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**APPLICATION OF BOUNDARY EDGE EFFECT CORRECTIONS
TO THE
VIBRATION OF BEAMS AND RECTANGULAR PLATES**

By

Rocco Maso

B.Eng.(MECH.), Carleton University

**A Thesis submitted to the Faculty of Graduate Studies in partial
fulfilment of the requirements for the degree of
Master of Engineering**

**Ottawa-Carleton Institute of Mechanical and Aerospace Engineering
Department of Mechanical and Aerospace Engineering
Carleton University
Ottawa, Ontario
March, 1993**

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APPLICATION OF BOUNDARY EDGE EFFECT CORRECTIONS

TO THE

VIBRATION OF BEAMS AND RECTANGULAR PLATES

submitted by Rocco Maso

in partial fulfilment of the requirements for
the degree of Master of Engineering

Thesis Supervisor

Chairman,

Department of Mechanical and Aerospace Engineering
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ABSTRACT

Simple expressions for the frequencies and buckling loads of beams with mixed pinned, clamped, free, or sliding supports, are obtained from the fundamental pinned-pinned support case, by using a corrected value of the number of halfwaves in the mode shape. The method is applied to deep (Timoshenko) beams, to beams with axial loads, and to beams on an elastic foundation, and is shown to provide simple, accurate predictions, adequate for preliminary design calculations.

Rectangular plates with mixed, simple, clamped, and free supports are then considered. The method is successfully applied to the vibration of plates with mixed supported or clamped edges. It was not found possible, however, to successfully treat plates having free edges.

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I greatly appreciate the support and guidance provided by my supervisor Professor James Kirkhope during the course of my research work and in preparation of this thesis. His knowledge, suggestions, and intuitiveness were instrumental in achieving the objectives of this thesis.

NOMENCLATURE

<u>Symbol</u>	<u>Terminology</u>
a, b	length of flat plate in x & y axis, respectively
A	cross sectional area of beam
C, d, p	equation constants
D	Plate's flexural rigidity $[= Eh^3/12(1-\nu^2)]$
E	Young's Modulus
G	Shear Modulus of Elasticity
h	plate thickness
I	moment of Inertia for beam
k	cross-sectional shear factor for beams
K_B	beam stiffness
k_F	foundation stiffness (per unit length)
K_G	load stiffness
L	beam length
L_{eff}	effective beam length
m, n	no. of halfwaves parallel to x & y axis, respectively
m_B	effective mode number for beams under buckling
m_E	effective mode number for Euler beams
m_e	effective mode length for plate in x-axis
N	applied axial load
N_{cr}	Euler's critical buckling load for beams
N_{cs}	critical buckling load for thick beams including shear effects
n_e	effective mode length for plate in y-axis

r	modal parameter [= mb/na]
U_G	Axial load energy of a beam
U_E	Strain energy of a beam
T	Kinetic energy of a beam
a/b	plate's aspect ratio
r/L	beam's inverse slenderness ratio
α	critical load ratio
β	eigenvalue (square root of λ)
β/β_{cc}	frequency ratio for a Clamped-Clamped beam
β/β_{cs}	frequency ratio for a Clamped-Pinned beam
γ	foundation constant ($=k_p L^4/\pi^4 EI$)
Δm	edge correction factor for m -halfwaves for flat plates
Δn	edge correction factor for n -halfwaves for flat plates
Δm_E	edge correction factor for Euler beams
Δm_B	edge correction factor for beams under buckling
Δm_S	edge correction factor for Timoshenko beams
Δm_{S1}	edge correction (fixed) factor for Timoshenko beams
Δm_{S2}	edge correction (variable) factor for Timoshenko beams
κ^*	foundation constant [= $k_p L^4/EI$]
ζ_m	bending wavelength equivalency factor for $m-1/2$ waves
ζ_n	bending wavelength equivalency factor for $n-1/2$ waves
λ_{ex}	Exact Frequency Parameter for Euler Beams
λ_E	Natural Frequency Parameter for Euler Beams
λ_N	Frequency Parameter for Axial Loading of Euler Beams
ω_E	Natural frequency of vibration of Euler Beams

ω_F	Natural frequency of vibration of a beam on an elastic foundation
ω_K	Natural frequency of vibration for a rigid beam on an elastic foundation
ω_N	Natural frequency of vibration of Euler beams under Axial loading
ω_S	Natural frequency of vibration of Timoshenko beams
μ_B	Buckling Parameter of Euler beams
ρ	plate mass density (per unit area)
$\%[\sigma]$	absolute value of standard deviation in percentage error
$\%[\lambda]$	absolute value of mean in percentage error
μ	beam shear factor
ν	Poisson's Ratio

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CHAPTER 1

1.0 INTRODUCTION

Beams and rectangular flat plates are perhaps the most common structural elements, and during design incorporating these elements it is often necessary to determine the natural frequencies of flexural vibration and the buckling loads.

A vast literature exists dealing with these problems and includes a considerable number of complicating effects such as, transverse shear in thick elements; the effects of axial loads on frequencies; the effects of an elastic foundation on both buckling loads and frequencies; and so on.

For simpler arrangements of loading geometry, and boundary conditions this literature provides exact solutions, or methodologies to obtain approximate solutions. For complex geometry, loading, and boundary supports, recourse nowadays is made to Finite Element Solutions readily carried out on a variety of commercially available software.

In many design situations, it is none-the-less desirable to be able to estimate the natural frequencies and buckling loads with the minimum of computational effort.

For beams recourse to well known texts [refs 1-5] provides tables of exact solutions. In the case of flat rectangular plates the works of Leissa[6,7] are perhaps the classic important references. The text by Blevins[10] is also a

notable collection of non-dimensional natural frequency data for a variety of beam and plate configurations.

The cases of beams and plates with simple (or pinned) supports result in very simple exact expressions for the natural frequencies and buckling loads. Thus for pinned-pinned thin Euler beams;

$$\omega_E = \lambda_E \left(\frac{EI}{\rho AL^4} \right)^{1/2} \dots (1.1)$$

$$N_{cr} = \mu_B \frac{EI}{L^2} \dots (1.2)$$

where $\lambda_E = (m\pi)^2$

$$\mu_B = (m\pi)^2$$

and "m" is the number of 1/2 waves in the vibration or buckled mode shape. When other variations of support conditions from the fundamental pinned-pinned (P-P) case are considered such as Clamped (C) or Free (F), the non-dimensional coefficients λ_E and μ_B are apparently not so simple. In the case of buckling the simplicity of the fundamental case is often retained by recourse to the concept of "effective length", L_{eff} , for other boundary conditions, Figure (1.1).

Thus,

$$N_{cr} = \mu_B \frac{EI}{L_{eff}^2} \dots (1.3)$$

and μ_B is still given by equation (1.2). This approach is well known and commonly appears in texts. Two questions arise from this.

First, could this approach be usefully adapted to the expression for the natural frequency? Second, it can be seen by referring to Figure (1.1) that the effect of the boundary support on the mode shape could be considered by either an effective length L_{eff} or by the addition of corrections Δm to the number of halfwaves m . Which of these alternatives is best? Chapter 2 of this thesis addresses these questions. It is shown that the best approach is the use of a mode number correction Δm , which depends only on the boundary support conditions, and is independent, in its simplest form, of the mode number m , unlike the effective length which is strongly dependant on mode number m . The method is readily adapted successfully to the natural frequencies, and results in simple expressions identical to the fundamental (P-P) case, but uses a corrected mode number $m + \Delta m$ in equation (1.2). It is further shown that complicating effects such as transverse shear and rotary inertia (Timoshenko Beams), axial, tensile or compressive forces, elastic foundation

support are all simply accommodated by this method. No previous references to this approach for beam buckling and vibration have been found in the literature.

In the development of simple expressions for the natural frequencies of thin, flat, rectangular plates, a most useful and elegant work is that of Warburton[8]. In this Warburton uses the exact beam functions, which correspond to the plate boundary conditions along the two edge directions, in an approximate Rayleigh Method which results in a simple closed form equation for the frequency. The coefficients required in the equation are tabulated for 15 different combinations of supported (S), clamped (C), and free (F) supports. Dickinson[9] extended this work to consider orthotropic plates. Leissa[7] compares and discusses the accuracy of Warburton's solution, compared with exact (series) solutions. The results are impressive, with accuracy generally of the order of 1% or better, except in configurations involving free boundaries. In cases with free boundaries, errors up to 5% are observed.

The frequency expression for the simply supported plate is, as in the case of beams, particularly simple, Figure (1.2),

$$\omega = [(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2] (\frac{D}{\rho h})^{1/2} \dots (1.4)$$

In this, m and n are the numbers of 1/2 waves along the

edges having lengths a and b respectively. If support conditions other than simply supported are applied, can this equation still be used by incorporating either "effective lengths" a' or b' or by corrections Δm and Δn to the mode number of $1/2$ waves? Both approaches have appeared in the literature for a C-C-C-C panel. First, Szechenyi[11] used an "effective length" approach for a C-C-C-C panel. Here the effective lengths are expressed as,

$$\begin{aligned} a' &= \xi_m a \dots\dots \\ b' &= \xi_n b \dots\dots (1.5) \end{aligned}$$

where

$$\begin{aligned} \xi_m &= \frac{d}{1 + \frac{0.5}{m}} - \frac{1.05}{1 + \frac{0.5}{m}} \dots\dots \\ \xi_n &= \frac{d}{1 + \frac{0.5}{n}} - \frac{1.05}{1 + \frac{0.5}{n}} \dots\dots (1.6) \end{aligned}$$

and d varies slightly with m, n in the range 1.0-1.11. A value of 1.05 is recommended by Szechenyi[11], who calls ξ_m and ξ_n "bending wavelength equivalency factors". Satisfactory results are claimed for this approach, but no detailed assessment of the accuracy is made by Szechenyi. Perhaps the greatest deficiency is that the corrections of equation (1.6) do not account for the effect of wave number $m\pi/a$ and $n\pi/b$ and the work of Hazell and Mitchell[12] and

this thesis shows that the corrections are a function of the wave numbers.

Hazell and Mitchell also considered a clamped all around, C-C-C-C, plate. As a result of holographic experiments these authors concluded that the frequency equation (1.4) for the simply supported case could be applied as,

$$\omega = \lambda \left(\frac{D}{\rho a^4 h} \right)^{1/2} \dots (1.7)$$

$$\lambda = \pi^2 [m^2 + n^2 \left(\frac{a}{b} \right)^2] \dots (1.8)$$

$$\text{and } m' = m + \Delta m$$

$$n' = n + \Delta n$$

where Δm and Δn are corrections to the number of 1/2 waves along the edges a and b respectively. For the simply supported plate these corrections are zero, while for the opposite edges C-C, they were found to be,

$$\Delta m = \frac{1}{r^{-2} + 2} + \frac{.017}{m} \dots (1.9)$$

and

$$\Delta n = \frac{1}{r^2 + 2} + \frac{.017}{n}$$

with

$$r = \frac{mb}{na}$$

Hazell and Mitchell present detailed examination of their equation, comparing results with exact frequencies for plates having a wide range of aspect ratios from $a/b = 0.1$ to 1.0. The accuracy demonstrated is very impressive and generally less than 1% for all modes and aspect ratios. This work of Hazell and Mitchell indicates that this method of correcting mode numbers m and n to account for the boundary conditions works extremely well for clamped plates C-C-C-C.

The application to other supports requires investigation, and is the topic addressed in Chapters 3 and 4 of this thesis. In Chapter 3 an examination of the method applied to plates having mixed supported and clamped boundaries is presented. It is found that the method works well for all cases of mixed clamped and supported edges. Chapter 4 considers plates having one or more free edges in combination with clamped and supported edges. Here it is found that addition of boundary corrections is complicated by cross-coupling effects between adjacent edges. It appears

that no simple method of applying the boundary correction method is possible for such cases. Finally Chapter 5 presents conclusions and makes recommendations for further studies.

Figure 1.1 Effective Mode Length and Edge Correction for a Clamped Beam

Figure 1.2 Mode Number (3,2) for a Clamped Rectangular Plate

CHAPTER 2

2.0 BOUNDARY EDGE EFFECT CORRECTION METHOD FOR BEAMS.

2.1 Transverse Vibration of Euler-Bernoulli Beams

In the vibration of thin beams, described by Euler-Bernoulli theory, the frequency of pinned-pinned (P-P) uniform beams is given by the simple expression,

$$\omega_E = \lambda_E \sqrt{\frac{EI}{\rho AL^4}} \quad . \quad . \quad . \quad (2.1)$$

with

$$\lambda_E = (m\pi)^2$$

In this "m" is the number of halfwaves in the vibration mode shape. When other variations of support conditions are considered such as clamped (C), free (F) or sliding (S) the non-dimensional frequency coefficients, λ are apparently not so simple, but are known from exact solutions [1,2,3].

In the application of a boundary edge correction it is assumed that the "effective mode number", m_E , can be obtained where

$$m_E = m + \Delta m_E \quad . \quad . \quad . \quad (2.2)$$

and Δm_E , is a correction to the number of 1/2 waves "m" and is dependant upon the support condition, see figure (2.1).

The frequency of the beam is then given by

$$\lambda_E = (m_E \pi)^2 \quad (2.3)$$

i.e. the same as the fundamental P-P case. By examining the exact solutions for the beams with various support conditions, the edge correction Δm_E required is quickly apparent and are tabulated in Table 2.1 .

The values of Δm_E to be used in equation (2.2) are obtained by adding the values corresponding to the two support conditions of the beam. For example for a beam frequency with free-sliding supports (F-S),

$$m_E = m - 3/4 - 1/2 = m - 5/4$$

$$\lambda_E = (m - 5/4)^2 \pi^2 \text{ for } m = 2, 3, 4, \dots$$

Table 2.2 summarizes the values of m_E for all ten possible combinations of the boundary conditions P, C, F, and S, while Table 2.3 summarizes the frequency expressions λ_E , and the exact values of λ_{ex} , for the first four modes of vibration. A comparison of the calculated and exact values is made in Table 2.4.

It is seen that with the exception of the 1st cantilever mode (C-F), the resulting frequencies are essentially exact for preliminary design purposes. The combination of equation (2.3) with the corrections of Table 2.1 gives a very simple means of remembering and estimating the frequencies of uniform beams. The values of Δm_E of Table 2.1 become evident by working backwards from the exact values of λ_E to get the required value of Δm_E .

Thus,

	<u>Case</u>	<u>Mode</u>	<u>Δm_E</u>
1)	P-P	m=1	0
		2	0
		--	0
2)	C-C	m=1	0.5056
		2	0.5000
		---	0.5000
3)	F-F	m=3	-1.4944
		4	-1.5000
		---	-1.5000
4)	S-S	m=2	-1.0000
		3	-1.0000
		---	-1.0000
5)	P-C	m=1	0.2500
		2	0.2500
		3	0.2500
7)	P-S	m=2	-0.5000
		3	-0.5000
		---	-0.5000
8)	C-F	m=1	-0.4031
		2	-0.4945
		3	-0.5000
		4	-0.5000

Cases 2, 3, and 4 indicate corrections Δm_E of $1/4$, $-3/4$, and $-1/2$ for a C, F, and S support condition, respectively. The values of $1/4$, and $-1/2$ for C, and S supports are confirmed by cases 5 and 7. Case 8 gives Δm_E for C + F supports. By removing the confirmed correction of $1/4$ for the C support, the corrections indicated for the F

support are,

m=1	-0.653
2	-0.744
3	-0.750
4	-0.750

It is evident that for lower values of the mode number m , the F support correction should not be $-3/4$. Assuming a correction of the form;

$$\Delta m_E = -3/4 + 1/km^\alpha$$

then with $m = 1$:

$$-0.653 = -0.750 + 1/k$$

$$\therefore k = 0.097 \approx 1/10$$

and with $m = 2$:

$$-0.744 = -0.75 + 1/10m^\alpha$$

$$\therefore 2^\alpha = 16.7, \alpha \approx 4$$

The revised correction for a free (F) support is,

$$\Delta m_E = -3/4 + 1/10m^4 \dots (2.4)$$

The revised edge corrections for beams with free boundaries are given in Table 2.5, and the resulting accuracy of λ_E compared with the exact values appears in Table 2.6.

It is noted that all frequencies are given accurately now by equations;

$$\lambda_E = (m_E \pi)^2$$

$$m_E = m + \Delta m_E$$

$$\Delta m_E = 0 \text{ for P}$$

$$= 1/4 \text{ for C}$$

$$= -3/4 + 1/10m^4 \text{ for F}$$

$$= -1/2 \text{ for S} \dots (2.5)$$

2.2 Buckling of Euler-Bernoulli Beams

In the buckling of thin slender beams, figure (2.2), the pinned-pinned (P-P) beam is often referred to as the fundamental case, for which the critical buckling load, N_{cr} is given by;

$$N_{cr} = \mu \left(\frac{EI}{L^2} \right) \dots \dots$$

where

$$\mu = (m\pi)^2 = \frac{N_{cr} L^2}{EI} \dots \dots (2.6)$$

where "m" is the number of 1/2 waves in the buckled mode shape.

When other variations of support conditions are considered it is again assumed that an "effective" mode number m_B can be obtained by a correction to m, which is independent on the supports.

Thus,

$$\begin{aligned} m_B &= m + \Delta m_B \dots \dots \\ \mu_B &= (m_B \pi)^2 \dots \dots (2.7) \end{aligned}$$

By examining the exact solutions for the non-dimensional buckling coefficient μ_B , the required correction Δm_B becomes quickly apparent, and thus is summarized in Table 2.7.

Table 2.8 lists the effective buckling mode number m_B for beams with ten possible combinations of supports- P, C, F

and S. In Table 2.9 the approximate values (equation 2.7) and exact values of μ_B are summarized and the resulting comparison of percentage accuracy of the approximate value is given in Table 2.10. With the exceptions of C-C and P-C cases equation (2.7) gives exact buckling loads μ_B .

For the C-C case the exact values of μ_B are:

$$\pi^2(m + 1)^2 = (m_B \pi)^2 \text{ for } m = 1, 3, 5, \dots$$

$$= 80.73 \text{ for } m = 2$$

$$= 238.72 \text{ for } m = 4$$

$$\pi^2(m + 1)^2 = (m_B \pi)^2 \text{ for } m = 6, 8, 10, \dots$$

For the P-C case the exact values of μ_B are,

$$20.19 \text{ for } m = 1$$

$$59.68 \text{ for } m = 2$$

$$\pi^2(m + 1/2)^2 = (m_B \pi)^2 \text{ for } m = 3, 4, \dots$$

Thus for the important critical buckling load - corresponding to the 1st mode, μ_B is given exactly by equation (2.7) except for the P-C case which has an error of 10.0 %

2.3 Vibration of Euler-Bernoulli Beams under Axial Loading

The flexural vibration of thin slender beams in the presence of steady compressive or tensile axial forces is an important problem often encountered in practical design, figure (2.3). Exact and approximate solutions are available for these problems and recently Bokaian[13,14] has presented exact solutions, and detailed discussion of approximations by the so-called Galef formula [15]. Stephen[16] has shown lower (Galef) and upper bound solutions are obtained by energy methods.

The Galef formula, is easily derived for a pinned-pinned beam subject to an axial load N . Thus assuming a mode shape,

$$y = a \sin\left(\frac{m\pi x}{L}\right) \quad \dots (2.8)$$

and employing the energy functions,

Kinetic Energy,

$$T = \frac{1}{2} \int_0^L \rho A \dot{y}^2 dx \quad \dots$$

Flexural Strain Energy,

$$U_F = \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx \quad \dots$$

Axial Load Energy,

$$U_G = \frac{1}{2} \int_0^L (\pm N) \left(\frac{\partial y}{\partial x} \right)^2 dx \quad \dots (2.9)$$

where N is positive if tensile and negative if compressive. Evaluating these, and applying the Lagrangian method gives;

$$m = \frac{\partial^2 T}{\partial \dot{a}^2} = \frac{1}{2} \rho AL \quad . \quad . \quad . \quad .$$

$$k_E = \frac{\partial^2 U_E}{\partial a^2} = \frac{1}{2} (m\pi)^4 \frac{EI}{L^3} \quad . \quad . \quad . \quad .$$

$$k_G = \frac{\partial^2 U_G}{\partial a^2} = \frac{1}{2} \frac{(\pm N)}{L} (m\pi)^2 \quad . \quad . \quad . \quad . (2.10)$$

The frequency of vibration is then;

$$\omega_N = \left(\frac{k_E + k_G}{m} \right)^{\frac{1}{2}} = \omega_E \left(1 \pm \frac{N}{N_{cr}} \right)^{\frac{1}{2}} \quad . \quad . \quad . \quad .$$

where

$$\omega_E = (m\pi)^2 \left(\frac{EI}{\rho AL^4} \right)^{\frac{1}{2}} \quad . \quad . \quad . \quad .$$

$$N_{cr} = (m\pi)^2 \left(\frac{EI}{L^2} \right) \quad . \quad . \quad . \quad . (2.11)$$

Thus for a P-P beam

$$\frac{\omega_N}{\omega_E} = \sqrt{1 \pm \frac{N}{N_{cr}}} \quad . \quad . \quad . \quad . (2.12)$$

which is the Galef formula.

This equation can be applied to beams having other supports

using the boundary correction equations of (2.1) and (2.2) above.

Thus,

$$\begin{aligned}\lambda_E &= (m_E \pi)^2, \quad m_E = m + \Delta m_E \dots \dots \dots \\ \mu_B &= (m_B \pi)^2, \quad m_B = m + \Delta m_B \dots \dots \dots \\ \omega_E &= \lambda_E (EI / \rho A L^4)^{1/2} \dots \dots \dots \\ N_{cr} &= \mu_B (EI / L^2) \dots \dots \dots (2.13)\end{aligned}$$

Table 2.11 demonstrates the accuracy of equations (2.12) and (2.13) used to calculate λ_N in the presence of high compressive axial loads, namely $N/N_{cr} = 0.5$ & 0.7 .

The accuracy of the approximate calculations is better than 2% for both load cases, with the exception of the P-C and P-F supports. The error in the case of P-C arises from the fact that μ_B is in error by about 10% for this case. If the exact value of μ_B is used, the errors in λ_N for this P-C case become -0.6% and -0.7% respectively. The error in the P-F case is discussed at length by Bokaian and Stephen. This is the one case of support conditions for which the Galerkin equation (2.12) breaks down, and gives inaccurate results. They concluded that the problem was the dissimilarity of mode shapes in vibration and buckling for this case as shown in figure (2.4). Additional comparisons for high tensile and compressive loads can be made with the exact solutions given by Gorman[4] for C-C and P-C beams. Figures (2.5) and (2.6) compare Gorman's results with those obtained by using equations (2.12) and (2.13). In these figures β/β_{cr} is the

ratio $(\lambda_N/\lambda_E)^{1/2}$ in equation (2.12), while $\alpha = NL^2/\pi^2 EI$, and is negative for "tensile" axial loads. Thus it can easily be shown that $N/N_{cr} = -\alpha/m_b^2$, and this is used to calculate the comparison data in figures (2.5) and (2.6). The agreement is seen to be excellent in figure (2.5) for the C-C beam for the range of loads used, and for vibration mode 1 to 6. For this beam $\alpha = -10$ corresponds to $N/N_{cr} = 2.5$, while $\alpha = 2$ corresponds to $N/N_{cr} = -0.5$.

In general $N/N_{cr} = -\alpha/(m + 1)^2$ here.

For the P-C beam, shown in figure (2.6), good agreement is evident again, except for mode 1 at high compressive loads. This discrepancy results from the error in the 1st buckling load predicted by $(m_b\pi)^2$ for this case. The exact value is $\mu_b = 20.19$, while the approximate value $\mu_b = (m + 1/2)^2\pi^2 = 22.21$. Using the correct value of μ_b , shown by Δ 's in figure (2.6), removes the discrepancy with Gorman's results. For all other modes up to mode 6, the agreement of the approximate equations with the exact results is excellent. For this case $N/N_{cr} = -\alpha/(m + 1/2)^2$; hence for $\alpha = -10$, $N/N_{cr} = 4.4$, while for $\alpha = 1.8$, $N/N_{cr} = -0.8$.

2.4 Vibration of Euler-Bernoulli Beams on a Uniform Elastic Foundation

Thin beams vibrating on an uniform elastic foundation of stiffness k_f per unit length, figure (2.7), are also easily treated by support corrections.

For a pinned-pinned beam the additional strain energy stored in the foundation is,

$$U_f = \frac{1}{2} \int_0^L k_f a^2 \sin^2 \left(\frac{m\pi x}{L} \right) dx \quad (2.14)$$

so that the generalized stiffness,

$$K_f = \frac{\partial^2 U_f}{\partial a^2} = \frac{1}{2} k_f L \quad (2.15)$$

This stiffness together with the previously derived flexural stiffness and inertia, equations (2.10), give the vibration frequency as,

$$\omega_f^2 = \frac{(m\pi)^4 \frac{EI}{L^3} + k_f L}{\rho AL} \quad (2.16)$$

or,

$$\frac{\omega_f}{\omega_E} = \left[1 + \frac{k_f L^4}{(m\pi)^4 EI} \right] \quad$$

or,

$$\frac{\omega_F}{\omega_E} = \sqrt{1 + \frac{\gamma}{m^4}} \quad . \quad . \quad . \quad (2.17)$$

where

$$\gamma = \left(\frac{k_p L^4}{\pi^4 EI} \right)$$

and is called the "foundation constant".

A more rigorous analysis, eg, Gorman in reference [4], shows that for any end support, the relation (2.17) applies in the form,

$$\frac{\rho A \omega_F^2 L^4}{EI} = \beta^4 + \kappa^* \quad . \quad . \quad . \quad (2.18)$$

where β is the eigenvalue for the beam without elastic foundation, and, $\kappa^* = k_p L^4 / EI$.

With a P-P beam $\beta = m\pi$, resulting in equation (2.17). For other end supports we approximate β as,

$$\beta = m_E \pi$$

$$\text{with } m_E = m + \Delta m_E$$

giving ,

$$\frac{\omega_F}{\omega_E} = \sqrt{1 + \frac{\gamma}{m_E^4}} \quad . \quad . \quad . \quad (2.19)$$

Table 2.12 lists the non-dimensional frequencies λ , for P-P, C-F, and C-C beams on foundations with $\gamma = 0, 3, 4.5$.

Figures (2.8) and (2.9) compare the results calculated using equation (2.19) with exact values, and a FEM solution obtained by Abbas and Thomas[17] for the P-P and cantilever C-F cases. The agreement is seen to be excellent. Abbas and Thomas also report numerical values for a C-C beam where the foundation constant $\gamma = 4.5$. Thus

	Ref[17]	Equation(2.19)	% Error
mode 1	30.64	30.52	-0.4%
mode 2	65.24	65.14	-0.2%

Again, the two results are essentially identical, and demonstrate the effectiveness of equation (2.19) used with boundary corrections in predicting the vibration frequencies of thin beams on elastic foundations.

2.5 Vibration of Thick (Timoshenko) Beams.

When thick beams, described by Timoshenko beam theory including transverse shear and rotary inertia, are considered, there is no simple exact solution available, even for the fundamental P-P case. However, a simple approximate expression can be derived easily, for example reference[2], and has been widely used for this case,

$$\frac{\omega_s}{\sqrt{\frac{EI}{\rho AL^4}}} = \frac{(m\pi)^2}{\sqrt{1 + (m\pi)^2 \left(\frac{r}{L}\right)^2 \left(1 + \frac{E}{kG}\right)}} \dots (2.20)$$

In (2.20), (r/L) is the inverse of the slenderness ratio, where r is the section radius of gyration, and k is the Timoshenko shear coefficient. If equation (2.20) is generalized to use support corrections to the number of $1/2$ waves in the mode shape, it takes the form,

$$\frac{\omega_s}{\omega_E} = \frac{1}{\sqrt{1 + (m_s\pi)^2 \left(\frac{r}{L}\right)^2 \left(1 + \frac{E}{kG}\right)}} \dots (2.21)$$

where ω_E is given by the previous equations for the thin beam frequencies, equations (2.5), and

$$m_s = m + \Delta m_s \dots (2.22)$$

where Δm_s is the support correction for a Timoshenko beam. Exact solutions for various support conditions have been given by Carr[18]. By working backwards from these exact solutions

simple constant corrections Δm_{s1} , independent of mode m , are quickly evident. These, while they will be seen to yield reasonably accuracy, can be markedly improved by corrections Δm_{s2} which are dependant on the mode m and slenderness ratio, (r/L) . The resulting corrections Δm_{s1} and Δm_{s2} are summarized in Table 2.13, while Tables 2.14 and 2.15 list the values of Δm_{s1} and Δm_{s2} applicable to all ten variations of support, P, C, F, and S. The accuracy of equation (2.21) together with the corrections of Table 2.13 has been compared with the exact solutions obtained by Carr for beams having P, C, or F supports. For beams having S supports the basis of comparison has been an FEM solution using 40 elements, and this was verified to give exact solutions for P, C, F supports. The tabulated frequency ratios for the first 5 modes, for all 10 variations of support, and for inverse slenderness ratios $r/L = 0.025, 0.05$ and $.075$, are detailed in Tables 2.16 to 2.25. In calculating these the following applied;

-for P, C, F boundaries, i.e. comparison with Carr[18]:

$$E/G = 8/3$$

$$k = 2/3$$

$$(1 + E/kG) = 5$$

-for FEM solution used with S boundaries;

$$E/G = 2.6$$

$$k = 0.844$$

$$(1 + E/kG) = 4.081$$

Examination of the tables of results shows that equation (2.21) can be remarkably effective in predicting the frequencies of thick beams. With the simple correction Δm_{s1} the accuracy is better than 2% provided the reduction in frequency is no more than a ratio of 0.7. When the correction Δm_{s2} is used an accuracy of 2% or better is evident for all modes provided the reduction in frequency is no more than a ratio of 0.4.

Table 2.26 compares the accuracy using Δm_{s1} and Δm_{s2} for the first 5 modes of P-P, C-C, F-F, S-S, beams having $(r/L) = 0.05$. It is seen that use of correction Δm_{s2} greatly reduces the error--for the above beam to less than 2% for all 5 modes in all cases. These results are for a relatively stubby beam with length/depth ratio of approximately 6, and where the mode 5 frequency can be less than 40% of the corresponding thin beam frequency. In this context the use of equation (2.21) with the corrections of Table 2.15 provides a simple accurate prediction of Timoshenko beam frequencies, adequate for most engineering design calculations.

2.6 Buckling of Thick Beams Including Transverse Shear Effects

A slight modification of the equations for buckling of thin beams, equation (2.6) together with (2.7), allows consideration of the effect of transverse shear on buckling loads. Timoshenko[5] determined the effect of transverse shear on the buckling load, obtaining,

$$N_{sc} = \frac{N_{cr}}{1 + \frac{N_{cr}}{kGA}} \quad (2.23)$$

where N_{sc} is the buckling load including shear effect, and N_{cr} is the critical buckling load for thin beams, with no shear effects. Thus using support corrections,

$$N_{cr} = \mu_B \frac{EI}{L^2} = (m_B \pi)^2 \frac{EI}{L^2} \quad (2.24)$$

with $m_B = m + \Delta m_B$

Rewriting equation (2.23) gives,

$$N_{sc} = \frac{EI}{L^2} \frac{(m_B \pi)^2}{[1 + (m_B \pi)^2 (\frac{L}{EI})^2 (\frac{E}{kG})]} \quad (2.25)$$

or,

$$\frac{N_{sc}}{N_{cr}} = \frac{1}{[1 + (m_B \pi)^2 (\frac{L}{EI})^2 (\frac{E}{kG})]} \quad (2.26)$$

The form of equation (2.26) can be compared with that of

equation (2.21) for the Timoshenko beam frequency ratio. The modification in the E/kG term results from the absence of rotary inertia in the buckling equation (2.26). The approximation in (2.25) results from the use of $\mu_B = (m_B \pi)^2$ with $m_B = m + \Delta m_B$ for the various support conditions. Thus exact solutions are obtained, except in the C-C and P-C cases. Exact solutions are given for the 1st mode critical buckling load, except in the P-C case. For this case, the exact value is,

$$N_{sc} = \frac{EI}{L^2} \frac{20.19}{[1 + 20.19 (\frac{r}{L})^2 (\frac{E}{kG})]} \dots (2.27)$$

while the approximate value is,

$$N_{sc} = \frac{EI}{L^2} \frac{22.21}{[1 + 22.21 (\frac{r}{L})^2 (\frac{E}{kG})]} \dots (2.28)$$

Defining a shear factor η , such that,

$$\eta = (\frac{r}{L})^2 (\frac{E}{kG})$$

a typical value will be $\eta = 2.6/k(r/L)^2 \cong 3(r/L)^2$. For a thick beam $(r/L) \geq 0.05$, and the error in the approximate equation (2.28) is as follows,

η	0.0	0.02	0.05	0.10
% error	10.0	6.9	4.7	3.1

Thus for deep beams with P-C supports the error in the approximate critical buckling load given by equation (2.25) or (2.28) is reduced substantially from the maximum value of 10% pertaining to a thin beam.

Finally, it is noted that Ang and Wang[19] present equations for the critical buckling load, $m = 1$, for beams with P, C, F supports, which are the same as those obtained above, although they do not approach the problem from the support correction relation. Their work also does not include sliding supports, or consideration of buckling in the higher modes.

2.7 Vibration of Timoshenko Beams on a Uniform Elastic Foundation

An expression for the frequency of a Timoshenko beam vibrating on a uniform elastic foundation is derived as follows.

Recalling equation (2.16) for a P-P thin beam vibrating on an elastic foundation of stiffness k_f per unit length,

$$\omega_F^2 = \frac{(m\pi)^4 \frac{EI}{L^3} + k_f L}{\rho AL}$$

or,

$$\omega_F^2 = \omega_E^2 + \omega_K^2 \quad . \quad . \quad . \quad (2.29)$$

where

$$\omega_K^2 = \frac{k_f L}{\rho AL}$$

Thus the frequency of the beam on the foundation, ω_F , can be obtained by combining the frequency of a rigid beam on the foundation, ω_K , by a Rayleigh-Southwell procedure, as in equation (2.29).

Applying this procedure to the Timoshenko beam, gives,

$$\omega_F^2 = \omega_S^2 + \frac{k_f L}{\rho AL} \quad . \quad . \quad . \quad .$$

$$\omega_F^2 = \omega_S^2 + \frac{\gamma \omega_E^2}{m^4} \quad . \quad . \quad . \quad .$$

$$\omega_F^2 = \omega_s^2 + \frac{\gamma \omega_s^2}{m^4 \left(\frac{\omega_s}{\omega_E} \right)^2} \quad \dots \quad (2.30)$$

In the above ω_E is the thin beam frequency, equations (2.1), (2.3), ω_s is the thick beam frequency, equations (2.21) and Table 2.15. Finally considering other end supports equation (2.30) can be written as,

$$\frac{\omega_F}{\omega_s} = \sqrt{1 + \frac{\gamma}{m_E^4 \left(\frac{\omega_s}{\omega_E} \right)^2}} \quad \dots \quad (2.31)$$

where (ω_s/ω_E) is given by equation (2.21) and m_E is given by equation (2.2).

Equation (2.31) may be compared with equations (2.19) which is the corresponding equation for a thin beam, and is seen that the effects of shear and rotary inertia are accounted for by the factor $(\omega_s/\omega_E)^2$ in the denominator of equation (2.31). The accuracy of the above has been assessed by comparison with F.E.M. solutions, and also by Abbas and Thomas in reference[17].

First, FEM calculations for beams with P-P, C-C, F-F, S-S, and C-F, supports were carried out using $(r/L) = 0.08$ and $E/kG = 3.064$. A model using 40 elements was used and verified to give exact solutions for the beam alone. Non-dimensional frequencies λ of modes 1 to 5 were determined for foundation constant $\gamma = 0, 3, \text{ and } 5$.

The comparison of FEM results and those calculated using equations (2.31) and (2.21) are given in Tables 2.28 to (2.32) for the five support conditions noted above. These results are for a relatively stubby beam of length/depth ratio approximately $L/D = 3$. The approximate solutions are seen to be in very good agreement with the FEM values for all supports and modes. The error is largest in the F-F case, with the largest error being approximately 3%.

The results for P-P and C-F supports are also compared with FEM results reported by Abbas and Thomas in figures (2.10) and (2.11), which are taken from reference [17]. It is seen that the approximate frequencies are very close indeed to these independently obtained FEM values, giving considerable confidence in the use of the approximation for design calculations.

2.8 Summary and Conclusion

Simple methods of determining the frequencies and buckling loads of uniform thin and thick beams have been described which use as the fundamental cases the solutions for a beam with P-P supports. The solutions for beams having other variations of C, F, and S supports are then obtained by adding a correction to the number of 1/2 waves "m" in the mode shape. These corrections and the resulting equations for the various problems may be summarized as follows;

-Support Corrections

Support	Δm_E	Δm_B	Δm_S
P	0	0	$-3/4(m-1)(r/L) \dots$
C	1/4	1/2	$1/2-(m-3)/40 \dots$
F	$-3/4+1/10m^4$	-1	$-3/4-3/8(m-1)(r/L) \dots$
S	-1/2	-1/2	$-1/2-3/4(m-1)(r/L) \dots (2.32)$

-Euler Beam Frequencies

$$\omega_E = \lambda_E(EI/\rho AL^4)^{1/2} \dots (2.33)$$

$$\lambda_E = (m_E \pi)^2 \dots (2.34)$$

$$m_E = m + \Delta m_E \dots (2.35)$$

-Euler Beam Buckling

$$N_{cr} = \mu_B(EI/L^2) \dots (2.36)$$

$$\mu_B = (m_B \pi)^2 \dots (2.37)$$

$$m_B = m + \Delta m_B \dots (2.38)$$

-Beam Frequency with Tensile/Compressive Axial Load

$$\omega_N/\omega_E = (1 \pm N/N_{cr})^{1/2} \dots \dots \dots (2.39)$$

-Euler Beam Frequency with Elastic Foundation

$$\omega_F/\omega_E = (1 + \gamma/m_E^4)^{1/2} \dots \dots \dots (2.40)$$

$$\text{where } \gamma = K_F L^4 / \pi^4 EI$$

-Timoshenko Beam Frequencies

$$\frac{\omega_s}{\omega_E} = \frac{1}{\sqrt{1 + (m_s \pi)^2 \left(\frac{I}{L}\right)^2 \left(1 + \frac{E}{kG}\right)}} \dots \dots \dots (2.41)$$

$$\text{where } m_s = m + \Delta m_s \dots \dots \dots (2.42)$$

-Shear Beam Buckling

$$\frac{N_{sc}}{N_{cr}} = \frac{1}{[1 + (m_s \pi)^2 \left(\frac{I}{L}\right)^2 \left(\frac{E}{kG}\right)]} \dots \dots \dots (2.42)$$

-Timoshenko Beam Frequency with Elastic Foundation

$$\frac{\omega_F}{\omega_s} = \sqrt{1 + \frac{\gamma}{m_E^4 \left(\frac{\omega_s}{\omega_E}\right)^2}} \dots \dots \dots (2.43)$$

The accuracy of the above equations has been examined by comparison with exact and FEM solutions. In general the resulting errors were found to be very small, even when high mode frequencies or buckling loads are considered. These equations thus provide a compact, easily used (and even memorized) method of predicting uniform beam frequencies and buckling loads, including a variety of complicating factors,

and which is sufficiently accurate for preliminary design calculations.

Table 2.1

Edge Effect Correction, Δm_E for,
Euler-Bernoulli Beams
with various Boundary Conditions

Boundary Condition	BC	Δm_E	Representation
Pinned	P	0
Clamped	C	0.25
Free	F	-0.75
Sliding	S	-0.5

Table 2.2

Effective Mode Number, m_E for
Euler-Bernoulli Beams
with various Boundary Conditions

1) P-P	$\Delta m_E = 0.0 + 0.0$	$m_E = m$	for $m=1, 2, \dots$
2) C-C	$\Delta m_E = 0.25 + 0.25$	$m_E = m + 1/2$	for $m=1, 2, \dots$
3) F-F	$\Delta m_E = -0.75 + -0.75$	$m_E = m - 3/2$	for $m=3, 4, \dots$
4) S-S	$\Delta m_E = -0.50 - 0.50$	$m_E = m - 1.0$	for $m=2, 3, \dots$
5) P-C	$\Delta m_E = 0.0 + 0.25$	$m_E = m + 1/4$	for $m=1, 2, \dots$
6) P-F	$\Delta m_E = 0.0 - 0.75$	$m_E = m - 3/4$	for $m=2, 3, \dots$
7) P-S	$\Delta m_E = 0.0 - 0.50$	$m_E = m - 1/2$	for $m=1, 2, \dots$
8) C-F	$\Delta m_E = 0.25 - 0.75$	$m_E = m - 1/2$	for $m=1, 2, \dots$
9) C-S	$\Delta m_E = 0.25 - 0.50$	$m_E = m - 1/4$	for $m=1, 2, \dots$
10) F-S	$\Delta m_E = -0.75 - 0.50$	$m_E = m - 5/4$	for $m=2, 3, \dots$

Table 2.3

Exact λ_{ex} and Calculated λ_E for First Four
Flexural Modes of Euler-Bernoulli Beams

B.C's	Calculated λ_E	Mode Number	1st Mode λ_{ex}	2nd Mode λ_{ex}	3rd Mode λ_{ex}	4th Mode λ_{ex}
1) P-P	$(\pi m)^2$	$m=1,2,\dots\text{etc}$	9.87	39.48	88.83	157.9
2) C-C	$\pi^2(m + 1/2)^2$	$m=1,2,\dots\text{etc}$	22.37	61.69	120.9	199.9
3) F-F	$\pi^2(m - 3/2)^2$	$m=3,4,\dots\text{etc}$	22.37	61.69	120.9	199.9
4) S-S	$\pi^2(m - 1)^2$	$m=2,3,\dots\text{etc}$	9.87	39.48	88.83	157.9
5) P-C	$\pi^2(m + 1/4)^2$	$m=1,2,\dots\text{etc}$	15.42	49.96	104.2	178.3
6) P-F	$\pi^2(m - 3/4)^2$	$m=2,3,\dots\text{etc}$	15.42	49.96	104.2	138.3
7) P-S	$\pi^2(m - 1/2)^2$	$m=1,2,\dots\text{etc}$	2.47	22.21	61.69	120.9
8) C-F	$\pi^2(m - 1/2)^2$	$m=1,2,\dots\text{etc}$	3.52	22.03	61.69	120.9
9) C-S	$\pi^2(m - 1/4)^2$	$m=1,2,\dots\text{etc}$	5.59	30.23	74.64	138.8
10) F-S	$\pi^2(m - 5/4)^2$	$m=2,3,\dots\text{etc}$	5.59	30.23	74.64	138.8

Table 2.4

Percentage Error of Calculated λ_E
Using the Boundary Correction

Mode no	P-P	C-C	F-F	S-S	P-C	P-F	P-S	C-F	C-S	F-S
1	exact	-.7 %	-.7 %	exact	exact	exact	exact	-29.9%	-.7 %	-.7 %
2	exact	exact	exact	exact	exact	exact	exact	0.8 %	exact	exact
3,4 etc	exact	exact	exact	exact	exact	exact	exact	exact	exact	exact

Table 2.5

Improved Edge Effect Correction, Δm_E for,
Euler-Bernoulli Beams with Free Boundaries

- 3) F-F ; $\Delta m_E = -3/2 + 2/10m^4$ for $m = 3, 4, \dots$ etc.
 6) P-F ; $\Delta m_E = -3/4 + 1/10m^4$ for $m = 2, 3, \dots$ etc.,
 8) C-F ; $\Delta m_E = -1/2 + 1/10m^4$ for $m = 1, 2, \dots$ etc.,
 10) F-S ; $\Delta m_E = -5/4 + 1/10m^4$ for $m = 2, 3, \dots$ etc.

Table 2.6

Percentage Error of Calculated λ_E
Using the Boundary Corrections

Mode no	P-P	C-C	F-F	S-S	P-C	P-F	P-S	C-F	C-S	F-S
1	exact	-.7 %	-.4 %	exact	exact	1.0%	exact	0.9%	-.7 %	1.0%
2	exact	exact	exact	exact	exact	0.1%	exact	1.6%	exact	exact
3, 4 etc	exact	exact	exact	exact	exact	exact	exact	exact	exact	exact

Table 2.7

Edge Effect Correction, Δm_E , for
Buckling of Euler-Bernoulli Beams
with various B.C.'s

Boundary Condition	BC	Δm_E	Representation
Pinned	P	0
Clamped	C	0.5
Free	F	-1.0
Sliding	S	-0.5

Table 2.8

Effective Buckling Mode Number, m_B ,
for Euler-Bernoulli Beams with various B.C.'s

1) P-P	$\Delta m_B = 0.0 + 0.0$	$m_B = m$
2) C-C	$\Delta m_B = 0.50 + 0.50$	$m_B = m + 1.0$
3) F-F	$\Delta m_B = -1.0 - 1.0$	$m_B = m - 2.0$
4) S-S	$\Delta m_B = -0.50 - 0.50$	$m_B = m - 1.0$
5) P-C	$\Delta m_B = 0.0 + 0.50$	$m_B = m + 1/2$
6) P-F	$\Delta m_B = 0.0 - 1.0$	$m_B = m - 1.0$
7) P-S	$\Delta m_B = 0.0 - 0.50$	$m_B = m - 1/2$
8) C-F	$\Delta m_B = 0.50 - 1.0$	$m_B = m - 1/2$
9) C-S	$\Delta m_B = 0.50 - 0.50$	$m_B = m$
10) F-S	$\Delta m_B = -1.0 - 0.50$	$m_B = m - 3/2$

Table 2.9

Buckling of Euler-Bernoulli Beams, μ_B ,
with various Boundary Conditions

B.C	Approximate μ_B	Exact μ_B	Mode No
1) P-P	$\pi^2(m + 1)^2$	$\pi^2(m + 1)^2$	$m=1, 3, \dots$
2) C-C	$\pi^2(m + 1)^2$	$\pi^2(m + 1)^2$	$m=1, 3, \dots$
C-C	$\pi^2(m + 1)^2$	$\tan(\mu^{1/2}/2) = \mu^{1/2}/2$	$m=2, 4, \dots$
3) F-F	$\pi^2(m - 2)^2$	$\pi^2(m - 2)^2$	$m=3, 4, \dots$
4) S-S	$\pi^2(m - 1)^2$	$\pi^2(m - 1)^2$	$m=2, 3, \dots$
5) P-C	$\pi^2(m - 1/2)^2$	$\tan(\mu^{1/2}) = \mu^{1/2}$	$m=1, 2, \dots$
6) P-F	$\pi^2(m - 1)^2$	$\pi^2(m - 1)^2$	$m=2, 3, \dots$
7) P-S	$\pi^2(m - 1/2)^2$	$\pi^2(m - 1/2)^2$	$m=1, 2, \dots$
8) C-F	$\pi^2(m - 1/2)^2$	$\pi^2(m - 1/2)^2$	$m=1, 2, \dots$
9) C-S	$\pi^2 m^2$	$\pi^2 m^2$	$m=1, 2, \dots$
10) F-S	$\pi^2(m - 5/4)^2$	$\pi^2(m - 3/2)^2$	$m=2, 3, \dots$

Table 2.10

Percentage Error of Calculated μ_B
In Terms of Exact Value

Mode No	P-P	C-C	F-F	S-S	P-C	P-F	P-S	C-F	C-S	F-S
1	exact	exact	exact	exact	10 %	exact	exact	exact	exact	exact
2	exact	10 %	exact	exact	3.4 %	exact	exact	exact	exact	exact
3	exact	exact	exact	exact	exact	exact	exact	exact	exact	exact
4	exact	3.4%	exact	exact	exact	exact	exact	exact	exact	exact

Table 2.11

Exact vs Calculated Frequency Parameter, λ_N ,
for Beams Under Compressive Axial Loading

B.C	1st Mode	$N/N_{cr} = 0.5$			$N/N_{cr} = 0.7$		
	* λ_E	λ_N Exact [16]	λ_N Eqt'n 2.12	% error	λ_N Exact [16]	λ_N Eqt'n 2.12	% error
1) P-P	9.87	6.98	6.98	0	5.41	5.41	0
2) C-C	22.37	15.95	15.70	-1.6	12.40	12.25	-1.2
3) F-F	22.37	15.92	15.75	-1.1	12.37	12.20	-1.4
4) S-S	9.87	6.98	6.98	0	5.41	5.41	0
5) P-C	15.42	10.97	10.90	-.6	8.52	9.30	9.1
6) P-F	15.42	10.13	10.90	7.6	7.17	8.53	19.0
7) P-S	2.47	1.74	1.75	0.6	1.35	1.35	0
8) C-F	3.52	2.53	2.49	-1.6	1.98	1.94	-2.0
9) C-S	5.59	3.99	3.95	-1.0	3.10	3.04	-2.0
10) F-S	5.59	3.98	3.95	-.8	3.09	3.09	0

* values from Table 2.3

Table 2.12

$\lambda = \omega \sqrt{EI/\rho A L^4}$ vs γ for Euler-Bernoulli
Beams on Elastic Foundations

B.C	Mode No	$\gamma=0$	$\gamma=3$	$\gamma=4.5$
P-P	1	9.87	19.7	23.1
P-P	2	39.5	43.0	44.7
P-P	3	88.8	90.5	91.3
C-F	1	2.47	17.3	22.2
C-F	2	22.2	28.0	31.3
C-F	3	61.7	64.6	65.5
C-C	1	----	----	30.5
C-C	2	----	----	65.1

Table 2.13

Δm_{s1} and Δm_{s2} for Timoshenko Beams
for Various Boundary Conditions

Boundary Condition	BC	Δm_{s1}	Δm_{s2}
Pinned	P	0	$-.75(m-1)(r/L)$
Clamped	C	0.5	$0.50+0.025(3-m)$
Free	F	-0.75	$-0.75-0.375(m-1)(r/L)$
Sliding	S	-0.5	$-0.50-0.75(m-1)(r/L)$

Table 2.14

Equations for Two Boundary Conditions
for Timoshenko Beams Using Δm_{s1}

1) P-P	$m_{s1}=m$	for $m=1,2,\dots\text{etc.},$
2) C-C	$m_{s1}=m + 1.0$	for $m=1,2,\dots\text{etc.},$
3) F-F	$m_{s1}=m - 3/2$	for $m=3,4,\dots\text{etc.},$
4) S-S	$m_{s1}=m - 1.0$	for $m=2,3,\dots\text{etc.},$
5) P-C	$m_{s1}=m + 1/2$	for $m=1,2,\dots\text{etc.},$
6) P-F	$m_{s1}=m - 3/4$	for $m=2,3,\dots\text{etc.},$
7) P-S	$m_{s1}=m - 1/2$	for $m=1,2,\dots\text{etc.},$
8) C-F	$m_{s1}=m - 1/4$	for $m=1,2,\dots\text{etc.},$
9) C-S	$m_{s1}=m$	for $m=1,2,\dots\text{etc.},$
10) F-S	$m_{s1}=m - 5/4$	for $m=2,3,\dots\text{etc.},$

Table 2.15

Equations for Two Boundary Conditions
for Timoshenko Beams Using Δm_{32}

1) P-P	$\Delta m_{32} = \begin{matrix} -.75(m-1)(r/L) \\ -.75(m-1)(r/L) \end{matrix}$	$m_{32} = m - 1.5(m-1)(r/L)$
2) C-C	$\Delta m_{32} = \begin{matrix} .50 + .025(3-m) \\ .50 + .025(3-m) \end{matrix}$	$m_{32} = m + 1.0 + .05(3-m)$
3) F-F	$\Delta m_{32} = \begin{matrix} -.75 -.375(m-1)(r/L) \\ -.75 -.375(m-1)(r/L) \end{matrix}$	$m_{32} = m - 1.5 -.75(m-1)(r/L)$
4) S-S	$\Delta m_{32} = \begin{matrix} -.50 -.75(m-1)(r/L) \\ -.50 -.75(m-1)(r/L) \end{matrix}$	$m_{32} = m - 1.0 - 1.5(m-1)(r/L)$
5) P-C	$\Delta m_{32} = \begin{matrix} .50 -.75(m-1)(r/L) \\ +.025(3-m) \end{matrix}$	$m_{32} = m + 0.50 + .75(m-1)(r/L) \\ +.025(3-m)$
6) P-F	$\Delta m_{32} = -.75 - 1.125(m-1)(r/L)$	$m_{32} = m -.75 - 1.125(m-1)(r/L)$
7) P-S	$\Delta m_{32} = -.50 - 1.5(m-1)(r/L)$	$m_{32} = m -.50 - 1.5(m-1)(r/L)$
8) C-F	$\Delta m_{32} = \begin{matrix} -.25 - 0.375(m-1)(r/L) \\ +0.025(3-m) \end{matrix}$	$m_{32} = m -.25 -.375(m-1)(r/L) \\ +0.025(3-m)$
9) C-S	$\Delta m_{32} = \begin{matrix} -.75(m-1)(r/L) \\ +.025(3-m) \end{matrix}$	$m_{32} = m -.75(m-1)(r/L) \\ +.025(3-m)$
10) F-S	$\Delta m_{32} = -.125 - 1.125(m-1)(r/L)$	$m_{32} = m -.125 - 1.125(m-1)(r/L)$

Table 2.16

Comparison of Exact vs Approximate ω_s/ω_E
 Timoshenko Beams with Transverse Shear & Rotary
 Inertia Effects for Pinned-Pinned Case

r/L	m	ω_s/ω_E (exact)	Using Δm_{s1}	% error	Using Δm_{s2}	% error
0.025	1	0.9850	0.9849	- 0.01	0.9849	- 0.01
	2	0.9444	0.9435	- 0.10	0.9454	0.11
	3	0.8881	0.8847	- 0.38	0.8895	0.16
	4	0.8257	0.8183	- 0.90	0.8259	0.02
	5	0.7635	0.7514	- 1.58	0.7613	-0.29
0.050	1	0.9444	0.9435	- 0.10	0.9435	- 0.10
	2	0.8257	0.8183	- 0.90	0.8284	0.33
	3	0.7049	0.6884	- 2.34	0.7067	0.26
	4	0.6034	0.5799	- 3.90	0.6021	- 0.21
	5	0.5220	0.4948	- 5.21	0.5181	- 0.75
0.075	1	0.8881	0.8847	- 0.38	0.8847	- 0.38
	2	0.7049	0.6884	- 2.34	0.7091	0.59
	3	0.5604	0.5347	- 4.59	0.5646	0.74
	4	0.4573	0.4287	- 6.26	0.4601	0.62
	5	0.3745	0.3549	- 5.23	0.3850	2.80

Table 2.17

Comparison of Exact vs Approximate ω_s/ω_E
 Timoshenko Beams with Transverse Shear & Rotary
 Inertia Effects for Clamped-Free Case

r/L	m	ω_s/ω_E (exact)	Using Δm_{s1}	% error	Using Δm_{s2}	% error
0.025	1	0.9928	0.9914	-0.14	0.9903	-0.25
	2	0.9531	0.9559	0.29	0.9551	0.21
	3	0.8994	0.9005	0.12	0.9016	0.25
	4	0.8361	0.8352	-0.11	0.8387	0.31
	5	0.7729	0.7679	-0.65	0.7737	0.10
0.050	1	0.9723	0.9670	-0.54	0.9627	-0.99
	2	0.8447	0.8519	0.86	0.8511	0.76
	3	0.7234	0.7193	-0.57	0.7240	0.08
	4	0.6161	0.6047	-1.85	0.6131	-0.49
	5	0.5318	0.5141	-3.33	0.5242	-1.43
0.075	1	0.9412	0.9300	-1.19	0.9215	-2.09
	2	0.7255	0.7352	1.34	0.7358	1.42
	3	0.5796	0.5680	-2.00	0.5760	-0.62
	4	0.4671	0.4516	-3.32	0.4623	-1.03
	5	0.3877	0.3711	-4.29	0.3823	-1.39

Table 2.18

Comparison of Exact vs Approximate ω_s/ω_E
Timoshenko Beams with Transverse Shear & Rotary
Inertia Effects for Free-Free Case

r/L	m	ω_s/ω_E (exact)	Using Δm_{s1}	% error	Using Δm_{s2}	% error
0.025	3	0.9709	0.9670	-0.40	0.9686	- 0.24
	4	0.9210	0.9156	-0.58	0.9189	- 0.22
	5	0.8615	0.8519	-1.11	0.8569	- 0.53
	6	0.7985	0.7846	-1.74	0.7909	- 0.96
	7	0.7376	0.7193	-2.49	0.7264	- 1.52
0.050	3	0.8999	0.8847	-1.69	0.8942	- 0.63
	4	0.7709	0.7514	-2.53	0.7662	- 0.60
	5	0.6577	0.6310	-4.05	0.6476	- 1.54
	6	0.5628	0.5347	-5.00	0.5510	- 2.11
	7	0.4863	0.4597	-5.47	0.4750	- 2.33
0.075	3	0.8150	0.7846	-3.73	0.8073	- 0.95
	4	0.6319	0.6047	-4.31	0.6314	- 0.08
	5	0.5057	0.4767	-5.73	0.5014	- 0.84
	6	0.3988	0.3886	-2.55	0.4103	2.88

Table 2.19

Comparison of Exact vs Approximate ω_s/ω_E
Timoshenko Beams with Transverse Shear & Rotary
Inertia Effects for Clamped-Clamped Case

r/L	m	ω_s/ω_E (exact)	Using Δm_{s1}	% error	Using Δm_{s2}	% error
0.025	1	0.9409	0.9435	0.28	0.9382	- 0.28
	2	0.8776	0.8847	0.81	0.8815	0.45
	3	0.8125	0.8183	0.71	0.8183	0.71
	4	0.7489	0.7514	0.34	0.7547	0.77
	5	0.6903	0.6884	- 0.28	0.6944	0.60
0.050	1	0.8113	0.8183	0.86	0.8048	- 0.81
	2	0.6783	0.6884	1.49	0.6824	0.60
	3	0.5792	0.5799	0.12	0.5799	0.12
	4	0.5021	0.4948	- 1.45	0.4986	- 0.70
	5	0.4421	0.4287	- 3.03	0.4346	- 1.70
0.075	1	0.6804	0.6884	1.18	0.6705	- 1.45
	2	0.5263	0.5347	1.60	0.5284	0.39
	3	0.4340	0.4287	- 1.22	0.4287	- 1.22
	4	0.3661	0.3549	- 3.06	0.3580	- 2.21

Table 2.20

Comparison of Exact vs Approximate ω_s/ω_E
 Timoshenko Beams with Transverse Shear & Rotary
 Inertia Effects for Clamped-Pinned Case

r/L	m	ω_s/ω_E (exact)	Using Δm_{s1}	% error	Using Δm_{s2}	% error
0.025	1	0.9643	0.9670	0.28	0.9649	0.06
	2	0.9122	0.9156	0.38	0.9153	0.34
	3	0.8502	0.8519	0.20	0.8544	0.50
	4	0.7864	0.7846	- 0.23	0.7900	0.46
	5	0.7257	0.7193	- 0.89	0.7272	0.21
0.050	1	0.8776	0.8847	0.81	0.8783	0.08
	2	0.7489	0.7514	0.34	0.7531	0.56
	3	0.6376	0.6310	- 1.03	0.6392	0.26
	4	0.5492	0.5347	- 2.65	0.5465	- 0.49
	5	0.4795	0.4597	- 4.13	0.4732	- 1.31
0.075	1	0.7750	0.7846	1.23	0.7745	- 0.06
	2	0.6073	0.6047	- 0.43	0.6095	0.36
	3	0.4907	0.4767	- 2.85	0.4888	- 0.39
	4	0.4084	0.3886	- 4.85	0.4033	- 1.25

Table 2.21

Comparison of Exact vs Approximate ω_s/ω_E
 Timoshenko Beams with Transverse Shear & Rotary
 Inertia Effects for Free-Pinned Case

r/L	m	ω_s/ω_E (exact)	Using Δm_{s1}	% error	Using Δm_{s2}	% error
0.025	2	0.9778	0.9767	- 0.11	0.9777	- 0.01
	3	0.9332	0.9300	- 0.34	0.9331	- 0.01
	4	0.8749	0.8685	- 0.73	0.8740	- 0.10
	5	0.8121	0.8014	- 1.32	0.8090	- 0.38
	6	0.7504	0.7352	- 2.03	0.7443	- 0.81
0.050	2	0.9210	0.9156	- 0.58	0.9222	0.13
	3	0.7985	0.7846	- 1.74	0.7997	0.15
	4	0.6809	0.6589	- 3.23	0.6786	- 0.33
	5	0.5830	0.5566	- 4.54	0.5775	- 0.95
	6	0.5042	0.4767	- 5.45	0.4972	- 1.40
0.075	2	0.8474	0.8352	- 1.44	0.8521	0.56
	3	0.6691	0.6448	- 3.63	0.6738	0.71
	4	0.5320	0.5043	- 5.21	0.5351	0.57
	5	0.4308	0.4078	- 5.34	0.4365	1.32

Table 2.22

Comparison of Exact vs Approximate ω_s/ω_E
Timoshenko Beams with Transverse Shear & Rotary
Inertia Effects for Sliding-Sliding Case

r/L	n	ω_s/ω_E FEM	Using Δm_{s1}	% error	Using Δm_{s2}	% error
0.025	2	0.988	0.9876	- 0.04	0.9885	0.05
	3	0.954	0.9532	- 0.09	0.9564	0.25
	4	0.906	0.9029	- 0.34	0.9091	0.35
	5	0.852	0.8443	- 0.90	0.8534	0.16
	6	0.797	0.7834	- 1.70	0.7948	- 0.28
0.050	2	0.954	0.9532	- 0.09	0.9595	0.58
	3	0.851	0.8443	- 0.79	0.8624	1.34
	4	0.741	0.7243	- 2.26	0.7505	1.28
	5	0.645	0.6189	- 4.05	0.6484	0.53
	6	0.566	0.5332	- 5.79	0.5631	- 0.51
0.075	2	0.906	0.9029	- 0.34	0.9212	1.68
	3	0.741	0.7243	- 2.26	0.7639	3.09
	4	0.603	0.5736	- 4.87	0.6194	2.73
	5	0.500	0.4650	- 7.02	0.5093	1.84

Table 2.23

Comparison of Exact vs Approximate ω_s/ω_E
Timoshenko Beams with Transverse Shear & Rotary
Inertia Effects for Clamped-Sliding Case

r/L	n	ω_s/ω_E FEM	Using Δm_{s1}	% error	Using Δm_{s2}	% error
0.025	1	0.987	0.9876	0.07	0.9864	- 0.06
	2	0.951	0.9532	0.23	0.9529	0.20
	3	0.902	0.9029	0.10	0.9050	0.33
	4	0.847	0.8443	- 0.32	0.8492	0.26
	5	0.790	0.7834	- 0.83	0.7910	0.13
0.050	1	0.952	0.9532	0.12	0.9487	- 0.34
	2	0.842	0.8443	0.28	0.8458	0.46
	3	0.730	0.7243	- 0.78	0.7329	0.40
	4	0.634	0.6189	- 2.39	0.6322	- 0.29
	5	0.556	0.5332	- 4.10	0.5488	- 1.29
0.075	1	0.902	0.9029	0.10	0.8945	- 0.83
	2	0.725	0.7243	- 0.10	0.7297	0.65
	3	0.587	0.5736	- 2.28	0.5883	0.23
	4	0.488	0.4650	- 4.72	0.4832	0.98

Table 2.24

Comparison of Exact vs Approximate ω_s/ω_B
Timoshenko Beams with Transverse Shear & Rotary
Inertia Effects for Pinned-Sliding Case

r/L	m	ω_s/ω_B FEM	Using Δm_{s1}	% error	Using Δm_{s2}	% error
0.025	1	0.997	0.9969	0.01	0.9969	0.01
	2	0.973	0.9728	0.02	0.9741	-0.11
	3	0.931	0.9295	0.16	0.9333	-0.03
	4	0.879	0.8742	0.54	0.8808	-0.20
	5	0.824	0.8139	1.23	0.8230	0.12
0.050	1	0.988	0.9876	0.04	0.9876	0.04
	2	0.906	0.9029	0.34	0.9112	-0.57
	3	0.795	0.7834	1.40	0.8016	-0.84
	4	0.691	0.6691	3.17	0.6934	-0.34
	5	0.603	0.5736	4.87	0.6002	0.47
0.075	1	0.973	0.9728	0.02	0.9728	0.02
	2	0.823	0.8139	1.11	0.8344	-1.39
	3	0.667	0.6434	3.55	0.6784	-1.71
	4	0.547	0.5147	5.91	0.5533	-1.16

Table 2.25

Comparison of Exact vs Approximate ω_s/ω_B
Timoshenko Beams with Transverse Shear & Rotary
Inertia Effects for Free-Sliding Case

r/L	m	ω_s/ω_B FEM	Using Δm_{s1}	% error	Using Δm_{s2}	% error
0.025	2	0.9930	0.9930	0.00	0.9935	0.05
	3	0.9640	0.9635	-0.05	0.9657	0.18
	4	0.9190	0.9166	-0.27	0.9210	0.22
	5	0.8650	0.8594	-0.65	0.8661	0.13
	6	0.8100	0.7986	-1.41	0.8072	-0.35
0.050	2	0.9740	0.9728	-0.12	0.9766	0.27
	3	0.8790	0.8742	-0.54	0.8874	0.95
	4	0.7680	0.7535	-1.89	0.7736	0.73
	5	0.6670	0.6434	-3.55	0.6665	-0.08
	6	0.5820	0.5528	-5.01	0.5763	-0.98
0.075	2	0.9450	0.9418	-0.34	0.9533	0.88
	3	0.7810	0.7684	-1.62	0.7990	2.30
	4	0.6320	0.6071	-3.94	0.6438	1.87
	5	0.5180	0.4888	-5.64	0.5243	1.21

Table 2.26

Percentage Error in Frequency Ratio, ω_s/ω_B ,
Using Δm_{s1} and Δm_{s2} , $r/L=0.05$

Mode	P-P		C-C		F-F		S-S	
	Δm_{s1}	Δm_{s2}	Δm_{s1}	Δm_{s2}	Δm_{s1}	Δm_{s2}	Δm_{s1}	Δm_{s2}
1	-0.1	-0.1	0.9	-0.7	-1.7	-0.7	-0.1	-0.1
2	-1.0	0.2	1.5	0.6	-2.6	-0.6	-0.8	1.3
3	-2.4	0.3	0.2	0.2	-4.1	-1.5	-2.3	1.3
4	-3.8	0.2	-1.4	1.0	-5.0	-2.1	-4.0	0.5
5	-5.2	-0.8	-2.9	-1.4	-5.3	-2.3	5.8	-0.5

Table 2.27

Average Percentage Error of ω_s/ω_B for
the First Five Modes, $r/L = 0.05$

Support	Δm_{s1}	Δm_{s2}
P-P	2.5	0.3
C-C	1.4	0.8
F-F	3.7	1.4
S-S	2.6	0.8
P-C	1.8	0.4
P-F	3.1	0.6
P-S	2.0	0.5
C-F	1.4	0.6
C-S	1.5	0.5
F-S	3.4	1.4

Table 2.28

Comparison of Exact (FEM) vs Approximate Value of $\lambda = \omega / \sqrt{EI / \rho A L^4}$
for a Timoshenko Beam on an Elastic Foundation
for Pinned-Pinned Case

Mode	$\gamma = 0$		$\gamma = 3$		$\gamma = 5$	
	FEM	Eqt'n 2.31	FEM	Eqt'n 2.31	FEM	Eqt'n 2.31
1	8.8	8.8	18.9	19.2	23.3	23.8
2	28.4	28.5	32.8	33.2	35.4	36.0
3	51.5	51.6	54.0	54.4	55.6	56.1
4	75.4	75.2	77.2	77.1	78.3	78.4
5	99.5	99.5	99.7	99.9	99.7	100.8

Table 2.29

Comparison of Exact (FEM) vs Approximate $\lambda = \omega / \sqrt{EI / \rho A L^4}$
for a Timoshenko Beam on an Elastic Foundation
for Clamped-Clamped Case

Mode	$\gamma = 0$		$\gamma = 3$		$\gamma = 5$	
	FEM	Eqt'n 2.31	FEM	Eqt'n 2.31	FEM	Eqt'n 2.31
1	15.7	15.2	23.1	22.9	26.9	26.9
2	33.8	33.4	37.6	37.5	40.0	40.0
3	54.8	53.4	57.2	56.1	58.7	57.8
4	76.7	74.0	78.3	76.0	79.4	78.3
5	99.5	94.6	99.7	96.1	99.8	97.1

Table 2.30

Comparison of Exact (FEM) vs Approximate $\lambda = \omega \sqrt{EI/\rho A L^4}$
for a Timoshenko Beam on an Elastic Foundation
for Free-Free Case

Mode	$\gamma = 0$		$\gamma = 3$		$\gamma = 5$	
	FEM	Eqt'n 2.31	FEM	Eqt'n 2.31	FEM	Eqt'n 2.31
1	18.2	18.2	23.9	25.0	27.0	28.6
2	39.4	40.5	42.1	44.0	43.8	46.1
3	62.3	64.2	64.0	66.4	65.2	67.9
4	81.6	88.0	82.6	89.6	83.3	90.7
5	99.6	111.4	100.1	112.7	100.5	113.6

Table 2.31

Comparison of Exact (FEM) vs Approximate $\lambda = \omega \sqrt{EI/\rho A L^4}$
for a Timoshenko Beam on an Elastic Foundation
for Sliding-Sliding Case

Mode	$\gamma = 0$		$\gamma = 3$		$\gamma = 5$	
	FEM	Eqt'n 2.31	FEM	Eqt'n 2.31	FEM	Eqt'n 2.31
1	8.8	8.8	18.9	19.2	23.3	23.8
2	28.4	28.5	32.8	33.2	35.4	36.0
3	51.5	51.6	54.0	54.4	55.6	56.1
4	75.4	75.2	77.2	77.1	78.2	78.4
5	99.5	99.5	99.7	99.9	99.7	100.8

Table 2.32

Comparison of Exact (FEM) vs Approximate $\lambda = \omega / \sqrt{EI / \rho A L^4}$
for a Timoshenko Beam on an Elastic Foundation
for Clamped-Free Case

Mode	$\gamma = 0$		$\gamma = 3$		$\gamma = 5$	
	FEM	Eqt'n 2.31	FEM	Eqt'n 2.31	FEM	Eqt'n 2.31
1	3.33	2.31	17.2	17.3	22.0	22.2
2	16.3	17.0	22.9	24.1	26.4	27.8
3	36.7	37.5	40.0	41.2	42.0	43.5
4	58.2	58.2	60.3	61.9	61.6	63.5
5	80.2	81.8	81.6	83.5	82.6	84.7

Figure 2.1 Effective Mode Number for Prismatic Beam with Clamped Edges

Figure 2.2 Buckling of Thin Beams

Figure 2.3 Vibration of Beams under Axial Loading

Figure 2.4 Dissimilarity of Buckling & Vibration Modes for Beams under Pinned-Free Condition

Figure 2.7 Vibration of a Thin Beam on a Uniform Elastic Foundation

Figure 2.5 Exact and Approximate β/β_{cc} for Clamped Beam

• indicates calculated β/β_{cc} values

Figure 2.6 Exact and Approximate β/β_{cs} for Clamped-Pinned Beam

Δ indicates calculated β/β_{cs} values

Figure 2.8 Effect of Elastic Foundation on Natural Frequencies of Pinned-Pinned Euler-Bernoulli Beams

. calculated λ values

Figure 2.9 Effect of Elastic Foundation on Natural Frequencies for Clamped-Free Euler-Bernoulli Beams

. calculated λ values

Figure 2.10 Effect of Elastic Foundation on Natural Frequencies of Pinned-Pinned Timoshenko Beams

. calculated λ values

Figure 2.11 Effect of Elastic Foundation on Natural Frequencies of Clamped-Free Timoshenko Beams

. calculated λ values

CHAPTER 3

3.0 BOUNDARY EDGE EFFECT CORRECTIONS FOR RECTANGULAR PLATES WITH SIMPLY SUPPORTED AND CLAMPED EDGES

3.0 Introduction

The successful application of boundary support corrections in beam problems suggests investigation of application of the method to rectangular plate problems.

Leissa[7] has published a detailed study of the vibration of rectangular plates. His solutions, obtained by using series of beam functions in a Rayleigh-Ritz procedure, can be considered as exact. Warburton[8] used only one function comprising of a pair of beam functions corresponding to the support conditions along the plate edges, x , y , figure (3.1) This Rayleigh solution resulted in a simple equation for the plate frequencies, and Warburton tabulates the required parameters in the frequency equation for some 15 variations of edge support. This solution thus provides a simple design calculation for the prediction of rectangular plate frequencies. The accuracy of the Warburton solution is impressive, and has been discussed by Leissa[7] in his work. For supports involving simple or clamped cases the accuracy is generally better than 1%. In cases involving free boundaries errors as high as 5% are observed. In applying the edge support correction method, the fundamental case is taken as a plate simply-supported on all edges.

For this case, S-S-S-S,

$$\omega = [(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2] (\frac{D}{\rho h})^{1/2} \quad (3.1)$$

or, rewriting,

$$\lambda = \frac{\omega}{\sqrt{\frac{D}{(\rho a^4 h)}}} = \pi^2 [m^2 + n^2 (\frac{a}{b})^2] \quad (3.2)$$

In this m and n are the number of $1/2$ waves along the x -direction (length a) and y -direction (length b), respectively. For other boundary supports than simply supported (S), such as clamped (C), or free (F), it is assumed that corrections Δm and Δn can be used to obtain effective mode numbers m_e and n_e . Thus

$$m_e = m + \Delta m \quad$$

$$n_e = n + \Delta n \quad (3.3)$$

$$\text{and } \lambda = \pi^2 [m_e^2 + n_e^2 (a/b)^2]$$

By using Leissa's exact solutions and working backwards, the required support corrections Δm , Δn can be obtained.

Hazell and Mitchell[12] have already used this approach to obtain a simple frequency for a clamped, C-C-C-C, plate.

They found expressions for the corrections Δm , Δn , as follows;

$$\Delta m = \frac{1}{r^{-2} + 2} + \frac{.017}{m} \quad$$

and

$$\Delta n = \frac{1}{r^2 + 2} + \frac{.017}{n} \dots (3.4)$$

where they define the modal parameter,

$$r = mb/na \dots (3.5)$$

The work described below re-examines equation (3.4), and extends the application to plates having mixed supported (S) and clamped (C) boundaries.

The notation used below to identify edge support conditions is as follows. The dimensions of the plate are "a" in the x-direction, and "b" in the y-direction. It is assumed that mode (m,n) defines a mode which has "m" 1/2 waves (or m + 1 node lines) along the x-direction and "n" 1/2 waves (or n + 1 node lines) along the y-direction. The symbolism S-C-S-F, for example, will define supports as follows;

along x = 0, simply supported

along y = 0, clamped

along x = a, simply supported

along y = b, free

This representation is shown in figure (3.1).

3.1 Corrections for a Clamped Edge

The equations used by Hazell and Mitchell[12] for an all around clamped rectangular flat plate are curvefit polynomials used to predict the effective mode numbers for the m and n halfwaves for plates under transverse vibration. Recalling their equation for Δm in the m-direction;

$$\Delta m = \frac{1}{\left(\frac{na}{mb}\right)^2 + 2} + \frac{.017}{m} \dots (3.6)$$

and Δn for the n-direction

$$\Delta n = \frac{1}{\left(\frac{na}{mb}\right)^{-2} + 2} + \frac{.017}{n} \dots (3.7)$$

where mb/na is the modal parameter as defined by Hazell and Mitchell.

These equations can be rewritten as in the form of equation (3.4) where the Modal Parameter is

$$r = \frac{mb}{na}$$

A simpler approach to Hazell and Mitchell's approximation would be to express these equations in terms of only one variable, the Modal Parameter, r . The advantages in doing this is are:

- 1) equations would be independent from either "m" or "n" mode number,
- 2) equations in one variable are easier to remember,
- 3) computations would be easier and quicker to perform,
- 4) comparisons between other equations (i.e Δ_m and Δ_n) from other boundary conditions having the same variable is simpler.

Considering the Δ_m equation in the form;

$$\Delta_m = \frac{1}{r^{-2}+2} + Cr^p \quad (3.8)$$

where the constant C and power p , are unknown.

A "trial-and-error" process was used to evaluate the C and coefficients by using the exact Δ_m values Hazell and Mitchell used for a C-C square plate.

Table 3.1 shows that by setting $p = -1$ and $C \cong 1/60$, the Δ_m 's generated are within 5.5% of the exact values, thus establishing the following equation,

$$\Delta m = \frac{1}{r^{-2}+2} + \frac{1}{60r} \cdot \cdot \cdot (3.9)$$

and Δn is the same equation as Δm except that r is "inverted" so that,

$$\Delta n = \frac{1}{r^2+2} + \frac{r}{60} \cdot \cdot \cdot (3.10)$$

3.2 Comparison with the Hazell and Mitchell Equations for a Clamped Rectangular Plate C-C-C-C.

Equations (3.6) and (3.7) are used as the "edge corrections" in the m and n directions used in calculating the Modal Frequency Parameter, λ .

Hazell and Mitchell[12] also use equations derived by Bolotin[20] to give the exact values of Δm and Δn ;

$$\Delta m = \frac{2}{\pi} \arctan[1+2r^{-2}]^{-1/2}$$

and 3.11

$$\Delta n = \frac{2}{\pi} \arctan[1+2r^2]^{-1/2}$$

The calculated Δm and Δn from Eqt'ns (3.9) and (3.10) are compared to;

- i) Eqt'ns 3.6 and 3.7 (Hazell and Mitchell)
- ii) Eqt'ns 3.11 (Bolotin)

Table 3.1 compares the Δm calculations and figure (3.2) is a graphical plot of these values.

As in the case for thin beams this "correction" is added to the mode number, m, which gives the "effective mode number", m_E , for the mode case. In turn, equation (3.12) below is used to generate natural frequency parameters, λ 's for any boundary conditions of a rectangular plate.

$$\lambda = \pi^2 [m_e^2 + n_e^2 (\frac{a}{b})^2] \quad (3.12)$$

where $m_e = m + \Delta m$ and $n_e = n + \Delta n$

Table 3.2 lists the λ 's for a clamped plate for $a/b = 0.4, 0.667, 1.5$ and 2.5 for different mode shapes where the exact values Leissa[7] are compared to the following three approximations;

- i) Eqt'ns 3.6 and 3.7
- ii) * Warburton equations[8]
- iii) Eqt'ns 3.9 and 3.10

* Leissa[7] only gives a comparison of errors when using the Warburton formulas.

3.3 Application of Correction to Mixed Clamped-Supported Conditions

3.3.1 Supported-Clamped-Supported-Supported (SCSS)

To establish the correction for the C-S edge, exact (Leissa) λ 's for a rectangular plate having S-C-S-S edges, were used to determine the Δn values. Noting that the Δm for the S-S edge is zero, Eq't'n 3.11 reduces to;

$$\lambda = \pi^2 [m^2 + n^2 (\frac{a}{b})^2] \quad . . . (3.13)$$

By substituting m , n , a/b , and λ values for each mode case m , and thus Δn was found.

Knowing that the "supporting edge" does not contribute any correction ($\Delta m=0$), it's assumed then at the total correction for the C-S side should be one-half of C-C side equation (3.10), thus;

$$\Delta n = \frac{1}{2} \left[\frac{1}{r^2 + 2} + \frac{r}{60} \right] \quad . . . (3.14)$$

The corresponding Δm correction for a C-S edge would also be one-half of equation (3.9), thus;

$$\Delta m = \frac{1}{2} \left[\frac{1}{r^2 + 2} + \frac{1}{60r} \right] \quad . . . (3.15)$$

Table 3.3 compares the exact vs calculated Δn as a function of r and figure (3.4) is a plot of these values.

Table 3.4 gives a listing of the exact and calculated λ 's using the above equations.

3.3.2 Supported-Clamped-Supported-Clamped (SCSC)

For a rectangular plate with S-S edge in the m -direction, $\Delta m = 0$, and for the C-C edge in the n -direction the Δn correction is Eqn (3.10) and figure (3.3) is a plot of these values.

Table 3.5 compares the exact and approximate λ 's.

For the 9th mode case $(m,n)=(3,2)$ for $a/b = 2/3$, Leissa lists $\lambda = 101.08$, which would have put the calculated error out-of-line with values for other modes.

To verify Leissa's value a FEM analysis was run on PC/MSC-PAL2 to obtain the first nine modal frequencies for a flat plate with $a/b = 2/3$ and SCSC. The resulting 9th mode shape for the ninth mode is given in figure (3.5) and the extracted natural frequencies from the FEM analysis are listed in Table 3.6.

Examination of the errors in Table 3.6 strongly suggests that the correct value should be 110.08, and that the published value is a typographical error.

3.3.3 Clamped-Clamped-Supported-Supported (CCSS)

For a rectangular plate with C-S edge in the m-direction, $\Delta_m = 1/2 \text{ Eqt}'n$ (3.9), and for the C-S edge in the n-direction the Δ_n correction is $1/2 \text{ Eqt}'n$ (3.10) as discussed above for the SCSS case, equations (3.14) and (3.15).

Table 3.7 compares the exact and approximate λ 's.

3.3.4 Clamped-Clamped-Clamped-Supported (CCCS)

For a rectangular plate with C-C edge in the m-direction, $\Delta_m = \text{Eqt}'n$ (3.9), and for the C-S edge in the n-direction the Δ_n correction is $1/2 \text{ Eqt}'n$ (3.10), i.e. equation (3.15). Table 3.8 compares the exact and approximate λ 's.

3.4 Discussion of Accuracy of Results

When using equations (3.9) and (3.10) to generate λ 's, the following comments can be made:

In Table 3.2.(CCCC)

- i) 17 out of 30 mode cases showed better accuracy when compared to Hazell and Mitchell's eqt'ns (3.6) and (3.7),
- ii) 13 out of 30 mode cases showed better accuracy when compared to the Warburton results.

In Table 3.4.(SCSS)

- i) 31 out of 45 mode cases showed better accuracy when compared to Hazell and Mitchell's eqt'n (3.7),

In Table 3.5.(SCSC)

- i) 36 out of 45 mode cases showed better accuracy when compared to Hazell and Mitchell's eqt'n (3.6) and (3.7)..,

In Table 3.7.(CCSS)

- i) 20 out of 30 mode cases showed better accuracy when compared to Hazell and Mitchell's eqt'n (3.6) and 1/2 of eqt'n (3.7),
- ii) 27 out of 30 mode cases showed better accuracy when compared to the Warburton results.

In Table 3.8, (CCCS)

- i) 11 out of 30 mode cases showed better accuracy when compared to Hazell and Mitchell's eqt'n (3.6) and 1/2 eqt'n (3.7),
- ii) 15 out of 30 mode cases showed better accuracy when compared to the Warburton results.

In all cases, except one, more than 50 % show better accuracy by using equations (3.9) and (3.10) than equations (3.6) and (3.7).

In Tables 3.9-3.11 $\%([\lambda] + [\sigma_\lambda])$ is used as an average measure of the maximum percentage error for the calculated λ 's. Therefore, this value gives an indication how accurate any set of equations are relative to the exact λ values.

For all mode cases in Table 3.9 the λ 's generated from eqt'ns (3.9) and (3.10) showed smaller errors (i.e $<1.29\%$) than those by Hazell and Mitchell (Table 3.10). However, only the CCCS case in Table 3.9 was able to match the CCCS case in Table 3.11 where errors derived from using the Warburton formulas proved to be very low.

It is seen that all three methods Warburton[8], Hazell and Mitchell[12], and the method of this thesis using equations (3.9) and (3.10), give excellent results when predicting frequency of plates with mixed supported (S) and clamped (C) edges.

The Warburton results are generally the most accurate, but followed very closely by this thesis, which shows marked improvement in accuracy over the Hazell and Mitchell results.

3.5 Summary

The frequency of plates with mixed supported (S) and clamped (C) edges can be determined accurately by applying a support correction Δm , Δn to the mode number m and n as follows;

$$\lambda = \pi^2 [m_o^2 + n_o^2 (\frac{a}{b})^2] \quad (3.16)$$

$$m_o = m + \Delta m \quad$$

$$n_o = n + \Delta n \quad (3.17)$$

The corrections are;

<u>Support</u>	<u>Δm</u>	<u>Δn</u>
S	0	0 (3.18)
C	$1, (2+r^2)+1/(60r)$	$1/(2+r^2)+r/60$

where $r = mb/na$

The resulting frequencies are essentially as accurate as the Warburton equations[8], and somewhat easier calculated. The accuracy obtained is sufficient for most engineering design estimates of mode frequency.

Table 3.1

Comparison of Exact, Hazell and Mitchell and
Approximate Δm for a Clamped Square Plate

m	n	r	Eqt'n 3.11	Eqt'n 3.6	Error	Eqt'n 3.9	Error
1	3	0.3333	0.1436	0.1079	-24.84%	0.1409	-1.85%
2	5	0.4000	0.1692	0.1297	-23.32%	0.1629	-3.72%
1	2	0.5000	0.2048	0.1837	-10.33%	0.2000	-2.36%
5	8	0.6250	0.2446	0.2227	-8.94%	0.2460	0.58%
2	3	0.6667	0.2566	0.2438	-4.99%	0.2603	1.44%
3	4	0.7500	0.2789	0.2704	-3.07%	0.2869	2.87%
4	5	0.8000	0.2913	0.2850	-2.17%	0.3015	3.53%
5	6	0.8333	0.2991	0.2941	-1.66%	0.3107	3.89%
1	1	1.0000	0.3333	0.3503	5.10%	0.3500	5.00%
6	5	1.2000	0.3656	0.3740	2.29%	0.3850	5.32%
5	4	1.2500	0.3724	0.3822	2.63%	0.3921	5.30%
8	5	1.6000	0.4094	0.4204	2.70%	0.4287	4.73%
4	5	2.0000	0.4359	0.4487	2.93%	0.4528	3.87%
5	2	2.5000	0.4560	0.4664	2.28%	0.4696	3.00%
3	1	3.0000	0.4681	0.4794	2.40%	0.4792	2.38%

Eqt'n 3.11 = Exact Δm , Bolotin[20]

Eqt'n 3.6 = Hazell and Mitchell[12]

Eqt'n 3.9 = Approximate Δm

Table 3.2

Comparison of Frequency Parameters, λ ,
for a Clamped Rectangular Plate

a/b	0.4	2/3	1.0	1.5	2.5	
1	11	11	11	11	11	Mode
	23.648	27.010	35.992	60.772	147.800	Leissa[7]
	23.663	26.951	35.992	60.640	147.894	(3.6) & (3.7)
	23.452	26.874	35.975	60.466	146.575	(3.9) & (3.10)
	-0.06%	-0.50%	-0.13%	0.22%	-0.06%	(3.6) & (3.7)
	0.83%	0.50%	0.05%	0.50%	0.83%	(3.9) & (3.10)
	-0.23%	-0.31%	-0.33%	-0.31%	-0.23%	Warburton[8]
2	12	12	21	21	21	
	27.817	41.716	73.413	93.860	173.850	
	27.504	41.453	73.213	93.270	171.897	
	27.492	41.670	73.589	93.757	171.826	
	1.13%	0.63%	0.27%	0.63%	1.12%	
	1.17%	0.11%	-0.24%	0.11%	1.16%	
	-0.35%	-0.44%	-0.44%	-0.44%	-0.35%	
3	13	21	12	12	31	
	35.446	66.143	73.413	148.820	221.540	
	34.884	66.193	73.213	148.935	218.027	
	35.050	66.375	73.589	149.343	219.060	
	1.58%	-0.08%	0.27%	-0.08%	1.59%	
	1.12%	-0.35%	-0.24%	-0.35%	1.12%	
	-0.31%	-0.35%	-0.44%	-0.35%	-0.31%	
4	14	13	22	31	41	
	46.702	66.552	108.270	149.740	291.890	
	45.971	66.042	108.253	148.593	287.321	
	46.294	66.507	109.010	149.640	289.339	
	1.56%	0.77%	0.02%	0.77%	1.57%	
	0.87%	0.07%	-0.68%	0.07%	0.87%	
	-0.40%	-0.45%	-0.53%	-0.45%	-0.40%	
5	15	22	31	22	51	
	61.554	79.850	131.640	179.660	384.710	
	60.695	79.770	131.595	179.482	379.345	
	61.164	80.220	132.320	180.496	382.277	
	1.40%	0.10%	0.03%	0.10%	1.39%	
	0.63%	-0.46%	-0.52%	-0.47%	0.63%	
	-0.43%	-0.50%	-0.64%	-0.50%	-0.43%	

6	21	14	13	41	12
	63.100	100.850	132.240	226.920	394.370
	63.375	100.231	131.595	225.520	396.094
	63.348	100.928	132.320	227.087	395.927
	-0.44%	0.61%	0.49%	0.62%	-0.44%
	-0.39%	-0.08%	-0.06%	-0.07%	-0.39%
	-0.20%	-0.47%	-0.18%	-0.47%	-0.20%

Mean %	Std %	
0.62%	0.54%	(3.6) & (3.7)
0.49%	0.26%	(3.9) & (3.10)
0.38%	0.11%	Warburton[8]

Note: () denotes equation number used in calculations.

Table 3.3

**Edge Effect Correction, Δn vs r
for Rectangular Plate with
SCSS Edges**

a/b	0.4	2/3	1.0	1.5	2.5	
	(11) 2.5000	(11) 1.5000	(11) 1.0000	(11) 0.6667	(11) 0.4000	Mode r
1	0.0913	0.1408	0.1815	0.2127	0.2348	1/2 of (3.12)
	0.0691	0.1261	0.1752	0.2130	0.2400	1/2 of (3.7)
	0.0814	0.1301	0.1750	0.2101	0.2348	1/2 of (3.10)
	(12) 1.2500	(12) 0.7500	(21) 2.0000	(21) 1.3333	(21) 0.8000	
2	0.1527	0.1986	0.1116	0.1531	0.2002	
	0.1446	0.1994	0.0918	0.1409	0.1979	
	0.1508	0.2014	0.1000	0.1435	0.1961	
	(13) 0.8333	(21) 3.0000	(12) 0.5000	(31) 2.0000	(31) 1.2000	
3	0.1878	0.0768	0.2231	0.1116	0.1638	
	0.1884	0.0540	0.2265	0.0918	0.1538	
	0.1925	0.0705	0.2264	0.1000	0.1553	
	(14) 0.6250	(13) 0.5000	(22) 1.0000	(12) 0.3333	(41) 1.6000	
4	0.2080	0.2215	0.1742	0.2370	0.1341	
	0.2113	0.2251	0.1709	0.2411	0.1181	
	0.2144	0.2264	0.1750	0.2396	0.1230	
	(21) 5.0000	(22) 1.5000	(31) 3.0000	(22) 0.6667	(12) 0.2000	
5	0.0462	0.1344	0.0768	0.2069	0.2451	
	0.0270	0.1219	0.0540	0.2088	0.2493	
	0.0602	0.1301	0.0705	0.2101	0.2468	
	(22) 2.5000	(23) 1.0000	(13) 0.3333	(41) 2.6667	(51) 2.0000	
6	0.0878	0.1717	0.2361	0.0859	0.1116	
	0.0649	0.1695	0.2397	0.0634	0.0918	
	0.0814	0.1750	0.2396	0.0771	0.1000	

	(15)	(14)	(32)	(32)	(22)
	0.5000	0.3750	1.5000	1.0000	0.4000
7	0.2201	0.2322	0.1344	0.1742	0.2318
	0.2239	0.2357	0.1219	0.1709	0.2357
	0.2264	0.2367	0.1301	0.1750	0.2348
	(23)	(31)	(23)	(13)	(32)
	1.6667	4.5000	0.6667	0.2222	0.6000
8	0.1222	0.0514	0.2048	0.2435	0.2135
	0.1075	0.0310	0.2074	0.2468	0.2161
	0.1185	0.0600	0.2101	0.2458	0.2169
	(24)	(32)	(41)	(42)	(61)
	1.2500	2.2500	4.0000	1.3333	2.4000
9	0.1491	0.0965	0.0579	0.1462	0.0948
	0.1425	0.0750	0.0363	0.1366	0.0729
	0.1508	0.0895	0.0611	0.1435	0.0844

Note: () denotes equation number used in calculations.

Table 3.4

Comparison of Frequency Parameters, λ ,
for a SCSS Rectangular Plate

a/b	0.4	2/3	1	1.5	2.5	
1	11	11	11	11	11	Mode
	11.7502	15.5783	23.6463	42.5278	103.9227	Leissa[7]
	11.6745	15.4326	23.4997	42.5462	104.7137	1/2 (3.6) & (3.7)
	11.7164	15.4722	23.4958	42.3877	103.9249	1/2 (3.9) & (3.10)
	-0.64%	-0.94%	-0.62%	0.04%	0.76%	1/2 (3.6) & (3.7)
	-0.29%	-0.68%	-0.64%	-0.33%	0.00%	1/2 (3.6) & (3.10)
2	12	12	21	21	21	
	17.1872	31.0724	51.6743	65.0031	128.3382	
	17.1325	31.0881	51.2440	68.3813	127.9933	
	17.1744	31.1267	51.4206	68.5138	127.7226	
	-0.32%	0.05%	-0.83%	-0.90%	-0.27%	
	-0.07%	0.17%	-0.49%	-0.71%	-0.48%	
3	13	21	12	31	31	
	25.9171	44.5644	58.6464	116.2671	172.3804	
	25.9229	44.3510	58.7950	115.2989	170.9519	
	25.9644	44.5048	58.7913	115.6964	171.1655	
	0.02%	-0.48%	0.25%	-0.83%	-0.83%	
	0.18%	-0.13%	0.25%	-0.49%	-0.70%	
4	14	13	22	12	41	
	37.8317	55.3926	86.1345	120.9956	237.2502	
	37.8753	55.4934	85.9927	121.4022	235.0358	
	37.9164	55.5312	86.1678	121.2557	235.7040	
	0.12%	0.18%	-0.16%	0.34%	-0.93%	
	0.22%	0.25%	0.04%	0.21%	-0.65%	
5	21	22	31	22	12	
	41.2070	59.4627	100.2698	147.6353	320.7921	
	41.1440	59.2284	99.7898	147.8195	321.9691	
	41.2534	59.3822	100.1358	147.9476	321.2526	
	-0.15%	-0.39%	-0.48%	0.12%	0.37%	
	0.11%	-0.14%	-0.13%	0.21%	0.14%	
6	22	23	13	41	51	
	46.3620	83.6060	113.2281	184.1006	322.9647	
	46.2113	83.5439	113.4560	183.0243	320.2748	
	46.3199	83.6970	113.4525	183.6766	321.3790	
	-0.33%	-0.07%	0.20%	-0.58%	-0.83%	
	-0.09%	0.11%	0.20%	-0.23%	-0.49%	

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7	15	14	32	32	22
	52.9007	88.4384	133.7910	193.8025	346.7382
	52.9632	88.5684	133.2638	193.4835	347.8107
	53.0039	88.6055	133.6100	193.8776	347.5579
	0.12%	0.15%	-0.39%	-0.16%	0.31%
	0.20%	0.19%	-0.14%	0.04%	0.24%

8	23	31	23	13	32
	54.8720	93.6758	140.8456	243.4964	391.0659
	54.7273	93.4889	141.0098	243.9666	391.7717
	54.8360	93.7548	141.1822	243.8251	391.9768
	-0.26%	-0.20%	0.12%	0.19%	0.18%
	-0.07%	0.08%	0.24%	0.13%	0.23%

9	24	32	41	42	61
	66.6637	108.1069	168.9585	260.2020	429.2420
	66.5766	107.7139	168.5124	259.2884	426.3166
	66.6852	107.9788	169.0264	259.9406	427.8470
	-0.13%	-0.36%	-0.26%	-0.35%	-0.68%
	0.03%	-0.12%	0.04%	-0.10%	-0.32%

Mean %	Std %	
0.38%	0.27%	1/2 (3.6) & (3.7)
0.21%	0.14%	1/2 (3.9) & (3.10)

Note: () denotes equation number used in calculations.

Table 3.5

Comparison of Frequency Parameters, λ ,
for a SCSC Rectangular Plate

a/b	0.4	2/3	1	1.5	2.5	
1	11	11	11	11	11	Mode
	12.1347	17.3730	28.9509	56.3481	145.4839	Leissa[7]
	11.9154	16.7487	27.8658	55.0320	144.9777	(3.6) & (3.7)
	12.0051	16.8358	27.8570	54.6598	143.0976	(3.9) & (3.10)
	-1.81%	-3.59%	-3.75%	-2.34%	-0.35%	(3.6) & (3.7)
	-1.07%	-3.09%	-3.78%	-3.00%	-1.64%	(3.9) & (3.10)
2	12	12	21	21	21	
	18.3647	35.3445	54.7431	78.9836	164.7387	
	18.1450	35.1094	53.3064	75.9588	159.6547	
	18.2344	35.1936	53.6906	76.2566	159.0241	
	-1.20%	-0.67%	-2.62%	-3.83%	-3.09%	
	-0.71%	0.43%	-1.92%	-3.45%	-3.47%	
3	13	21	12	31	31	
	27.9657	45.4294	69.3270	123.1719	202.2271	
	27.8762	44.8627	69.2544	119.9394	194.3124	
	27.9640	45.1882	69.2463	120.8040	194.7969	
	-0.32%	-1.25%	-0.10%	-2.62%	-3.91%	
	-0.01%	-0.53%	-0.12%	-1.92%	-3.67%	
4	14	13	22	12	41	
	40.7500	62.0544	94.5853	146.2677	261.1053	
	40.7559	62.0832	93.6051	146.6898	252.1951	
	40.8421	62.1639	93.9833	146.3654	253.6753	
	0.01%	0.05%	-1.04%	0.29%	-3.41%	
	0.23%	0.18%	-0.64%	0.07%	-2.85%	
5	21	22	31	22	51	
	41.3782	62.3131	102.2162	170.1112	342.1442	
	41.2328	61.5627	100.9410	169.2704	333.1650	
	41.4606	61.8887	101.6734	169.5509	335.5665	
	-0.35%	-1.20%	-1.25%	-0.49%	-2.62%	
	0.20%	-0.68%	-0.53%	-0.33%	-1.92%	
6	22	23	13	41	12	
	47.0009	88.8047	129.0955	189.1219	392.8746	
	46.6409	88.3831	129.3499	186.1067	394.9990	
	46.8657	88.7058	129.3422	187.4968	393.4079	
	-0.77%	-0.47%	0.20%	-1.59%	0.54%	
	-0.29%	-0.11%	0.19%	-0.86%	0.14%	

7	23	31	32	32	22
	56.1782	94.2131	140.2045	212.8169	415.6906
	55.8004	93.7732	138.5161	210.6116	416.2585
	56.0257	94.3283	139.2496	211.4624	415.6997
	-0.67%	-0.47%	-1.20%	-1.04%	0.14%
	-0.27%	0.12%	-0.68%	-0.64%	0.00%

8	15	14	23	51	61
	56.6756	97.4254	154.7757	276.0012	444.9682
	56.7368	97.5707	154.5636	273.2824	436.2987
	56.8217	97.6492	154.9309	275.1875	439.5828
	0.11%	0.15%	-0.14%	-0.99%	-1.95%
	0.26%	0.23%	0.10%	-0.29%	-1.21%

9	24	* 32	41	42	32
	68.7486	110.0788	170.3465	276.0125	455.3054
	68.4726	109.1047	169.2674	272.6655	453.7388
	68.6975	109.6555	170.3433	274.0551	454.1891
	-0.40%	-0.88%	-0.63%	-1.21%	-0.34%
	0.07%	0.38%	0.00%	0.71%	0.25%

Mean %	Std %	
1.40%	1.52%	(3.6) & (3.7)
0.44%	0.26%	(3.9) & (3.10)

* Leissa value verified through FEM model.

Note: () denotes equation number used in calculations.

Table 3.6

Frequencies λ for Flat Plate with $a/b = 2/3$
SCSC Sides for the first Nine Modes

Mode No	Modes (m,n)	FEM λ	Exact λ (Leissa[7])	% Error
1	11	17.33	17.37	-0.25
2	12	35.22	35.35	-0.38
3	21	45.31	45.43	-0.26
4	13	61.82	62.05	-0.37
5	22	61.84	62.31	-0.75
6	23	87.89	88.81	-1.04
7	31	94.03	94.21	-0.19
8	14	97.09	97.43	-0.35
*9	32	109.20	101.08	+8.03
9	32	109.20	110.78	-0.80

* Leissa's Typographical Error

Table 3.7

Comparison of Frequency Parameters, λ ,
for a CCSS Rectangular Plate

a/b	0.4	2/3	1.0	1.5	2.5	
1	11	11	11	11	11	Mode
	16.849	19.952	27.056	44.893	105.310	Leissa[7]
	16.980	20.086	27.260	45.193	106.125	1/2 (3.6 & 3.7)
	16.896	20.055	27.252	45.124	105.598	1/2 (3.9 & 3.10)
	0.78%	0.67%	0.75%	0.67%	0.77%	1/2 (3.6 & 3.7)
	0.28%	0.52%	0.73%	0.51%	0.27%	1/2 (3.9 & 3.10)
	0.41%	0.55%	0.58%	0.55%	0.41%	Warburton[8]
2	12	12	12	21	21	
	21.363	34.024	60.544	76.554	133.520	
	21.425	34.064	60.691	76.645	133.908	
	21.424	34.162	60.864	76.864	133.899	
	0.29%	0.12%	0.24%	0.12%	0.29%	
	0.28%	0.40%	0.53%	0.40%	0.28%	
	0.36%	0.52%	0.76%	0.52%	0.36%	
3	13	21	12	12	31	
	29.236	54.370	60.791	122.330	182.730	
	29.193	54.443	60.691	122.496	182.459	
	29.269	54.531	60.864	122.695	182.931	
	-0.15%	0.13%	-0.16%	0.14%	-0.15%	
	0.11%	0.30%	0.12%	0.30%	0.11%	
	0.44%	0.44%	0.35%	0.44%	0.44%	
4	14	13	22	31	41	
	40.509	57.517	92.865	129.410	253.180	
	40.345	57.389	93.029	129.126	252.158	
	40.493	57.604	93.379	129.608	253.083	
	-0.40%	-0.22%	0.18%	-0.22%	-0.40%	
	-0.04%	0.15%	0.55%	0.15%	-0.04%	
	0.55%	0.58%	0.28%	0.58%	0.55%	
5	21	22	13	22	12	
	51.457	67.815	114.570	152.580	321.600	
	51.602	67.902	114.550	152.778	322.510	
	51.596	68.112	114.892	153.253	322.476	
	0.28%	0.13%	-0.02%	0.13%	0.28%	
	0.27%	0.44%	0.28%	0.44%	0.27%	
	0.24%	0.28%	0.54%	0.28%	0.24%	

6	15	14	31	41	51
	55.117	90.069	114.720	202.660	344.480
	54.859	89.859	114.550	202.183	342.870
	55.077	90.186	114.892	202.919	344.229
	-0.47%	-0.23%	-0.15%	-0.24%	-0.47%
	-0.07%	0.13%	0.15%	0.13%	-0.07%
	0.59%	0.55%	0.42%	0.55%	0.59%

Mean %	Std %	
0.31%	0.22%	1/2 (3.6 & 3.7)
0.26%	0.14%	1/2 (3.9 & 3.10)
0.47%	0.12%	Warburton[8]

Note: () denotes equation number used in calculations.

Table 3.8

Comparison of Frequency Parameters, λ ,
for a CCCS Rectangular Plate

a/b	0.4	2/3	1.0	1.5	2.5	
1	11	11	11	11	11	Mode
	23.440	25.861	31.829	48.167	107.070	Leissa[7]
	23.422	25.635	31.626	48.154	107.630	(3.6) & 1/2 (3.7)
	23.163	25.509	31.614	48.194	107.402	(3.9) & 1/2 (3.10)
	-0.08%	-0.87%	-0.64%	-0.03%	0.52%	(3.6) & 1/2 (3.7)
	-1.18%	-1.36%	-0.68%	0.06%	0.31%	(3.9) & 1/2 (3.10)
	0.23%	0.34%	0.42%	0.46%	0.40%	Warburton[8]
2	12	12	12	21	21	
	27.022	38.102	63.347	85.507	139.660	
	26.491	37.432	62.753	85.692	140.236	
	26.432	37.603	63.134	86.014	140.524	
	-1.96%	-1.76%	-0.94%	0.22%	0.41%	
	-2.18%	-1.31%	-0.34%	0.59%	0.62%	
	0.22%	0.38%	0.49%	0.55%	0.54%	
3	13	13	21	12	31	
	33.799	60.325	71.084	123.990	194.410	
	32.931	59.452	71.150	123.647	194.666	
	33.050	59.874	71.319	124.233	195.429	
	-2.57%	-1.45%	0.09%	-0.28%	0.13%	
	-2.22%	-0.75%	0.33%	0.20%	0.52%	
	0.29%	0.50%	0.48%	0.45%	0.45%	
4	14	21	22	31	41	
	44.131	65.516	100.830	143.990	270.480	
	43.091	65.682	100.641	143.953	270.161	
	43.369	65.691	101.194	144.532	271.368	
	-2.36%	0.25%	-0.19%	-0.03%	-0.12%	
	-1.73%	0.27%	0.36%	0.38%	0.33%	
	0.41%	0.34%	0.41%	0.51%	0.52%	
5	15	22	13	22	12	
	58.034	77.563	116.400	158.360	322.550	
	56.922	77.435	115.701	158.031	323.065	
	57.347	77.714	116.430	158.892	323.772	
	-1.92%	-0.16%	-0.60%	-0.21%	0.16%	
	-1.18%	0.19%	0.03%	0.34%	0.38%	
	0.49%	0.30%	0.49%	0.50%	0.27%	

6	21	14	31	32	22
	62.971	92.154	130.370	214.780	353.430
	63.286	91.229	130.444	214.693	353.098
	63.141	91.884	130.782	215.812	354.250
	0.50%	-1.00%	0.06%	-0.04%	-0.09%
	0.27%	-0.29%	0.32%	0.48%	0.23%
	0.18%	0.52%	0.40%	0.23%	0.44%

Mean %	Std %	
0.81%	0.84%	(3.6) & 1/2 (3.7)
0.68%	0.61%	(3.9) & 1/2 (3.10)
0.66%	0.59%	Warburton[8]

Note: () denotes equation number used in calculations.

Table 3.9

Mean Error $[\lambda]$, Standard Deviation $[\sigma_\lambda]$
 & Maximum Error $[\lambda]+[\sigma_\lambda]$
 for Approximate Equations (3.9) & (3.10)

Table no	B.Cs	Δm (3.9)	Δn (3.10)	% $[\lambda]$ Mean error	% $[\sigma_\lambda]$ STD error	% $([\lambda]+[\sigma_\lambda])$ Max error
3.2	CCCC	1	1	.49	.26	.75
3.4	SCSS	0	1/2	.21	.14	.35
3.5	SCSC	0	1	.44	.26	.70
3.7	CCSS	1/2	1/2	.26	.14	.50
3.8	CCCS	1	1/2	.68	.61	1.29

Table 3.10

Mean Error $[\lambda]$, Standard Deviation $[\sigma_\lambda]$
 and Maximum Error $[\lambda]+[\sigma_\lambda]$
 for the Hazell and Mitchell Equations[12]

Table no	B.Cs	Δm (3.6)	Δn (3.7)	% $[\lambda]$ Mean error	% $[\sigma_\lambda]$ STD error	% $([\lambda]+[\sigma_\lambda])$ Max error
3.2	CCCC	1	1	.62	.54	1.18
3.4	SCSS	0	1/2	.38	.27	.65
3.5	SCSC	0	1	1.40	1.52	2.92
3.7	CCSS	1/2	1/2	.31	.22	.53
3.8	CCCS	1	1/2	.81	.84	1.65

Note: () denotes equation number used in calculations.

Table 3.11

Mean Error $[\lambda]$, Standard Deviation $[\sigma_\lambda]$
and Maximum Error $[\lambda]+[\sigma_\lambda]$
from Warburton[8] Equations

Table no	B.Cs	$\% [\lambda]$ Mean error	$\% [\sigma_\lambda]$ STD error	$\% ([\lambda]+[\sigma_\lambda])$ Max error
3.2	CCCC	.38	.11	.49
3.4	SCSS	*N/A	*N/A	*N/A
3.5	SCSC	*N/A	*N/A	*N/A
3.7	CCSS	.47	.12	.59
3.8	CCCS	.66	.59	1.25

* N/A (Not Available)

**Figure 3.1 Coordinate Axes
for Rectangular Plate**

Figure 3.2 Δm vs r for CCCC Case

Figure 3.3 Δn vs r for SCSC Case

Figure 3.4 Δn vs r for SCSS and SCSS Case

Figure 3.5 9th Mode Shape for a Flat Rectangular Plate
with $a/b=2/3$ and SCSC Sides

CHAPTER 4

4.0 BOUNDARY EDGE EFFECT CORRECTIONS FOR RECTANGULAR PLATES WITH FREE EDGES

4.1 Derivation of Correction for One Free Edge

4.1.1 Supported-Supported-Supported-Free (SSSF)

To establish the correction for the S-F edge, exact values of λ (Leissa[7]) for a rectangular plate with S-S-S-F edges were used to determine the exact Δ_n values. Again, ($\Delta_m = 0$) for the S-S edge and by substituting the known values of m , n , a/b and λ into Eqt'n 3.12, Δ_n is found.

Table 4.1 is a listing of the exact Δ_n derived from using Eqt'n 3.12 for different mode cases for $a/b = 0.4, 2/3, 1.0, 1.5$ and 2.5 , and figure (4.1) is the exact graphical plot of the Δ_n vs r .

Referring to figure (4.1) a polynomial "Curvefit" function was used to try and match the exact Δ_n values.

Unlike the C-S edge condition the derived Δ_n for all mode cases are negative. Figure (4.1) reveals that the amount of scatter in Δ_n is considerably higher than the previous C-S conditions, This results in a more complex equation then previously derived for the C-S condition, having the form,

$$-\Delta n = \frac{3/8}{(r^{-4}+1.5)^3} + \frac{1}{(15r)^2} + \frac{9}{16} \quad \dots \quad (4.1)$$

Table 4.2 compares the exact and approximate (Eq't'n 4.1) λ for various mode cases for a SSSF rectangular plate. The resulting errors, when compared with the exact values, are seen to be much higher than in the C-S edge cases considered in Chapter 3. Thus at high aspect ratio, $a/b = 2.5$, errors exceeding 20% (e.g. mode (1,2)) are noted, and the errors which typically result are too large to be acceptable in design calculations.

4.1.2 Supported-Clamped-Supported-Free (SCSF)

By the principles of superposition the Δn correction for the C-S edge (1/2 of Eq't'n 3.7) and the S-F edge (Eq't'n 4.1) when added should produce the Δn correction for the C-F edge. This principle is applied and the new equation for Δn for the C-F edge becomes,

$$-\Delta n = \frac{3/8}{(r^{-4}+1.5)^3} + \frac{1}{(15r)^2} + \frac{9}{16} - \frac{r}{120} - \frac{1/2}{r^2+2} \quad \dots \quad (4.2)$$

To calculate the exact Δn values for the C-F edge, known values of m , n , a/b and λ are substituted back into Eq't'n (3.12) and Δn is found.

Table 4.3 lists the exact Δn vs r values compared to those calculated when using Eq't'n (4.2).

Figure (4.2) shows these Δn values for various r .

Alternatively a more complex equation than that derived by Superposition above can be derived to simulate the exact Δn pattern and has the form,

$$-\Delta n = \frac{1/16}{(r^{-3}+1)^2} + \frac{1}{(7r)^2} + .425 \quad \dots \quad (4.3)$$

Figure (4.3) represents this curve from Eqt'n (4.3) and compares it with the series of the exact $(\Delta n, r)$ points derived from using exact values.

Table 4.4 compares the exact and the approximate λ for a rectangular plate with SCSF edges.

In both cases figures (4.2) and (4.3) do not accurately predict Δn values in the areas of large scatter. However, figure (4.3) does show a much better correlation with the exact Δn values than those derived through superposition. Superposition shows large discrepancies for Δn in the range $0 \leq r \leq 1.5$ where values tend to upperbound and for $r \geq 2.0$ values are mostly lowerbound.

Table 4.4 shows much higher percentage errors (e.g. 46.6 % for Mode (1,2) for $a/b = 2.5$) than previously shown with C-S mixed edge conditions.

Large errors can result if;

- a) the point on the curvefit equation deviates from the exact value,
- b) the point on the curvefit equation lies in between the scatter areas.

The failure of superposition to accurately predict the C-F edge condition seems to indicate a strong presence of cross-coupling effects that alter the mode shape for the C-F edge condition.

This interaction is difficult to determine and is not taken into consideration when adding two independent equations.

Therefore, Eqt'n (4.1) and (4.3) will be used to calculate λ for the remaining "one free" edge conditions;

i) CSCF	iii) CCSF
ii) CSSF	iv) CCCF

4.1.3 Clamped-Supported-Clamped-Free (CSCF)

For a rectangular plate with C-C edge in the m-direction, $\Delta_m = \text{Eq't'n (3.6)}$, and the S-F edge for the n-direction the Δ_n correction is Eqt'n (4.1).

Table 4.5 compares the exact and approximate λ derived from using these equations.

4.1.4 Clamped-Supported-Supported-Free (CSSF)

For a rectangular plate with C-S edge in the m-direction, $\Delta_m = 1/2 \text{ Eq't'n (3.6)}$, and the S-F edge for the n-direction the Δ_n correction is Eqt'n (4.1).

Table 4.6 compares the exact and approximate λ derived from using these equations.

4.1.5 Clamped-Clamped-Supported-Free (CCSF)

For a rectangular plate with C-S edge in the m-direction, $\Delta_m = 1/2$ Eqt'n (3.6), and the C-F edge for the n-direction the Δ_n correction is Eqt'n (4.3).

Table 4.7 compares the exact and approximate λ derived from using these equations.

4.1.6 Clamped-Clamped-Clamped-Free (CCCF)

For a rectangular plate with C-C edge in the m-direction, $\Delta_m =$ Eqt'n (3.6), and the C-F edge for the n-direction the Δ_n correction is Eqt'n (4.3).

Table 4.8 compares the exact and approximate λ derived from using these equations.

4.1.7 Discussion and Conclusions

It is evident from the above that the use of a boundary correction for plates with one free edge did not result in a simple accurate prediction of the plate frequencies.

First, the equation developed for the correction was not unique to a single free edge condition. It was found necessary to obtain different equations for S-F and C-F opposing edges, and these equations were complex and cumbersome to use.

Despite these different equations, the resulting accuracy was very poor, and quite unsatisfactory for practical predictions. Thus the boundary correction method failed here both on accuracy and complexity, by comparison with the Warburton equations [8].

Some preliminary investigation of plates with two free boundaries, in particular S-S-F-F, indicated that the problem becomes more difficult and there is noted a further interaction between the corrections required along m and n directions. Further investigation of the boundary edge correction for plates with free edges was therefore abandoned in this thesis work.

Table 4.1
Exact and Approximate Δn for SSSF Case

r	Eqt'n 3.12 (Exact)	Eqt'n 4.1 (Approx)	% Error
0.2	-1.2870	-0.6736	-91.06%
0.4	-0.7462	-0.5903	-26.41%
0.6	-0.6260	-0.5753	-8.81%
0.8	-0.5932	-0.5756	-3.06%
1.0	-0.5833	-0.5909	1.29%
1.2	-0.5795	-0.6137	5.58%
1.4	-0.5769	-0.6335	8.94%
1.6	-0.5746	-0.6473	11.23%
1.8	-0.5720	-0.6562	12.84%
2.0	-0.5694	-0.6619	13.98%
2.2	-0.5668	-0.6656	14.84%
2.4	-0.5642	-0.6679	15.53%
2.6	-0.5617	-0.6695	16.11%
2.8	-0.5593	-0.6706	16.60%
3.0	-0.5570	-0.6714	17.04%
3.2	-0.5548	-0.6720	17.43%
3.4	-0.5528	-0.6723	17.78%
3.6	-0.5509	-0.6726	18.10%
3.8	-0.5491	-0.6729	18.39%
4.0	-0.5473	-0.6730	18.68%
4.2	-0.5457	-0.6732	18.93%
4.4	-0.5442	-0.6732	19.17%
4.6	-0.5428	-0.6733	19.39%
4.8	-0.5415	-0.6734	19.59%
5.0	-0.5402	-0.6734	19.78%

Table 4.2

Comparison of Frequency Parameters, λ ,
for a SSSF Rectangular Plate

a/b	0.4	2/3	1.0	1.5	2.5	
	11	11	11	11	11	Mode
1	10.016	10.410	11.521	14.056	21.638	(4.1)
	10.126	10.671	11.685	13.711	18.801	Leissa[7]
	-1.09%	-2.45%	-1.40%	2.52%	15.09%	(4.1)
2	12	12	12	21	21	
	12.863	18.856	30.231	42.470	50.806	
	13.057	18.300	27.756	43.572	50.541	
	-1.49%	3.04%	8.91%	-2.53%	+0.52%	
3	13	13	21	12	31	
	19.165	35.906	40.498	55.726	97.873	
	18.839	33.697	41.197	47.857	100.232	
	1.73%	6.56%	-1.70%	16.44%	-2.35%	
4	14	21	22	22	12	
	28.503	39.845	59.074	85.156	137.304	
	27.558	40.131	59.066	81.479	110.226	
	3.43%	-0.71%	0.01%	4.51%	24.57%	
5	15	22	13	31	22	
	40.948	47.483	68.485	91.121	166.818	
	39.338	48.408	61.861	92.693	147.632	
	4.09%	-1.91%	10.71%	-1.70%	13.00%	
6	21	14	31	13	41	
	39.552	61.683	89.651	141.784	165.172	
	39.612	57.593	90.294	124.564	169.103	
	-0.15%	7.10%	-0.71%	13.82%	-2.32%	
7	22	23	23	32	32	
	42.164	64.936	97.959	132.916	215.924	
	42.696	64.728	94.484	132.897	203.730	
	-1.25%	0.32%	3.68%	0.01%	5.99%	
8	23	24	32	23	51	
	48.114	91.124	106.836	171.320	253.113	
	48.775	89.186	108.919	158.918	257.479	
	-1.35%	2.17%	-1.91%	7.80%	-1.70%	
9	16	31	14	41	42	
	56.546	89.073	126.474	159.900	283.794	
	54.250	89.573	115.686	161.421	277.428	
	4.23%	-0.56%	9.33%	-0.94%	2.29%	

Table 4.3

Exact and Approximate Δn for SCSF Case

r	Eq't'n 3.12 (Exact)	Eq't'n 4.2 (Approx)	% Error
0.200	-0.4720	-0.9352	98.14%
0.222	-0.6520	-0.8391	28.70%
0.333	-0.5750	-0.6091	5.93%
0.375	-0.5590	-0.5703	2.02%
0.400	-0.5500	-0.5528	0.50%
0.400	-0.5070	-0.5528	9.03%
0.500	-0.5250	-0.5074	-3.35%
0.600	-0.4720	-0.4837	2.47%
0.625	-0.5040	-0.4797	-4.83%
0.667	-0.4710	-0.4741	0.67%
0.667	-0.4980	-0.4741	-4.79%
0.750	-0.4880	-0.4668	-4.35%
0.800	-0.4620	-0.4641	0.44%
0.833	-0.4790	-0.4628	-3.38%
1.000	-0.4560	-0.4610	1.10%
1.000	-0.4660	-0.4610	-1.07%
1.200	-0.4550	-0.4642	2.03%
1.250	-0.4510	-0.4654	3.19%
1.333	-0.4550	-0.4674	2.72%
1.333	-0.4580	-0.4674	2.05%
1.500	-0.4400	-0.4713	7.11%
1.500	-0.4600	-0.4713	2.45%
1.600	-0.4630	-0.4733	2.24%
1.667	-0.4610	-0.4746	2.95%
2.000	-0.4670	-0.4795	2.67%
2.000	-0.6820	-0.4795	-29.69%
2.250	-0.4740	-0.4819	1.66%
2.400	-0.4790	-0.4829	0.81%
2.500	-0.4200	-0.4835	15.11%
2.500	-0.4760	-0.4835	1.57%
2.667	-0.4810	-0.4843	0.68%
3.000	-0.4820	-0.4854	0.70%
4.000	-0.4910	-0.4869	-0.84%
4.500	-0.4920	-0.4872	-0.98%
5.000	-0.4960	-0.4873	-1.75%

Table 4.4

Comparison of Frequency Parameters, λ ,
for a SCSF Rectangular Plate

a/b	0.4	2/3	1.0	1.5	2.5	Mode
1	11	11	11	11	11	Leissa[7] (4.2)
	10.291	11.096	12.737	16.009	22.207	
	10.189	10.975	12.687	16.823	30.628	
	1.00%	1.10%	0.39%	-4.83%	-27.49%	
2	12	12	12	21	21	
	13.588	20.181	31.858	45.778	57.197	
	13.604	20.336	33.065	45.302	58.080	
	-0.11%	-0.76%	-3.65%	1.05%	-1.52%	
3	13	13	21	12	31	
	20.035	37.123	42.152	52.852	106.532	
	20.097	37.955	41.702	61.018	105.547	
	-0.31%	-2.19%	1.08%	-13.38%	0.93%	
4	14	21	22	22	12	
	29.440	40.640	62.854	91.178	79.807	
	29.622	40.272	63.015	92.307	149.457	
	-0.62%	0.91%	-0.26%	-1.22%	-46.60%	
5	21	22	13	31	41	
	39.245	49.730	66.305	94.843	175.023	
	39.638	49.732	72.398	93.829	173.106	
	-0.99%	-0.00%	-8.42%	1.08%	1.11%	
6	15	14	31	32	22	
	41.742	61.468	91.440	141.421	168.675	
	42.243	64.189	90.611	141.783	182.811	
	-1.18%	-4.24%	0.91%	-0.26%	-7.73%	
7	22	23	23	13	32	
	43.110	67.755	102.444	113.643	230.658	
	42.999	67.899	103.162	149.606	235.016	
	0.26%	-0.21%	-0.70%	-24.04%	-1.85%	
8	23	31	32	41	51	
	49.549	89.980	111.892	163.820	263.453	
	49.574	89.357	111.896	162.241	260.637	
	-0.05%	0.70%	-0.00%	0.97%	1.08%	
9	16	24	14	23	42	
	56.897	94.238	114.018	175.099	303.437	
	58.002	94.515	131.429	181.187	305.222	
	-1.90%	-0.29%	-13.25%	-3.36%	-0.58%	

Table 4.5
Comparison of Frequency Parameters, λ ,
for a CSCF Rectangular Plate

a/b	0.4	2/3	1.0	1.5	2.5	Mode
1	11	11	11	11	11	
	22.490	20.838	19.639	19.519	24.323	(3.6) & (4.1)
	22.544	22.855	23.460	24.775	28.564	Leissa[7]
	-0.24%	-8.83%	-16.29%	-21.21%	-14.85%	(3.6) & (4.1)
2	12	12	12	12	21	
	22.327	25.086	33.987	57.722	63.268	
	24.296	27.971	35.612	53.731	70.561	
	-8.10%	-10.31%	-4.56%	7.43%	-10.34%	
3	13	13	21	21	12	
	26.093	39.663	61.613	60.432	138.117	
	28.341	40.683	63.126	64.959	114.000	
	-7.93%	-2.51%	-2.40%	-6.97%	21.16%	
4	14	21	13	22	31	
	33.558	63.226	70.481	95.483	122.545	
	35.345	62.310	66.808	97.257	130.840	
	-5.06%	1.47%	5.50%	-1.82%	-6.34%	
5	15	14	22	31	22	
	44.705	64.106	74.100	121.666	172.028	
	45.710	62.695	77.502	124.480	159.540	
	-2.20%	2.25%	-4.39%	-2.26%	7.83%	
6	16	22	23	13	41	
	59.408	66.480	108.286	142.768	202.260	
	59.562	68.683	108.990	127.920	210.320	
	-0.26%	-3.21%	-0.65%	11.61%	-3.83%	

Note: () denotes equation number used in calculations.

Table 4.6
Comparison of Frequency Parameters, λ ,
for a CSSF Rectangular Plate

a/b	0.4	2/3	1.0	1.5	2.5	
	11	11	11	11	11	Mode
1	15.222	15.017	15.278	16.639	21.898	1/2 (3.6) & (4.1)
	15.649	16.067	16.865	18.540	23.067	Leissa[7]
	-2.73%	-6.53%	-9.41%	-10.25%	-5.07%	1/2 (3.6) & (4.1)
	12	12	12	12	21	
2	17.130	21.823	31.834	54.679	56.767	
	17.946	22.449	31.138	50.442	59.969	
	4.55%	2.79%	-2.23%	-8.40%	-5.34%	
	13	13	21	21	12	
3	22.444	37.624	50.050	50.909	119.616	
	22.902	36.703	51.631	53.715	111.950	
	-2.00%	2.51%	-3.06%	-5.22%	6.85%	
	14	21	13	22	31	
4	30.975	49.979	68.040	89.951	109.796	
	30.892	50.696	64.043	88.802	115.110	
	0.27%	-1.42%	6.24%	1.29%	-4.62%	
	15	22	22	31	22	
5	42.788	56.308	66.285	105.277	165.346	
	42.108	57.908	67.646	108.190	153.240	
	1.61%	-2.76%	-2.01%	-2.69%	7.90%	
	21	14	23	13	41	
6	49.990	62.334	102.879	133.464	182.965	
	50.222	59.840	101.210	126.090	189.490	
	-0.46%	4.17%	1.65%	5.85%	-3.44%	

Note: () denotes equation number used in calculations.

Table 4.7
Comparison of Frequency Parameters, λ ,
for a CCSF Rectangular Plate

a/b	0.4	2/3	1.0	1.5	2.5	Mode
1	11	11	11	11	11	
	15.470	15.679	16.493	18.746	23.880	1/2 (3.6) & (4.3)
	15.696	16.287	17.615	21.035	33.578	Leissa[7]
	-1.44%	-3.73%	-6.37%	-10.88%	-28.88%	1/2 (3.6) & (4.3)
2	12	12	12	21	21	
	17.838	23.216	33.930	54.128	63.373	
	18.373	24.201	36.046	55.184	66.612	
	-2.91%	-4.07%	-5.87%	-1.91%	-4.86%	
3	13	13	21	12	31	
	23.340	39.196	51.596	54.292	118.298	
	23.987	40.701	52.065	63.178	119.900	
	-2.70%	-3.70%	-0.90%	-14.07%	-1.34%	
4	14	21	22	22	12	
	32.016	50.667	70.065	96.483	81.031	
	32.810	50.822	71.194	99.007	150.830	
	-2.42%	-0.31%	-1.59%	-2.55%	-46.28%	
5	15	22	13	31	22	
	43.815	58.460	67.744	108.755	171.955	
	44.862	59.071	74.349	109.220	187.610	
	-2.33%	-1.03%	-8.88%	-0.43%	-8.34%	
6	21	14	31	13	41	
	50.236	63.048	106.197	114.862	192.401	
	50.251	66.262	106.280	150.900	193.230	
	-0.03%	-4.85%	-0.08%	-23.88%	-0.43%	

Note: () denotes equation number used in calculations.

Table 4.8

Comparison of Frequency Parameters, λ ,
for a CCCF Rectangular Plate

a/b	0.4	2/3	1.0	1.5	2.5	Mode
1	11	11	11	11	11	
	21.738	21.133	20.854	21.816	25.684	(3.6) & (4.3)
	22.577	23.015	24.020	26.731	37.656	Leissa[7]
	-3.72%	-8.18%	-13.18%	-18.39%	-31.79%	(3.6) & (4.3)
2	12	12	12	12	21	
	22.846	26.657	36.200	55.829	69.998	
	24.623	29.427	40.039	65.916	76.407	
	-7.22%	-9.41%	-9.59%	-15.30%	-8.39	
3	13	13	21	21	31	
	27.121	41.466	62.051	63.273	130.795	
	29.244	44.363	63.493	66.219	135.150	
	-7.26%	-6.53%	-2.27%	-4.44%	-3.22%	
4	14	21	13	22	12	
	34.892	61.826	69.282	102.123	82.326	
	37.059	62.417	76.761	106.800	152.470	
	-5.85%	-0.95%	-9.74%	-4.38%	-46.01%	
5	15	14	22	31	22	
	46.085	64.746	77.880	123.679	175.367	
	42.283	68.887	80.713	125.400	193.010	
	8.99%	-6.01%	-3.51%	-1.37%	-9.14%	
6	21	22	23	13	41	
	61.781	68.061	113.389	116.152	210.687	
	61.922	69.696	116.800	152.480	213.740	
	-0.23%	-2.35%	-2.92%	-23.82%	-1.43%	

Note: () denotes equation number used in calculations.

Figure 4.1 Exact and Approximate Δn for SSSF Case

Figure 4.2 Exact and Approximate Δn for SCSF (Superposition) Case

Figure 4.3 Exact and Approximate Δn for SCSF (Curvefit) Case

5.0 CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

- 1) A method has been developed which obtains the effective number of halfwaves resulting when clamped or free supports are present in beams or rectangular plates.
- 2) By using this effective wave number, the simple equation pertaining to simple supports (P-P) can then be used to obtain frequencies for the other variations of support.
- 3) The method has been applied successfully to both thin (Euler) and thick (Timoshenko) beams. The support corrections for the clamped, free, and sliding supports are given by simple equations, and the separate corrections for the two boundaries can be added to obtain the total correction. The method applies successfully to both flexural vibration and buckling problems, including complicating effects such as axial load, and an elastic foundation, and provides a simple method of prediction suitable for design purposes.
- 4) The method has been successfully applied to rectangular plates with mixed supported and clamped edges. First a modification to the correction suggested by Mitchell and Hazell[12] is made which improves the accuracy they obtained for the fully clamped, CCCC, plate. The revised corrections are then applied to plates with mixed supports, and a detailed comparison is made with the exact values[7], and the

approximations of Mitchell and Hazell[12], and of Warburton[8]. Very high accuracy is exhibited by all three approximate methods, with the method of the thesis slightly less accurate than Warburton, but also more convenient to use.

5) The application of the boundary correction method to plates having free edges was attempted unsuccessfully. It was found that three serious difficulties arose with free edges.

a) there was considerable scatter in the required correction Δn making it difficult to use a simple equation for the correction.

b) the corrections could not be added or superimposed, as found for beams. Thus the correction for C-F $\neq$ correction for C-S + correction for S-F.

c) the corrections required for given supports along the x-direction were not independent of the supports present along the y-direction, i.e. a cross-coupling was evident.

As a result of the above, further work on the effect of free edges was abandoned in the thesis.

5.2 Recommendations

The following additional studies are suggested.

1) The development for beams presented appears to be quite complete. The prediction of buckling loads for both thin and thick beams on an elastic foundations is one possible extension of the method.

2) The successful application of the method to the vibration of rectangular plates with mixed supported-clamped edges, suggests that the method might also be applied successfully to buckling of such plates.

3) Notwithstanding the failure to successfully treat plates with free edges in this thesis, further study of these problems is suggested.

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