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Full Name of Author — Nom complet de l'auteur

SANTIAGO MENZIE CHACÓN

Date of Birth — Date de naissance

APRIL 27, 1955

Country of Birth — Lieu de naissance

COSTA RICA

Permanent Address — Résidence fixe

APARTADO 6659
1000 - SAN JOSE
SAN JOSE COSTA RICA

Title of Thesis — Titre de la thèse

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Name of Supervisor — Nom du directeur de thèse

DR. VLASTIMIL DLAB

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Santiago Menzie Chacón

**A CATALOGUE OF INDECOMPOSABLE
PRÉPROJECTIVE REPRESENTATIONS OF
EUCLIDEAN DIAGRAMS**

by

© Santiago Menzie Chacón, B.Sc., Lic. Mat.

A thesis submitted to the Faculty of
Graduate Studies and Research in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Canada

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The undersigned recommend to the Faculty
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the degree of Doctor of Philosophy.

Dr. V. Dlab, Thesis Supervisor

Dr. B. M. Puttaswamaiah, Chairman

Department of Mathematics and Statistics

Dr. D. Happel, External Examiner

Carleton University

August, 1984

ABSTRACT

A representation $\mathbf{X} = (X_i, \tilde{\varphi}_i)$ of a diagram $\Delta = (\Delta_0, \Delta_1)$ with modulation $\mathbf{M} = (F_i, M_j)$ and orientation Ω is a set of finite dimensional F_i -vector spaces X_i , $i \in \Delta_0$ together with F_i -linear mappings

$$\tilde{\varphi}_i : X_i \longrightarrow X_i \otimes M_j$$

for $(i, j) \in \Delta_1$.

All the indecomposable preprojective representations $\mathbf{X}$ (up to isomorphism) have been catalogued by giving the vector spaces X_i and the maps $\tilde{\varphi}_i$ when Δ is an Euclidean diagram and Ω a subspace orientation.

The main tools used in this description are: K -structures, admissible subgroups, extended lattice polynomials and extension diagrams.

To my Parents

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INTRODUCTION

In recent years, the diagramatic methods gave birth to a new trend in the representation theory of finite dimensional algebras, and allowed the classification of certain classes of algebras according to their so called representation type (finite, tame or wild). For instance, the hereditary algebras of finite and tame representation type were classified in terms of valued graphs.

In the case of the finite representation type, several classes of algebras were described in terms of graphs. Results on categories of representations of diagrams associated to algebras with radical square zero were reported by Gabriel in [G-72] and extended by Dlab-Ringel in [DR-75] to finite dimensional K -algebras (K a field not necessarily algebraically closed) which are hereditary or with radical square zero. Furthermore, selfinjective algebras were studied by Riedtmann in [Rd-79] and [Rd-80], and Estrada in [E-83], and tilted algebras by Happel and Ringel in [HR-81] and [HR-82]. With respect to decomposition of modules, it was proven that if an algebra is of finite type then any module is a direct sum of modules of finite length, cf.: Auslander [A-74] and Ringel and Tachikawa [RT-74], and by Azumaya [Az-50] we know that such decomposition is unique up to isomorphism.

In the case of the wild representation type, it was proven that any finite dimensional K -algebra is the algebra of all endomorphisms of some representation of the graph D^5 , i.e. a vector space endowed with five unrelated subspaces. So a classification of the representations of D^5

contains in some sense a classification of all finite dimensional K -algebras, and is therefore presently unattainable. Cf.: [G-67], [B-67].

In the case of tame representation type, only few classes of algebras have been studied in detail; among these are the hereditary algebras. Our aim is to catalogue the indecomposable representations of the two sided indecomposable, hereditary, finite dimensional K -algebras of tame type, i.e., the tensor algebras associated with the Euclidean diagrams (also called extended Dynkin diagrams). In what follows they are called Euclidean-Tensor-algebras.

Euclidean diagrams:

Moreover, $X \rightarrow \dim X$ induces a bijection between the isomorphism classes of indecomposable representations of discrete dimension type (i.e., $\dim X$ is a positive root of the corresponding quadratic form) and the positive roots of the diagram. The positive roots are of the form $x = x_0 + rgn$ where r is a non-negative integer, x_0 is a positive root satisfying $x_0 \leq gn$, n the canonic vector and g is in $\{1, 2, 3\}$ called the tier number and depends only on the diagram. Cf. [M-69] and [DR-76a].

The positive roots x_0 are called small and the set $\{x_0 + r\alpha_n/r \in N\}$ is called a tier. Two small positive roots are said to be symmetric if there is a non-trivial automorphism of the diagram which moves one to the other.

Finally, if $\delta X \neq 0$, then $e^m \dim X = \dim X + s(\delta X)n$, where s is an integer and m is the order of $\bar{z}$ in the Weyl group $\bar{W}$. Cf: [DR-76a]. X is called regular if it is a finite direct sum of representations of defect zero. The regular modules form an abelian category which is the product of the category of homogeneous representations and h ($h \leq 3$) uniserial subcategories $R^{(t)}$ ($0 \leq t \leq h$), whose objects are determined by their simple socle and regular length. Moreover, the homogeneous representations can be obtained from a bimodule ${}_F M_G$ of $\widetilde{A}_{11}$ or $\widetilde{A}_{12}$ and a full exact embedding T from the homogeneous representations of the bimodule into the category of regular A -modules, which are listed in [DR-76a].

If the affine bimodule ${}_F M_G$ is non simple, then $M = M(\epsilon, \partial)$, where ϵ is an endomorphism of F and ∂ is an $(\epsilon - 1)$ -derivation. And for these bimodules the category of homogeneous representations is the product of an uniserial category and the category of all (right) R -modules of finite length, where R is the skew polynomial ring $F[X; \epsilon, \partial]$. Cf: [R-76].

If the affine bimodule ${}_F M_G$ is simple and $\dim_F M = \dim M_G = 2$, then the category of homogeneous representations of the bimodule ${}_F M_G$ is the category of all R -modules of finite length, where R is a certain bounded Noetherian domain [DR-to appear].

Therefore, one can see the complexity involved in the category of the homogeneous representations, of which only a few examples are well understood. For instance, L. Kronecker in [K-1890] classified pairs of complex matrices under a certain similarity relation, solving a problem posed by Weierstrass. An account of this can be found in [G-59]. Gelfand and Ponomarev in [GP-70] obtained the indecomposable representations of finite length of $\widetilde{D}_4$ with subspace orientation.

Extending Kronecker, Dlab and Ringel in [DR-77] obtained the indecomposable representations of finite length of the graph $\widetilde{A}_{12}$ with bimodule ${}_C C_R \otimes {}_R C_C$ which leads to the study of the skew polynomial ring over C in one variable with $\epsilon = id_C$ and ∂ the complex conjugation. Here the representations are used to solve a matrix problem.

In [DR-78], the indecomposable representations of finite length of the graph $\widetilde{A}_{14}$ with bimodule ${}_R H_H$ are constructed and used to solve a geometric problem of systems of subspaces.

In [DR-79], another matrix problem is solved studying $\widetilde{A}_{12}$ with the bimodule $H \oplus H$. In [D-80], Dlab constructed the indecomposable representations of finite length of $\widetilde{BC}_3$ with modulation

$$\begin{array}{ccc} & C & \\ & \downarrow & \\ R & \xrightarrow{{}_R C_C} & C \leftarrow C \end{array}$$

and gave an application of the theory to the classification of torsion free abelian groups of finite rank.

In summary, for modules of finite length, those in the uniserial categories are "easily" constructed following the algorithms given in the papers. Further, the homogeneous part has the complexity of the categories of R -modules where R is a skew polynomial ring $F[X; \epsilon, \partial]$ or a bounded Noetherian domain.

Therefore, we restrict ourselves to representations of non-zero defect. Some remarks and reductions are in place here.

Using partial Coxeter functors as in [DR-76], it is possible to go from one orientation to another (in practice, this process could be bothersome) so we restrict the study to systems of

subspaces.

One interesting feature is that the representations of non-zero defect are independent of the field K and due to this characteristic, the calculations can be done in any field. To fix ideas we can think of $K = \mathbb{Z}/2\mathbb{Z}$ and its extensions.

Note that for an arrow $i \rightarrow j$ in an Euclidean diagram with a subspace orientation, if $V = (V_i, V_0)$ is an indecomposable system of subspaces such that $\dim_{F_i} V_i = \dim_{F_i}(V_j \otimes {}_j M_i)$, then ${}_j \tilde{\varphi}_i: V_i \rightarrow V_j \otimes {}_j M_i$ is an isomorphism. In particular, we can use the constructions of "specialization" given in [R-79b].

These observations reduce the study to one algebra in each one of the 7 infinite families of Euclidean diagrams.

Next, we observe that using dual K -structures we can obtain almost all (except those which do not correspond to systems of subspaces and in general are easily constructed) the preinjective indecomposable representations out of the preprojective ones when the partial ordered set associated with the K -structure is "equal" to its opposite and the function between lattices is constant on chains.

The thesis is divided in three chapters. In chapter 1, the method used in the constructions is presented along with notations and results for later use. Section 1 of this chapter deals with K -structures, systems of subspaces and a review of the reduction steps mentioned in this introduction. It should be noted that all these, and more reductions are in force if we want to keep this catalogue

within reasonable dimensions. In subsections 3 and 7, the concepts which are the cornerstone of this work, namely, admissible subgroups, extended lattice polynomials and extension diagrams can be found.

The extension diagrams are used to find candidates when we are building up the indecomposable representations and later on they provide the key to read the tables presented in this catalogue. The introduction of these diagrams was motivated by the result in subsection 4 and the way indecomposable representations are built in the uniserial subcategories of the category of regular representations.

The extended lattice polynomials are used as a major tool in the reduction step in proofs done by induction and for the construction of enough admissible subgroups (sort of diagram chasing). The admissible subgroups are linked to the decomposition of a representation and great deal of the proofs are concerned about the existence of enough of them. The rest of the subsections bind all these ideas together in an example.

Section 2 of chapter 1 is intended to be a bridge between Euclidean diagrams with trivial and with non-trivial valuations. The main concepts to be reviewed are the adjoint maps which are obtained through an adjoint isomorphism and the basis of vector spaces which involve tensor products.

Section 3 gathers together some observations on the internal structure of systems of subspaces, and how compatible vectors are linked. It also contains a note on indecomposable amalgamated sums.

In chapter 2 we study the classical Euclidean diagrams. Each section contains one diagram and the information is divided as follows: general data, a reduction step (when applicable), the proofs and finally the indecomposable representations.

Chapter 3 is devoted to Euclidean diagrams with non-trivial valuation and it follows a format similar to chapter 2.

The beginning of the preprojective component of the Auslander-Reiten graph of Euclidean diagrams is given at the end in an appendix. The core of this work is the explicit construction of indecomposable preprojective representations as systems of subspaces of Euclidean diagrams.

Throughout this investigation the terminology (when not defined) is as in [DR-76a] and/or [R-79b].

Chapter 1

PRELIMINARIES

Representation Theory of an algebra A is concerned with the structure of the category of A -modules and thus with the building blocks of this category: the indecomposable modules.

In this chapter, we define basic concepts and bring together results which are helpful in subsequent chapters. Section 1 presents the method with an example. Section 2 deals with tensor products and the last one with some observations which are used in the study of indecomposable representations.

Throughout this work, the fields are commutative, skew-fields are not necessarily commutative fields. $K \subseteq G \subseteq F \subseteq H$ is a tower of skew-field extensions of the field K . The ET -algebras are over a field, they are associative and have units. An Euclidean diagram Δ identifies the corresponding ET -algebra. The terms module over the algebra Δ and representation of the graph Δ are interchangeable.

1.1 Motivation of the method

From the explicit description of the representations of *ET*-algebras available in the literature it is observed that they follow certain pattern; which is described somehow by the extension diagrams. Therefore, it is interesting to know if all the *ET*-algebras conform to this model. In this section, we study systems of subspaces and the extension diagrams.

1.1.1 *K*-structures

Let K be a field and F a skewfield containing K in its center. A K -structure for F is a pair (S, l) where S is a finite partially ordered set and l is an order preserving map from S into the lattice $L(F; K) = \{L/L \text{ is a finite skew-field extension of } K \text{ with } K \text{ in its center and } L \text{ is related to } F, \text{ i.e., } L \subseteq F \text{ or } F \subseteq L\}$. Denote $F_i = l(i)$, $i \in S$. For example $S = \{a\} \cup \{b\}$ the disjoint union of two unrelated points and $l: S \rightarrow L(\mathbb{C}; \mathbb{R})$ given by $l(a) = \mathbb{R}$ and $l(b) = \mathbb{H}$, the quaternions, define a real structure for the complex numbers.

An S -space (system of subspaces or filtered vector space) $W = (W_i, W)_{i \in S}$ of a K -structure for F is an F -vector space W_F , called the representation space, and F_i -subspaces W_i of $W \otimes L_i$ where $L_i = {}_F F_{F_i}$ if $F_i \subseteq F$ or $L_i = {}_F F_i F_i$ if $F \subseteq F_i$, such that $W_i \rightarrow W_j$ is an inclusion if $i \leq j$ in S .

The S -spaces form an additive category in which the morphisms $(W_i, W) \rightarrow (V_i, V)$ are F -linear mappings $\alpha: W \rightarrow V$ such that for each i in S , $(\alpha \otimes 1)(W_i) \subseteq V_i$.

Observe that $W = X \oplus Y$ means $W = X \oplus Y$ and $W_i = X_i \oplus Y_i$, for all i in S . Thus

$W = X \oplus Y$ if and only if $W = X \oplus Y$ and for each i in S :

$$W_i = W_i \cap (X \otimes L_i) \oplus W_i \cap (Y \otimes L_i)$$

1.1.2 Duality and preinjective representations

Let $S = \sqcup I_n$ be the partially ordered set which is the finite disjoint union of the chains $I_n = \{s_{1n} < s_{2n} < \dots < s_{kn}\}$ and $l: S \rightarrow L(F; K)$ be constant on each chain. Then (S, l) is a K -structure for F . Define its dual K -structure by $(S, l)^* = (S^{op}, l^{op})$ where S^{op} is the set S with the opposite order and $l^{op}: S^{op} \rightarrow L(F^{op}; K)$ is given by $l^{op}(i) = (l(i))^{op} = F_i^{op}$.

Every finite dimensional S -space W defines a S^{op} -space, namely: $(W_i, W)^* = (W_i^\perp, W^*)$ where $W^* = \text{Hom}(W, F)$ and $\{w \in (W \otimes L_i)^* / w(W_i) = 0\} = W_i^\perp \subseteq L_i^* \otimes W^*$ so $\dim W = \dim W^*$ and $\dim W_i^\perp = (\dim W_F)(\dim L_{iF_i}) - (\dim W_{iF_i})$.

Observe that $(W_i, W)^{**} \simeq (W_i, W)$ therefore $*$ defines a duality, thus it preserves indecomposability.

As was pointed out in the introduction, if $S = S^{op}$ is the disjoint union of chains, l is constant on each chain and F is commutative, then (W_i, W) and $(W_i, W)^*$ are representations of the same diagram. In these cases we can recover the preinjective representations out of the preprojective ones. As an example of the procedure we refer to [D-80]. For Euclidean diagrams, in each family there is at least one satisfying the above hypothesis, for instance, the smallest diagram in each family with the subspace orientation. In larger diagrams, W and W^* are not necessarily

representations of the same diagram, but a great deal of information is obtained when we consider subdiagrams in which $*$ is an endofunctor.

1.1.3 Admissible subgroups

A subgroup U of the abelian group $W \oplus L$ is said to be compatible with the decomposition $W = X \oplus Y$ if

$$U = U \cap (X \oplus L) \oplus U \cap (Y \oplus L)$$

U is admissible in W if it is compatible with any decomposition of W .

A subset U of $W \oplus L$ is said to be (totally) compatible with the decomposition $W = X \oplus Y$ if each element of U is either in X or in Y .

In particular if $W = (W_i, W)$ is an S -space, then

- i) For each i in S , W_i is admissible in W
- ii) If U and V are admissible subgroups in W_F and f is an element in F , then the following subgroups are admissible in W

- 1) $U + V$ sum of subgroups
- 2) $U \cap V$ intersection of subgroups
- 3) $Uf = \{w \in W / \exists u \in U, w = uf\}$
- 4) $U^i = \{u \in U / uF \subseteq U\}$ the F -interior of U
- 5) $U^c = \sum_{u \in U} uF$ the F -closure of U

Figure 1-1: The representations V^j

Cf.: [DR-75]

Given an S -space $W = (W, W)$, let $L(W)$ be the lattice of subgroups of W and $A(W)$ the sublattice of admissible subgroups in W . An extended lattice polynomial or an e -polynomial is a function $p : (L(W))^n \rightarrow L(W)$, where n is a positive integer and the only operations involved are sums, intersections, products by scalars in the skew-field F , interiors and closures.

1.1.4 Deleting one dimensional spaces

Remarks:

- i) If W is an indecomposable representation, then $L(W) = A(W)$.
- ii) The image of an e -polynomial evaluated in admissible subgroups is admissible.

iii) If U is an admissible subspace of W , then $U = (U_i, U)$, where $U_i = (U \oplus L_i) \cap W_i$ is a sub- S -space of W and it is called the U -subsystem in W . Any S -space isomorphic to U is called admissible subsystem in W .

Let $U = (U_i, U)$ be an indecomposable S -space for which $F_i = F$ for i in all S . Define the S -space W by $W = U, W_j \oplus aF = U_j$ and $W_k = U_k$ if $k \neq j$. If aF is admissible in W , then W is indecomposable. For, suppose $W = X \oplus Y$, then aF is in one of the summands, say $aF \subseteq X$, because aF is admissible in W . If we prove that $U = X' \oplus Y$, where X' is defined by $X' = X, X'_j = X_j \oplus aF, X'_k = X_k$ if $k \neq j$, then this decomposition must be trivial and then $Y = 0$. Therefore W is indecomposable. We know that if $k \neq j, U_k = W_k$ is compatible with the decomposition. So we only need to prove $U_j = U_j \cap X \oplus U_j \cap Y$. Clearly $U_j \cap X \oplus U_j \cap Y \subseteq U_j$ and $U_j \cap X \oplus U_j \cap Y = (0)$. Now $U_j \subseteq U_j \cap X \oplus U_j \cap Y$, for if $u \in U_j = W_j \oplus aF$, then $u = w + a\lambda$ for some w in W_j and λ in F . Since $W_j = W_j \cap X \oplus W_j \cap Y$, then $w = x + y$ where $x \in W_j \cap X \subseteq U_j \cap X$ and $y \in W_j \cap Y \subseteq U_j \cap Y$, so $u = (x + a\lambda) + y \in U_j \cap X + U_j \cap Y$.

1.1.5 Example $\widetilde{D}_4$: defect -1

The process described in the previous subsection allows us to delete a one-dimensional vector space which is admissible from one of the subspaces of an S -space. In the following example, we get preprojective indecomposable representations out of regular ones.

Example. Consider the graph $\widetilde{D}_4$ with subspace orientation and the generating set $O = \{E_1, E_2\}$, where E_1 and E_2 are the simple regular S -spaces of dimension types $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Figure 1-2: Representations of defect -1

respectively. Next, construct the S -spaces $V^{(2k)}$ and $V^{(2k+1)}$ of dimension types $\begin{pmatrix} k & 2k & k \\ k & 2k+1 & k+1 \end{pmatrix}$ and

where, $V^{(2k)} = \text{span}\{a_1, b_1, a_2, b_2, \dots, b_k\}$ and $V^{(2k+1)} = \text{span}\{a_1, b_1, a_2, b_2, \dots, b_k, a_{k+1}\}$.

Each vertical layer of squares corresponds to a vector in the basis of $V^{(j)}$ and each horizontal layer to a subspace, which is labeled on the left. The distinguished vectors form a basis for the subspaces. Sometimes we require the sum of several vectors in the basis of $V^{(j)}$. This is represented by a line joining the squares and called a diagonal. For instance, $V_1 = \sum_{j=1}^k a_j F$, $V_4 = \sum_{j=1}^k (a_j + b_j) F$ are the first and fourth subspaces of $V^{(2k)}$.

Let $W_i = V_i$, if $i \neq 3$ and $W_3 = \sum (b_j + a_{j+1}) F$. That is, we delete $a_1 F$ from V_3 . Define the e -polynomial

$$p(X) = (((X \cap W_2) + W_4) \cap W_1)$$

then

i) for $V^{(2k)}$, $p^{k-1}(W_1) = a_1 F$

ii) for $V^{(2k+1)}$, $p^k(W_1) = a_1 F$, where, p^j is p composed as a function with itself j times.

Observe that the effect of $p(W_1)$ is to delete the last subspace in figure 1, namely, $a_{k+1} F$ in $V^{(2k+1)}$ or $a_k F$ in $V^{(2k)}$; and each time it is used, the dimension of the representation space is reduced by one.

Now we apply a result in (1.1.4) and obtain all the indecomposable isomorphism classes of S -spaces W of defect -1 up to a permutation of the spaces W_i ($i = 1, 2, 3, 4$), which are the families $S_3(2k, -1)$ and $S_1(2k+1, -1)$ in [GP-70] of dimension types $\begin{pmatrix} k & 2k & k-1 \\ k & & \end{pmatrix}$ and $\begin{pmatrix} k+1 & 2k+1 & k \\ k & & \end{pmatrix}$ respectively.

1.1.5 Example $\widetilde{D}_4$: defect -2

It seems plausible that using again (1.1.4) we can get the family $S(2k+1, -1)$. So, if we examine our procedure in example (1.1.5) we observe that in order to prove $a_1 F$ admissible we used a certain e -polynomial. Now, is any admissible subgroup (or subspace) of W the image of an e -polynomial evaluated in the subspaces W_i ?

We have a negative answer and it is based on (1.1.4). An example will clarify this.

Example. With the hypothesis of example in (1.1.5) consider the indecomposable S -space $U = S_1(3, -1)$ of dimension type $\begin{pmatrix} 2 & 3 & 1 \\ 1 & & \end{pmatrix}$ as in figure 3. If we take $a = a_1 + a_2$, then aF

Figure 1-3: $S_1(3, -1)$

is admissible in W because W is indecomposable, but aF is not the image of an e -polynomial evaluated in the W_i 's.

1.1.7 Extension diagrams

All this motivated us to examine closely the formation of the new representations and find a connection to the old ones.

First, we need to identify the constituents of the construction as well as the steps performed.

On one hand, we started with a generating set O from which we took the factors of the S -spaces.

On the other hand, the S -spaces are middle terms in a non-split short exact sequence (ses) with a sub- S -space which was constructed previously on the left hand side, and a regular S -space with same dimension type as a direct sum of elements of O on the right hand side. Now we are going to test this and determine what else is required.

(1) Start with a preprojective representation $P(k, h) = C^{-h}P(k)$ where C^{-1} is the inverse of the Coxeter functor, $P(k)$ is the projective at the vertex k and h is a positive integer.

(2) Using the formula:

$$\dim \operatorname{Ext}^1(R, P) = - \sum_{i \in \Gamma_0} r_i p_i + \sum_{j \rightarrow i} r_j p_i$$

where R is regular, P is preprojective, with (r_i, r_0) and (p_i, p_0) their dimension types, (Γ, v) the valued graph associated with the K -structure (S, l) with Γ_0 its set of vertices. We find the set:

$$I = \{i \in \mathbb{Z} / E_i \in O \text{ and } \operatorname{Ext}^1(E_i, P(k, h)) \neq 0\}$$

then form the vectors

$$x = \dim P(k, h) + \sum_{i \in I} \alpha_i \dim E_i$$

where, $\alpha_i = 0$ or 1 and look for indecomposables with these dimension types (which must be of same defect as $P(k, h)$ because the defect is additive and each E_i has zero defect).

If there exists $P(m, n)$, a preprojective indecomposable representation whose dimension type is x , then we write:

$$\dim P(k, h) \xrightarrow{*} \dim P(m, n)$$

where $*$ stands for the elements of I . It is called an extension diagram. Since C^{-1} is a functor, it is enough to find these diagrams for small positive roots.

(3) In example (1.1.5), we have the following extension diagrams, where $P(k, h) = \dim P(k, h)$

$$P(1, 0) \xrightarrow{2} P(3, 1) \xrightarrow{1} P(1, 2) \xrightarrow{2} P(3, 3) \xrightarrow{1} \dots$$

$$P(0, 0) \xrightarrow{1,2} P(0, 1) \xrightarrow{1,2} P(0, 2) \xrightarrow{1,2} P(0, 3) \xrightarrow{1,2} \dots$$

Figure 1-4: Orbit of $W=P(0,0)$

the orbit of $P(1,0)$ is found in figure 2. We start with $P(1,0)$, extend by the regular E_2 and obtain $P(3,1)$, then extend by E_1 to obtain $P(1,2)$ and so on. The number i below the picture in figure 2 indicates the simple regular E_i .

The orbit of $P(0,0)$ is in figure 4, with the subspaces arrayed as horizontal layers corresponding to the same relative position. Here at each step we extend by a regular S -space with same dimension type as the direct sum $E_1 \oplus E_2$.

(4) The S -spaces $P(0, k)$ ($k \geq 0$) described in figure 4 are indecomposable. This is proved by induction on k . First we produce an e -polynomial which when evaluated in a certain admissible subspaces will allow us to "erase" the latest extension. For instance, p defined by

$$p((W_i)_{i \in S}) = (W_1 + W_3) \cap (W_2 + W_4) =: U$$

yields the U -subsystem in $P(0, k)$ isomorphic to $P(0, k-1)$.

If $P(0, k) = X \oplus Y$, then this decomposition induces a decomposition in U , but since U is indecomposable, it must be in one of the summands of $P(0, k)$, say in X .

Next, we consider the subspaces W_1 and W_3 and observe that there are two vectors

$$\begin{aligned} v_1 &= a_k + \sum_{j=1}^{k-1} a_j \alpha_j \\ v_2 &= a_k + b_{k-1} + \sum_{j=1}^{k-1} (a_j + b_{j-1}) \beta_j \end{aligned}$$

with α_j, β_j in F , which must lie either in X or in Y and are linearly independent. Thus a dimension argument shows that they must be in X , so a_k is in X . By the same token, b_k is also in X . Therefore, $W = X$ and $Y = 0$, which means that the decomposition is trivial.

(5) Our working hypothesis is that with this pattern we can obtain the preprojective indecomposable S -spaces W . In summary, we need to show that:

- i) the extension diagrams cover all the dimension types of the preprojective indecomposable S -spaces.
- ii) there is an e -polynomial which reduces the dimension of the representation space to obtain a subspace U for which the U -subsystem in W is indecomposable.

Figure 1-5: Subgraph of the AR-graph of $\widetilde{E}_8$

iii) there is a complement subspace V of $\mathcal{U}$ in W in the same summand as U ,

iv) W is indecomposable through a dimension argument.

Part i) follows from the lists of positive roots of negative defect. Techniques dealing with the other parts are placed in subsequent sections.

(6) In $\widetilde{E}_8$, it is observed that there is no monomorphism from $C^{-5}P(0)$ to $C^{-5-j}P(0)$ for any $j \geq 1$ therefore no ses as described above. So $C^{-5-j}P(0)$ is constructed in a different way, namely as an amalgamated sum of previously constructed indecomposable representations.

We will prove the previous remark when $K = \mathbb{Z}/2\mathbb{Z}$ and for our purpose this is enough.

Consider the subgraph of the preprojective component of the AR-graph of $\widetilde{E}_8$ as in figure 5.

Using the mesh relations in the graph we see that any map $\theta : C^{-5}P(0) \longrightarrow C^{-5-j}P(0)$ can be written as $\theta = \psi \circ \sum_{i=1}^3 (\beta_i \circ \alpha_i) k_i$ where, $\psi : C^{-5+1}P(0) \longrightarrow C^{-5-j}P(0)$ and $k_i \in K$ ($i = 1, 2, 3$). Now if θ is a monomorphism, then so is $\sum_{i=1}^3 (\beta_i \circ \alpha_i) k_i$. From the mesh relation in the

first diamond we have $\sum_{i=1}^3 \beta_i \circ \alpha_i = 0$. This says that not all the k_i 's are 1. But if this is so, then one of the maps $\beta_j \circ \alpha_j$ is a monomorphism and therefore α_j is a monomorphism. Thus α_j is an isomorphism. This is a contradiction because α_j was irreducible and hence non-invertible.

1.1.8 Specializations

As was mentioned in the introduction, the specializations given in [R-79b] by Ringel are useful to construct indecomposables. One of them is the shrinking of arrows. Namely, if we have an arrow $a \xrightarrow{\alpha} b$ in an oriented valued graph Γ such that the valuation on α is trivial, then we can obtain another oriented valued graph Γ' by deleting α and identifying the vertices a and b . Denote the new vertex by c .

It can be seen that a modulation $M = (F_i, {}_iM_j)$ of Γ induces a modulation M' of Γ' ; if the bimodule ${}_aM_b$ is erased and F_a is identified with F_b . This yields a surjection between the path algebra of Γ , $A(M, \Gamma)$, onto the path algebra of Γ' , $A(M', \Gamma')$, which in turn gives a full exact embedding in their categories of modules of finite length

$$T : \text{mod } A(M, \Gamma) \longrightarrow \text{mod } A(M', \Gamma')$$

Observe that the image of T is dense in the set of representations $V = (V_i, {}_i\varphi_j)$ of Γ such that ${}_a\varphi_b$ is an isomorphism (i.e., there is a representation W of Γ such that $T(W) \simeq V$).

Moreover, if W is an indecomposable representation of Γ' , then TW is indecomposable. For, $\Theta : \text{End}(W) \longrightarrow \text{End}(TW)$ given by $\Theta(\eta_j) = (\zeta_k)$ where, $\zeta_a = \zeta_b = \eta_c$ and $\zeta_k = \eta_k$ if $k \neq a, b$, is a ring isomorphism.

Figure 1-6: Space endowed with different subspaces

1.1.9 Space endowed with different systems of subspaces

Given an F -space W , we can endow it with different systems of subspaces, i.e., if (S, I) and (S', I') are K -structures for F . Let $W = (W_i, W)_{i \in S}$ and $W' = (W'_i, W)_{i \in S'}$, such that $W = W'$, be an S - and S' -space respectively. Suppose that for any j in S' there is an e -polynomial p_j such that $W'_j = p_j((W_i)_i)$. If W' is indecomposable, then so is W .

This is because, any decomposition $W = X \oplus Y$ induces a decomposition of W' which must be trivial. So one of the representation spaces X or Y is the zero space. Therefore X or Y is zero.

Example: Let W' be the indecomposable representation of $\widetilde{D}_4$ of dimension type $\begin{pmatrix} 2 & 4 & 2 \\ 2 & 4 & 1 \end{pmatrix}$ and W' be the representation of $\widetilde{D}_5$ of dimension type $\begin{pmatrix} 1 & 2 & 2 & 4 \\ 1 & 2 & 2 & 4 \end{pmatrix}$ as in the pictures, where

$$W'_1 = W_1 \quad W'_2 = W_2 \quad W'_3 = W_4 + W_5$$

$$W'_4 = (W_1 + W_4) \cap (W_2 + W_5) \cap W_3$$

Since W' is indecomposable, W is also indecomposable.

1.2 Properties of tensor products

Here, basic properties of the tensor product are assembled, the adjoint mappings and the ascent and descent concepts are of particular importance.

The usual notation for right, left and bimodules is used. Every ring has an identity and every module is unitary. Rings are denoted by R , S or T .

1.2.1 Definitions

A tensor product of a right R -module A_R and a left R -module ${}_R B$ is an abelian group $A_R \otimes_R B$ and an R -biadditive function h which solves the following universal mapping problem:

$$\begin{array}{ccc} A \times B & \xrightarrow{h} & A \otimes B \\ f \downarrow & \nearrow f' & \\ G & & \end{array}$$

For every abelian group G and every R -biadditive function f from $A \times B$ into G there exists a unique homomorphism f' making the diagram commute.

We know that such a tensor product $A_R \otimes_R B$ always exists and any two of them are isomorphic. If $f : A_R \rightarrow A'_R$ and $g : {}_R B \rightarrow {}_R B'$ are R -maps, then there is a unique homomorphism $A_R \otimes {}_R B \rightarrow A'_R \otimes {}_R B'$ with $a \otimes b \rightarrow f(a) \otimes g(b)$. The composition of these homomorphisms is given by $g' \otimes f' \circ g \otimes f = g' \circ g \otimes f' \circ f$.

Given the bimodules ${}_S A_R$, ${}_R B_T$ then ${}_S ({}_S A_R \otimes {}_R B_T) {}_T$.

If R is commutative, then $A_R \otimes_R B$ solves the universal problem posed by R -bilinear functions as well as that posed by R -biadditive functions.

Recall that the functors $A \otimes_R -$ and $- \otimes_R B$ are right exact. A module A_R is flat if $A \otimes_R -$ is exact. For instance, every projective module is flat, in particular R is a flat R -module. The sum $\sum_k A_k$ is a flat module if and only if for all k , A_k is flat. Vector spaces are flat.

1.2.2 Some isomorphisms

We have the following isomorphisms:

- i) For any R -module B , $R \otimes_R B \simeq B$
- ii) Associativity: $(A \otimes_R B) \otimes_S C \simeq A \otimes_R (B \otimes_S C)$
- iii) If R is commutative: $A \otimes_R B \simeq B \otimes_R A$.
- iv) $A \otimes_R (\sum_j {}_R B_j) \xrightarrow{\sim} \sum_j (A \otimes_R B_j)$ defined by $a \otimes (b_j) \longrightarrow (a \otimes b_j)$.
- v) Adjoint isomorphisms:

$$\text{If } {}_R A, {}_S B_R, {}_S C \text{ then } \text{Hom}_S(B \otimes_R A, C) \xrightarrow{\sim} \text{Hom}_R(A, \text{Hom}_S(B, C)).$$

$$\text{If } A_R, {}_S B_R, {}_S C \text{ then } \text{Hom}_S(A \otimes_R B, C) \xrightarrow{\sim} \text{Hom}_R(A, \text{Hom}_S(B, C)).$$

- vi) If ${}_R B_R$ is flat and A is finitely related, then for any module C there is a natural isomorphism

$$B \otimes_R \text{Hom}_R(A, C) \xrightarrow{\sim} \text{Hom}_R(A, C \otimes_R B) \text{ given by } b \otimes g \longrightarrow g_b \text{ where } g_b(a) = g(a) \otimes b.$$

Recall that a module A is finitely related if there is an exact sequence $F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$

where F_1 and F_0 are finitely generated free modules.

1.2.3 Adjoint maps

Let (Γ, v) be a valued graph with admissible orientation Ω and K -modulation $\mathbf{M} = (F_i, {}_iM_j)$.

Since

$${}_jM_i \simeq \text{Hom}_{F_i}({}_iM_j, F_i) \simeq \text{Hom}_{F_j}({}_iM_j, F_j)$$

then

$$\Theta : \text{Hom}_{F_i}(X_j \otimes {}_jM_i, X_i) \xrightarrow{\sim} \text{Hom}_{F_j}(X_j, X_i \otimes {}_iM_j)$$

where X_i, F_i, X_j, F_j are vector spaces.

If $\mathbf{X} = (X_i, {}_i\varphi_j)$ is a representation of Γ , then the F_i -map ${}_i\varphi_j : X_j \otimes {}_jM_i \rightarrow X_i$ can be interchanged by its adjoint

$${}_i\tilde{\varphi}_j := \Theta({}_i\varphi_j) : X_j \rightarrow X_i \otimes {}_iM_j$$

If the underlying graph of (Γ, v) is a tree with a distinguished vertex and Ω is the subspace orientation (i.e., all the arrows are directed to this vertex), the representation is a system of subspaces or filtered vector space if all the F_j -maps ${}_i\tilde{\varphi}_j$ are embeddings.

1.2.4 Ascent

Let K be a field and $L \supseteq F$ be finite skew-field extensions of K . For any finite dimensional F -vector space V_F , we can extend scalars tensoring by L , and the L -vector space $V_L = V_F \otimes L$ is referred as obtained from V_F by ascent from F to L . If $B = \{b_1, b_2, \dots, b_n\}$ is an F -basis for V_F ,

then $B \otimes 1 = \{b_1 \otimes 1, \dots, b_n \otimes 1\}$ is an L -basis for V_L .

Moreover: $V_L^* \simeq L^* \otimes V_F^*$, where $X^* := \text{Hom}(X, Y)$. For,

$$\begin{aligned} V_L^* &= \text{Hom}(V \otimes L, K) \\ &\simeq \text{Hom}(V, \text{Hom}(L, K)) \\ &\simeq \text{Hom}(V, L^*) \\ &\simeq L^* \otimes \text{Hom}(V, K) \\ &= L^* \otimes V^*. \end{aligned}$$

1.2.5 Descent

If we start with an L -vector space V_L , it can be regarded as an F -vector space $(V_L)_F$. Let U_F be a subspace of $(V_L)_F$ and extend scalars to L : $U_L = U_F \otimes_F L$, so we have the situation:

$$\begin{array}{ccc} U_F \times L & \xrightarrow{\otimes} & U_F \otimes L \\ \beta \downarrow & \nearrow \alpha & \\ & V_L & \end{array}$$

where $\beta(u, l) = ul$ is the product by scalars in V_L and α is the L -map for which the diagram commutes. U_F is said to be an F -form of V_L if α is an isomorphism. The process to pass from V_L to an F -form is called descent from L to F .

The following are equivalent if U_F is an F -subspace of $(V_L)_F$

- i) U_F is an F -form of V_L
- ii) An F -basis for U_F is an L -basis for V_L .

Because, if U_F is an F -form of V_L and B is an F -basis for U_F , then $B \otimes 1$ is an L -basis for $U_F \otimes L$ which is isomorphic to V_L by α . Since $B = \alpha(B \otimes 1)$, then B is an L -basis for V_L .

Conversely, since $B = \alpha(B \otimes 1)$ for any F -basis B for U_F , then α is an L -map which transforms a basis into a basis and so it is an isomorphism.

1.3 Decomposition of representations

We prove here some properties used throughout the paper. These are based on the concept of compatibility given in (1.1.3) and the fact that the lattice $L(V)$ of subspaces of a finite dimensional vector space V is a modular lattice. A lattice L is modular (or Dedekind) if for any a, b, c in L such that $a \leq b$, then $b(a + c) = a + bc$.

If $W_F = X_F \oplus Y_F$ is a decomposition of a vector space W_F , the dimension of Y_F is called the codimension of X_F in W_F . An hyperplane is a subspace of codimension 1.

We study also subspaces of low codimension and the existence of compatible sets of vectors.

1.3.1 Admissible indecomposable S -spaces

Let $W = X \oplus Y$ be a decomposition of the S -space W and U an admissible subspace of W . Then there is a basis B of U whose elements are either in X or in Y . Moreover, if the U -subsystem in W is indecomposable, then $U \subseteq X$ or $U \subseteq Y$. Therefore, any 'subset' of U is (totally) admissible in W . For, the decomposition of W induces a decomposition in U since the subspaces U , W_i and $U_i = U \cap W_i$ are admissible in W . Thus, since U is indecomposable, then the induced decomposition must be trivial, therefore $U \cap X = U$ or $U \cap Y = U$.

The fact that U is admissible is essential. For example in A_3 : $\circ \rightarrow \circ \leftarrow \circ$ with trivial modulation, consider the filtered vector spaces

$$\begin{aligned} W = X \oplus Y &= (F \rightarrow F \leftarrow 0) \oplus (0 \rightarrow F \leftarrow F) \\ &= (F \oplus 0 \rightarrow F \oplus F \leftarrow 0 \oplus F) \\ &\supseteq U = (0 \rightarrow (1, 1)F \leftarrow 0) \end{aligned}$$

and $(1, 1)F$ is neither in $X = (1, 0)F$ nor in $Y = (0, 1)F$.

1.3.2 Intersecting systems

If U and V are indecomposable admissible subsystems in W such that $U_F + V_F = W_F$ and $U \cap V \neq (0)$, then W is indecomposable. For, if $W = X \oplus Y$, it follows from (1.3.1) that $(U \subseteq X \text{ or } U \subseteq Y)$ and $(V \subseteq X \text{ or } V \subseteq Y)$. So, if U and V are in different summands, then $U \cap V \subseteq X \cap Y = (0)$, which contradicts our hypothesis. Therefore, $W_F = X_F \oplus Y_F$ is in one of the summands and the decomposition is trivial.

We get the same conclusion if we have a finite family of indecomposable $U^{(j)}$ -subsystems in W ($j = 1, 2, \dots, n$), for which $W = \sum_j U^{(j)}$ and for each pair of numbers j and k in $\{1, 2, \dots, n\}$ there is a sequence $j = j_1, \dots, j_{i+1} = k$, such that $U^{(j_i)} \cap U^{(j_{i+1})} \neq (0)$, ($i = 1, \dots, l$).

1.3.3 Compatible F -vectors

Let $W = X \oplus Y$ be a decomposition of the S -space W . A vector in W is said to be (totally) compatible with the decomposition if it is either in X or in Y .

If U_F, V_F are admissible subspaces of W_F such that $V = U \oplus wF$ for some w in W , then

there exists u in U such that the vector $u + w$ is compatible with the decomposition $W = X \oplus Y$

For,

$$\begin{aligned}
 V &= (U \oplus wF) \cap X \oplus (U \oplus wF) \cap Y \\
 &= ((U \cap X \oplus U \cap Y) \oplus wF) \cap X \oplus ((U \cap X \oplus U \cap Y) \oplus wF) \cap Y \\
 &= (U \cap X) \oplus ((U \cap Y \oplus wF) \cap X) \oplus (U \cap Y) \oplus ((U \cap X \oplus wF) \cap Y) \\
 &= U \oplus (U \cap Y \oplus wF) \cap X \oplus (U \cap X \oplus wF) \cap Y
 \end{aligned}$$

and by dimensions, exactly one of the last two summands is different from (0) . Therefore, there is

u' in U and λ in F such that $u' + w\lambda$ is a compatible vector. Take $u = u'\lambda^{-1}$ if $\lambda \neq 0$, otherwise $u = 0$.

1.3.4 Compatible sets

Let n be the dimension of the representation space W . Suppose there is an indecomposable U -subsystem in W and a linearly independent set C of vectors in $W \setminus U$, compatible with the decomposition $W = X \oplus Y$ such that $\text{card}(C) \geq (n - \dim U) + 1$ and any $n - \dim U$ vectors in C belong to a complementary space of U . Then the decomposition is trivial. For, since U is indecomposable, we can suppose $U \subseteq X$. So if exactly $m < n - \dim U$ of the elements of C are in X , then

$$\begin{aligned}
 \dim X &\geq \dim U + m \\
 \dim Y &\geq \text{card}(C) - m
 \end{aligned}$$

since C is compatible, so

$$\begin{aligned}
 n &= \dim W \\
 &= \dim X + \dim Y \\
 &\geq \dim U + m + (n - \dim U) + 1 - m \\
 &= n + 1.
 \end{aligned}$$

Contradiction, therefore C is in X .

1.3.5 Compatible basis

If b_1, b_2 are linearly independent vectors in W such that $b_1 F$, $b_2 F$ and $(b_1 + b_2)F$ are admissible in W , then b_1 and b_2 are in the same summand of any decomposition of W .

Let $B = \{b_0, b_1, \dots, b_n\}$ be a linearly independent set of vectors:

1) If $b_i F$ and $(b_j + b_{j+1})F$ ($i = 0, 1, \dots, n$), ($j = 0, 1, \dots, n-1$) are admissible in W , then B is in only one summand of any decomposition of W .

2) If $b_i F$ and $(b_0 + b_i)F$ ($i = 0, 1, \dots, n$) are admissible in W , then B is in only one summand of any decomposition of W .

3) If B is a basis of W with any of the above two properties, then W is indecomposable.

4) Let $F = G \oplus fG$ such that f^2 is in G and f commutes with the elements of G . If the following subspaces are admissible in W :

$$\begin{aligned} U &= aF \oplus bF \oplus cF \\ U_1 &= aG \oplus (b + cf)G \\ U_2 &= (a + bf)G \oplus (cf)G \end{aligned}$$

and $W = X \oplus Y$, then there exist totally compatible basis of U_i , say $\{w_1^i, w_2^i\}$ ($i = 1, 2$); and any three vectors of the set $\{w_1^1, w_2^1, w_1^2, w_2^2\}$ are F -linearly independent. The first part follows because $\dim_F U_i^c = 2$ and U_1, U_2 are admissible. The second part follows because $U = (U_i^c)_G \oplus (U_j)_G$, $i \neq j$.

1.3.6 Hyperplanes

Let U be an hyperplane of W such that the U -subsystem in W is indecomposable. Suppose

there exist admissible subspaces V_1, V_2, U_1, U_2 in W such that $V_i = U_i \oplus w_i F$ ($i = 1, 2$), where $w_i \in W \setminus U$ and $U_i \subseteq U$.

If the vectors $w_i + u_i$ for any u_i in U_i ($i = 1, 2$) are linearly independent, then any decomposition of W is trivial.

This follows because, given a decomposition $W = X \oplus Y$, take $C = \{v_1, v_2\}$ where $v_i = w_i + u_i$ and u_i is the vector obtained in (1.3.3).

1.3.7 Example

The previous observations lead us to look for totally compatible vectors outside an admissible group. For instance, the orbit of $P(0, 0)$ of $\widetilde{D}_4$ in (1.1.6) was obtained in this form.

The representations of defect -1 of $\widetilde{D}_4$ given in (1.1.5) can be seen to be indecomposable using (1.3.4) with U an hyperplane and $\text{card } C = 2$. First we delete $a_{k+1}F$ from W . Namely, $U = W_2 + W_4 = \sum_{j=1}^k a_j F$, which is admissible and $\text{codim } U = 1$. Next, given a decomposition of W there exist compatible vectors in $W \setminus U$:

$$u_2 = a_{k+1} + \sum_{j=1}^k a_j \alpha_j \in W_2$$

$$u_4 = (a_{k+1} + b_k) + \sum_{j=1}^k (a_j + b_j) \beta_j \in W_4$$

with α_j, β_j in F , which are linearly independent. Therefore, if U is indecomposable, then W is indecomposable. A similar remark occurs when we delete $b_k F$.

Thus, the indecomposability of the representations pictured in (1.1.5), follows by induction.

The following observation is useful to identify linearly independent vectors. Let $W = X \oplus Y$,

and $\{x_i\}_{i \in I}$ be a linearly independent set of vectors in X and $\{y_i\}_{i \in I}$ be a subset of Y , then $\{x_i + y_i\}_{i \in I}$ is linearly independent.

1.3.8 Compatible G -vectors

Let $K \subseteq G \subseteq F$ be a tower of finite skew-field extensions of K and $W = X \oplus Y$ be a decomposition of the S -space W . Observe that the result in (1.3.3) holds if U and V are G -vector spaces.

Suppose there exists an indecomposable U -subsystem in W such that U_F is an F -hyperplane of W_F , say $W = U \oplus wF$. If there are G -vector spaces V_1, V_2, U_1, U_2 admissible in W such that

$$\begin{aligned} V_1 &= U_1 \oplus wG \\ V_2 &= U_2 \oplus (w + z)G \end{aligned}$$

where $z \in U_F \setminus (U_1 + U_2)_G$ and $U_i \subseteq U$ ($i = 1, 2$), then the decomposition of W is trivial.

It follows from (1.3.4) and the fact that the vectors $w + u_1$ and $w + z + u_2$ for any u_i in U_i ($i = 1, 2$) are F -linearly independent.

1.3.9 Observations on G -vectors

Let B be a set of F -linearly independent vectors in X_F and suppose w_1, w_2 are two F -linearly independent vectors in W in the G -space spanned by the set B such that the space $V = (B \cup \{w_1, w_2\})^F$ is compatible with the decomposition $W = X \oplus Y$.

If $\dim_F V_F = \dim_F(BF) + 1$, then w_1 and w_2 are in X . For, clearly $\dim_F V \cap X \geq \text{card}(B)$ and if w_1, w_2 are in Y , then $\dim_F V \cap Y \geq 2$ so

$$\begin{aligned} \text{card } B + 1 &= \dim V \\ &= \dim V \cap X + \dim V \cap Y \\ &\geq \text{card } B + 2. \end{aligned}$$

Contradiction, therefore one of w_1, w_2 is in X , then $V \subseteq X$ and so both w_1, w_2 are in X .

1.3.10 G -forms

Now, suppose that U is an indecomposable U -subsystem in W such that U_F is an F -hyperplane of W_F and V_G an admissible G -form of a subspace of W such that $\dim_G(U \cap V) = \dim_G V_G + 2$.

Then it follows from (1.3.9) that the decomposition is trivial.

As an example of this observation we have the case when $B = \{b_1, \dots, b_n\}$ is a basis of W , $B' = \{b_2, \dots, b_n\}$ is a basis of U and f is in $F \setminus G$. If $V = b_1 G \oplus (b_1 + b_2)fG$ is compatible with the decomposition, then the result holds.

1.3.11 More linearly independent vectors

Let $F = G \oplus fG$ with f^2 in G and f commutes with the elements of G , $B = \{a, c, b_1, \dots, b_n\}$ a basis of W ,

$$\begin{aligned} U_1 &= a fG \oplus cG \\ U_2 &= (a + b_1)fG \oplus (b_n + c)G \end{aligned}$$

admissible subspaces in $W = X \oplus Y$. Then

a) If $\{w_1, w_2\}$ and $\{w_3, w_4\}$ are totally compatible basis of U_1 and U_2 respectively, then any three vectors of $\{w_1, w_2, w_3, w_4\}$ are F -linearly independent. For, if $b_1 = b_n$, this is (1.3.5). If

$n > 1$, then $\dim_F(U_1^e + U_2^e) = 4$ and so all are F -linearly independent.

b) If $U = \text{span}\{b_1, \dots, b_n\}$ and U is an indecomposable subsystem of W , then W is indecomposable. This follows from a) and (1.3.1).

1.3.12 Indecomposable amalgamated sums

Let Δ be an Euclidean diagram with trivial valuation (recall, this is to be applied in $\widetilde{E}_8$), P, P_1, P_2 indecomposable preprojective representations such that:

- 1) $\mu_i: P \rightarrow P_i$ ($i = 1, 2$) are proper inclusions.
- 2) $\text{Hom}(P_i, P_j) = 0$, ($i \neq j$).
- 3) $\text{Hom}(C^{-1}P, P_i) = 0$, ($i = 1, 2$).

Then the pushout X of the inclusions μ_i ($i = 1, 2$) is indecomposable. For, we have $0 \rightarrow P \xrightarrow{\mu} \oplus P_i \xrightarrow{\eta} X \rightarrow 0$ is exact. Then applying the functor $\text{Hom}(P_i, -)$ we get:

$$0 \rightarrow \text{Hom}(P_i, P) \rightarrow \text{Hom}(P_i, \oplus P_i) \rightarrow \text{Hom}(P_i, X) \rightarrow \text{Ext}(P_i, P)$$

Now:

- i) By (1) and P, P_i indecomposable preprojective representations, it follows $\text{Hom}(P_i, P) = 0$.
- ii) By (2), it follows $\text{Hom}(P_i, \oplus_k P_k) = \text{End}(P_i) \simeq F$. Since Δ is an Euclidean diagram with trivial valuation.
- iii) $\text{Ext}(P_i, P) \simeq D\text{Hom}(C^{-1}P, P_i) = 0$, follows from (i) and the fact that there are no cycles in the preprojective component of the AR-graphs of ET-algebras.

Therefore $F \simeq \text{Hom}(P_i, X)$.

Now, suppose $X = X_1 \oplus X_2$, then $\text{Hom}(P_i, X_j) \simeq F$ for some j (w.l.o.g. : $j = i$), and therefore

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} : P_1 \oplus P_2 \longrightarrow X_1 \oplus X_2$$

so $\text{Ker} \eta = \text{Im} \mu$ is either 0, P_1 , P_2 or $P_1 \oplus P_2$ which contradicts (1).

More on indecomposable amalgamated sums can be found in [GG-77].

Chapter 2

Classical Euclidean Diagrams

We consider now the case of the classical Euclidean diagrams, i.e., those with trivial valuation.

These are $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$. The representation theory for these was studied by Donovan and Freislich in [DF-73].

The subspace orientation in $\widetilde{A}_n$ means that there exist exactly one source and one sink.

The preprojective indecomposable representations of $\widetilde{A}_n$ are obtained from Kronecker's classification of pairs of complex matrices which in turn can be deduced from the study of the graph $\widetilde{A}_{12}$. Cf.: [D-80].

The study of $\widetilde{D}_n$ is based on that of $\widetilde{D}_6$.

Finally the indecomposable preprojective representations of $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ are constructed following the pattern of $\widetilde{D}_6$.

2.1 Preprojective representations of $\tilde{A}_n$

Donovan and Freislich in [DF-73] studied $\tilde{A}_n$ for $n = 2m - 1$ and an orientation where every vertex is either a source or a sink. They proved that the indecomposable preprojective representations are characterized by two monomorphisms.

$\tilde{A}_n$ has some special interest because its underlying graph is not a tree. This enhances its complexity though it is reflected in its category of regular modules. This peculiarity was brought to light by Dlab and Ringel in [DR-76a]. A generalization is studied in [DR-76b].

Back in 1890, by solving a problem proposed by Weierstrass, L. Kronecker in [K-1890] classified pairs of complex matrices under a certain similarity relation. An account of this can be found in [G-59], and the way it fits our purpose is in [D-80].

2.1.1 Reduction step to $\tilde{A}_{12}$

Given the diagram $\tilde{A}_n$, then a straightforward computation gives $Q(x) = \sum_{(i,j) \in \Gamma_1} (x_i - x_j)^2$ where Γ_1 is the set of arrows of the diagram. The canonical vector $n = (1, 1, \dots, 1, 1)$, the tier number $g = 1$ and the defect vector $\delta_e = ((n+1), 0, \dots, 0, -(n+1))$.

We know that the positive roots α of negative defect are $\alpha = \alpha_0 + t\alpha$ where t is a natural number and $\alpha_0 = (0, 0, \dots, 0, 1, 1) \leq n\alpha$. Therefore, for all the arrows of the diagram but two, the corresponding morphisms φ_j in a preprojective indecomposable representation $V = (V_i, \varphi_j)$ are

isomorphisms and the remaining two are monomorphisms whose images are hyperplanes in the codomain.

Thus V is determined by these two monomorphisms. Define them as the maps in the preprojective indecomposable representations in Kronecker's classification. Namely, $U \rightrightarrows V$ where U, V are F -vector spaces, of dimensions k and $k+1$ respectively. And with respect to fixed basis in U and V , α and β are defined by the matrices:

$$[\alpha] = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad [\beta] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Now by (1.3.11), V is indecomposable. Shrink all the arrows for which φ_i is an isomorphism.

2.1.2 Example

In $\widetilde{A}_3$ ($\circ \rightrightarrows \circ \rightrightarrows \circ$) the small positive roots α_0 defined in (2.1.1) are: $(\begin{smallmatrix} 0 & 0 & 1 \end{smallmatrix})$, $(\begin{smallmatrix} 0 & 1 & 1 \end{smallmatrix})$, $(\begin{smallmatrix} 0 & 0 & 1 \end{smallmatrix})$ and $(\begin{smallmatrix} 0 & 1 & 1 \end{smallmatrix})$.

The representation V of dimension type $(2 \ 2 \ 3)$ is given by:

$$\left(\begin{array}{ccc} U & \xrightarrow{1} & U \\ & \searrow & \nearrow \\ & U & \xrightarrow{\beta} V \end{array} \right) = \left(\begin{array}{ccc} F \oplus F & \xrightarrow{1} & F \oplus F \\ & \searrow & \nearrow \\ & F \oplus F & \xrightarrow{\beta} F \oplus F \oplus F \end{array} \right)$$

where

$$[\alpha] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad [\beta] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

with respect to the canonical basis. Pictorially:

V	$//$	$//$	$//$			
αU		$//$	$//$	$\alpha(1,0) = (0,1,0)$	$\alpha(0,1) = (0,0,1)$	
βU	$//$	$//$		$\beta(1,0) = (1,0,0)$	$\beta(0,1) = (0,1,0)$	

In a similar way the preinjective indecomposable representations of $\widetilde{A}_n$ are constructed.

2.2 Preprojective representations of $\widetilde{D}_n$

Gelfand and Ponomarev studied $\widetilde{D}_4$ and gave a list of the finite dimensional indecomposable representations of it in [GP-70].

A few years later, they studied the diagram D' in [GP-74], [GP-76] and [GP-77] corresponding to a vector space V endowed with r unrelated subspaces. This was done in the setting of free modular lattices where they constructed some important sublattices, called the cubicles, whose elements are perfect. [GP-77] ends with a characterization of the completely irreducible representations.

A bound for the number of perfect elements not in the cubicles up to q -linear equivalence is given in [DR-80].

In [GP-76], [GP-77] and [D-79], a certain infinite dimensional graded algebra A' is constructed and the representation from the free modular lattice on r generators into the lattice of right ideals of A' is proven to be a direct sum of "preprojective" representations and a combinatorial description of them is given.

A predominant idea in the proofs is that the preprojective representations are partially ordered and that they appear in levels or ranks. Moreover, this order is given by the Coxeter functor.

In this section, we construct the indecomposable preprojective representations of $\widetilde{D}_n$ out of those of $\widetilde{D}_6$.

2.2.1 Data

Let $\widetilde{D}_n$ be the oriented diagram $\circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \circ$, then $Q(x) = \sum_{(i,j) \in \Gamma_1} (x_i - \alpha_j x_j)^2$

where Γ_1 is the set of arrows of $\widetilde{D}_n$ and $\alpha_j = 2$ if j is a source in $\widetilde{D}_n$, otherwise $\alpha_j = 1$. The

canonic vector is $n = \begin{pmatrix} 1 & 2 & 2 \cdots 2 & 2 & 1 \\ 1 & & & & & \end{pmatrix}$, the tier number $g = 1$. The order of the Coxeter element

$\bar{c}$ in the Weyl group is $n - 2$ if n is even, $2n - 4$ if n is odd. Therefore the defect vector is:

$\delta_c = \begin{pmatrix} 1 & 0 & 0 \cdots 0 & -2 & 1 \\ 1 & & & & \end{pmatrix}$ if n is even, $\delta_c = \begin{pmatrix} 2 & 0 & 0 \cdots 0 & -4 & 2 \\ 2 & & & & \end{pmatrix}$ if n is odd.

Figure 2-1: Representations of $\widetilde{D}_6$: defect -1

2.2.2 Representations of $\widetilde{D}_6$

First we construct the preprojective indecomposable representations of $\widetilde{D}_6$ where we fix the following admissible order: $(\begin{smallmatrix} 5 \\ 0 \end{smallmatrix} \xrightarrow{4} 3 \xrightarrow{2} 0 \xleftarrow{1} 2)$. Table 1 contains the small positive roots x_0 of negative defect, up to symmetry.

$P(0,0) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$P(0,1) = \begin{pmatrix} 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}$	$P(0,2) = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 1 \end{pmatrix}$	
$P(1,0) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$	$P(1,1) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$	$P(1,2) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$	$P(1,3) = \begin{pmatrix} 1 & 2 & 2 & 2 & 0 & 1 \end{pmatrix}$
$P(3,0) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$	$P(3,1) = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 & 1 \end{pmatrix}$		
$P(4,0) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$			
$P(5,0) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$	$P(5,1) = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 & 1 \end{pmatrix}$	$P(5,2) = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 1 \end{pmatrix}$	$P(5,3) = \begin{pmatrix} 0 & 2 & 2 & 2 & 1 & 1 \end{pmatrix}$

Table 2-1: Small roots of $\widetilde{D}_6$

Observe that the number of small positive roots x_0 is 22 and that $P(2,h)$ is obtained from $P(1,h)$ by an automorphism of the oriented graph ($h \in \mathbb{N}$). $P(5,h)$ and $P(6,h)$ are obtained in a similar way. So, it is enough to construct one representation from each pair. Thus, we need to consider only 14 of the roots x_0 .

Extension diagrams

$$\begin{aligned}
 P(0,0) &\xrightarrow{1,2} P(0,3) \xrightarrow{1,2} \dots \\
 P(0,1) &\xrightarrow{1,2} P(0,4) \xrightarrow{1,2} \dots \\
 P(0,2) &\xrightarrow{1,2} P(0,5) \xrightarrow{1,2} \dots \\
 P(3,0) &\xrightarrow{1,2} P(3,2) \xrightarrow{1,2} \dots \\
 P(3,1) &\xrightarrow{1,2} P(3,3) \xrightarrow{1,2} \dots \\
 P(4,0) &\xrightarrow{1,2} P(4,1) \xrightarrow{1,2} \dots \\
 P(1,0) &\xrightarrow{2} P(5,1) \xrightarrow{1} \dots \\
 P(2,1) &\xrightarrow{2} P(6,3) \xrightarrow{1} \dots \\
 P(1,2) &\xrightarrow{2} P(5,3) \xrightarrow{1} \dots \\
 P(5,0) &\xrightarrow{2} P(1,3) \xrightarrow{1} \dots
 \end{aligned}$$

Representations of defect -1: There are 8 roots x_0 of this defect, and they fall into one of the four patterns which differ only in the "first" subspace as in figure 1. We identify along 'a' the block on the right hand side with each of the small blocks on the left hand side in figure 1. They are clearly indecomposable, because one can delete the "last" subspace using $U = W_2 + W_5$ or $U = W_1 + W_6$, and then apply (1.3.4) as in (1.3.7).

Representations of defect -2: There are 6 roots x_0 of this defect and the families differ only at the beginning, as in figure 2 where we identify along 'x' and 'y' the blocks on the right hand side with each one of the blocks on the left hand side. Note that the dimension of the representation space increases by two at each step. The proof of indecomposability is as in (1.1.7) where the reduction step is given by: $U = (W_1 + W_5) \cap (W_2 + W_6)$. An alternative proof can be given using (1.3.12) and it is straightforward.

Figure 2-2: Representations of $\widetilde{D}_6$: defect -2

2.2.3 Reduction step to $\widetilde{D}_6$

We are now ready to proceed in general. But before doing so, we observe that the preprojective indecomposable representations of $\widetilde{D}_5$ are obtained by shrinking the arrow $4 \rightarrow 3$. This follows because the defect is independent of the spaces in the positions 3 and 4.

Now in general: $n \geq 6$. Since $x_0 \leq n\theta$, then x_0 has the form (up to an automorphism of the oriented graph $\widetilde{D}_n$)

$$1) \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & \dots & 1 & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 1 & 1 & 0 \end{pmatrix}$$

$$2) \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & \dots & 1 & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 1 & 1 & 1 \end{pmatrix}$$

Figure 2-3: Space endowed with different subspaces

$$3) \begin{pmatrix} 0 & 1 & \dots & 1 & 0 \\ 1 & & & & 0 \end{pmatrix}$$

$$4) \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & \dots & 1 & 2 & \dots & 2 & 2 \\ 0 & & & & & & & & & & 1 \end{pmatrix}$$

$$5) \begin{pmatrix} 0 & 1 & \dots & 1 & 2 & \dots & 2 & 2 \\ 1 & & & & & & & 1 \end{pmatrix}$$

$$6) \begin{pmatrix} 1 & 2 & \dots & 2 & 2 \\ 1 & & & & 0 \end{pmatrix}$$

We know that the positive roots of negative defect are of the form $x = x_0 + t n$, ($t \in \mathbb{N}$). Then for any indecomposable preprojective representation

$$\begin{array}{ccccccc}
 & X_n & & & & & X_1 \\
 & \downarrow \varphi_n & & & & & \downarrow \varphi_1 \\
 X & = & X_{n-1} & \xrightarrow{\varphi_{n-1}} & X_{n-2} & \longrightarrow & \dots \longrightarrow X_3 \xrightarrow{\varphi_3} X_0 \\
 & \uparrow \varphi_{n+1} & & & & & \uparrow \varphi_2 \\
 & X_{n+1} & & & & & X_2
 \end{array}$$

all but at most 2 of the maps φ_i , ($i = 3, 4, \dots, n-1$) are isomorphisms. Therefore, it is determined by the spaces and morphisms of an indecomposable representation of $\widetilde{D}_6$, see (1.3.11).

2.2.4 Example

If we wish the indecomposable preprojective X of $\widetilde{D}_8$ of dimension type $\begin{pmatrix} 1 & 2 & 3 & 4 & 4 & 2 \\ 1 & & & & & 2 \end{pmatrix}$, then we identify the vertices 0 and 3, 4 and 5. Thus we get $x = \begin{pmatrix} 1 & 2 & 3 & 4 & 2 \\ 1 & & & & 2 \end{pmatrix}$ which is the dimension type of an indecomposable representation Y of $\widetilde{D}_8$, and $x_0 = -t_0 g + x = \begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & & & & 1 \end{pmatrix} = P(0, 1)$

See figure 3.

2.3 Preprojective representations of $\widetilde{E}_6$

2.3.1 Data

$$\Omega \quad \begin{array}{c} 2 \rightarrow 1 \\ 4 \rightarrow 3 \\ 6 \rightarrow 5 \end{array} \Rightarrow 0 \quad n = \begin{pmatrix} 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 \end{pmatrix} \quad \delta_e = \begin{pmatrix} 1 & 1 \\ 1 & 1 & -3 \\ 1 & 1 \end{pmatrix}$$

$$Q(x) = \frac{1}{2} \sum_{i=1}^3 [3(2x_{2i} - x_{2i-1})^2 + (3x_{2i-1} - 2x_0)^2]$$

$$c^{-1} = \begin{bmatrix} 2 & 0 & -1 & 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad m = 6$$

$$P(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P(0,1) = \begin{pmatrix} 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$P(1,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad P(1,1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad P(1,2) = \begin{pmatrix} 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$P(2,0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad P(2,1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad P(2,2) = \begin{pmatrix} 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad P(2,3) = \begin{pmatrix} 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$P(2,4) = \begin{pmatrix} 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \quad P(2,5) = \begin{pmatrix} 0 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Table 2-2: Small roots of $\widetilde{E}_6$

2.3.2 Extension diagrams

(All other small roots of negative defect are symmetric to the ones listed).

$$P(2, 0) \xrightarrow{2} P(6, 2) \xrightarrow{1} P(4, 4) \xrightarrow{3} \dots$$

$$P(2, 1) \xrightarrow{1} P(6, 3) \xrightarrow{3} P(4, 5) \xrightarrow{2} P(2, 7) \xrightarrow{1} \dots$$

$$P(1, 0) \xrightarrow{2} P(3, 1) \xrightarrow{3} P(5, 2) \xrightarrow{1} \dots$$

$$P(0, 0) \xrightarrow{1,2,3} P(0, 2) \xrightarrow{1,2,3} \dots$$

$$P(0, 1) \xrightarrow{1,2,3} P(0, 3) \xrightarrow{1,2,3} \dots$$

where: $E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ $E_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

2.3.3 Proof:

The reduction step is given in the following table. To delete the block i use the e-polynomial

p_i .

$$1 \quad p_1 = W_2 + W_3$$

$$2 \quad p_2 = W_1 + W_6$$

$$3 \quad p_3 = W_4 + W_5$$

and the construction of one-dimensional admissible spaces follows from the fact that:

$$W_2 \cap W_3 = (0)$$

$$W_1 \cap W_6 = (0)$$

$$W_4 \cap W_5 = (0)$$

In defect -3 , the reduction is given by $p_0 = p_1 \cap p_2 \cap p_3$.

Figure 2-4: Representations of defect -1

Figure 2-5: Representations of defect -2

Figure 2-6: Representations of defect -3

2.4 Preprojective Representations of $\widetilde{E}_7$

2.4.1 Data

$$\Omega \quad \begin{array}{c} 1 \\ 4 \rightarrow 3 \rightarrow 2 \rightarrow 0 \\ 7 \rightarrow 6 \rightarrow 5 \rightarrow \end{array} \quad n = \begin{pmatrix} 2 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix} \quad \delta_c = \begin{pmatrix} 2 \\ 1 & 1 & 1 & -4 \\ 1 & 1 & 1 \end{pmatrix}$$

$$Q(x) = \sum_{i=1}^2 (x_{3i+1} - \frac{1}{2}x_{3i})^2 + \sum_{i=1}^2 \left[\frac{3}{4}(x_{3i} - \frac{2}{3}x_{3i-1})^2 + \frac{2}{3}(x_{3i-1} - \frac{3}{4}x_0)^2 \right]$$

$$c^{-1} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad m = 12$$

Consider the following roots of zero defect:

$$E_1 = \begin{pmatrix} 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P(0,0) = \begin{pmatrix} 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad P(0,1) = \begin{pmatrix} 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad P(0,2) = \begin{pmatrix} 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

$$P(1,0) = \begin{pmatrix} 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad P(1,1) = \begin{pmatrix} 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad P(1,2) = \begin{pmatrix} 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad P(1,3) = \begin{pmatrix} 1 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$$

$$P(1,4) = \begin{pmatrix} 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \quad P(1,5) = \begin{pmatrix} 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$$

$$P(2,0) = \begin{pmatrix} 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad P(2,1) = \begin{pmatrix} 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad P(2,2) = \begin{pmatrix} 1 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \quad P(2,3) = \begin{pmatrix} 2 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$P(3,0) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} 1 \quad P(3,1) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} 2 \quad P(3,2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} 2 \quad P(3,3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} 3$$

$$P(3,4) = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix} 4 \quad P(3,5) = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \end{pmatrix} 4$$

$$P(4,0) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} 1 \quad P(4,1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} 1 \quad P(4,2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} 1 \quad P(4,3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} 2$$

$$P(4,4) = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} 2 \quad P(4,5) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} 2 \quad P(4,6) = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} 3 \quad P(4,7) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} 3$$

$$P(4,8) = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix} 3 \quad P(4,9) = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} 4 \quad P(4,10) = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} 4 \quad P(4,11) = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} 4$$

Table 2-3: Small roots of $\widetilde{E}_7$

then we have the following extension diagrams:

Defect - 1

$$\begin{aligned} \text{a)} & P(7,0) \xrightarrow{4} P(4,3) \xrightarrow{1} P(7,6) \xrightarrow{2} P(4,9) \xrightarrow{3} \dots \\ \text{b)} & P(7,1) \xrightarrow{1} P(4,4) \xrightarrow{2} P(7,7) \xrightarrow{3} P(4,10) \xrightarrow{4} \dots \\ \text{c)} & P(7,2) \xrightarrow{2} P(4,5) \xrightarrow{3} P(7,8) \xrightarrow{4} P(4,11) \xrightarrow{1} \dots \end{aligned}$$

Defect - 2

$$\begin{aligned} \text{d)} & P(6,0) \xrightarrow{4} P(1,2) \xrightarrow{3} P(3,3) \xrightarrow{1} P(1,5) \xrightarrow{2} \dots \\ \text{e)} & P(1,0) \xrightarrow{3} P(6,1) \xrightarrow{1} P(1,3) \xrightarrow{2} P(6,4) \xrightarrow{4} \dots \\ \text{f)} & P(1,1) \xrightarrow{4} P(6,2) \xrightarrow{2} P(1,4) \xrightarrow{3} P(6,5) \xrightarrow{1} \dots \end{aligned}$$

Defect - 3

$$\text{g)} P(5,0) \xrightarrow{4} P(5,1) \xrightarrow{3} P(5,2) \xrightarrow{2} P(5,3) \xrightarrow{1} \dots$$

Defect - 4

$$\begin{aligned} \text{h)} & P(0,0) \xrightarrow{1,2,3,4} P(0,0) + n \dots \\ \text{i)} & P(0,1) \xrightarrow{1,2,3,4} \dots \\ \text{j)} & P(0,2) \xrightarrow{1,2,3,4} \dots \end{aligned}$$

2.4.2 Reduction step

From the above information we have:

Defect	A	B	C
-4	3	0	3
-3	8	4	1
-2	18	6	3
-1	24	12	3

[Totals: 53 22 10]

where:

A is the number of small roots with defect as in the first column.

B is the number of pairs of symmetric roots.

C is the number of infinite families defined by the extension diagrams.

The last row contains the totals for each column, therefore from these 53 tiers (column A), because of the symmetry of the graph, it is enough to study $31 (= 53 - 22)$ of them which are divided into 10 families.

2.4.3 Proof:

The reduction step is given in the following table. To delete the block i use the e -polynomial

$\hat{p}_i$

$$p_1 = W_1 + W_6$$

$$p_2 = W_2 + W_7$$

$$p_3 = W_3 + W_1$$

$$p_4 = W_4 + W_5$$

And for defect -4 we take $p_0 = p_1 \cap p_2 \cap p_3 \cap p_4$.

Figure 2-7: $\tilde{E}_7$: $P(7,0)$, $P(7,1)$, $P(7,2)$

Figure 2-8: $\tilde{E}_7$: $P(0,0)$

Figure 2-9: $\tilde{E}_7$: $P(1,0)$

Figure 2-10: $\widetilde{E}_7$; $P(1,1)$

Figure 2-11: $\tilde{E}_7$: P(5,0)

Figure 2-12: $\tilde{E}_7$; $P(0,0)$

Figure 2-13: $\tilde{E}_7$; $P(0,1)$

Figure 2-14: $\widetilde{E}_7: P(0,2)$

2.5 Preprojective Representations of $\tilde{E}_8$

2.5.1 Data

$$\Omega \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & & \nearrow & & & \\ & & 3 & \rightarrow 2 & \rightarrow & 0 & \\ 8 & \rightarrow & 7 & \rightarrow & 6 & \rightarrow & 5 & \rightarrow & 4 \end{array} \quad n = \begin{pmatrix} & 3 & & & \\ & 2 & 4 & 6 & \\ 1 & 2 & 3 & 4 & 5 & \end{pmatrix} \quad \delta = \begin{pmatrix} & 3 & & & \\ & 2 & 2 & 4 & 6 & \\ 1 & 1 & 1 & 1 & 1 & \end{pmatrix}$$

$$Q(x) = (x_1 - \frac{1}{2}x_0)^2 + (x_3 - \frac{1}{2}x_2)^2 + \frac{3}{4}(x_2 - \frac{2}{3}x_0)^2 + (x_8 - \frac{1}{2}x_7)^2 + \frac{3}{4}(x_7 - \frac{2}{3}x_6)^2 \\ + \frac{2}{3}(x_6 - \frac{3}{4}x_5)^2 + \frac{5}{8}(x_5 - \frac{4}{5}x_4)^2 + \frac{3}{5}(x_4 - \frac{5}{6}x_0)^2$$

$$e^{-1} = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad m = 30$$

Next we consider the following roots of zero defect:

$$E_1 = \begin{pmatrix} & 1 & & \\ & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} & 0 & & \\ & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \\ E_3 = \begin{pmatrix} & 1 & & \\ & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} & 1 & & \\ & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$P(0,0) = \begin{pmatrix} 0 \\ 000001 \end{pmatrix}$$

$$P(0,1) = \begin{pmatrix} 1 \\ 000011 \end{pmatrix}$$

$$P(0,2) = \begin{pmatrix} 1 \\ 000122 \end{pmatrix}$$

$$P(0,3) = \begin{pmatrix} 2 \\ 001234 \end{pmatrix}$$

$$P(0,4) = \begin{pmatrix} 2 \\ 012345 \end{pmatrix}$$

$$P(1,0) = \begin{pmatrix} 1 \\ 000001 \end{pmatrix}$$

$$P(1,1) = \begin{pmatrix} 0 \\ 000011 \end{pmatrix}$$

$$P(1,2) = \begin{pmatrix} 1 \\ 000112 \end{pmatrix}$$

$$P(1,3) = \begin{pmatrix} 1 \\ 001122 \end{pmatrix}$$

$$P(1,4) = \begin{pmatrix} 1 \\ 011223 \end{pmatrix}$$

$$P(1,5) = \begin{pmatrix} 2 \\ 112234 \end{pmatrix}$$

$$P(1,6) = \begin{pmatrix} 2 \\ 011234 \end{pmatrix}$$

$$P(1,7) = \begin{pmatrix} 2 \\ 112234 \end{pmatrix}$$

$$P(1,8) = \begin{pmatrix} 2 \\ 012345 \end{pmatrix}$$

$$P(1,9) = \begin{pmatrix} 2 \\ 123456 \end{pmatrix}$$

$$P(2,0) = \begin{pmatrix} 0 \\ 000001 \end{pmatrix}$$

$$P(2,1) = \begin{pmatrix} 1 \\ 000011 \end{pmatrix}$$

$$P(2,2) = \begin{pmatrix} 1 \\ 000012 \end{pmatrix}$$

$$P(2,3) = \begin{pmatrix} 1 \\ 001122 \end{pmatrix}$$

$$P(2,4) = \begin{pmatrix} 2 \\ 012234 \end{pmatrix}$$

$$P(2,5) = \begin{pmatrix} 2 \\ 122345 \end{pmatrix}$$

$$P(2,6) = \begin{pmatrix} 2 \\ 112234 \end{pmatrix}$$

$$P(2,7) = \begin{pmatrix} 2 \\ 012234 \end{pmatrix}$$

$$P(3,0) = \begin{pmatrix} 0 \\ 000001 \end{pmatrix}$$

$$P(3,1) = \begin{pmatrix} 1 \\ 000011 \end{pmatrix}$$

$$P(3,2) = \begin{pmatrix} 0 \\ 000011 \end{pmatrix}$$

$$P(3,3) = \begin{pmatrix} 1 \\ 001112 \end{pmatrix}$$

$$P(3,4) = \begin{pmatrix} 1 \\ 011122 \end{pmatrix}$$

$$P(3,5) = \begin{pmatrix} 1 \\ 111223 \end{pmatrix}$$

Table 2-4: Small roots of $\widetilde{E}_8$

$$P(3,6) = \begin{pmatrix} 2 \\ 001123 \end{pmatrix} \quad P(3,7) = \begin{pmatrix} 1 \\ 011123 \end{pmatrix} \quad P(3,8) = \begin{pmatrix} 2 \\ 112123 \end{pmatrix}$$

$$P(3,9) = \begin{pmatrix} 2 \\ 012123 \end{pmatrix} \quad P(3,10) = \begin{pmatrix} 2 \\ 122234 \end{pmatrix} \quad P(3,11) = \begin{pmatrix} 3 \\ 112124 \end{pmatrix}$$

$$P(3,12) = \begin{pmatrix} 2 \\ 012234 \end{pmatrix} \quad P(3,13) = \begin{pmatrix} 3 \\ 123245 \end{pmatrix} \quad P(3,14) = \begin{pmatrix} 3 \\ 123456 \end{pmatrix}$$

$$P(4,0) = \begin{pmatrix} 0 \\ 000011 \end{pmatrix} \quad P(4,1) = \begin{pmatrix} 1 \\ 000011 \end{pmatrix} \quad P(4,2) = \begin{pmatrix} 1 \\ 001112 \end{pmatrix}$$

$$P(4,3) = \begin{pmatrix} 2 \\ 011123 \end{pmatrix} \quad P(4,4) = \begin{pmatrix} 2 \\ 112123 \end{pmatrix} \quad P(4,5) = \begin{pmatrix} 3 \\ 012234 \end{pmatrix}$$

$$P(5,0) = \begin{pmatrix} 0 \\ 000011 \end{pmatrix} \quad P(5,1) = \begin{pmatrix} 1 \\ 001011 \end{pmatrix} \quad P(5,2) = \begin{pmatrix} 1 \\ 011112 \end{pmatrix}$$

$$P(5,3) = \begin{pmatrix} 2 \\ 111123 \end{pmatrix} \quad P(5,4) = \begin{pmatrix} 2 \\ 001123 \end{pmatrix} \quad P(5,5) = \begin{pmatrix} 2 \\ 012234 \end{pmatrix}$$

$$P(5,6) = \begin{pmatrix} 3 \\ 123456 \end{pmatrix}$$

$$P(6,0) = \begin{pmatrix} 0 \\ 001111 \end{pmatrix} \quad P(6,1) = \begin{pmatrix} 1 \\ 011111 \end{pmatrix} \quad P(6,2) = \begin{pmatrix} 1 \\ 111112 \end{pmatrix}$$

$$P(6,3) = \begin{pmatrix} 2 \\ 000123 \end{pmatrix} \quad P(6,4) = \begin{pmatrix} 1 \\ 001123 \end{pmatrix} \quad P(6,5) = \begin{pmatrix} 2 \\ 012123 \end{pmatrix}$$

Table 2-4: $\widetilde{E}_3$

$P(6,6) = \begin{pmatrix} & 2 \\ 123 & 34 & 5 \end{pmatrix}$	$P(6,7) = \begin{pmatrix} & 3 \\ 122 & 24 & 5 \end{pmatrix}$	$P(6,8) = \begin{pmatrix} & 3 \\ 112 & 24 & 5 \end{pmatrix}$
$P(6,9) = \begin{pmatrix} & 3 \\ 012 & 24 & 5 \end{pmatrix}$		

$P(7,0) = \begin{pmatrix} & 0 \\ 011 & 00 & 1 \end{pmatrix}$	$P(7,1) = \begin{pmatrix} & 1 \\ 111 & 01 & 2 \end{pmatrix}$	$P(7,2) = \begin{pmatrix} & 1 \\ 000 & 12 & 2 \end{pmatrix}$
$P(7,3) = \begin{pmatrix} & 1 \\ 000 & 12 & 2 \end{pmatrix}$	$P(7,4) = \begin{pmatrix} & 1 \\ 001 & 02 & 2 \end{pmatrix}$	$P(7,5) = \begin{pmatrix} & 1 \\ 012 & 12 & 2 \end{pmatrix}$
$P(7,6) = \begin{pmatrix} & 2 \\ 122 & 12 & 4 \end{pmatrix}$	$P(7,7) = \begin{pmatrix} & 2 \\ 111 & 12 & 4 \end{pmatrix}$	$P(7,8) = \begin{pmatrix} & 2 \\ 001 & 22 & 4 \end{pmatrix}$
$P(7,9) = \begin{pmatrix} & 2 \\ 012 & 12 & 4 \end{pmatrix}$	$P(7,10) = \begin{pmatrix} & 2 \\ 123 & 12 & 5 \end{pmatrix}$	$P(7,11) = \begin{pmatrix} & 3 \\ 123 & 14 & 5 \end{pmatrix}$
$P(7,12) = \begin{pmatrix} & 3 \\ 122 & 24 & 5 \end{pmatrix}$	$P(7,13) = \begin{pmatrix} & 3 \\ 112 & 24 & 5 \end{pmatrix}$	$P(7,14) = \begin{pmatrix} & 3 \\ 012 & 24 & 5 \end{pmatrix}$

$P(8,0) = \begin{pmatrix} & 0 \\ 111 & 00 & 1 \end{pmatrix}$	$P(8,1) = \begin{pmatrix} & 1 \\ 000 & 00 & 1 \end{pmatrix}$	$P(8,2) = \begin{pmatrix} & 0 \\ 000 & 01 & 1 \end{pmatrix}$
$P(8,3) = \begin{pmatrix} & 1 \\ 000 & 00 & 1 \end{pmatrix}$	$P(8,4) = \begin{pmatrix} & 0 \\ 001 & 01 & 1 \end{pmatrix}$	$P(8,5) = \begin{pmatrix} & 1 \\ 011 & 11 & 2 \end{pmatrix}$

Table 2-4: $\tilde{E}_3$

$P(8,6) = \begin{pmatrix} 1 \\ 111012 \\ 2 \end{pmatrix}$	$P(8,7) = \begin{pmatrix} 1 \\ 000112 \\ 2 \end{pmatrix}$	$P(8,8) = \begin{pmatrix} 1 \\ 001112 \\ 2 \end{pmatrix}$
$P(8,9) = \begin{pmatrix} 1 \\ 011012 \\ 2 \end{pmatrix}$	$P(8,10) = \begin{pmatrix} 1 \\ 112112 \\ 2 \end{pmatrix}$	$P(8,11) = \begin{pmatrix} 2 \\ 011112 \\ 2 \end{pmatrix}$
$P(8,12) = \begin{pmatrix} 1 \\ 111112 \\ 2 \end{pmatrix}$	$P(8,13) = \begin{pmatrix} 2 \\ 001122 \\ 2 \end{pmatrix}$	$P(8,14) = \begin{pmatrix} 1 \\ 012112 \\ 2 \end{pmatrix}$
$P(8,15) = \begin{pmatrix} 2 \\ 122112 \\ 2 \end{pmatrix}$	$P(8,16) = \begin{pmatrix} 2 \\ 112112 \\ 2 \end{pmatrix}$	$P(8,17) = \begin{pmatrix} 2 \\ 011112 \\ 2 \end{pmatrix}$
$P(8,18) = \begin{pmatrix} 2 \\ 112112 \\ 2 \end{pmatrix}$	$P(8,19) = \begin{pmatrix} 2 \\ 012112 \\ 2 \end{pmatrix}$	$P(8,20) = \begin{pmatrix} 2 \\ 122112 \\ 2 \end{pmatrix}$
$P(8,21) = \begin{pmatrix} 3 \\ 122112 \\ 2 \end{pmatrix}$	$P(8,22) = \begin{pmatrix} 2 \\ 112112 \\ 2 \end{pmatrix}$	$P(8,23) = \begin{pmatrix} 3 \\ 012112 \\ 2 \end{pmatrix}$
$P(8,24) = \begin{pmatrix} 2 \\ 123112 \\ 2 \end{pmatrix}$	$P(8,25) = \begin{pmatrix} 3 \\ 123112 \\ 2 \end{pmatrix}$	$P(8,26) = \begin{pmatrix} 3 \\ 123112 \\ 2 \end{pmatrix}$
$P(8,27) = \begin{pmatrix} 3 \\ 122112 \\ 2 \end{pmatrix}$	$P(8,28) = \begin{pmatrix} 3 \\ 113112 \\ 2 \end{pmatrix}$	$P(8,29) = \begin{pmatrix} 3 \\ 023112 \\ 2 \end{pmatrix}$

Table 2-4: $\widetilde{E}_2$

2.5.2 Extension diagrams

Defect -1

$$\begin{aligned}
 P(8,0) &\xrightarrow{1} P(8,6) \xrightarrow{2} P(8,12) \xrightarrow{3} P(8,18) \xrightarrow{4} P(8,24) \xrightarrow{5} \dots \\
 P(8,1) &\xrightarrow{2} P(8,7) \xrightarrow{3} P(8,13) \xrightarrow{4} P(8,19) \xrightarrow{5} P(8,25) \xrightarrow{1} \dots \\
 P(8,2) &\xrightarrow{3} P(8,8) \xrightarrow{4} P(8,14) \xrightarrow{5} P(8,20) \xrightarrow{1} P(8,26) \xrightarrow{2} \dots \\
 P(8,3) &\xrightarrow{4} P(8,9) \xrightarrow{5} P(8,15) \xrightarrow{1} P(8,21) \xrightarrow{2} P(8,27) \xrightarrow{3} \dots \\
 P(8,4) &\xrightarrow{5} P(8,10) \xrightarrow{1} P(8,16) \xrightarrow{2} P(8,22) \xrightarrow{3} P(8,28) \xrightarrow{4} \dots \\
 P(8,5) &\xrightarrow{1} P(8,11) \xrightarrow{2} P(8,17) \xrightarrow{3} P(8,23) \xrightarrow{4} P(8,29) \xrightarrow{5} \dots
 \end{aligned}$$

Defect -2

$$\begin{array}{ccccccc}
 & & P(7,5) & & P(7,11) & & \\
 & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \\
 P(3,0) & \xrightarrow{3} P(3,3) & \xrightarrow{1} P(3,6) & \xrightarrow{4} P(3,9) & \xrightarrow{2} P(3,12) & \xrightarrow{5} P(3,0) + n & \xrightarrow{3} P(3,3) + n \dots \\
 & \searrow & \searrow & \searrow & \searrow & \searrow & \\
 & P(7,2) & & P(7,8) & & P(7,14) &
 \end{array}$$

$$\begin{array}{ccccccc}
 & & P(7,6) & & P(7,12) & & \\
 & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \\
 P(3,1) & \xrightarrow{4} P(3,4) & \xrightarrow{2} P(3,7) & \xrightarrow{5} P(3,10) & \xrightarrow{3} P(3,13) & \xrightarrow{1} P(3,1) + n & \xrightarrow{4} P(3,4) + n \dots \\
 & \searrow & \searrow & \searrow & \searrow & \searrow & \\
 & P(7,3) & & P(7,9) & & P(7,0) + n &
 \end{array}$$

$$\begin{array}{ccccccc}
 & & P(7,7) & & P(7,13) & & \\
 & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \\
 P(3,2) & \xrightarrow{5} P(3,5) & \xrightarrow{3} P(3,8) & \xrightarrow{1} P(3,11) & \xrightarrow{4} P(3,14) & \xrightarrow{2} P(3,2) + n & \xrightarrow{5} P(3,5) + n \dots \\
 & \searrow & \searrow & \searrow & \searrow & \searrow & \\
 & P(7,4) & & P(7,10) & & P(7,1) + n &
 \end{array}$$

Defect -3

Defect -4

$$P(5,0) \xrightarrow{1} P(2,2) \xrightarrow{5} P(5,3) \xrightarrow{4} P(2,5) \xrightarrow{3} P(5,6) \xrightarrow{2} P(2,0) + n \dots$$

$$P(2,0) \xrightarrow{3} P(5,1) \xrightarrow{2} P(2,3) \xrightarrow{1} P(5,4) \xrightarrow{5} P(2,6) \xrightarrow{4} P(5,0) + n \dots$$

$$P(2,1) \xrightarrow{4} P(5,2) \xrightarrow{3} P(2,4) \xrightarrow{2} P(5,5) \xrightarrow{1} P(2,7) \xrightarrow{5} P(2,1) + n \dots$$

Defect -5

$$P(4,k) \xrightarrow{1,2,3,4,5} P(4,k+5) \quad \dots \quad k \in \mathbb{N}$$

Defect -6 Amalgamated sums.

$C^{-j+5}P(0,0) \simeq X$ where X is the pushout of:

$$\begin{array}{ccc}
 C^{-j}P(8,0) & \longrightarrow & C^{-j+5}P(2,0) \\
 \downarrow & & \\
 C^jP(1,0) & &
 \end{array}$$

2.5.3 Proofs

For Defect -6 it follows from (1.3.12) for any $j \geq 0$, the first five in the c^{-1} -orbit are given in a separate figure and are clearly indecomposable. The reduction step in the rest of the representations

is given by:

Block polynomial

$$1 \quad p_1 = W_3 + W_5$$

$$2 \quad p_2 = W_1 \dot{+} W_6$$

$$3 \quad p_3 = W_2 + W_7$$

$$4 \quad p_4 = W_1 + W_3 + W_8$$

$$5 \quad p_5 = (p_6 \cap W_2) + (p_6 \cap W_4)$$

$$p_6 = W_1 + (W_2 \cap W_4)$$

(a)

(b)

Figure 15.: $\tilde{E}_8$: $P(8,0)$, $P(8,1)$

(82)

(83)

Figure 16.: E_8 : $P(8, 2)$, $P(8, 3)$.

Figure 17. $\tilde{E}_8$: $P(8,4)$, $P(8,5)$

Figure 18.: $\tilde{E}_2: P(3,0)$

Figure 19.: $E_8, P(3,1)$

Figure 20.: $\tilde{E}_3$: P(3,2)

Figure 21.: E_8 : $P(1,0)$

Figure 22.: $\tilde{E}_8$: P(1,1)

Figure 23.: $E_3: P(5,0)$

Figure 24.: $\tilde{E}_8$: $P(2,0)$

Figure 25.: $\widetilde{E}_8$: $P(2,1)$

Figure 26.: $\tilde{E}_8: P(4,0)$

The letters identify the blocks on the right hand side to those they are linked, as in the previous figure. Note that the point b appears in two different subspaces namely W_3 and W_5 , so we should change the second block in the example accordingly.

Figure 27.: E_8 : Small blocks for defect -5

Figure 28.: $\tilde{E}_3$: Defect -6, kernel of type P(8,0)

Figure 29.: $\tilde{E}_3$: Defect -6, kernel of type P(8,1)

Figure 30.: $\tilde{E}_8$: Defect -6, kernel of type $P(8,2)$

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Figure 31.: E_3 : Defect -6, kernel of type P(8,3)

Figure 32.: $\tilde{E}_8$: Defect -6, kernel of type $R(8, 4)$

Figure 33. $\tilde{E}_8$: Defect -6, kernel of type P(8,5)

Figure 34.: $\tilde{E}_3$: Defect -6, $P(0,0)$, $P(0,1)$, $P(0,2)$, $P(0,3)$, $P(0,4)$

Chapter 3.

More Euclidean Diagrams

This chapter follows the same format as the previous one. Here we study the diagrams with non trivial valuations, therefore, the subspaces of the representation space are sometimes tensor products which is translated in the pictures, as follows, if $V \otimes L$ is a subspace of W and $v \in V$, $l_i \in L$, then the vectors $v \otimes l_i$ are in the same vertical layer and are separated by dashed lines (whenever it is clear, we omit them). The reader is referred to the figures for examples.

3,1 Preprojective representations of $\widetilde{A}_{12}$

$$\Omega : \quad 1 \xrightarrow{(2,2)} 0 \quad n = (1,1) = E_1 \quad \delta = (2,-2) \quad g = 1$$

$$M = F \xrightarrow{F M_F} F \quad \dim(M_F) = \dim({}_F M) = 2$$

$$Q(x) = (x_1 - x_0)^2 \quad m = 1$$

Small positive roots: $P(0,0) = (0,1)$ and $P(1,0) = (1,2)$

Extension diagrams: $P(0,0) \xrightarrow{1} P(1,0) \xrightarrow{1} P(0,1) \xrightarrow{1} \dots$

Recall if: $W = (V_F, W_F)$, then $V_F \longrightarrow W_F \otimes_F M_F \longleftarrow W_F$

Preprojective representations: The reduction step is given by: $p = (W_F + V_F)^i$. Cf.: [DR-77].

Figure 3-1: $\tilde{A}_{12}$

3.2 Preprojective representations of $\tilde{A}_{22}$

$$\Omega \quad 1 \xrightarrow{(1,4)} 0 \quad n = (2,1) = E_1 \quad \delta = (1,-2) \quad g = 2$$

$$M = G \xrightarrow{G M_F} F \quad \dim(G M) = 4 \quad \dim(M_F) = 1$$

$$Q(x) = (2x_1 - x_0)^2 \quad m = 1$$

$$\text{Small positive roots: } P(0,0) = (0,1) \quad P(1,0) = (1,1) \quad P(1,1) = (3,2)$$

Extension diagrams:

$$P(1,0) \xrightarrow{1} P(1,1) \xrightarrow{1} P(1,2) \xrightarrow{1} \dots$$

$$P(0,0) \xrightarrow{1,1} P(0,1) \xrightarrow{1,1} P(0,2) \xrightarrow{11} \dots$$

Note that the G -basis of F induces a G -basis in $(W_F)_G$ which is described by the horizontal layers and identified on the right hand side of the representations of defect -1. Besides the G -basis of F is a basis of the quaternions. In the following e-polynomials just substitute f^m ($m = 1, 2, 3$) for the corresponding element if F is a quaternion skew-field extension of G .

Reduction Step: To delete the right hand side extension we use $p_1(X_1) = (X_1 + f^2 X_1)^i$. In defect -1, two extensions on the left hand side can be deleted with $p_2(X_1) = (X_1 + f X_1)^i$ and if $\dim X_0 \geq 4$, by (1.3.2), it follows that X is indecomposable. Cf.: [DR-78].

Figure 3-2: $\tilde{A}_{22}$

3.3 Preprojective representations of $\tilde{B}_n$

$$\Omega \quad (n-1) \xrightarrow{(1,2)} (n-2) \longrightarrow \cdots \longrightarrow 1 \longrightarrow 0 \xleftarrow{(2,1)} n$$

$$M = G \xrightarrow{G M_F} F \longrightarrow \cdots \longrightarrow F \longrightarrow F \xleftarrow{G M_F} G \quad [F : G] = 2$$

$$Q(x) = (x_n - x_0)^2 + \sum_{i=0}^{n-2} (x_i - x_{i+1})^2$$

$$n = \begin{pmatrix} 1 & & \\ & 1 & \\ 1 & 1 & \dots & 1 \end{pmatrix} \quad \delta = \begin{pmatrix} 2 & & \\ & 2 & \\ 2 & 0 & \dots & 0 \end{pmatrix} \quad g = 2 \quad m = n$$

Reduction: If X is a preprojective indecomposable representation such that $\dim X \leq 2n$, then the maximum number of morphisms φ_{i+1} ($i = 0, \dots, n-2$) which are not isomorphisms is 2, and therefore it is enough to study $\tilde{B}_4$.

$$P(0,0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad P(0,1) = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad P(0,2) = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$P(1,0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad P(1,1) = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$P(2,0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$P(3,0) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad P(3,1) = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \quad P(3,2) = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix} \quad P(3,3) = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

$$P(4,0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad P(4,1) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad P(4,2) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad P(4,3) = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$

Table 3-1: Small roots of $\tilde{B}_4$

Extension diagrams: with $E_1 = n$

Defect - 1

$$a) P(4, 0) \xrightarrow{1} P(4, 4) \xrightarrow{1} \dots$$

$$b) P(4, 1) \xrightarrow{1} P(4, 5) \xrightarrow{1} \dots$$

$$c) P(4, 2) \xrightarrow{1} P(4, 6) \xrightarrow{1} \dots$$

$$d) P(4, 3) \xrightarrow{1} P(4, 7) \xrightarrow{1} \dots$$

$$e) P(3, 0) \xrightarrow{1} P(3, 4) \xrightarrow{1} \dots$$

$$f) P(3, 1) \xrightarrow{1} P(3, 5) \xrightarrow{1} \dots$$

$$g) P(3, 2) \xrightarrow{1} P(3, 6) \xrightarrow{1} \dots$$

$$h) P(3, 3) \xrightarrow{1} P(3, 7) \xrightarrow{1} \dots$$

Defect - 2

$$i) P(0, 0) \xrightarrow{1,1} P(2, 1) \xrightarrow{1,1} P(0, 4) \xrightarrow{1,1} \dots$$

$$j) P(0, 1) \xrightarrow{1,1} P(2, 2) \xrightarrow{1,1} P(0, 5) \xrightarrow{1,1} \dots$$

$$k) P(0, 2) \xrightarrow{1,1} P(2, 3) \xrightarrow{1,1} P(0, 6) \xrightarrow{1,1} \dots$$

$$l) P(2, 0) \xrightarrow{1,1} P(0, 3) \xrightarrow{1,1} P(2, 4) \xrightarrow{1,1} \dots$$

$$m) P(1, 0) \xrightarrow{1,1} P(1, 2) \xrightarrow{1,1} P(1, 4) \xrightarrow{1,1} \dots$$

$$n) P(1, 1) \xrightarrow{1,1} P(1, 3) \xrightarrow{1,1} P(1, 5) \xrightarrow{1,1} \dots$$

More reductions: In defect -2, the indecomposability depends only on the representation space and the G -subspaces (cf: the proofs), which are the indecomposable representations of defect -2 in $\widetilde{B}_2$. This in turn is symmetric, and therefore we only need to construct one of the families whose dimension types are:

$$\begin{pmatrix} 1 & & \\ 0 & 1 & \\ & & 1 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 2 & & \\ 1 & 2 & \\ & & 2 \end{pmatrix}$$

In defect -4, the representations can be divided into two groups:

- 1) Those whose indecomposability depend on the G -subspaces:

$$\begin{pmatrix} 0 & & \\ & 2 & \\ 0 & x & 1 \end{pmatrix} \xrightarrow{1,1} \begin{pmatrix} & 2 & \\ & x & 3 \end{pmatrix} \xrightarrow{1,1} \dots$$

which are the families in i), j) and k).

- 2) Those whose indecomposability depend on the G -subspaces and one F -subspace:

$$\begin{pmatrix} & 2 & \\ 0 & a & b \end{pmatrix} \xrightarrow{1,1} \begin{pmatrix} & 4 & \\ x & a+2 & b+2 \end{pmatrix} \xrightarrow{1,1} \dots$$

where a or b is 1, the other can be either 0, 1 or 2 with the constrain $a \leq b$. The general case is when $b = 1$.

Proofs: The reduction step is given by $p = (X_3 + X_4)^i$. In defect -4, X_1 is used to prove the indecomposability of the inner block, i.e. the subsystem defined by p . By (1.3.5) it follows that they are indecomposable.

Figure 3-3: Representations of $\tilde{B}_4$

3.4 Preprojective representations of $\widetilde{BC}_n$

$$\Omega \quad (n-1) \xrightarrow{(1,2)} (n-2) \longrightarrow \cdots \longrightarrow 1 \longrightarrow 0 \xleftarrow{(1,2)} n$$

$$M = G \xrightarrow{cM_F} F \longrightarrow \cdots \longrightarrow F \longrightarrow F \xleftarrow{H} H \quad [H : F] = [F : G] = 2$$

$$Q(x) = \frac{1}{2} \left[(2x_n - x_0)^2 + \sum_{i=0}^{n-2} (x_i - x_{i+1})^2 \right]$$

$$n = \begin{pmatrix} 1 & & \\ 2 & 2 & \cdots & 2 \end{pmatrix} \quad \delta = \begin{pmatrix} 2 & & \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad g = 2 \quad m = n$$

Reduction: The maximum number of maps φ_{i+1} ($i = 0, 1, \dots, n-2$) which are not isomorphisms is 2, and therefore it is enough to study $\widetilde{BC}_4$. This follows because for a small root x of negative defect $x \leq 2n$ so $x_0 \leq 4$. Now, in $\widetilde{BC}_6$ if we take $x \leq 2n$ such that $x_4 \leq x_3 \leq x_2 \leq x_1 \leq x_0$ with at least three strict inequalities, then a straightforward computation shows that there is no $0 \leq h \leq 6 (= m)$ for which $c^{-1}x$ is the dimension type of a projective representation.

$$P(0,0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad P(0,1) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad P(0,2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad P(0,3) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

$$P(1,0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad P(1,1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad P(1,2) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \quad P(1,3) = \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}$$

$$P(2,0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad P(2,1) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad P(2,2) = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} \quad P(2,3) = \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}$$

$$P(3,0) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad P(3,1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad P(3,2) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad P(3,3) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

$$P(4,0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad P(4,1) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad P(4,2) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad P(4,3) = \begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix}$$

Table 3-2: Small roots of $\widetilde{BC}_4$

Next we consider the following roots of defect zero:

$$E_1 = \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & \\ 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 0 & \\ 0 & 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & \\ 1 & 1 \end{pmatrix}$$

And to avoid repetitions in constructions we use the following

Extension diagrams:

Defect - 1

$$P(3, 0) \xrightarrow{1} P(3, 1) \xrightarrow{2} P(3, 2) \xrightarrow{3} P(3, 3) \xrightarrow{4} P(3, 4) \xrightarrow{1} \dots$$

Defect - 2

$$a) P(0, 0) \xrightarrow{1} P(4, 0) \xrightarrow{2} P(0, 1) \xrightarrow{2} P(4, 1) \xrightarrow{3} P(0, 2) \xrightarrow{3} \dots$$

$$b) P(1, 0) \xrightarrow{3} P(2, 0) \xrightarrow{1} P(1, 1) \xrightarrow{4} P(2, 1) \xrightarrow{2} P(1, 2) \xrightarrow{1} \dots$$

Proofs. To delete the block i we use the e-polynomial p_i :

$$p_1 = X_3^e$$

$$p_2 = X_2$$

$$p_3 = X_3^e \cap X_1$$

$$p_4 = (X_0 + X_4)^{i(H)}$$

In defect -2, family b); p_1 is either X_1 or X_2 , the other cases are the same.

Figure 3-5: $\widetilde{BC}_1$: defect -2

3.5 Preprojective representations of $\widetilde{BD}_n$

$$\begin{array}{c}
 \Omega \quad \begin{array}{c} n \\ n-1 \rightarrow 0 \\ \begin{array}{c} (1,2) \\ n-2 \rightarrow n-3 \rightarrow \dots \rightarrow 3 \rightarrow 2 \rightarrow 1 \end{array} \end{array} \\
 \\
 M = \quad \begin{array}{c} F \\ F \rightarrow F \\ G \xrightarrow{GM_F} F \rightarrow \dots \rightarrow F \rightarrow F \end{array} \quad [F:G] = 2 \\
 \\
 n = \begin{pmatrix} 1 \\ 1 & 2 \\ 2 & 2 \dots 2 \end{pmatrix} \quad \delta = p \begin{pmatrix} 1 \\ 1 & -2 \\ 1 & 0 \dots 0 \end{pmatrix} \quad g = 1 \quad m = p(n-1) \\
 \\
 \text{where } p \text{ is } 1 \text{ if } n \text{ is odd or } 2 \text{ if even} \\
 Q(x) = \frac{1}{2} \sum_{i=n-1}^n (2x_i - x_0)^2 + \sum_{i=0}^{n-1} (x_i - x_{i+1})^2
 \end{array}$$

Reduction: Since $x_0 \leq 2$ for a small root then the maximum number of maps φ_{i+1} ($i = 0, \dots, n-4$) which are not isomorphisms is 2, and therefore it is enough to study $\widetilde{BD}_5$.

$$P(0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P(0,1) = \begin{pmatrix} 1 \\ 0 & 0 & 1 \\ 2 \end{pmatrix} \quad P(0,2) = \begin{pmatrix} 1 \\ 0 & 1 & 2 \\ 2 \end{pmatrix}$$

$$P(1,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 \end{pmatrix} \quad P(1,1) = \begin{pmatrix} 1 \\ 0 & 1 & 2 \\ 2 \end{pmatrix}$$

$$P(2,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 \end{pmatrix}$$

$$P(3,0) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 \end{pmatrix} \quad P(3,1) = \begin{pmatrix} 1 \\ 1 & 1 & 2 \\ 2 \end{pmatrix} \quad P(3,2) = \begin{pmatrix} 1 \\ 1 & 1 & 2 \\ 2 \end{pmatrix} \quad P(3,3) = \begin{pmatrix} 1 \\ 1 & 2 & 2 \\ 2 \end{pmatrix}$$

$$P(4,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 \end{pmatrix} \quad P(4,1) = \begin{pmatrix} 1 \\ 0 & 0 & 1 \\ 1 \end{pmatrix} \quad P(4,2) = \begin{pmatrix} 0 \\ 0 & 1 & 1 \\ 1 \end{pmatrix} \quad P(4,3) = \begin{pmatrix} 1 \\ 2 & 2 & 2 \\ 2 \end{pmatrix}$$

Table 3-3: Small roots of $\widetilde{BD}_5$

By symmetry of the graph, $P(4, k)$ and $P(5, k)$ differ only in the position of the subspaces, so

It is enough to build only one of them for each k . Next, we consider the following roots of defect

zero:

$$E_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ 0 & 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & & \\ & 0 & \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ 0 & 1 & 0 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & & \\ & 0 & \\ 2 & 1 & 1 \end{pmatrix}$$

Extension Diagrams:

Defect -1

$$\begin{aligned} P(4, 0) &\xrightarrow{2} P(5, 1) \xrightarrow{3} P(4, 2) \xrightarrow{4} P(5, 3) \xrightarrow{1} P(4, 4) \xrightarrow{2} \dots \\ P(3, 0) &\xrightarrow{1} P(3, 1) \xrightarrow{2} P(3, 2) \xrightarrow{3} P(3, 3) \xrightarrow{4} P(3, 4) \xrightarrow{1} \dots \end{aligned}$$

Defect -2

$$\begin{array}{ccccc} P(0, 0) & P(0, 1) & P(0, 2) & P(0, 3) & P(0, 4) \\ \downarrow 2 \nearrow 1 & \downarrow 3 \nearrow 2 & \downarrow 4 \nearrow 3 & \downarrow 1 \nearrow 4 & \downarrow 2 \\ P(1, 0) & P(1, 1) & P(x, x) & P(x, x) & P(x, x) \\ \downarrow 3 \nearrow 1 & \downarrow 4 \nearrow 2 & \downarrow 1 \nearrow 3 & \downarrow 2 \nearrow 4 & \downarrow 3 \\ P(2, 0) & P(x, x) & P(x, x) & P(x, x) & P(x, x) \end{array}$$

Proofs: When the extension is by the blocks 1 or 4, the e-polynomial used is: $p_1 = X_1$, $p_4 = X_4 + X_5$. In the other cases the previous subsystem is obtained as follows:

If the extension is by the block 2: $U_1 = X_2$, $U_i = X_i$ otherwise

If the extension is by the block 3: $U_2 = X_3$, $U_i = X_i$ otherwise.

X_1	2	"	2	"			2	"
X_2	3	"	3	"			3	"
X_3	—	—	—	—			—	—
X_4	"	"	"	"			"	"
X_5	—	—	—	—			—	—
	4	1	4	1			4	1

(7,0)

X_1	"	2	"	2	"			2	"
X_2	"	3	"	3	"			3	"
X_3	—	—	—	—	—			—	—
X_4	"	"	"	"	"			"	"
X_5	—	—	—	—	—			—	—
	1	4	1	4	1			1	4

(3,0)

Figure 3-6: $\widetilde{BD}_5$: Defect -1

Figure 3-7: $\widetilde{BD}_5$: Defect -2

3.6 Preprojective representations of $\widetilde{C}_n$

$$\Omega \quad (n-1) \xrightarrow{(2,1)} (n-2) \longrightarrow \dots \longrightarrow 0 \xleftarrow{(1,2)} n$$

$$M = F \xrightarrow{F^{M_G}} G \xrightarrow{F^{M_G}} \dots \longrightarrow G \longrightarrow G \longrightarrow G \xleftarrow{F^{M_G}} F \quad [F : G] = 2$$

$$Q(x) = \frac{1}{2} \left[(2x_{n-1} - x_{n-2})^2 + (2x_n - x_0)^2 + \sum_{i=0}^{n-2} (x_i - x_{i+1})^2 \right]$$

$$n = \begin{pmatrix} 1 & & \\ 1 & 2 & \dots & 2 \end{pmatrix} \quad \delta = \begin{pmatrix} 2 & & \\ 2 & 0 & \dots & 0 & -2 \end{pmatrix} \quad y = 1 \quad m = n$$

Reduction: By the same token as in $\widetilde{B}_n$, it is enough to study $\widetilde{C}_4$, whose small positive roots of negative defect are:

$$P(0,0) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \quad P(0,1) = \begin{pmatrix} 0 & 0 & 1 & 2 \end{pmatrix} \quad P(0,2) = \begin{pmatrix} 0 & 1 & 2 & 2 \end{pmatrix}$$

$$P(1,0) = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \quad P(1,1) = \begin{pmatrix} 0 & 1 & 1 & 2 \end{pmatrix}$$

$$P(2,0) = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}$$

$$P(3,0) = \begin{pmatrix} 0 & 0 & 2 & 2 \end{pmatrix}$$

$$P(4,0) = \begin{pmatrix} 0 & 0 & 0 & 2 \end{pmatrix} \quad P(4,1) = \begin{pmatrix} 0 & 0 & 1 & 2 \end{pmatrix} \quad P(4,2) = \begin{pmatrix} 0 & 2 & 1 & 2 \end{pmatrix}$$

Table 3-4: Small roots of $\widetilde{C}_5$

Extension Diagrams:

where:

$$E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

If we want the indecomposable representation of dimension type $x = x_0 + rn$, then we first build the indecomposable representation of dimension type x_0 , then follow one path in the graph of extension diagrams which joins x_0 with x .

Reduction step:

A	B
1 or 1-1	$p_1 = X_2$
2 or 2-2	$p_2 = (X_2 + X_3)^i$
3 or 3-3	$p_3 = (X_0 + X_4)^i \cap X_3$
4 or 4-4	$p_4 = (X_0 + X_3)^i \cap X_4$

where in column:

A are the extensions we wish to delete.

B are the e-polynomials to be used.

Note that when we delete extensions of type 1, 1-1, 2 or 2-2, all the subspaces remain the same except X_1 or X_2 which are changed using p_1 or p_2 accordingly.

Figure 3-8: Representations of $\widetilde{C}_4$

3.7 Preprojective representations of $\widetilde{CD}_n$

$$\begin{array}{c}
 \Omega \\
 \begin{array}{c}
 \begin{array}{c}
 \xrightarrow{(2,1)} \\
 n-2 \rightarrow n-3 \rightarrow \dots \rightarrow 3 \rightarrow 2 \rightarrow 1
 \end{array}
 \end{array}
 \end{array}
 \begin{array}{c}
 n \\
 \begin{array}{c}
 \xrightarrow{\quad} \\
 n-1 \rightarrow 0
 \end{array}
 \end{array}$$

$$\bullet \quad M = \begin{array}{c}
 \begin{array}{c}
 \xrightarrow{F} \\
 F \xrightarrow{M_G} G \rightarrow \dots \rightarrow G \rightarrow G
 \end{array}
 \end{array}
 \begin{array}{c}
 G \\
 \begin{array}{c}
 \xrightarrow{\quad} \\
 G \rightarrow G
 \end{array}
 \end{array}
 \quad [F : G] = 2$$

$$n = \begin{pmatrix} 1 & & \\ & 1 & \\ 1 & 2 & 2 \end{pmatrix} \quad \delta = p \begin{pmatrix} 1 & & \\ & 1 & \\ 2 & 0 & -2 \end{pmatrix} \quad g = 2 \quad m = p(n-1)$$

$$Q(x) = \frac{1}{16} \left[2 \sum_{i=0}^{n-4} (x_i - x_{i+1})^2 + 2(2x_{n-2} - x_{n-3})^2 + (2x_{n-1} - x_0)^2 + (2x_n - x_0)^2 \right]$$

where p is 1 if n is odd and 2 otherwise.

Reduction: Since $x_0 \leq 4$ for a small positive root, then the maximum number of maps φ_{i+1} ($i = 0, \dots, n-4$) which are not isomorphisms is 2. Cf. $\widetilde{BC}_n$.

$P(0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$P(0,1) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$P(0,2) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$P(0,3) = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$
$P(1,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$P(1,1) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$P(1,2) = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$	$P(1,3) = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$
$P(2,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$P(2,1) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	$P(2,2) = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$	$P(2,3) = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$
$P(3,0) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}$	$P(3,1) = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$	$P(3,2) = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$	$P(3,3) = \begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix}$
$P(4,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$P(4,1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$P(4,2) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$P(4,3) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

Table 3-5: Small roots of $\widetilde{CD}_5$

Let:

$$E_1 = \begin{pmatrix} 0 & & \\ & 0 & \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & & \\ & 0 & \\ 0 & 1 & 0 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & 1 & 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 1 & & \\ & 1 & \\ 0 & 0 & 0 \end{pmatrix}$$

By symmetry of the graph, it is enough to build only one of the representations $P(5, k)$ or $P(4, k)$

for each k .

Extension Diagrams:

Defect - 1

$$P(4, 0) \xrightarrow{1} P(5, 1) \xrightarrow{2} P(4, 2) \xrightarrow{3} P(5, 3) \xrightarrow{4} P(4, 3) \xrightarrow{1} P(5, 4) \xrightarrow{2} \dots$$

Defect - 2

$$P(0, 0) \quad P(0, 1) \quad P(0, 2) \quad P(0, 3) \quad P(x, x)$$

$$\downarrow 1 \nearrow 4 \quad \downarrow 2 \nearrow 3 \quad \downarrow 3 \nearrow 2 \quad \downarrow 4 \nearrow 1$$

$$P(1, 0) \quad P(1, 1) \quad P(1, 2) \quad P(1, 3) \quad P(x, x)$$

$$\downarrow 2 \nearrow 4 \quad \downarrow 3 \nearrow 3 \quad \downarrow 4 \nearrow 2 \quad \downarrow 1 \nearrow 1$$

$$P(2, 0) \quad P(2, 1) \quad P(2, 2) \quad P(2, 3) \quad P(x, x)$$

$$\downarrow 3 \nearrow 4 \quad \downarrow 4 \nearrow 3 \quad \downarrow 1 \nearrow 2 \quad \downarrow 2 \nearrow 3$$

$$P(3, 0) \quad P(3, 1) \quad P(3, 2) \quad P(3, 3) \quad P(x, x)$$

Proof: First we note that the role of the extensions by E_1 and E_2 is to define the extension by

E_3 , i.e., the indecomposability of X is 'independent' of X_1 and X_2 . And the reduction step is

given by:

$$= X_4 + X_5$$

$$= (X_0 + X_3)$$

Figure 3-9: Representations of CD_5 .

3.8 Preprojective representations of $\widetilde{F}_{41}$

$$\begin{array}{c}
 \Omega \quad \quad \quad \begin{array}{c} 1 \\ \swarrow \\ 4 \rightarrow 3 \rightarrow 2 \rightarrow 0 \end{array} \\
 M = G \rightarrow G \rightarrow G \xrightarrow{G^{Mr}} F \leftarrow F \quad [F : G] = 2 \\
 n = \begin{pmatrix} 1 & \\ & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 2 & \\ & -4 \\ 1 & 1 & 1 \end{pmatrix} \quad g = 2 \quad m = 6 \\
 Q(x) = \frac{1}{12} [6(2x_1 - x_0)^2 + 3(2x_4 - x_3)^2 + 2(2x_2 - 3x_0)^2]
 \end{array}$$

Here we use two different generating sets, though not essential, but gives a new insight on how to build the representations; this is done for defects -1 and -2.

$$\begin{array}{l}
 O_1: \quad E_1 = \begin{pmatrix} 1 & \\ & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & \\ & 1 \\ 1 & 1 & 2 \end{pmatrix} \\
 O_2: \quad E_3 = \begin{pmatrix} 1 & \\ & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & \\ & 1 \\ 0 & 2 & 2 \end{pmatrix} \quad E_5 = \begin{pmatrix} 1 & \\ & 2 \\ 2 & 2 & 2 \end{pmatrix}
 \end{array}$$

$$P(0,0) = \begin{pmatrix} 0 & \\ & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad P(0,1) = \begin{pmatrix} 1 & \\ & 2 \\ 0 & 0 & 2 \end{pmatrix} \quad P(0,2) = \begin{pmatrix} 1 & \\ & 3 \\ 0 & 2 & 4 \end{pmatrix}$$

$$P(1,0) = \begin{pmatrix} 1 & \\ & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad P(1,1) = \begin{pmatrix} 0 & \\ & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad P(1,2) = \begin{pmatrix} 1 & \\ & 2 \\ 0 & 2 & 2 \end{pmatrix} \quad P(1,3) = \begin{pmatrix} 1 & \\ & 3 \\ 2 & 2 & 4 \end{pmatrix}$$

$$P(1,4) = \begin{pmatrix} 2 & \\ & 3 \\ 0 & 2 & 4 \end{pmatrix} \quad P(1,5) = \begin{pmatrix} 1 & \\ & 4 \\ 2 & 4 & 6 \end{pmatrix}$$

$$P(2,0) = \begin{pmatrix} 0 & \\ & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad P(2,1) = \begin{pmatrix} 1 & \\ & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$P(3,0) = \begin{pmatrix} 0 & \\ & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad P(3,1) = \begin{pmatrix} 1 & \\ & 2 \\ 1 & 1 & 2 \end{pmatrix} \quad P(3,2) = \begin{pmatrix} 1 & \\ & 3 \\ 0 & 1 & 3 \end{pmatrix}$$

$$P(4,0) = \begin{pmatrix} 0 & \\ & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad P(4,1) = \begin{pmatrix} 1 & \\ & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad P(4,2) = \begin{pmatrix} 0 & \\ & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad P(4,3) = \begin{pmatrix} 1 & \\ & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

$$P(4,4) = \begin{pmatrix} 1 & \\ & 2 \\ 1 & 1 & 3 \end{pmatrix} \quad P(4,5) = \begin{pmatrix} 1 & \\ & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

Table 3-6: Small roots of $\widetilde{F}_{41}$

Extension Diagrams:

Defect -1: (We identify the first and last rows)

$$\begin{array}{ccccccc}
 P(4,0) & \xrightarrow{1} & P(4,3) & \xrightarrow{2} & P(4,0) + n & \xrightarrow{1} & P(4,3) + n \xrightarrow{2} P(4,0) + 2n \xrightarrow{1} \dots \\
 & \searrow 3 & & \searrow 3 & & \searrow 3 & \\
 P(4,1) & \xrightarrow{1} & P(4,4) & \xrightarrow{2} & P(4,1) + n & \xrightarrow{1} & P(4,4) + n \xrightarrow{2} P(4,1) + 2n \xrightarrow{1} \dots \\
 & \searrow 4 & & \searrow 4 & & \searrow 4 & \\
 P(4,2) & \xrightarrow{1} & P(4,5) & \xrightarrow{2} & P(4,2) + n & \xrightarrow{1} & P(4,5) + n \xrightarrow{2} P(4,2) + 2n \xrightarrow{1} \dots \\
 & \searrow 5 & & \searrow 5 & & \searrow 5 & \\
 P(4,0) & \xrightarrow{1} & P(4,3) & \xrightarrow{2} & P(4,0) + n & \xrightarrow{1} & P(4,3) + n \xrightarrow{2} P(4,0) + 2n \xrightarrow{1} \dots
 \end{array}$$

Defect -2: (We identify the first and last rows)

$$\begin{array}{ccccccc}
 P(3,0) & \xrightarrow{1} & P(1,2) & \xrightarrow{2} & P(3,0) + n & \xrightarrow{2} & P(1,5) \xrightarrow{1} P(3,0) + 2n \xrightarrow{1} \dots \\
 & \searrow 3 & & \searrow 3 & & \searrow 3 & \\
 P(1,1) & \xrightarrow{1} & P(3,2) & \xrightarrow{1} & P(1,4) & \xrightarrow{2} & P(3,2) + n \xrightarrow{2} P(1,1) + 2n \xrightarrow{1} \dots \\
 & \searrow 5 & & \searrow 5 & & \searrow 5 & \\
 P(1,0) & \xrightarrow{2} & P(3,1) & \xrightarrow{2} & P(1,3) & \xrightarrow{1} & P(3,1) + n \xrightarrow{1} P(1,0) + 2n \xrightarrow{2} \dots \\
 & \searrow 4 & & \searrow 4 & & \searrow 4 & \\
 P(3,0) & \xrightarrow{1} & P(1,2) & \xrightarrow{2} & P(3,0) + n & \xrightarrow{2} & P(1,5) \xrightarrow{1} P(3,0) + 2n \xrightarrow{1} \dots
 \end{array}$$

Defect -3

$$[P(2,0) \xrightarrow{1} P(2,1) \xrightarrow{2} P(2,0) + n \xrightarrow{1} \dots]$$

the tables start with $P(2,1)$.

Defect -4

$$P(0,0) \xrightarrow{1,2,1,2} P(0,0) + 2n \dots$$

$$P(0,1) \xrightarrow{1,2,1,2} P(0,1) + 2n \dots$$

$$P(0,2) \xrightarrow{1,2,1,2} P(0,2) + 2n \dots$$

Proof:

block	e-polynomial
-------	--------------

1 or 1-1	$p_1 = (X_2 + fX_4)^i$
----------	------------------------

2 or 2-2	$p_2 = (X_1 + fX_3)^i$
----------	------------------------

3	$p_3 = X_3^e$
---	---------------

4	$p_4 = X_1 + X_4^e$
---	---------------------

5	$p_5 = [f(X_1 \cap X_2) + X_2]^i$
---	-----------------------------------

For defects -1 and -2, the numbers above and below the spaces identify the extensions. The indecomposability follows from (1.3.2) and using the given e-polynomials. For, they erase the subspaces corresponding to the first (left hand side) and last (right hand side) extensions. For defect -3, the e-polynomial $p = p_1 \cap p_2$ deletes the last three extensions (ie.: 1-1-2 or 2-1-1), and this happens if the representation space has dimension at least 6, for lower dimensions it is easy to see that they are indecomposable. For defect -4, we use $p = p_1 \cap p_2$.

Figure 3-10: $\widetilde{F}_{41}$: Defect -1

Figure 8-11: F_{41} : Defect -2

Figure 3-12: F_{41} : Defect -3

Figure 3-13: F_{41} : Defect -4

3.9 Preprojective representations of $\widetilde{F}_{42}$

$$\Omega \quad \begin{array}{c} 2 \rightarrow 1 \\ 4 \rightarrow 3 \end{array} \rightarrow 0$$

$$M = G \rightarrow G \xrightarrow{GM_F} F \leftarrow F \leftarrow F \quad [F : G] = 2$$

$$n = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \quad \delta = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad g = 1 \quad m = 6$$

$$Q(x) = \frac{1}{4}(2x_2 - x_1)^2 + \frac{1}{12}(3x_1 - 4x_0)^2 + \frac{1}{2}(2x_4 - x_3)^2 + \frac{1}{6}(3x_3 - 2x_0)^2$$

$$P(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad P(0,1) = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$$

$$P(1,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad P(1,1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad P(1,2) = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$$

$$P(2,0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad P(2,1) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad P(2,2) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad P(2,3) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$P(2,4) = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \quad P(2,5) = \begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}$$

Table 3-7: Small roots of $\widetilde{F}_{42}$

Next we consider:

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then we have the following:

Extension diagrams: Note that some roots are repeated. We have chosen this instead of changes of basis of the representation spaces, in order to make the proofs easier.

Defect -1

$$\begin{aligned}
 a) & P(2,0) \xrightarrow{3} P(4,2) \xrightarrow{1} P(2,4) \xrightarrow{2} P(2,0) + n \xrightarrow{3} P(4,2) + n \xrightarrow{1} P(2,4) + n \xrightarrow{2} \dots \\
 b) & P(2,0) \xrightarrow{1} P(2,2) \xrightarrow{2} P(4,4) \xrightarrow{3} P(2,0) + n \xrightarrow{1} P(2,2) + n \xrightarrow{2} P(4,4) + n \xrightarrow{3} \dots \\
 c) & P(4,0) \xrightarrow{2} P(2,2) \xrightarrow{3} P(2,4) \xrightarrow{1} P(4,0) + n \xrightarrow{2} P(2,2) + n \xrightarrow{3} P(2,4) + n \xrightarrow{1} \dots \\
 d) & P(2,1) \xrightarrow{1} P(4,3) \xrightarrow{2} P(2,5) \xrightarrow{3} P(2,1) + n \xrightarrow{1} P(4,3) + n \xrightarrow{2} P(2,5) + n \xrightarrow{3} \dots \\
 e) & P(2,1) \xrightarrow{2} P(2,3) \xrightarrow{3} P(4,5) \xrightarrow{1} P(2,1) + n \xrightarrow{2} P(2,3) + n \xrightarrow{3} P(4,5) + n \xrightarrow{1} \dots \\
 f) & P(4,1) \xrightarrow{3} P(2,3) \xrightarrow{1} P(2,5) \xrightarrow{2} P(4,1) + n \xrightarrow{3} P(2,3) + n \xrightarrow{1} P(2,5) + n \xrightarrow{2} \dots
 \end{aligned}$$

Defect -2:

$$\begin{aligned}
 g) & P(3,0) \xrightarrow{2} P(1,1) \xrightarrow{1} P(1,2) \xrightarrow{3} P(3,0) + n \xrightarrow{2} P(1,1) + n \xrightarrow{1} P(1,2) + n \xrightarrow{3} P(3,0) + 2n \dots \\
 h) & P(1,0) \xrightarrow{1} P(3,1) \xrightarrow{3} P(1,2) \xrightarrow{2} P(1,0) + n \xrightarrow{1} P(3,1) + n \xrightarrow{3} P(1,2) + n \xrightarrow{2} P(1,0) + 2n \dots \\
 i) & P(1,0) \xrightarrow{3} P(1,1) \xrightarrow{2} P(3,2) \xrightarrow{1} P(1,0) + n \xrightarrow{3} P(1,1) + n \xrightarrow{2} P(3,2) + n \xrightarrow{1} P(1,0) + 2n \dots
 \end{aligned}$$

Defect -3:

$$\begin{aligned}
 j) & P(0,0) \xrightarrow{1,2,3} P(0,2) \xrightarrow{1,2,3} \dots \\
 k) & P(0,1) \xrightarrow{1,2,3} P(0,3) \xrightarrow{1,2,3} \dots
 \end{aligned}$$

Proofs: For defects -1 and -2, the representations can be constructed following different paths in the extension diagrams and this is reflected by the numbers on top and bottom of the representations themselves.

The e-polynomials given below will delete: a) only the latest extension if the first and last of them written on the bottom of each representation are different, b) the first and last extensions if they are equal.

For defect -3, there is an inner block whose representation space has dimension either one or two. We extend by $E = E_1 \oplus E_2 \oplus E_3$ and at this first stage the e-polynomial p is not useful but is straightforward that it is indecomposable. Each new extension by E is written on the other

side of the previous one. The e -polynomial p deletes the two latest extensions by E , namely a six dimensional vector space.

block	e -polynomial
1 or 1-1	$p_1 = (X_1 + fX_2)^i$
2 or 2-2	$p_2 = (X_2 + X_3)^i$
3 or 3-3	$p_3 = (X_1 + X_4)^i$

and $p = p_1 \cap p_2 \cap p_3$. Recall: $F = 1G \oplus fG$ and $f^2 \in G$.

Figure 3-14: $\tilde{F}_{42}$, families a), b) and c)

Figure 3-16: F_{42} , defect -2

Figure 3-17: $\widehat{F}_{42}$, defect -3

3.10 Preprojective representations of $\widetilde{G}_{21}$

$$\Omega \quad 2 \xrightarrow{(1,3)} 1 \xrightarrow{} 0$$

$$M = G \rightarrow G \xrightarrow{GM_F} F \quad [F : G] = 3$$

$$n = (1 \ 2 \ 1) \quad \delta = (1 \ 1 \ -2) \quad g = 3 \quad m = 2$$

$$Q(x) = \frac{1}{4}(2x_2 - x_1)^2 + \frac{3}{4}(x_1 - 2x_0)^2$$

$$P(0,0) = (0 \ 0 \ 1) \quad P(0,1) = (0 \ 3 \ 2)$$

$$P(1,0) = (0 \ 1 \ 1) \quad P(1,1) = (1 \ 3 \ 2) \quad P(1,2) = (2 \ 5 \ 3)$$

$$P(2,0) = (1 \ 1 \ 1) \quad P(2,1) = (0 \ 2 \ 1) \quad P(2,2) = (2 \ 3 \ 2)$$

Table 3-8: Small roots of $\widetilde{G}_{21}$

Extension diagrams:

Defect -1

$$a) P(3,0) \xrightarrow{1} P(3,2) \xrightarrow{1} P(3,4) \xrightarrow{1} \dots$$

$$b) P(3,1) \xrightarrow{1} P(3,3) \xrightarrow{1} P(3,5) \xrightarrow{1} \dots$$

Defect -2:

$$c) P(1,0) \xrightarrow{1} P(1,1) \xrightarrow{1} P(1,2) \xrightarrow{1} \dots$$

Defect -3:

$$d) P(0,0) \xrightarrow{1,1,1} P(0,2) \xrightarrow{1,1,1} \dots$$

$$e) P(0,1) \xrightarrow{1,1,1} P(0,1) \xrightarrow{1,1,1} \dots$$

where $E_1 = (1 \ 2 \ 1)$. Observe that $\{E_1\}$ is not a generating set as defined in [DR 76a].

Figure 3-18: G_{21} , families a), b) and c)

Figure 3-19: G_{21} , defect -3

3.11 Preprojective representations of $\widetilde{G}_{22}$

$$\Omega \quad \begin{array}{c} \nearrow (1,3) \\ 1 \rightarrow 0 \leftarrow 2 \end{array}$$

$$M = G \xrightarrow{cM_F} F \leftarrow F \quad [F : G] = 3$$

$$n = (3 \ 2 \ 1) \quad \delta(1 \ -2 \ 1) \quad g = 1 \quad m = 2$$

$$Q(x) = \frac{1}{4}(2x_1 - 3x_0)^2 + \frac{3}{4}(2x_2 - x_0)^2$$

$$P(0,0) = (0 \ 1 \ 0) \quad P(0,1) = (3 \ 3 \ 1)$$

$$P(1,0) = (1 \ 1 \ 0) \quad P(1,1) = (2 \ 2 \ 1)$$

$$P(2,0) = (0 \ 1 \ 1) \quad P(2,1) = (3 \ 2 \ 0)$$

Table 3-9: $\widetilde{G}_{22}$

Extension diagrams:

Defect -1

$$a) P(1,0) \xrightarrow{1} P(2,1) \xrightarrow{2} P(1,2) \xrightarrow{1} P(2,3) \xrightarrow{2} \dots$$

$$b) P(2,0) \xrightarrow{1} P(1,1) \xrightarrow{2} P(2,2) \xrightarrow{1} P(1,3) \xrightarrow{2} \dots$$

$$c) P(1,0) \xrightarrow{2} P(1,1) \xrightarrow{1} P(1,2) \xrightarrow{2} P(1,3) \xrightarrow{1} \dots$$

Defect -2

$$d) P(0,0) \xrightarrow{1,2} P(0,1) \xrightarrow{1,2} P(0,2) \xrightarrow{1,2} \dots$$

where $E_1 = (2 \ 1 \ 0)$ and $E_2 = (1 \ 1 \ 1)$. The series c) is built in order to avoid change of basis.

Proof: The e-polynomials used in the reduction step are:

block	e-polynomial
1	$p_1 = (X_1 + X_2)^i$
2	$p_2 = (X_1 + fX_2)^i$

the e-polynomials delete repeated extensions of the same kind which are at the left or right hand side of the representation. For instance in the series a) if the latest extension is of type 1 and we apply p_1 , then we delete the first two and last one extensions of this type, as written below the representation. For defect -2, take $p = p_1 \cap p_2$ which deletes a six dimensional vector space. If the representation space has dimension less than six, the indecomposability follows by a straightforward computation.

Figure 3-20: $\widetilde{G}_{22}$, families a), b) and c)

We identify the letters on the dashed rectangles. The formation is as follows: first extend by $E_1 \oplus E_2$ on the top branches then middle and afterwards bottom to return to the top.

Figure 3-21: $\widetilde{G}_{22}$, defect -2

$\tilde{E}_7$

$\widetilde{E}_8$

A_{12} and A_{22} up to valuations.

$\widetilde{B}_4$ and $\widetilde{F}_{4,1}$ up to valuations.

$\widetilde{BC}_4$ and $\widetilde{C}_4$ up to valuations.

$\widetilde{BD}_3$ and $\widetilde{CD}_3$ up to valuations.

$\widetilde{F}_{12}$

$\widetilde{G}_{21}$ $\widetilde{G}_{22}$

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