). In an early version of this result, Richter and Vitray noticed an important mistake in the original bound, and obtained the exact bound above with a different approach [RV2]. We are grateful to them for pointing out this mistake.

In this chapter, we will show a more general result: we will get a bound that depends on the representativity of the complete partition that contains the loop $[\gamma]_1$. This will be the object of Section 2. In Section 3, we will look at the implications of Theorem 3.2.1, and we will show that the bounds of Brunet-Mohar-Richter and Richter-Vitray are consequences of this Theorem.

3.2 Noncontractible separating parallels.

Definition 3.2.1. Let Σ be a non-trivial surface, and let γ be an O-circuit of Σ . Suppose that γ' is a separating O-circuit of Σ . Then γ' isolates γ if γ is in one and only one component of $\Sigma \setminus \gamma'$.

Theorem 3.2.1. Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a 3-connected graph G onto a nontrivial orientable surface Σ . Suppose that $\rho \geq 3$, and let $(\mathcal{C}, \mathcal{E})$ be a complete partition of $\overrightarrow{\Gamma}_\Sigma$. Let γ be a face-represented O-circuit in $\Omega_\mathcal{E}$ of length $\ell_\gamma(\Psi)$ in G_Ψ .

- (1) Suppose that $\gamma \in \mathcal{E} \cap \mathcal{C}_s$. Then G_Ψ has at least $\lfloor \frac{\rho(\Psi) + \rho_\mathcal{E}(\Psi) - \ell(\gamma) - 1}{2} \rfloor$ noncontractible separating parallels all freely homotopic to γ , and belonging to $\Omega_\mathcal{E}$.
- (2) Suppose that $\gamma \in \mathcal{E} \cap \mathcal{E}_s$. Then G_Ψ has at least $\lfloor \frac{\rho(\Psi) + \rho_\mathcal{E}(\Psi) - \ell(\gamma) - 1}{8} \rfloor - 1$ separating parallels, all isolating a freely homotopic copy of γ .

Proof. 1) This is a direct consequence of Proposition 2.3.1.

2) Since γ is face-represented, there is a face cycle $A = v_0, f_1, v_1, f_2, v_2, \dots, f_l, v_l$ that face-represents γ with $l = \ell_\gamma(\Psi)$. Without loss of generality, we can assume that $\gamma \subseteq \bigcup_{i=1}^l \bar{f}_i$ with $\{v_1, \dots, v_l\} = \gamma \cap \bigcup_{i=1}^l \text{bd}(f_i)$.

Let $m = \lfloor \frac{\rho(\Psi) + \rho_\mathcal{E}(\Psi) - \ell_\gamma(\Psi) - 1}{2} \rfloor$, and $n = \lfloor \frac{\rho(\Psi) + \rho_\mathcal{E}(\Psi) - \ell_\gamma(\Psi) - 1}{8} \rfloor - 1$. Then, since $\ell_\gamma(\Psi) \geq \rho_\mathcal{E}(\Psi)$, we have that $\rho(\Psi) \geq 8n + 9$. Assume that $n \geq 1$. Then $m \geq 8$, and $\rho(\Psi) \geq 17$. Let $C_1, \dots, C_m$ be the noncontractible parallels freely homotopic to γ constructed in the proof of Theorem 2.2.2. By hypothesis, these polygons are nonseparating.

We will construct a set $\{C_1^*, C_2^*, \dots, C_n^*\}$ of separating parallels, all freely homotopic, and isolating a freely homotope of γ . The proof will be divided into the following steps.

I. Construction of C_1^* .

A) Construction of two arcs P_1 and P_2 .

B) P_1 and P_2 are disjoint.

C) $P_1 \cup P_1' \cup C_1 \cup C_2$ contains a separating polygon C_1^* that isolates a freely homotopic copy of γ .

II. Construction of C_t^* , for $t \leq n$.

A) Construction of two arcs P_{2t-1} and P_{2t} .

B) P_{2t-1} and P_{2t} are disjoint.

C) $P_{2t-1} \cap P_{2t-3} = P_{2t} \cap P_{2t-2} = \emptyset$.

D) $P_{2t-1} \cup P_{2t} \cup C_{2t-1} \cup C_{2t}$ contains a separating polygon C_t^* .

E) C_t^* is freely homotopic to C_{t-1}^* .

I. Construction of C_1^* .

A) Construction of two arcs P_1 and P_2 .

Let Q be the open cylinder bounded by C_1 and C_2 containing γ . Since these two polygons are not separating, $\Sigma \setminus Q$ is t -connected. Let $\Lambda' = u_0, g_1, u_1, \dots, g_k, u_k$ be the shortest face chain in $\Sigma \setminus Q$ between C_1 and C_2 , that is to say, $u_0 \in V(C_1)$ and $u_k \in V(C_2)$. Because of its minimality, Λ' is a face path.

Since G_Ψ is 3-connected and $\rho(\Psi) \geq 3$, Ψ is a wheel-neighbourhood embedding by Proposition 1.8.6. Let α be a simple path such that $\tilde{\alpha}(0) = u_0$, $\tilde{\alpha}(1) = u_k$, $\alpha \subseteq \bigcup_{i=1}^k \bar{g}_i$ and $\alpha \cap (\bigcup_{i=1}^k \text{bd}(g_i)) = \{u_0, \dots, u_k\}$. Let $\tilde{\alpha}_i$ be the subpath of $\tilde{\alpha}$ such that $\tilde{\alpha}_i(0) = u_{i-1}$, $\tilde{\alpha}_i(1) = u_i$ and $\alpha_i \subseteq \bar{g}_i$, for $i = 1, \dots, k$. Since Σ is orientable, we can define a right-hand side and a left-hand side to α according to the traversal $\tilde{\alpha}$. For $i = 1, \dots, k$, let $g_{i,R}$ be the t -component of $g_i \setminus \alpha_i$ on the right-hand side of α , and let $g_{i,L}$ be the t -component of $g_i \setminus \alpha_i$ on the left-hand side of α . By Corollary 1.8.7, we can choose the u_i so that $\bar{g}_{i,R} \cap \bar{g}_{i+1,R}$ contains no vertex other than u_i , for $i = 1, \dots, k-1$.

Since Ψ is a wheel-neighbourhood embedding, the closure of each face of G_Ψ is homeomorphic to a closed disc. Therefore, the boundary of each face of G_Ψ is a polygon.

Let $\vec{\eta}_i$ be a traversal of $(\partial(g_{i,R})) \setminus \alpha_i([0, 1])$ with $\vec{\eta}_i(0) = u_{i-1}$ and $\vec{\eta}_i(1) = u_i$, for $i = 1, \dots, k$. Let $\vec{\sigma}_1 = \vec{\eta}_1 \circ \vec{\eta}_2 \circ \dots \circ \vec{\eta}_k$. Since $\vec{\alpha}_i$ and $\vec{\eta}_i$ together bound $g_{i,R}$, we have that $\alpha \sim \sigma_1$.

For $i = 1, \dots, k$, let N_i be the wheel-neighbourhood of u_i in G_Ψ . Notice that, for $i = 1, \dots, k-1$, $N_i \setminus \alpha$ is formed of two t-components. Let $N_{i,L}$ be the graph of $(N_i \setminus \alpha) \cup \{u_{i-1}, u_i, u_{i+1}\}$ on the left-hand side of α , for $i = 1, \dots, k-1$. The subgraphs $N_{i,L} - \{u_{i-1}, u_i, u_{i+1}\}$ are nonnull graphs, since otherwise there would be a shortest face path in $\Sigma \setminus Q$ between C_1 and C_2 . We orient the arcs $N_{i,L} - \{u_i\}$ in a direction that goes from u_{i-1} to u_{i+1} . Respecting this order, we let x_i be the first vertex of $N_{i,L} - \{u_{i-1}, u_i, u_{i+1}\}$ that is a neighbour of u_i , and we let y_i be the last vertex of $N_{i,L} - \{u_{i-1}, u_i, u_{i+1}\}$, for $i = 1, \dots, k-1$. Let $\vec{\eta}_i$ be a simple path that is a traversal of $N_{i,L} - \{u_{i-1}, u_i, u_{i+1}\}$ with $\vec{\eta}_i(0) = u_{i-1}$, and $\vec{\eta}_i(1) = u_{i+1}$.

We also define similar paths in N_0 and N_k , but the construction is somewhat different. In N_0 , we traverse the rim of N_0 in a counterclockwise direction starting at u_1 , and we let y_0 be the first vertex met. We continue this traversal until we meet a neighbour x_0 of u_0 in N_0 that is also a vertex of C_1 , and then traverse their common incident edge and stop at u_0 . Let $(\vec{\eta}_0)^{-1}$ be the simple path that is the traversal made from y_0 to u_0 . Then $\vec{\eta}_0$ is a simple path that is the traversal made with $\vec{\eta}_0(0) = u_0$ and $\vec{\eta}_0(1) = y_0$.

In a similar pattern, we traverse the rim of N_k in a clockwise direction beginning at u_{k-1} . Let x_k be the first vertex met that is a neighbour of u_k . We continue the traversal until we meet a vertex that belongs to C_2 and that is a neighbour of u_k . We traverse their common incident edge, and stop at u_k . Let $\vec{\eta}_k$ be a simple path that is the traversal made with $\vec{\eta}_k(0) = u_{k-1}$ and $\vec{\eta}_k(1) = u_k$.

Let $\bar{\sigma}_2 = \bar{\eta}_0 \circ \bar{\eta}_2 \circ \cdots \circ \bar{\eta}_k$. Then $\bar{\sigma}_2$ is a path such that $\bar{\sigma}_2(0) = u_0$, and $\bar{\sigma}_2(1) = u_k$. Therefore, $\bar{\sigma}_1$ and $\bar{\sigma}_2$ have the same endpoints. Let J_1 be the skeleton of $\bar{\sigma}_1$, and let J_2 be the skeleton of $\bar{\sigma}_2$.

We now "extract" from J_1 and J_2 two arcs P_1 and P_2 respectively, whose endpoints are vertices of C_1 and C_2 .

We start with J_1 , and prove that J_1 is an arc. Suppose, on the contrary, that J_1 contains a vertex v of degree greater than 2 in J_1 . Let i be the smallest index such that v is a vertex of $\bar{g}_i \setminus \{u_i\}$, and let j be the greatest index such that v is a vertex of $\bar{g}_j \setminus \{u_{j-1}\}$. By Corollary 1.8.7, i and j cannot be consecutive.

Consider the face chain $\Lambda' = u_0, g_1, u_1, \dots, v_{i-1}, g_i$ (if $v \neq u_{i-1}$), v, g_j (if $v \neq v_j$), $u_j, \dots, g_k, u_k$. Now the length of Λ' is smaller than the length of Λ . This is a contradiction with the minimality of the length of Λ . Therefore, all the vertices of P_1 have degree at most 2, and therefore all the points of σ_1 have multiplicity at most 2. It follows that the subgraph P_1 , whose segment is σ_1 , is an arc. Since $\bar{\sigma}_1 \sim \bar{\alpha}$, we have that P_1 is homotopic as a segment to α , and so $P_1 \sim \alpha$.

We now extract from J_2 an arc A_2 . Suppose that J_2 contains a vertex v of degree greater than 2 in J_2 . Let s_1 and s_2 be respectively the smallest and largest numbers in $[0, 1]$ such that $\bar{\sigma}_2(s_1) = \bar{\sigma}_2(s_2) = v$. Let $\bar{\sigma}'_2$ be the path induced by $\bar{\sigma}_2|_{[0, s_1]}$, let $\bar{\gamma}$ be the path induced by $\bar{\sigma}_2|_{[s_1, s_2]}$, and let $\bar{\sigma}''_2$ be the path induced by $\bar{\sigma}_2|_{[s_2, 1]}$. Then $[\bar{\sigma}_2]_v$ is the sum of $[\bar{\sigma}'_2 \circ \bar{\sigma}''_2]_v$ and $[\bar{\gamma}]_v$ at v with respect to $\bar{\sigma}'_2$ and $\bar{\gamma}$.

Now let i be the smallest index such that $\bar{\sigma}_2(s_1) \in \eta'_i$, and let j be the greatest index such that $\bar{\sigma}_2(s_2) \in \eta'_j$. Let g be a face of $(F(N_{i,L}) \setminus \{g_{i,L}, g_{i+1,L}\}) \cup \{g_i, g_{i+1}\}$ such that $v \in V(\text{bd}(g))$ and such that $\text{bd}(g)$ shares at least one edge with η'_i (we put $g_0 = g_{k+1} = \emptyset$). Let g' be a face of $(F(N_{j,R}) \setminus \{g_{j,R}, g_{j+1,R}\}) \cup \{g_j, g_{j+1}\}$ such that $v \in V(\text{bd}(g'))$ and such that $\text{bd}(g')$ shares at least one edge with η'_j .

If $j - i \geq 3$, then we have a contradiction, since $u_0, g_1, u_1, \dots, u_i, g, v, g', u_j, g_{j+1}, \dots, u_k$ is a face path in $\Sigma \setminus Q$ between C_1 and C_2 shorter than Λ' . Therefore, we must have that $j - i \leq 2$. The face polygon $u_i, g, v, g', u_j, g_j, u_{j-1}, \dots, g_{i+1}, u_i$ has length at most 4, and so it face-represents a contractible 0-circuit, since $\rho(\Psi) \geq 17$ by hypothesis.

Let $\tilde{\beta}$ be a simple closed path such that $\tilde{\beta}(0) = \tilde{\beta}(1) = v$, and such that $\beta' \subset \bar{g} \cup \bar{g}' \cup (\bigcup_{k=i+1}^j \bar{g}_k)$ and $\beta' \cap (\text{bd}(g) \cup \text{bd}(g') \cup (\bigcup_{k=i+1}^j \text{bd}(g_k))) \subseteq \{v, u_i, \dots, u_{j-1}, u_j\}$. Then β' is a contractible closed path. Since $\tilde{\gamma}$ is homotopic to $\tilde{\beta}$ through the faces of the wheel-neighbourhoods of $v_i, v_{i+1}, \dots, v_j$, we also have that $\tilde{\gamma}$ is contractible. Therefore, $\tilde{\sigma}'_2 \circ \tilde{\gamma} \circ \tilde{\sigma}''_2 \sim \tilde{\sigma}'_2 \circ \tilde{\sigma}''_2 = \tilde{\sigma}'''_2$, and, since $\tilde{\sigma}'_2 \circ \tilde{\gamma} \circ \tilde{\sigma}''_2 \sim \tilde{\sigma}_2$, we obtain that $\tilde{\sigma}'''_2 \sim \tilde{\sigma}_2$. We discard $\tilde{\gamma}$, and repeat the whole process with $\tilde{\sigma}'''_2$ instead of $\tilde{\sigma}_2$: we scan for some vertex of degree greater than 2 in the skeleton of $\tilde{\sigma}'''_2$, and eliminate some part of it that is contractible. We end up with a subsegment σ'''_2 of σ_2 . Let A_2 be the skeleton of σ'''_2 . From the former process, $[\tilde{\sigma}_2]_s$ is the sum of $[\tilde{\sigma}'''_2]_s$ with some contractible loops. Therefore, since $\tilde{\sigma}'''_2$ and $\tilde{\sigma}_2$ have the same origin and terminus, $\tilde{\sigma}'''_2 \sim \tilde{\sigma}_2$ and so, by transitivity of $\sim$, $\tilde{\sigma}'''_2 \sim \tilde{\alpha}$. It follows that A_2 is homotopic as a segment to α .

Let s_1 be the largest index such that $\tilde{\sigma}'''_2(s_1) \in V(C_1)$, and let s_2 be the smallest index such that $\tilde{\sigma}'''_2(s_2) \in V(C_2)$. Let $\tilde{\sigma}^*_2$ be the path induced by $\tilde{\sigma}'''_2|_{[s_1, s_2]}$. Let P_1 be the skeleton of σ_1 . Let φ_1 be a path from u_0 along the part of C_1 on the left-hand side of u_0 until $\tilde{\sigma}^*_2(0)$. Let φ_2 be a path from $\tilde{\sigma}^*_2(1)$ along C_2 on the right-hand side of $\tilde{\sigma}^*_2$ until u_k .

B) P_1 and P_2 are disjoint.

We now show that the arcs P_1 and P_2 are disjoint. Suppose that there is a vertex $v \in V(P_1) \cap V(P_2)$. Then $v \in \sigma_1^* \cap \sigma_2^*$. Let i be the smallest index such that $v \in \bar{f}_i$ and let j be the largest integer such that $v \in \bar{g} \subset N_{j,L}$. If $j - i \geq 3$, then there is a shorter face chain between C_1 and C_2 through v , a contradiction. If $j - i \leq 2$, then the face chain $A' = v, g, u_i, g_{i+1}, \dots, g_j, u_j, g_{j+1}$ (if $v \neq v_j$), v is a face-polygon that face-represents an O-circuit γ_0 . If γ_0 is noncontractible, then, since its length is at most 4, we have a contradiction with the value of $\rho(\Psi)$. If $\tilde{\gamma}_0$ is contractible, then we also have a contradiction since there is a freely homotopic copy of $\tilde{\gamma}_0$ that intersects α transversely only once. Therefore, $V(P_1) \cap V(P_2) = \emptyset$.

C) $P_1 \cup P'_1 \cup C_1 \cup C_2$ contains a separating polygon C_1^* that isolates a freely homotopic copy of γ .

By construction and since $C_3 \sim_f C_1$ and $C_4 \sim_f C_2$, we have that $\bar{\varphi}_1 \circ \bar{\sigma}_2 \circ \bar{\varphi}_2 \circ \bar{\alpha}^{-1}$ and $\bar{\alpha} \circ \bar{\sigma}_1^{-1}$ are contractible. It follows that their composition, $\bar{\varphi}_1 \circ \bar{\sigma}_2 \circ \bar{\varphi}_2 \circ \bar{\sigma}_1^{-1}$, is also contractible. Therefore, $P_1 \cup P_2 \cup \varphi_1 \cup \varphi_2$ bounds a disc Δ_1 , and, since C_1 and C_2 bound a cylinder Q_1 , we obtain that $C_2^* = P_1 \cup P_2 \cup \{C_1 \setminus \varphi_1\} \cup \{C_2 \setminus \varphi_2\}$ is a polygon such that $\Delta_1 \cup Q_1$ is contained in one t-component of $\Sigma \setminus C_1^*$. Notice that $\Delta_1 \cup C_1^* \cup Q_1$ is homeomorphic to a punctured torus. Therefore, since Σ is a compact surface, C_1^* is a separating polygon, and one t-component of $\Sigma \setminus C_1^*$ contains a freely homotopic copy of γ .

II) Construction of C_t^* , for $t \leq n$.

A) Construction of two arcs P_{2t-1} and P_{2t} .

Suppose now that we have constructed $2t - 2$ arcs $P_1, \dots, P_{2t-2}$, such that for $1 \leq j \leq t$,

- (1) P_{2j-1} and P_{2j} each join a vertex of C_{2j-1} to a vertex of C_{2j} ;
- (2) P_{2j-1} and P_{2j} are disjoint from the cylinder Q_j bounded by C_{2j-1} and C_{2j} ;
- (3) the ordering is such that going away from α on its right-hand side we meet $P_1, P_3, \dots, P_{2j-1}$ and as we go away from α on its left-hand side we meet $P_2, P_4, \dots, P_{2j}$;
- (4) P_{2j-1} is a subgraph of the wheel-neighbourhoods of the vertices of P_{2j-3} ;
 P_{2j} is a subgraph of the wheel-neighbourhoods of the vertices of P_{2j-2} ;
- (5) $P_{2j-1} \cup P_{2j} \cup C_{2j-1} \cup C_{2j}$ contains a separating polygon C_j^* , and P_{2i-1}, P_{2i} and the part of C_{2i-1} and C_{2i} containing the endpoints of the paths $P_1, \dots, P_{2i-2}$ bounds a disc Δ_i^* containing these paths;
- (6) any two of these polygons C_j^* are disjoint and freely homotopic.

We now describe C_t^* . We begin by describing P_{2t-1} . Let $w_0, e_1, w_1, e_2, \dots, e_p, w_p$ be a traversal of P_{2t-3} in the same direction as $\vec{\alpha}$.

For $i = 0, \dots, p$, let N_i be the wheel-neighbourhood of w_i in G_Ψ . Let $N_{i,R}$ be the graph $(N_i \setminus C_{i-1}^*) \cup \{w_{i-1}, e_i, w_i, e_{i+1}, w_{i+1}\}$ on the right-hand side of C_{i-1}^* , for $i = 1, \dots, p-1$.

Notice that the arcs P_j we construct are subgraphs of the union of the wheel-neighbourhoods of the vertices of the preceding arc P_{j-2} . Therefore, given P_j with $j \leq t-1$ and $v \in V(P_j)$, there is a face incident to v and to a vertex v' of P_{j-2} . Repeat the same process with v' and P_{j-4} , and continue in this way until we reach one of the vertices of $X = \{u_0, \dots, u_k\} \cup (\bigcup_{i=1}^k V(\text{bd}(g_i)))$. We thus obtain a short face path A' from v to a vertex of X with inner vertices in $V(P_{j-2}), V(P_{j-4}), \dots, V(P_r)$ where r is 1 if j is odd and 2 if j is even. Moreover, A' has length at most $\lfloor \frac{j}{2} \rfloor$.

It might happen that some subgraphs $N_{i,R} - \{w_{i-1}, w_i, w_{i+1}\}$ are null (but not all of them since otherwise C_{2t-1} would have a vertex in the wheel-neighbourhood of a vertex of C_{2t} , a contradiction with the value of m). In that case, we keep only those that are nonnull, and the rest of the argument still holds. So, without loss of generality, we assume that the $N_{i,R}$ are nonnull graphs, for $i = 1, \dots, p-1$. We orient the arcs $N_{i,R} - \{w_i\}$ in a direction that goes from w_{i-1} to w_{i+1} , for $i = 1, \dots, p-1$. Respecting this ordering, let x_i be the first vertex of $N_{i,R} - \{w_{i-1}, w_i, w_{i+1}\}$ that is a neighbour of w_i , and let y_i be the last vertex of $N_{i,R} - \{w_{i-1}, w_i, w_{i+1}\}$.

Let $\tilde{\eta}_i$ be a simple path that is a traversal of the part of $N_{i,R}$ from $x_i = \tilde{\eta}_i(0)$ to $y_i = \tilde{\eta}_i(1)$, for $i = 1, \dots, p-1$. Notice that $x_i = y_{i-1}$, for $i = 2, \dots, p-1$.

We also define some paths in N_0 and N_p . We traverse the rim of N_0 in a counterclockwise direction starting at w_1 , and we let y_0 be the first vertex met. We continue this traversal until we meet a neighbour x_0 of w_0 in N_0 that is also a vertex of C_{2t-1} , and we then traverse their common incident edge and stop at w_0 . Let $\tilde{\eta}_0^{-1}$ be the simple path that is the traversal we made from y_0 to w_0 . Then $\tilde{\eta}_0$ is a simple path that is the backward traversal of the traversal we made with $\tilde{\eta}_0(0) = w_0$ and $\tilde{\eta}_0(1) = y_0$.

We traverse the rim of N_p in a clockwise direction beginning at w_{p-1} . Let x_p be the first vertex met that is a neighbour of w_p . We continue the traversal until we meet a vertex that belongs to C_{2t} and that is a neighbour of w_p . We traverse their common incident edge and we stop at w_p . Let $\tilde{\eta}_p$ be a simple path that is the traversal we made with $\tilde{\eta}_p(0) = w_{p-1}$ and $\tilde{\eta}_p(1) = w_p$.

Let $\tilde{\sigma}_{2t-1} = \tilde{\eta}_0 \circ \tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_p$. Then $\tilde{\sigma}_{2t-1}$ is a path such that $\tilde{\sigma}_{2t-1}(0) = w_0$ and $\tilde{\sigma}_{2t-1}(1) = w_p$.

Now let $w'_0, e'_1, u'_1, \dots, w'_q$ be a traversal of P_{2t-2} in the same direction as $\tilde{\alpha}$. For $i = 0, \dots, q$, let N'_i be the wheel-neighbourhood of w'_i in G_Φ . Let

$N'_{i,L} = (N'_i \setminus C_{i-1}^*) \cup \{w'_{i-1}, e'_i, w'_i, e'_{i+1}, w'_{i+1}\}$ on the left-hand side of C_{i-1}^* , for $i = 1, \dots, q-1$.

It might happen that some subgraphs $N_{i,L} - \{w'_{i-1}, w'_i, w'_{i+1}\}$ are null (but not all of them since then, as for P_{2i-1} , C_{2i-1} would have a vertex in the wheel-neighbourhood of a vertex of C_{2i} , a contradiction with the value of m assumed in the hypothesis). In this case, we keep only those that are nonnull, and the rest of the argument still holds. Therefore, without loss of generality, we can assume that the $N'_{i,L}$ are nonnull graphs, for $i = 1, \dots, q-1$. We orient the arcs $N'_{i,L} - \{w'_i\}$ in the direction that goes from w'_{i-1} to w'_{i+1} , for $i = 1, \dots, q-1$. Respecting this ordering, let x_i be the first vertex of $N'_{i,L} - \{w'_{i-1}, w'_i, w'_{i+1}\}$ that is a neighbour of w'_i and let y_i be the last vertex of $N'_{i,L} - \{w'_{i+1}, w'_i, w'_{i+1}\}$.

Let $\vec{\eta}_i$ be a simple path that is a traversal of the part of $N_{i,L}$ from $x_i = \vec{\eta}_i(0)$ to $y_i = \vec{\eta}_i(1)$, for $i = 1, \dots, q-1$. Notice that $x_i = y_{i-1}$, for $i = 2, \dots, q-1$.

We also define some paths in N'_0 and N'_q . We traverse the rim of N'_0 in a clockwise direction starting at w'_1 , and we let y_0 be the first vertex met. We continue this traversal until we meet a neighbour x_0 of w'_0 in N'_0 that is also a vertex of C_{2i-1} , and we then traverse their common incident edge and stop at w'_0 . Let $(\vec{\eta}_0)^{-1}$ be the simple path that is the traversal we made from y_0 to w'_0 . Then $\vec{\eta}_0$ is a simple path that is the backward traversal of the traversal we made, with $\vec{\eta}_0(0) = w'_0$ and $\vec{\eta}_0(1) = y_0$.

We traverse the rim of N'_q in a counterclockwise direction beginning at w'_{q-1} . Let x_q be the first vertex met that is a neighbour of w'_q . We traverse their common incident edge and we stop at w'_q . Let $\vec{\eta}_q$ be a simple path that is the backward traversal of the traversal we made with $\vec{\eta}_q(0) = w'_{q-1}$ and $\vec{\eta}_q(1) = w'_q$.

Let $\vec{\sigma}_{2i} = \vec{\eta}_0 \circ \vec{\eta}_1 \circ \dots \circ \vec{\eta}_q$. Then $\vec{\sigma}_{2i}$ is a simple path such that $\vec{\sigma}_{2i}(0) = w'_0$ and $\vec{\sigma}_{2i}(1) = w'_q$.

Let J_{2t-1} and J_{2t} be the skeletons of $\bar{\sigma}_{2t-1}$ and $\bar{\sigma}_{2t}$, respectively.

We now "extract" from J_{2t-1} and J_{2t} two arcs P_{2t-1} and P_{2t} , respectively, whose endpoints are vertices of C_{2t-1} and C_{2t} (see Figure XV).

We start with J_{2t-1} . We actually extract first from J_{2t-1} an arc A_{2t-1} . Suppose that J_{2t-1} contains a vertex v of degree greater than 2 in J_{2t-1} . Let s_1 and s_2 be, respectively, the smallest and largest numbers in $[0, 1]$ such that $\bar{\sigma}_{2t-1}(s_1) = \bar{\sigma}_{2t-1}(s_2) = v$. Let $\bar{\sigma}'_{2t-1}$ be the path induced by $\bar{\sigma}_{2t-1}|_{[0, s_1]}$, let $\bar{\gamma}$ be the path induced by $\bar{\sigma}_{2t-1}|_{[s_1, s_2]}$, and let $\bar{\sigma}''_{2t-1}$ be the path induced by $\bar{\sigma}_{2t-1}|_{[s_2, 1]}$. Then $[\bar{\sigma}_{2t-1}]_s$ is the sum of $[\bar{\sigma}'_{2t-1} \circ \bar{\sigma}''_{2t-1}]_s$ and $[\bar{\gamma}]_s$ at v with respect to $\bar{\sigma}'_{2t-1}$ and $\bar{\gamma}$.

Let i be the smallest index such that $\bar{\sigma}_{2t-1}(s_1) \in \bar{\eta}_i$, and let j be the greatest index such that $\bar{\sigma}_{2t-1}(s_2) \in \bar{\eta}_j$. Let g be a face of $N_{i,R}$ such that $v \in V(\text{bd}(g))$ and such that $\text{bd}(g)$ shares at least one edge with η_i . Correspondingly, let g' be a face of $N_{j,R}$ such that $v \in V(\text{bd}(g'))$, and such that $\text{bd}(g')$ shares at least one edge with η_j .

As we have seen before, there are two face paths Λ_0 and Λ_1 , through g and w_i , and g' and w_j respectively. Both face paths link v to a vertex of X and have their inner vertices in $V(P_{2t-3}), V(P_{2t-5}), \dots, V(P_3)$. Each of Λ_0 and Λ_1 have length at most t . Let $\Lambda_0 = z_0, h_1, z_1, \dots, h_r, z_r$ where $z_0 = v, h_1 = g, z_1 = w_i$ and $z_r \in \bigcup_{l=1}^k \bar{g}_l$. Let $\Lambda_1 = z'_0, h'_1, z'_1, \dots, h'_s, z'_s$ where $z'_0 = v, h'_1 = g', z'_1 = w_j$ and $z'_s \in \bigcup_{l=1}^k \bar{g}_l$.

Suppose first that there is no common inner vertex u to both face paths Λ_0 and Λ_1 . Let $\Lambda'' = z_r, h_r, z_{r-1}, \dots, z_1, h_1, z_0, h'_1, z'_1, \dots, g'_s, z'_s$. Suppose that $z_r \in V(\bar{g}_{i'} \setminus \{u_{i'}\})$, and that $z'_s \in V(\bar{g}_{j'} \setminus \{u_{j'-1}\})$.

If $j' - i' > 2t + 2$, the face path through w_0 until $u_{i'}$, then through Λ'' until $u_{j'}$ and then to u_k is a shorter face path between C_1 and C_2 than Λ' , a contradiction. Therefore, we must have that $j' - i' \leq 2t + 2$.

Figure XV

Let $\Lambda''' = u_{i'}, g_{i'+1}, \dots, g_{j'}$. Then the face cycle formed by Λ'' and Λ''' face-represents an O-circuit of length at most $4t$. Let β be such an O-circuit through the faces and vertices of Λ'' and Λ''' . Since $t \leq m$, and $\rho_{\mathcal{L}}(\Psi) - \ell(\gamma) - 1 \leq 0$, we obtain that the length of β is at most $\frac{\ell}{2}$, and therefore β is contractible.

Suppose on the contrary that Λ_0 and Λ_1 have a common inner vertex u . Let i' and j' be the smallest indices such that $u = z_{i'} = z_{j'}$. Let $\Lambda'' = z_0, h_1, z_1, \dots, z_{i'}, h'_{j'}, z'_{j'-1}, \dots, h'_1, z'_0$. Let β be an O-circuit through the faces and vertices of Λ'' . Then the length of β is at most $2t$. But then, since $t \leq m$ and $\rho_{\mathcal{L}}(\Psi) - \ell_{\gamma}(\Psi) - 1 \leq 0$, we have that the length of β is at most $\frac{\ell}{2}$, and so β must be contractible.

Since $\tilde{\gamma}$ is freely homotopic to any path $\tilde{\beta}$ of β through the faces of the wheel-neighbourhoods of the vertices of $P_1, P_2, \dots, P_{2t-3}$, we have that $\tilde{\gamma}$ is also contractible. Therefore, $\tilde{\sigma}'_{2t-1} \circ \tilde{\gamma} \circ \tilde{\sigma}''_{2t-1} \sim \tilde{\sigma}'_{2t-1} \circ \tilde{\sigma}''_{2t-1} = \tilde{\sigma}'''_{2t-1}$, and since $\tilde{\sigma}'_{2t-1} \circ \tilde{\gamma} \circ \tilde{\sigma}''_{2t-1} \sim \tilde{\sigma}_{2t-1}$, we obtain that $\tilde{\sigma}'''_{2t-1} \sim \tilde{\sigma}_{2t-1}$. We discard $\tilde{\gamma}$, and repeat the whole process with $\tilde{\sigma}'''_{2t-1}$ instead of $\tilde{\sigma}_{2t-1}$: we scan for some vertex of degree greater than 2 in the skeleton of $\tilde{\sigma}'''_{2t-1}$, and eliminate some part of it that is contractible.

We end up with a unique subsegment σ'''_{2t-1} of σ_{2t-1} . Let A_{2t-1} be the skeleton of σ'''_{2t-1} . From the former process, $[\sigma_{2t-1}]_s$ is the sum of $[\sigma'''_{2t-1}]_s$ with some contractible loops. Therefore, since σ'''_{2t-1} and σ_{2t-1} have the same origin and terminus, we have that $\sigma'''_{2t-1} \sim \sigma_{2t-1}$.

Let s_1 be the largest index such that $\sigma'''_{2t-1}(s_1) \in V(C_{2t-1})$, and let s_2 be the smallest index such that $\sigma'''_{2t-1}(s_2) \in V(C_{2t})$. Let $\tilde{\sigma}^*_{2t-1}$ be the path induced by $\sigma'''_{2t-1}|_{[s_1, s_2]}$. Let $\tilde{\varphi}_1$ be a path from w_0 along the part of C_{2t-1} on the right-hand side of w_0 until $\tilde{\sigma}^*_{2t-1}(0)$. Let $\tilde{\varphi}_2$ be a path from $\tilde{\sigma}^*_{2t-1}(1)$ along C_{2t} on the left-hand side of $\tilde{\sigma}^*_{2t-1}$ until w_p .

To be sure that the polygon C_t^* we are constructing is freely homotopic to its predecessors, we must prove that $\tilde{\sigma}_{2t-1}|_{[0, s_1]} \circ \tilde{\varphi}_1^{-1}$ and $\tilde{\sigma}_{2t-1}|_{[s_2, 1]} \circ \tilde{\varphi}_2^{-1}$ are

contractible. We show this for the first path, the proof for the second path being similar. Let $x = \bar{\sigma}_{2t-1}(s_1)$. There is a face path of length at most t joining x to a vertex w_i of Λ' . If we combine these two face paths with the part of the face path from w_i to C_2 , we get a face path of length at most $2t + k - i$ from C_1 to C_2 . This must be at least k , and so $2t \geq i$.

Now if we use the other part of the face path Λ' , we get a face path Λ'' between two vertices of Λ of length at most $2t + i + 1$. Using the inequality at the end of the last paragraph, we have a face path between two vertices of Λ of length at most $4t + 1 \leq 4m + 1 \leq \frac{n}{2} - 3$. By Proposition 1.7.9, the face cycle formed by the combination of this face cycle with the shorter part of Λ between the endpoints of Λ'' face-represents a contractible O-circuit. It follows that $\bar{\sigma}_{2t-1}|_{[0,s_1]} \circ \bar{\varphi}_1^{-1}$ is contractible. Therefore, if $\bar{\sigma}_{2t-3}^*$ is a traversal of P_{2t-3} from C_{2t-1} to C_{2t} , we have that $\bar{\varphi}_1 \circ \bar{\sigma}_{2t-1} \circ \bar{\varphi}_2^{-1} \sim \bar{\sigma}_{2t-3}^*$. Thus, σ_{2t-1} , σ_{2t-3} , φ_1 and φ_2 bound a disc Δ_1 .

The construction of P_{2t} is similar: from σ_{2t} we remove some contractible parts and end up with a unique subsegment σ_{2t}''' between C_{2t-1} and C_{2t} . There are two indices s'_1 and s'_2 such that $\bar{\sigma}_{2t}'''|_{[s'_1,s'_2]} = \bar{\sigma}_{2t}^*$ is a path from C_{2t-1} to C_{2t} , and only its endpoints belong to $V(C_{2t-1}) \cup V(C_{2t})$. Let $\bar{\varphi}_1'$ be a path from w'_0 along the part of C_{2t-1} on the left-hand side of w'_0 until $\bar{\sigma}_{2t}^*(0)$, and let $\bar{\varphi}_2'$ be a path from $\bar{\sigma}_{2t}^*(1)$ along C_{2t} on the right-hand side of $\bar{\sigma}_{2t}^*$ until w'_q . Then, as in the case $2t - 1$, we have that $\bar{\sigma}_{2t}|_{[0,s'_1]} \circ (\bar{\varphi}_1')^{-1}$ and $\bar{\sigma}_{2t}|_{[s'_2,1]} \circ (\bar{\varphi}_2')^{-1}$ are contractible. Therefore, $\bar{\varphi}_1' \circ \bar{\sigma}_{2t}^* \circ (\bar{\varphi}_2')^{-1} \sim \bar{\sigma}_{2t-2}^*$. Hence, σ_{2t}^* , φ_1' , φ_2' and σ_{2t-2}^* bound a disc Δ_2 .

B) P_{2t-1} and P_{2t} are disjoint.

Suppose on the contrary that there is a common vertex v to P_{2t-1} and P_{2t} . Then there are two face paths, one, Λ_0 , starting at v and ending at a vertex $x \in V(\bar{g}_i \setminus \{u_i\})$ for some i , with inner vertices in $V(P_{2t-3}), V(P_{2t-5}), \dots, V(P_3)$, and another, Λ_1 , from v to a vertex $y \in V(\bar{g}_j \setminus \{u_j\})$, for some j , with inner vertices in $V(P_{2t-2}), V(P_{2t-4}), \dots, V(P_2)$. Both Λ_0 and Λ_1 have length at most t . Let Λ'' be the combination of Λ_0 and Λ_1 .

If $j - i \geq 2t + 2$, then the face path from u_0 to u_i , then through Λ'' , and then through u_j until u_k would be a shorter face path between C_1 and C_2 than Λ' , a contradiction. Therefore, we must have that $j - i \leq 2t + 2$. But then the face cycle formed by Λ'' and the part of Λ' between u_i and u_j face-represents an O-circuit β of length at most $4t + 2$. Since $t \leq m$ and $\rho_\ell(\Psi) - \ell(\Psi) - 1 \leq 0$, we have that the length of β is at most $\frac{\ell}{2}$ and therefore β is contractible. But β has a freely homotopic copy that crosses α transversely only once, a contradiction. Therefore, $P_{2t} \cap P_{2t-1} = \emptyset$.

C) $P_{2t-1} \cap P_{2t-3} = P_{2t} \cap P_{2t-2} = \emptyset$.

P_{2t-1} is a subgraph of the symmetric difference of the boundary graphs of the faces of the wheel-neighbourhoods of the vertices of P_{2t-3} , and this symmetric difference has $\bar{\sigma}_{2t-1}^*$ in one component and $w_1, \dots, w_p$ in another component. Therefore, $\bar{\sigma}_{2t-1}^*$ contains none of the vertices $w_1, \dots, w_p$ and so $P_{2t-1} \cap P_{2t-3} = \emptyset$. A similar argument leads to the conclusion that $P_{2t} \cap P_{2t-2} = \emptyset$.

D) $P_{2i-1} \cup P_{2i} \cup C_{2i-1} \cup C_{2i}$ contains a separating polygon C_i^* .

Let φ_1'' be a traversal of the part of C_{2i-1} between the endpoints of P_{2i-1} and P_{2i} , containing the endpoints of P_j for $j = 1, \dots, 2i - 1$. Let φ_2'' be the corresponding traversal in C_{2i} . By hypothesis, P_{2i-1}, P_{2i} and the parts of C_{2i-1} and C_{2i} containing the endpoints of the paths $P_1, \dots, P_{2i-2}$ bound a disc Δ_i^* containing these paths. It follows that $P_{2i-1}, P_{2i}, \varphi_1''$ and φ_2'' bound a disc Δ_i^* that contains a subdisc of Δ_{i-1}^* , and also contains Δ_1 and Δ_2 (see A)).

Since C_{2i-1} and C_{2i} bound a cylinder Q_i , we obtain that $C_i^* = P_{2i-1} \cup P_{2i} \cup (C_{2i-1} \setminus \varphi_1'') \cup (C_{2i} \setminus \varphi_2'')$ is a polygon such that $\Delta_i^* \cup Q_i$ is contained in one component of $\Sigma \setminus C_i^*$. Notice that $\Delta_i^* \cup C_i^* \cup Q_i$ is homeomorphic to a punctured torus. Therefore, since Σ is a compact surface, C_i^* is a separating polygon and one component of $\Sigma \setminus C_i^*$ contains a free homotopic copy of γ .

E) C_i^* is freely homotopic to C_{i-1}^* .

First note that C_i^* and C_{i-1}^* are separated by $((Q_i \setminus Q_{i-1}) \cup \Delta_1 \cup \Delta_2) \setminus \Delta_{i-1}^*$, which is a cylinder. Therefore, the two polygons are freely homotopic.

The reader can verify that hypothesis (1)-(6) are satisfied with $P_{2i-1}, P_{2i}, C_{2i-1}$ and C_{2i} .

3.3 Some consequences of Theorem 3.2.1.

Theorem 3.2.1 has some very interesting consequences. We first point out that Theorem 3.2.1 can tell us a little bit more about the type of separating polygons (if we assume that the genus is at least 2).

Proposition 3.3.1. *Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a 3-connected graph G onto an orientable surface Σ of genus at least 2. Suppose that $\rho(\Psi) \geq 3$, and let $(C, \mathcal{E})$ be a complete partition of $\vec{\Gamma}_\Sigma$. Let γ be a face-represented 0-circuit in $\Omega_{\mathcal{E}}$ of length $\ell_\gamma(\Psi)$ in G_Ψ . Suppose that $\gamma \in \mathcal{E}_s$. Then G_Ψ has at least $\lfloor \frac{\rho(\Psi) + \rho_{\mathcal{E}}(\Psi) - \ell_\gamma(\Psi) - 1}{8} \rfloor - 1$ noncontractible separating parallels, all isolating a freely homotopic copy of γ .*

Proof. As we have seen in the proof of Theorem 3.2.1, the polygons $C_1^*, \dots, C_m^*$ (where $m = \lfloor \frac{\rho_{\mathcal{E}_1}(\Psi) + \rho_{\mathcal{E}_2}(\Psi) - \ell_{\gamma^1}(\Psi) - 1}{8} \rfloor - 1$) each bound a punctured torus. Since Σ is a surface of genus at least 2, the "capping" of the other t -component of $\Sigma \setminus C_i^*$ with a closed disc is a nontrivial surface. Therefore, C_i^* is noncontractible, for $i = 1, \dots, m$. $\square$

It is interesting to compare this last result with (1) of Theorem 3.2.1: if γ is separating, we get approximately 4 times as many noncontractible separating and freely homotopic parallels.

Remark. Suppose we have an embedding $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ of a graph G onto an orientable surface of genus at least 2. Proposition 3.3.1 tells us that G_Ψ has at least one noncontractible separating polygon if $\rho(\Psi) \geq 17$. However, if we want only one such polygon, we only need $\rho(\Psi) \geq 6$: one of the major reasons is that we don't need two more parallels C_3 and C_4 in the construction of only one separating polygon C_1^* . The reader can refer to the proof of Theorem 7 in [BMR] for more details.

One of the obvious consequences of Theorem 3.2.1, in the spirit of Theorem 2.3.2, is the following result, which was proved in a more elementary version in [BMR].

Proposition 3.3.2. *Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a connected graph G onto an orientable surface Σ of genus at least 2. Let $(\mathcal{C}, \mathcal{E})$ be a complete partition of Σ .*

- (1) *Suppose that $\mathcal{E} \subseteq \mathcal{C}_s$. Then G_Ψ contains at least $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ noncontractible separating parallels belonging to $\Omega_{\mathcal{E}}$.*
- (2) *Suppose that $\mathcal{E} \subseteq \mathcal{E}_s$. Then G_Ψ contains at least $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor - 1$ noncontractible separating parallels.*

Proof. (1) This is Proposition 2.3.2 with $(\mathcal{C}_1, \mathcal{E}_1) = (\mathcal{C}_0, \mathcal{E}_0)$ and $(\mathcal{C}_2, \mathcal{E}_2) = (\mathcal{C}, \mathcal{E})$.
 (2) We apply Proposition 3.3.1. Notice that $\rho(\Psi) = 1$ and $\rho(\Psi) = 2$ are trivial. So suppose that $\rho(\Psi) \geq 3$. By Proposition 1.8.12, G_Ψ has a unique noncontractible cleavage unit K^* . By Proposition 1.8.13, $\rho_{\mathcal{E}_1}(\Psi) = \rho_{\mathcal{E}_1}(\Psi|_{K^*})$. Therefore, without loss of generality, we can assume that G_Ψ is 3-connected and that Ψ is a wheel-neighbourhood embedding. By Theorem 1.6.14, there is a face-represented O-circuit of $\Omega_{\mathcal{E}_1}$ such that $\ell_{\gamma}^{\mathcal{C}_1}(\Psi) = \rho_{\mathcal{E}_1}(\Psi)$. $\square$

Now, let us consider the case where $(\mathcal{C}, \mathcal{E}) = (\mathcal{C}_0, \mathcal{E}_0)$. We then obtain the following result, which appeared in [BMR]:

Proposition 3.3.3. *Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a 3-connected graph G onto an orientable surface Σ of genus at least 2. Suppose that $\rho(\Psi) \geq 3$, and let γ be a noncontractible O-circuit of length $\ell_\gamma(\Psi)$ in G_Ψ .*

- (1) *Suppose that $\gamma \in \mathcal{C}_s$. Then G_Ψ has at least $\rho(\Psi) - \lfloor \frac{\ell_\gamma(\Psi)}{2} \rfloor - 1$ noncontractible separating parallels, all freely homotopic to γ .*
- (2) *Suppose that $\gamma \in \mathcal{E}_s$. Then G_Ψ has at least $\lfloor \frac{\rho(\Psi) - \lfloor \frac{\ell_\gamma(\Psi)}{2} \rfloor - 1}{2} \rfloor - 1$ noncontractible separating parallels, all isolating a freely homotopic copy of γ .*

Proof. We put $(\mathcal{C}, \mathcal{E}) = (\mathcal{C}_0, \mathcal{E}_0)$ in Proposition 3.2.1. The fact that the parallels obtained are noncontractible comes from an argument similar to the one used in the proof of Proposition 3.2.1. $\square$

Richter and Vitray [RV2] have given another proof of Proposition 3.3.3 using a re-embedding of the graph in a surface of lower genus.

Definition 3.3.1. Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a connected graph G onto a surface of genus at least 2. An $n \times m$ grid in G_Ψ is a set $\{\{C_1, \dots, C_m\}, \{C'_1, \dots, C'_m\}\}$ of two sets of polygons of G_Ψ such that

- (1) $\{C_1, \dots, C_n\}$ and $\{C'_1, \dots, C'_m\}$ are two sets of parallels of G_Ψ ; and
- (2) for $i = 1, \dots, n$ and $j = 1, \dots, m$, there are two O -circuits γ_i, γ'_j such that $\gamma_i \sim_f C_i$ and $\gamma'_j \sim_f C'_j$, and γ_i crosses γ'_j transversally.

We will now consider a result that is a hybrid of the results of Theorem 2.2.2 and Theorem 3.2.1.

Proposition 3.3.4. Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a connected graph onto an orientable surface Σ of genus at least 2. Suppose that Σ contains a nonseparating face-represented O -circuit γ such that $\ell_\gamma(\Psi) = \rho(\Psi)$. Then G_Ψ has an $n \times m$ grid with $n = \lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ and $m = \lfloor \frac{\rho(\Psi)-1}{4} \rfloor$.

Proof. We give a sketch of the proof, since the techniques involved are similar to those of the proofs of Theorem 2.2.2 and Theorem 3.2.1.

Let $\Lambda = v_0, e_1, v_1, \dots, e_\rho, v_\rho$ be a face cycle that face-represents the nonseparating O -circuit γ . Let $C_1, \dots, C_n$ be the n parallels of G_Ψ freely homotopic to γ constructed in Theorem 2.2.2 on both sides of Λ .

Since γ is nonseparating, there is face path $\Lambda' = u_0, g_1, u_1, \dots, u_{k-1}, g_k, u_k$ between a vertex u_1 of C_1 and a vertex u_k of C_2 that is contained in the open cylinder bounded by C_1 and C_2 , but that does not contain $C_3, \dots, C_n$. We take Λ' to be the shortest such face path.

Let r, s be the two indices in $[1, \dots, \rho(\Psi)]$ such that $u_0 \in V(\text{bd}(f_r) \setminus \{v_r\})$ and $u_k \in V(\text{bd}(f_s) \setminus \{v_{s-1}\})$. We assume that $u_0 \neq v_r$ and that $u_k \neq v_s$, the proof being similar if one of the conditions $u_0 = v_r$ and $u_k = v_s$ is satisfied.

Without loss of generality, suppose that $u_0, f_r, v_{r+1}, \dots, v_{s-1}, f_s, u_k$ is the shortest part of Λ between the two vertices v_r and v_s . Therefore, we have a face cycle $u_0, g_1, u_1, \dots, u_k, f_s, v_{s-1}, \dots, v_{r+1}, f_r, u_0$ (which we rewrite as $x_0, f'_1, x_2, \dots, x_{l-1}, f'_l, x_l$) that is transversal to Λ . Let Λ_0 be this face cycle, and let γ_0 be an O-circuit through this face cycle.

Now suppose that we have constructed $t - 1$ parallels to γ_0 . We construct a path $\bar{\sigma}_t$ with the part of the wheel-neighbourhoods of the vertices of C'_{t-2} on the side of C'_{t-2} opposite to C'_{t-4} , as in the proof of Theorem 2.2.2.

Now suppose that there is a vertex v of multiplicity greater than 2 in $\bar{\sigma}_t$. Then there are two face paths between v and vertices x_i and x_j of Λ_0 of length at most $\lfloor \frac{t}{2} \rfloor$. Assume that these two face paths have no internal vertex in common (the other case is similar to the corresponding case in the proof of Theorem 2.2.2), and let Λ_1 be the combination of these two face paths.

Suppose that $|j - i| > 2\lfloor \frac{t}{2} \rfloor + \lfloor \frac{\rho(\Psi)}{2} \rfloor$. Then we have a shorter face path through Λ_1 and a part of Λ' between C_1 and C_2 , a contradiction.

If, on the contrary, $|j - i| \leq 2\lfloor \frac{t}{2} \rfloor + \lfloor \frac{\rho(\Psi)}{2} \rfloor$, then we have a face cycle through Λ' and the short part of Λ_0 between x_i and x_j of length at most $\rho(\Psi) - 1$, and so this face cycle face-represents a contractible O-circuit. We can therefore delete a contractible subcircuit from σ_t . We repeat this argument until we end up with a unique noncontractible O-circuit that is disjoint from C'_{t-2} and that is freely homotopic to γ_0 , as in the proof of Theorem 2.2.2. If we suppose now that there is a common vertex v to C'_t and C'_{t-1} , then similar arguments to those above lead to the type of contradictions of part II B) in the proof of Theorem 3.2.1. $\square$

CHAPTER IV

APPLICATION TO HAMILTONICITY

4.1 Introduction.

We will now use the methods of Chapter III and apply them to the question of Hamiltonicity of 5-connected triangulations of the torus.

Note that we will now use vertical n -connectivity instead of n -connectivity, and this all through the present chapter. Thus each time we mention " n -connected", we really mean "vertically n -connected". As we have seen in Section 2 of Chapter I, there is no problem doing so.

Some answers have been brought to the problem of Hamiltonicity: Whitney [Wh] proved in 1931 that every 4-connected planar triangulation is Hamiltonian. In 1956, Tutte [Tu3] generalized Whitney's result to 4-connected planar graphs in general. Thomassen [Th1] showed in 1983 that 4-connected planar graphs are actually Hamiltonian-connected, that is, any two vertices are linked by a path containing all the vertices of the graph. The question has also been considered for other surfaces than the plane: Thomas and Yu [TY] have proved in 1991 that 4-connected projective planar graphs are Hamiltonian-connected. The same authors claim that 5-connected toroidal graphs are Hamiltonian-connected. The proof seems, however, quite long [Ths]; we present here a proof for an easier case that is relatively short. Triangulations are not "exotic" cases of toroidal embeddings: one can show, using Euler's characteristic, that every 5-connected toroidal graph has at least $\frac{2}{3}$ of its faces that are triangles.

In this chapter, we prove the following theorem:

Theorem 4.1.1. *Let $\Psi : G \rightarrow G_\Psi \subset \Sigma_1 \cup \mathcal{P}(\Sigma_1)$ be an embedding of a 5-connected graph G onto the torus Σ_1 such that G_Ψ is a triangulation. Then G_Ψ is Hamiltonian and has a contractible Hamilton polygon.*

Theorem 4.1.1 is an immediate consequence of the following theorems:

Theorem 4.1.2. *Let $\Psi : G \rightarrow G_\Psi \subset \Sigma_1 \cup \mathcal{P}(\Sigma_1)$ be an embedding of a 5-connected graph G onto the torus Σ_1 such that G_Ψ is a triangulation. Then G_Ψ has a spanning 2-connected internally 5-connected planar subgraph with at most four peripheral humps.*

Theorem 4.1.3. *Every 2-connected internally 5-connected planar graph with at most four peripheral humps is Hamiltonian.*

Theorem 4.1.2 will be proved in the next section, and Theorem 4.1.3 will be proved in section 4.3.

4.2 Planar subgraphs of toroidal embeddings.

Our first objective is to prove Theorem 4.1.2, that is, we construct a 2-connected spanning planar subgraph H of G .

Definition 4.2.1. *A graph $H(C_1, C_2)$ of G_Ψ is cylindrical if there are disjoint polygons C_1 and C_2 of $H(C_1, C_2)$ and a closed cylinder $Q = S^1 \times [0, 1] \subseteq \Sigma$, where S^1 is the unit circle, such that $G_\Psi(\Sigma) \cap Q = H(C_1, C_2)(\Sigma)$ and $C_1 \cup C_2 = \partial Q = S^1 \times \{0, 1\}$. The boundary of $H(C_1, C_2)$ is $\partial Q = C_1 \cup C_2$.*

Lemma 4.2.1. Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be a wheel-neighbourhood embedding such that G_Ψ has a cylindrical subgraph $H(C_1, C_2)$ with boundary $C_1 \cup C_2$.

Then:

- (1) $H(C_1, C_2)$ is 2-connected;
- (2) if $\{x, y\}$ is a pair of hinges of a 2-separation of $H(C_1, C_2)$, then either $\{x, y\} \subseteq V(C_1)$ or $\{x, y\} \subseteq V(C_2)$.
- (3) let (L, K) be a 2-separation of $H(C_1, C_2)$, and suppose that the corresponding pair of hinges $\{x, y\}$ is contained in $V(C_i)$, for some i in $\{1, 2\}$.

Then $L \cap C_i$ and $K \cap C_i$ are both arcs with at least three vertices each.

Proof. (1) Since the closures of all faces of G_Ψ are closed discs, it follows that the closures of all the faces of $H(C_1, C_2)$ in Q are also closed discs, and so the boundary walk of any face of $H(C_1, C_2)$ in Q contains no repeated vertex. If v is a cut vertex of $H(C_1, C_2)$, then there is a face of $H(C_1, C_2)$ in Q whose boundary walk contains v as a repeated vertex, a contradiction. Therefore, $H(C_1, C_2)$ is 2-connected.

(2) Let $H(C_1, C_2)$ contain a 2-separation (L, K) with corresponding pair of hinges $\{x, y\}$.

Suppose that $\{x, y\} \not\subseteq V(C_1)$ and $\{x, y\} \not\subseteq V(C_2)$. We first show that there are two faces $f, f' \subset Q$ of $H(C_1, C_2)$ and an O-circuit γ such that:

- (a) $\{x, y\} \subseteq \bar{f} \cap \bar{f}' \cap \gamma$;
- (b) $\gamma \subseteq f \cap f' \cup \{x, y\}$;
- (c) $Q \setminus \gamma$ is composed of exactly two t-components Σ_1 and Σ_2 such that $L \subseteq \bar{\Sigma}_1$ and $K \subseteq \bar{\Sigma}_2$.

Since $\{x, y\}$ is a pair of hinges of $H(C_1, C_2)$, y is a cut vertex of $(H(C_1, C_2) - \{x, y\}) \cup C_1 \cup C_2$. Therefore, there is a face $g \subseteq Q$ whose boundary walk contains y as a repeated vertex. Now, since $H(C_1, C_2)$ is 2-connected, there is an arc A

in $H(C_1, C_2)$ between two vertices of $\text{bd}(g)$ distinct from y , such that A divides g into two faces g_1 and g_2 whose boundary walks each contain an occurrence of y . Therefore, there are two faces $f \subseteq g$, $f' \subseteq g'$ in Q which are incident with x and y .

Now there is an O-circuit γ in $f \cup f'$ such that $\gamma \cap (\partial f \cup \partial f') = \{x, y\}$. Therefore (a) and (b) are proved.

Since $\gamma \cap Q$ has at most one point in each of C_1 and C_2 , (c) follows.

On the other hand, since Ψ is a wheel-neighbourhood embedding, f and f' belong to the wheel-neighbourhood of x and so one of $\bar{\Sigma}_1$ and $\bar{\Sigma}_2$ is a disc that contains x , y and the edge xy . This is a contradiction with (c). Therefore, if a 2-separation of $H(C_1, C_2)$ exists, its corresponding pair of hinges $\{x, y\}$ is either a subset of $V(C_1)$ or $V(C_2)$.

(3) The vertices x and y determine two arcs A_i and A'_i in C_i such that $C_i = A_i \cup A'_i$. Suppose that one of them, say A_i , contains only x and y as vertices. Then there is a face $f' \in Q$ such that $xy \in f'$. Now since $\{x, y\}$ is a pair of hinges of $H(C_1, C_2)$, and for similar reasons as (a), (b), (c), there is another face $f \in Q$, incident with x and y , and an O-circuit $\gamma \subseteq f \cup f' \cup \{x, y\}$ with $\{x, y\} \subseteq \gamma$ such that either:

(a') the closure of one of the t-component of $Q \setminus \gamma$ contains exactly one of K and L ; or

(b') γ is a noncontractible O-circuit of Σ .

But (a') implies that $\{x, y\}$ is a pair of hinges of G_Ψ , and (b') implies that $\rho(\Psi) = 2$. Each of these implications is a contradiction with the hypothesis that Ψ is a wheel-neighbourhood embedding, by Proposition 1.8.6. Therefore, each of A_i and A'_i contains at least three vertices.

Also, since Ψ is a wheel-neighbourhood embedding, $K - \{x, y\}$ and $L - \{x, y\}$ are both connected. Therefore, each of K and L contain exactly one of A_i and A'_i .

□

To construct a spanning cylindrical subgraph of G_Ψ , we will construct two non-contractible parallels, and these polygons will be the boundaries of a 3-connected spanning cylindrical subgraph of G_Ψ . We will then use this cylindrical graph to obtain a planar subgraph of G_Ψ satisfying the conclusion of Theorem 4.1.2.

The proof technique used will be similar to those used in the proofs of Theorem 2.2.2 and Theorem 2.3.1. However, the reader will notice that the arguments are greatly simplified by the fact that we have a triangulation, as we show in the following propositions.

Proposition 4.2.2. *Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding such that G_Ψ is a triangulation. If γ is a face-represented O-circuit of Σ , then G_Ψ contains a polygon freely homotopic to γ .*

Proof. Let $\Lambda = v_0, f_0, v_1, \dots, f_l, v_l$, with $v_l = v_0$, be a face cycle in G_Ψ that face-represents γ . Since G_Ψ is a triangulation, each face f_i has in its boundary an edge e_i such that $e_i = v_{i-1}v_i$, for $i = 1, \dots, l$. Let C be the subgraph of G_Ψ with traversal $v_0, e_1, v_1, \dots, e_l, v_l$. Because Λ is a face polygon, C is a polygon. Since e_i bounds a part of the face f_i between v_{i-1} and v_i for $i = 1, \dots, l$, we have that Λ face-represents C as an O-circuit. It follows that $C \sim_f \gamma$. $\square$

Corollary 4.2.3. $\rho(\Psi) \geq 3$.

Proof. Since each face of G_Ψ is a triangle and is therefore homeomorphic to a closed disc, it follows, from Proposition 1.8.4, that $\rho(\Psi) \geq 2$.

Suppose now that $\rho(\Psi) = 2$. Then there is an O-circuit γ and two vertices $\{u, v\}$ such that $\gamma \cap G_\Psi(\Sigma) = \{u, v\}$, and therefore, there is a face polygon $\Lambda = u, f_1, v, f_2, u$ face-representing γ . Hence, there are two distinct edges $e_1 \in \text{bd}(f_1)$ and $e_2 \in \text{bd}(f_2)$, and these two edges have the same endpoints u and v . But this is a contradiction with the fact that G_Ψ is simple. $\square$

Lemma 4.2.4. *Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding such that G_Ψ is a triangulation. Let C be a shortest noncontractible polygon in G_Ψ . Let $v, w \in V(C)$. Then:*

- (1) *if v and w are not adjacent in C , they are not adjacent in G_Ψ ; and*
- (2) *if v and w are not adjacent in C and have no common neighbour in C , then v and w have no common neighbour in G_Ψ .*

Proof. Let $\rho = \rho(\Psi) = |V(C)|$, and let $v_0, e_1, v_1, \dots, e_\rho, v_\rho$, with $v_\rho = v_0$, be a traversal of C . By hypothesis, there are two indices i and j such that $v = v_i$ and $w = v_j$.

1) Suppose, on the contrary, that $v_i v_j \in E(G_\Psi)$. Then by Proposition 1.7.9 (with $(\mathcal{C}_1, \mathcal{E}_1) = (\mathcal{C}_2, \mathcal{E}_2) = (\mathcal{C}_0, \mathcal{E}_0)$), one of the cycles $v_{i-1}, e_i, v_i, v_i v_j, v_j, e_{j+1}, v_{j+1}, \dots, v_{i-1}$ and $v_i, e_{i+1}, v_{i+1}, \dots, e_j, v_j, v_j v_i, v_i$ is the traversal of a noncontractible polygon. Since both polygons have length less than ρ , we have a contradiction with the minimality of ρ . Therefore, $v_i v_j \notin E(G_\Psi)$. This proves (1).

Suppose now that $v_i, v_j \in V(C)$, but suppose that they are not adjacent in C and have no common neighbour in C . Suppose also that v_i and v_j have a common neighbour u in G_Ψ . Then by Proposition 1.7.9, one of the cycles $v_{i-1}, e_i, v_i, v_i u, u, u v_j, e_{j+1}, v_{j+1}, \dots, v_{i-1}$ and $v_i, e_{i+1}, v_{i+1}, \dots, e_j, v_j, v_j u, u v_i, v_i$ is the traversal of a noncontractible polygon. Since both polygons have length less than ρ , we have a contradiction with the minimality of ρ . Therefore, v_i and v_j do not have a common neighbour in G_Ψ . This proves (2). $\square$

We now begin the construction of $H(C_1, C_2)$.

Lemma 4.2.5. *Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a 5-connected graph G onto the torus $\Sigma = S^1$ such that G_Ψ is a triangulation, and let C be a shortest noncontractible polygon in G_Ψ . Then there is another polygon C' such that:*

- (1) C and C' are disjoint;
- (2) $C \cup C'$ is the boundary of a 2-connected spanning cylindrical subgraph $H(C, C')$ of G_Ψ ;
- (3) C does not contain any pair of hinges of $H(C, C')$.

Proof. (1) and (2) Let $\rho = \rho(\Psi) = |V(C)|$, and let $v_0, e_1, v_1, \dots, e_\rho, v_\rho$, with $v_\rho = v_0$, be a traversal of C . Since Σ is orientable, this traversal determines a right-hand side and a left-hand side to C .

Since $\rho \geq 3$ by Corollary 4.2.3 and since G_Ψ is 5-connected by hypothesis, we have, by Proposition 1.8.6, that Ψ is a wheel-neighbourhood embedding. For $i = 1, \dots, \rho$, let N_i be the wheel-neighbourhood of v_i , and let A_i and A'_i be respectively the component of the graph $N_i - \{v_{i-1}, v_i, v_{i+1}\}$ on the left-hand side and on the right-hand side of C . Let $C' = \bigcup_{i=1}^\rho A_i$.

We first prove that C' is a polygon. Since G_Ψ is a triangulation, A_i and A_{i+1} have a common end, for $i = 1, \dots, \rho$. It follows that the subgraph C' is Eulerian (the degree of each vertex is even).

Suppose that C' is not 2-regular. Let v be a vertex of C' that has degree greater than 2 in C' . Let i be the smallest index such that v is a neighbour of v_i , and let j be the largest integer such that v is a neighbour of v_j . Suppose first that $j = i + 1$.

Let C'_0 be the polygon whose cycle is $(v_i, v_i v, v, v v_{i+1}, v_{i+1}, v_{i+1} v_i, v_i)$. First, suppose that $\rho \geq 4$. If C'_0 is noncontractible, then we have a contradiction since

the length of C'_0 is 3. Therefore, C'_0 is contractible and it bounds an open disc Δ that does not contain v_{i+2} (for otherwise all of C' would be contractible). Note that since v has degree greater than 2, G_Φ is a triangulation and since C' is a subgraph of the wheel-neighbourhoods of the vertices of C , there is at least one vertex v' of C' inside Δ . Therefore, $\{v, v_i, v_{i+1}\}$ is a 3-cut of G_Φ that separates v' from v_{i+2} . But this is a contradiction with the connectivity of G_Φ .

Now, suppose that $\rho = 3$. If C'_0 is contractible, then as in the last paragraph, there is an open disc Δ bounded by C'_0 . Since G_Φ is a triangulation, the part of C' outside Δ contains at least one vertex v'' . But then $\{v, v_i, v_{i+1}\}$ is a 3-cut of G_Φ that separates v' from v'' , a contradiction. If C'_0 is noncontractible, then the polygon C'_1 whose cycle is $(v, vv_i, v_i v_{i-1}, v_{i-1}, v_{i-1} v_{i+1}, v_{i+1}, v_{i+1} v, v)$ is contractible since Σ is the torus. Then C'_1 bounds a disc, and so $\{v, v_{i-1}, v_i, v_{i+1}\}$ is a 4-cut of G_Φ that separates v' from v'' , again a contradiction.

From the two last paragraphs, we conclude that i and j cannot be consecutive. However, by Lemma 4.2.4 (2), v_i and v_j cannot be at a distance greater or equal to 3 in C' from each other. Therefore, v_i and v_j must be at a distance 2 from each other, that is, they must have a common neighbour in C , say w . Without loss of generality, let $w = v_{i+1}$ and let $v_j = v_{i+2}$. Let C'_0 be the polygon whose cycle is $(v, vv_i, v_i, v_i v_{i+1}, v_{i+1}, v_{i+1} v_{i+2}, v_{i+2}, v_{i+2} v, v)$.

First, suppose that $\rho \geq 5$. Then C'_0 is contractible and it bounds an open disc Δ that does not contain v_{i+3} (for otherwise all of C' would be contractible). As for the case $j = i + 1$, there is at least one vertex v' of C' inside Δ . Therefore, $\{v, v_i, v_{i+1}, v_{i+2}\}$ is a 4-cut of G_Φ that separates v' from v_{i+3} . But this is a contradiction with the connectivity of G_Φ .

Now suppose that $3 \leq \rho \leq 4$. If C'_0 is contractible, then as in the last paragraph, $\{v, v_i, v_{i+1}, v_{i+2}\}$ is a 4-cut of G_Φ that separates v' from v'' , a contradiction. So

suppose that C'_0 is noncontractible. Then since Σ is the torus, the polygon $C'_1 = (C - \{v_{i+1}\}) \cup \{v, vv_i, vv_{i+2}\}$ is contractible. But then C'_1 bounds a disc that separates v' from v'' . Hence $V(C'_1)$ is a cut set of G_Φ of size at most 4, that separates v' from v'' . This is a contradiction with the connectivity of G_Φ .

It therefore follows from the previous discussion that v is a vertex of degree 2 in C' . Therefore, C' is 2-regular, hence a polygon.

Since C' comes from the wheel-neighbourhoods of vertices of C , we have that C and C' are disjoint. One can show, using arguments similar to those used in the proof of Theorem 2.2.2, that C and C' are freely homotopic as O-circuits. Since they are also non-contractible, they bound a cylindrical subgraph $H(C, C')$ of G_Φ . By construction, $H(C, C')$ is spanning and by Lemma 4.2.1 (1), $H(C, C')$ is 2-connected.

(3) By Lemma 4.2.1, if $\{x, y\}$ is a pair of hinges of a 2-separation of $H(C, C')$, then either $\{x, y\} \subseteq V(C)$ or $\{x, y\} \subseteq V(C')$.

Suppose that C contains $\{x, y\}$. Then there are indices i, j such that $v_i = x$ and $v_j = y$. If v_i and v_j are adjacent in C , then neither of v_i nor v_j would have a wheel-neighbourhood in G_Φ , a contradiction with the hypothesis. Suppose then that v_i and v_j are not adjacent in C . Since they form a pair of hinges, there is a face and therefore an edge that is incident with them. But this is a contradiction with Lemma 4.2.4 (1). We thus have proved that C contains not pair of hinges of $H(C, C')$. It follows that if (L, K) is a 2-separation of $H(C, C')$ and $\{x, y\}$ is the corresponding pair of hinges, then $\{x, y\} \subseteq V(C')$. $\square$

Remark. Notice, in the last proof, that since C is a connected subgraph with neither of the vertices x, y , it is contained in exactly one of L and K . Hence if (L, K) is a 2-separation of $H(C, C')$, then (1) either $C \subseteq L$ or $C \subseteq K$, and (2) $C' \cap L, C' \cap K$ are arcs.

Lemma 4.2.6. Let $\Psi : G \rightarrow G_\Psi \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a 5-connected graph G onto the torus $\Sigma = S^1$ such that G_Ψ is a triangulation. Then there are two polygons C_1 and C_2 in G_Ψ such that:

- (1) C_1 and C_2 are disjoint and freely homotopic;
- (2) C_1 and C_2 bound a 3-connected spanning cylindrical subgraph $H(C_1, C_2)$ of G_Ψ .

Proof. Choose two polygons C_1 and C_2 in G_Ψ with the following properties:

- (A) C_1 and C_2 are both freely homotopic to C as 0-circuits;
- (B) C_1 and C_2 bound a spanning cylindrical subgraph $H(C_1, C_2)$ of G_Ψ ;
- (C) C_1 does not contain a cut pair of $H(C_1, C_2)$;
- (D) the hump of $H(C_1, C_2)$ containing C_1 has as many vertices as possible.

Notice that, by Lemma 4.2.5 and the remark following it, there are at least two polygons that satisfy the conditions (A), (B) and (C). Therefore, there exists C_1 and C_2 in G_Ψ that satisfy (A), (B), (C) and (D).

We now show that $H(C_1, C_2)$ is 3-connected. Suppose, on the contrary, that $H(C_1, C_2)$ is not 3-connected. Then there is a 2-separation (K, L) of $H(C_1, C_2)$. Let x and y be the hinges of (K, L) . So, by (C), x and y are vertices of C_2 (see Figure XVI). Suppose that $C_1 \subseteq L$.

Since $\{x, y\}$ is a cut pair of $H(C_1, C_2)$, there is a face f incident with x, y . Because G_Ψ is a triangulation, we can assume that the part of the boundary of f in L is an edge, say e , with ends x and y . The other part of the boundary of f is an arc in K with traversal x, e', z, e'', y .

Suppose first that $|V(K)| \geq 4$. Let u be any vertex of L not in $V(C_1) \cup V(C_2)$. Since $G_\Psi - \{x, y\}$ is 3-connected, it follows from Menger's Theorem that there are three internally-disjoint arcs from z to u , and so there are distinct edges e_1, e_2, e_3 of G_Ψ that are not edges of $H(C_1, C_2)$, with corresponding end vertices $u_1, u_2, u_3 \in V(C_2 \cap K)$ and $u'_1, u'_2, u'_3 \in V(C_1 \cap L)$, and such that $u_1, u_2, u_3, u'_1, u'_2, u'_3$ are distinct.

Choose u_1 as in last paragraph such that it is the nearest to x but distinct from x , and choose also u_3 as in last paragraph such that it is nearest to y but distinct from y .

Figure XVI

We now modify C_1 by making it "enter" into K through e_1 and e_3 . Suppose that there is a vertex z' such that $z' \in V((N(x)) \cap V((N(y)) \cap K$ and $z' \neq z$. Then, in $G_\Phi - \{x, y, z'\}$, z would be disconnected from C_1 , a contradiction with the connectivity of G_Φ .

Now let $x_1, \dots, x_n$ be the neighbours of x in K in their order of appearance as we go from the polygon C_2 to the vertex z (we put $x_n = z$), and let $y_1, \dots, y_m$ be the neighbours of y in K in their order of appearance as we go from the polygon C_2 to the vertex z (we put $y_m = z$). Then there is an arc P from u_1 to x_1 in $C_2 \cap K$, and then through $x_2, \dots, x_m = z = y_m, \dots, y_1$ and to u_3 in $C_2 \cap K$. Let P' be the $u'_1 u'_3$ arc obtained from C_1 by removing the part of C_1 between u'_1 and u'_3 through u'_2 . Let $C'_1 = P' \cup e_1 \cup P \cup e_3$ and let C'_2 be the polygon obtained by removing $K \cap C_2$ and replacing it by $\{x, e, y\}$.

Suppose that there is a cut pair $\{x', y'\}$ of $H(C'_1, C'_2)$ such that $\{x', y'\} \subseteq V(C'_1)$. We cannot have $\{x', y'\} \subseteq V(P')$, for then $\{x', y'\}$ would be a cut pair of $H(C_1, C_2)$ in C_1 , a contradiction with (C). It follows that either $\{x', y'\} \subseteq V(P)$, or one of these vertices, say x' , is in $V(P)$ and y' is in $V(P')$.

Suppose that $\{x', y'\} \subseteq V(P)$. Then $\{x, y, x', y'\}$ is a 4-cut of G_Φ that separates z from L . This is impossible because of the connectivity of G_Φ .

Suppose that $x' \in V(P)$ and $y' \in V(P')$. Then there is an interior face of $H(C'_1, C'_2)$ incident with both x' and y' . But this can happen only if x' and y' are the ends of e_1 , and then $\{x', y'\}$ is not a cut pair.

Hence, C'_1 contains no cut pair of $H(C'_1, C'_2)$. Notice that we have replaced a part of C_2 in C'_2 by the edge e , so that if a new cut pair is created, it would have been in C_2 already. Hence the hump containing C'_1 has more vertices than the hump containing C_1 , and it is easy to see that C'_1 and C'_2 satisfy the conditions (A), (B) and (C). Thus, we have a contradiction with the choice of C_1 and C_2 . Therefore, $H(C_1, C_2)$ is 3-connected.

Finally, suppose that $V(K) = \{x, y, z\}$, and let u be a vertex of L but not in $V(C_1) \cup V(C_2)$. Since $G_\Phi - \{x, y\}$ is 3-connected, it follows from Menger's Theorem that there are three internally-disjoint arcs from z to u , and so there are three internally-disjoint edges e_1, e_2, e_3 from z , in G_Φ but not in $H(C_1, C_2)$, with corresponding end vertices $u'_1, u'_2, u'_3 \in V(C_1 \cap L)$. Rerouting C_1 through z and C_2 through the edge e makes the hump containing C_1 bigger, as above. We thus have once again a contradiction with the choice of C_1 and C_2 , and therefore $H(C_1, C_2)$ is 3-connected. This proves (1), (2). $\square$

Definition 4.2.2. Let H be a 2-connected planar graph embedded in the plane with outer cycle C . A hump K is peripheral relative to C if $K \cap C$ is an arc.

We can now proceed to the proof of Theorem 4.1.2.

Proof of Theorem 4.1.2. Let $H(C_1, C_2)$ be the 3-connected spanning cylindrical subgraph obtained from Lemma 4.2.6. Let P_1 be a shortest arc from C_2 to C_1 in $H(C_1, C_2)$, and let $u_0, e_1, u_1, \dots, e_k, u_k$ be a traversal of P_1 with $u_0 \in V(C_2)$ and $u_k \in V(C_1)$.

Let N_i be the wheel-neighbourhood of u_i in G_Φ , for $i = 0, \dots, k$. The indexing of the vertices u_i gives an orientation to P_1 , and therefore determines a right-hand side and a left-hand side to P_1 . Clearly, u_{i-1} and u_{i+1} belong to $V(N_i)$, and $N_i - \{u_{i-1}, u_i, u_{i+1}\}$ consists of two subgraphs A_i and A'_i , with A_i on the left-hand side of P_1 and A'_i on the right-hand side of P_1 , for $i = 1, \dots, k-1$. Notice that the indexing of the u_i allows us to describe $\bigcup_{i=0}^k A_i$ by a walk W . Let A be the connected subgraph of $\bigcup_{i=0}^k A_i$ that corresponds to the subwalk of W that starts at a vertex of C_2 , ends at a vertex of C_1 , and such that its internal vertices do not belong to $V(C_1) \cup V(C_2)$. Since A is built from the wheel-neighbourhoods of vertices of P_1 where we remove these vertices, it follows that A is an arc disjoint from P_1 . Let $P_2 = A$. If we delete the edges of $H(C_1, C_2)$ with one end in P_1

and the other end in P_2 , we obtain a 2-connected planar spanning subgraph H_0 whose outer boundary is a separating polygon $C_0^* \subseteq P_1 \cup P_2 \cup C_1 \cup C_2$. The arc P_1 contains no cut-pair and we can arrange for P_2 to have none, as in the second part of the proof of Lemma 4.2.3. We then obtain a new 2-connected planar spanning subgraph H . Let C^* be its outer boundary. Now H has at most four peripheral humps, that is, those which have one hinge in P_i and one hinge in C_j , for some $i, j \in \{1, 2\}$. $\square$

4.3 Hamiltonicity of 5-connected toroidal triangulations.

In the last section, we have seen that if G_Φ is a 5-connected toroidal triangulation of the torus, then G_Φ has a spanning 2-connected internally 5-connected planar subgraph H with at most four peripheral humps. In the present section, we construct a Hamilton cycle through H .

In order to prove Theorem 4.1.3, we will prove a stronger result.

Theorem 4.3.1. *Let H be a 3-connected planar graph. Let C be a face boundary of H , and suppose that H is internally 5-connected relative to C . Let e_1, e_2, e_3, e_4 be any edges of C . Then H has a Hamilton cycle containing all of e_1, e_2, e_3, e_4 .*

For the proof of Theorem 4.3.1, we use the following result due to Richter. For its proof, we refer the reader to [BR]. It is a straightforward generalization of the proof in [Th1] and therefore is omitted here.

Theorem 4.3.2. *Let H be a 2-connected planar graph and let C be a polygon in H that is a face boundary of some embedding of H in the plane. Let A be any arc in C , with ends u and v and let F be a nonempty subset of $E(A)$. Then there is an arc A^* in H with ends u and v such that:*

- (1) $F \subseteq E(A^*)$;
- (2) every bridge of A^* in H has at most $\max\{3, |F| + 1\}$ vertices of attachment;
- (3) every bridge of A^* in H that contains an edge of A has at most 2 vertices of attachment;
- (4) every bridge of A^* in H that contains an edge of C has at most $|F| + 1$ vertices of attachment.

Observe that Thomassen proved this for the case $|F| = 1$ [Th1], except that we require v to be in the boundary of C while Thomassen does not. Thomas and Yu proved Theorem 4.3.2 in the case $|F| = 2$.

Proof of Theorem 4.3.1. Let u and v be the ends of e_1 . By Theorem 4.3.2, there is an arc A^* containing e_2, e_3, e_4 , with ends u and v , such that every bridge of A^* has at most 4 vertices of attachment and every bridge of A^* containing an edge of $C - e_1$ has at most 2 vertices of attachment. Therefore, the polygon $P = A^* + e_1$ is such that every bridge of P has at most 4 vertices of attachment and every bridge of P containing an edge of C has at most 2 vertices of attachment.

By 3-connection, a bridge with at most 2 vertices of attachment is just an edge. Any bridge K of P with more than 2 vertices of attachment can have no edge in C . Therefore, $K - V(P)$ is disjoint from C . By internal 5-connection, K has at least 5 vertices of attachment, and so no such bridge of P exists. Therefore, the cycle of P is a Hamilton cycle. $\square$

Proof of Theorem 4.1.3. Let $K_1, K_2, \dots, K_n$ be the cleavage units of H . For $i = 1, \dots, n$, let m_i denote the number of virtual edges in K_i . Since each virtual edge e corresponds to a good 2-separation (H_e, L_e) of H and each of H_e and L_e contains a peripheral hump of H , it follows that $m_i \leq 4$.

If K_i is a bond, then it has exactly 2 virtual edges and a single edge of H . Moreover, whether or not K_i is a bond or not, in the embedding of K_i in the plane, all the virtual edges are in the polygon bounding the infinite face of K_i . Therefore, for $i = 1, \dots, n$, either by Theorem 4.3.1 or by a triviality, there is a Hamilton cycle P_i in K_i that contains all the virtual edges of K_i . Let P'_i be P_i with the virtual edges deleted. Then $\bigcup_{i=1}^n P'_i$ is a Hamilton cycle of H . $\square$

CHAPTER 5

SUMMARY AND CONCLUSIONS

We now give a brief summary of the new results contained in this thesis, and we will end by some open questions. Let Σ be a nontrivial surface. We have defined a loop in Σ to be a particular equivalence class of closed paths. We have introduced the notion of complete partition of loops: a partition $(\mathcal{C}, \mathcal{E})$ of the set of loops of Σ is said to be complete if $\mathcal{C}$ contains all freely homotopic copies of the loops corresponding to the elements of some normal subgroup of the fundamental group of Σ . Let $\Psi : G \longrightarrow G_\Psi \subset \Sigma$ be an embedding of a connected graph G onto Σ . Let $\rho(\Psi)$ be the representativity of the embedding.

It was previously known that if Σ is orientable, then G_Ψ has at least $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ disjoint freely homotopic noncontractible polygons. In Chapter II, we have improved this to show that G_Ψ has at least $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ disjoint freely homotopic noncontractible polygons **whose corresponding loops are in $\mathcal{E}$, for any complete partition $(\mathcal{C}, \mathcal{E})$** (the previous case was when $\mathcal{E}$ is the set of noncontractible loops of Σ). Specifically, there are $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ **nonseparating** disjoint freely homotopic polygons.

In Chapter III, we have slightly improved the known results to show that G_Ψ has at least $\lfloor \frac{\rho(\Psi)-1}{8} \rfloor - 1$ noncontractible **separating** disjoint freely homotopic polygons if the genus of the orientable surface Σ is at least 2. Bounds were also given for the case of nonorientable surfaces.

In Chapter IV, we have applied these results to show that any δ -connected toroidal triangulation is Hamiltonian.

Many open questions still remain regarding complete partitions. We can ask, for example, what are the normal subgroups of $\pi(\Sigma, x)$, and therefore, what are the corresponding complete partitions. Besides the fundamental, separating and orientation-preserving partitions, are there any other "natural" complete partitions?

Recently, Thomassen [Th3] has used arguments involving "cuts" in orientable surfaces in order to prove that 3-connected triangulations of these surfaces, with very large representativity, have spanning trees with small degree. This result was improved by Yu [Yu]. Can we use our methods to avoid the need for very large representativity? We believe that some global arguments about complete partitions are needed.

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FIN

Remarks. 1) Notice that the number of C_1 -parallels in Proposition 2.3.2 is independent of any γ and of $(C_2, \mathcal{E}_2)$.

2) There is another way to look at Proposition 2.3.2. By Theorem 1.7.3, there is a bijection between the set of complete sets of loops of Σ and the set of normal subgroups of $\pi(\Sigma, x)$ for a given $x \in \Sigma$. Therefore, for any two normal subgroups N_1 and N_2 of $\pi(\Sigma, x)$ such that $N_1 \leq N_2$, the conclusion of Proposition 2.3.2 follows with $C_i = \overrightarrow{\Gamma}_{N_i}$ and $\mathcal{E}_i = \overrightarrow{\Gamma}_{\Sigma} \setminus \overrightarrow{\Gamma}_{N_i}$, for $i = 1, 2$.

3) The bound of the conclusion of Proposition 2.3.2 is better than the corresponding bound of Theorem 2.2.2. However, in Proposition 2.3.2, Σ is orientable and we have no control on the homotopy type of the C_1 -parallels besides the fact that they belong to $\Omega_{\mathcal{E}_2}$.

Now suppose that, in Proposition 2.3.2, we take $(C_1, \mathcal{E}_1)$ to be the fundamental partition $(C_0, \mathcal{E}_0)$. Then we obtain the following interesting result.

Theorem 2.3.3. Let $\Psi : G \longrightarrow G_{\Psi} \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a connected graph G onto a nontrivial orientable surface Σ . Let $(C, \mathcal{E})$ be a complete partition of Σ . Then G_{Ψ} has a set of $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ noncontractible parallels belonging to $\Omega_{\mathcal{E}}$.

Proof. Take $(C_1, \mathcal{E}_1) = (C_0, \mathcal{E}_0)$ in Proposition 2.3.2. $\square$

Remark. Theorem 2.3.3 says that given any complete partition $(C, \mathcal{E})$ of an orientable surface Σ , G_{Ψ} contains $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ noncontractible disjoint polygons of $\Omega_{\mathcal{E}}$ all freely homotopic.

Theorem 2.3.4. Let $\Psi : G \longrightarrow G_{\Psi} \subset \Sigma \cup \mathcal{P}(\Sigma)$ be an embedding of a connected graph G onto a nontrivial orientable surface Σ . Then G_{Ψ} has a set of $\lfloor \frac{\rho(\Psi)-1}{2} \rfloor$ nonseparating parallels.

Proof. Take $(C, \mathcal{E}) = (C_0, \mathcal{E}_0)$ in Theorem 2.3.3. $\square$