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CROSSING NUMBERS OF SEQUENCES OF GRAPHS

by

Benny Pinontoan

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

School of Mathematics and Statistics

Carleton University

Ottawa, Canada

March 2002

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Benny Pinontoan

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for the degree of Doctor of Philosophy

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Abstract

The crossing number of a graph G is the least number of pairwise crossings of edges among all the drawings of G in the plane. There are only a few families of graphs whose crossing numbers are known exactly. In this thesis, first we compute the crossing number of a family of 5-regular graphs using a well-known method. Then we develop a method to determine the crossing number of big graphs by analyzing their small subgraphs. We do this by introducing the concept of *tile* which is a graph with special conditions. Tiles are divided into planar and non-planar tiles. We use planar tiles to construct sequences of graphs with constant crossing number. We describe a method for computing the crossing number in many such cases and apply this method to determine the crossing number of several known families of graphs. Similarly, we use the non-planar tiles to construct sequences of graphs with non-constant crossing number. In this case, we asymptotically determine the crossing numbers and apply this construction to determine, up to a constant depending only on the tile, the crossing number of some well-known families of graphs, including the Generalized Petersen Graphs $P(n, 3)$ and the Cartesian product $C_m \times C_n$ of two cycles.

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Ajarkanlah kepadaku kebijaksanaan dan pengetahuan yang baik

(Mazmur 119 : 66 a)

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List of Symbols

	Page
$acr(T)$	75
$cr(G)$	2
$dcr(\mathcal{D})$	2
$(G \cup H)\{X, Y\}$	54
$L^{\leftarrow}$	49
$pd(T)$	63
$tcr(T)$	50
$\tilde{T}$	56
$T^{\uparrow}$	69
$T^{\leftarrow}$	69
T^n	56
T_{π}	76
$T^{\bowtie}(m, n)$	59
$T^{\bowtie h}(m, n)$	69
$T^{\bowtie v}(m, n)$	69
$T^{\otimes}(n)$	56
$T^{\otimes h}(n)$	69
$T^{\otimes v}(n)$	69
$\circ(T)$	54

List of Terminology

	Page
Average crossing number	75
Block-tile	56
Compatibility	51
Close	54
Crossing-critical	2
Crossing number	2
Flip	69
Glue	54
Left-wall	49
Optimal drawing	2
Perfect tile	50
Planar tile	50
Pair-disjoint	63
Right-wall	49
Subdivision	2
Tile	49
Tile crossing number	50
Twist	56
Wall edge	50
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Chapter 1

Introduction

1.1 On Crossing Number

A *drawing* $\mathcal{D}(G)$ of a graph G in a plane is a mapping of the graph to the plane in such a way that vertices are mapped to distinct points of the plane and edges are mapped into arcs terminating at the appropriate vertex points. A *crossing* in the drawing is a point on the plane not corresponding with a vertex of the graph, where two arcs intersect. We assume further that no vertex point can be an interior point of an arc representing an edge. From now on, if we say “edges cross in the drawing”, we mean that the corresponding arcs of these edges intersect in the drawing. All the drawings under consideration are assumed to be *simple*, i.e.:

- i) no edge crosses itself;
- ii) adjacent edges never cross;

- iii) two non-adjacent edges cross at most once;
- iv) no more than two edges cross at a point of the plane.

Let $\mathcal{D}(G)$ be a drawing of graph G . The *drawing crossing number* $dcr(\mathcal{D}(G))$ of $\mathcal{D}(G)$ is the number of crossings in the drawing $\mathcal{D}(G)$. The *crossing number* $cr(G)$ of a graph G is the minimum number of crossings among all the drawings of G in the plane. If two graphs G and H are isomorphic, then $cr(G) = cr(H)$. A drawing $\mathcal{D}(G)$ of graph G is called *optimal* if $dcr(\mathcal{D}(G)) = cr(G)$. A graph G is called *crossing-critical* if $cr(G - e) < cr(G)$, for every edge e of G .

Example 1.1. Consider the drawings $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ of $K_{3,3}$ in Figure 1.1. Then $dcr(\mathcal{D}_1) = 9$, $dcr(\mathcal{D}_2) = 3$, and $dcr(\mathcal{D}_3) = cr(K_{3,3}) = 1$. So $\mathcal{D}_3$ is an optimal drawing of $K_{3,3}$.

Let G be a graph and let e be an edge of G with ends u and v . An *elementary subdivision of G at e* is the graph obtained from G by deleting e and adding a new vertex adjacent just to u and v . A graph H is a *subdivision of G* if there are graphs $G = G_0, G_1, \dots, G_n = H$ such that, for each $i = 1, 2, \dots, n$, G_i is an elementary subdivision of G_{i-1} . Two graphs G and H are *homeomorphic* if there is a graph K that is a subdivision of both G and H . It is an easy exercise to show that homeomorphism defines an equivalence relation on the set of all graphs. Let $\mathcal{C}(G)$ denote the equivalence class containing G .

Figure 1.1: *Different drawings of $K_{3,3}$*

It is routine to prove that if G and H are homeomorphic, then $cr(G) = cr(H)$ (this basically involves showing that if H is an elementary subdivision of G , then $cr(G) = cr(H)$). Furthermore, G is crossing-critical if and only if H is crossing-critical. When we discuss crossing-critical graphs and their properties, we will typically be referring to the unique element of the corresponding equivalence class that has fewest vertices. Thus, crossing-critical graphs will not have degree 2 vertices, since if G is a connected graph with $cr(G) > 0$, in the homeomorphism class $\mathcal{C}(G)$, there is a unique element that has no vertex of degree 2.

A graph G is *planar* if $cr(G) = 0$. Kuratowski [Ku30] characterized planar graphs: a graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5 . (It is a consequence of this that these two graphs are the only crossing-critical graphs with crossing number one.) In other words, given a graph, one can determine whether or not it is planar, but if a graph is not planar, it is not an easy matter to determine what its crossing number is; computing crossing numbers is hard ([GJ83], [Tu70]).

1.2 Previous Results on Crossing Number and Contributions of This Thesis

In this section, we relate the results of this thesis to other work in crossing numbers.

Richter and Thomassen [RT93] proved that if k is an integer and G is such that, for all $e \in E(G)$, $cr(G - e) < k$, then $cr(G) \leq (5/2)k + 16$. This implies that if $\{G_n\}$ is an infinite collection of k -crossing-critical r -regular simple graphs, for some fixed k , then $r \in \{4, 5\}$. They gave an example with $r = 4$ and $k = 3$. Salazar [Sal00] has

observed that the same argument shows that the *average degree* of the members of an infinite collection of k -crossing-critical simple graphs, for some fixed k , must be in the open interval $(3, 6)$, and has given examples to show that every rational $p \in [4, 6)$ can be achieved as the average crossing number of such a collection.

Kochol [Ko87] gave the first example of an infinite family of simple, 3-connected graphs G_n , each 2-crossing-critical. Intriguingly the average degree of each graph in this sequence is 3.2. More generally, for every $k \geq 2$, he has exhibited an infinite family of k -crossing critical graphs; for fixed k , the average degree goes to 3.2 as n goes to infinity.

Coming from the other direction, Petr Hliněný [Hl01] has proved that if G is crossing-critical and $cr(G) = k$, then there is an $f(k)$ such that G has path-width at most $f(k)$. We do not require this concept here, but it does tell us that, roughly speaking, if G_n is an infinite sequence of crossing-critical graphs with bounded crossing numbers, then the G_n are “long and thin”, that is, they are made up of small pieces (not necessarily connected) that are glued together in a path-like fashion.

The existing examples and Hliněný’s Theorem all suggest that in looking for sequences of graphs with bounded crossing numbers, it is worthwhile to look at graphs that are made up from small pieces glued together in some repetitive pattern. Besides the examples above, many, if not all, other examples of known infinite families of graphs having the same crossing number are constructed this way. Also, many “standard” families such as the Generalized Petersen graphs ([EHK81], [Fi86], [MR92], [Sar97], [Sal97]) or the Cartesian products of two cycles ([AD01], [ARR96], [ARR97], [DR95], [HKS73], [KRS96], [RB78]) can be viewed this way. This is the main moti-

vation for the work of this thesis.

Chapter 3 introduces the concept of a *tile* which is the “small piece” that will be glued together to construct sequences of graphs. Tiles can be either planar or non-planar, and in Chapter 3 we focus on planar tiles. The idea is to glue together many copies of the same tile and some other tiles (which may be just a modification of the original tile), in a circular fashion, to create a graph whose crossing number can be computed by considering the properties of the tile.

We prove a general result about the crossing numbers of graphs obtained from a single planar tile and use this result to prove that the graphs in many of the known infinite families of crossing-critical graphs are indeed crossing-critical — making easier the hard part, which is to prove that the crossing number is what it is expected to be.

In order to make explicit the comparison between previous methods of computing crossing numbers and the new method, we use “old methods” in Chapter 2 to show that certain graphs M_n all have crossing number 6. The essential part of this is reproduced in much shorter fashion in Chapter 3 to show that, for n sufficiently large, M_n has crossing number 6. This gives a very clear idea of the flavour of the kinds of results we aim for here.

In Chapter 4 we focus on non-planar tiles. We use a quite similar construction of family of graphs as in Chapter 3, but now we only glue copies of the same tile together. We provide asymptotic estimates for the crossing numbers of the family of graphs obtained by gluing these copies in a circular fashion. We do this by analyzing a block of n copies of the tile glued together in a linear fashion and show that its

crossing number gives a good estimate, as $n \rightarrow \infty$, to the crossing number of the graph obtained by gluing n copies of the same tile in a circular fashion. We then apply this technique to the Generalized Petersen's graphs, $P(n, k)$, in particular to $P(n, 3)$, and to the Cartesian product $C_m \times C_n$ of two cycles.

Finally, in Chapter 5 we outline directions for future research and summarize some conjectures related to our work.

Chapter 2

The Crossing Number of M_n

2.1 Introduction

As a comparison to the new method that will be developed in the next chapter, for the computation of crossing number for infinite sequences of graphs, we use a standard method to determine the crossing number of a sequence $\{M_n\}$, $n \geq 5$ and odd, of 5-regular graphs.

In Section 2.2, we define the graphs M_n , and then we discuss some of the properties of these graphs in Section 2.3. Finally, in Section 2.4, we show that for odd $n \geq 5$, $cr(M_n) = 6$. We assume that $cr(M_5) = 6$ (which was computed using a computer program [Bu99]) and prove $cr(M_n) \geq 6$ by ensuring that for $n \geq 7$, we can find some edges in M_n without crossing, then we contract these edges and delete some other edges to get M_{n-2} with a number of crossings no more than M_n has. We prove $cr(M_n) \leq 6$ by showing a drawing of M_n with six crossings.

2.2 The graph M_n

Let $n \geq 5$. We define M_n to be the 5-regular graph consisting of $2n$ vertices $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ and containing the cycle $a_1a_2\dots a_na_1$, and $a_kb_k, b_kb_{k+2}, b_ka_{k+1}, a_kb_{k+1}$, for each k , where the indices are read modulo n , are edges of M_n .

We are only going to consider odd n . To keep the symmetry, we draw the cycle $a_1a_2\dots a_n$ of M_n as a regular n -gon on the outer boundary and draw each vertex b_k ($1 \leq k \leq n$), inside the n -gon in the corresponding position as a_k . We call this drawing *the regular drawing of M_n* . Figure 2.1 show the regular drawing of M_7 .

We call the vertices a_k (b_k respectively) of M_n *a-vertices* (*b-vertices*) and the edges whose ends are both *a-vertices* (*b-vertices*), *a-edges* (*b-edges*). Furthermore, an edge incident with an *a-vertex* and a *b-vertex* is an *ab-edge*, a path that consists only of *a-vertices* (*b-vertices*) is an *a-path* (*b-path*) and a cycle that consists only of *a-vertices* (*b-vertices*) an *a-cycle* (*b-cycle*).

Immediately, we have the following observation.

Observation 2.1. In M_n ,

- a) there is an *a-cycle* C_a and a *b-cycle* C_b consisting of all the *a-vertices* and all the *b-vertices* respectively;
- b) every *a-vertex* is in five triangles and every *b-vertex* is in four; and
- c) every *b-edge* is in one triangle whereas any other edge is in two triangles. $\square$

Figure 2.1: *Regular drawing of M_7*

2.3 Properties of M_n

In this section, we discuss some properties of M_n .

Lemma 2.2. *For $1 \leq i < j \leq n$ ($n \geq 5$), there are three internally-disjoint (disjoint, except for their ends) paths in M_n joining $\{a_i, b_i\}$ and $\{a_j, b_j\}$ consisting only of the vertices a_k and b_k ($i \leq k \leq j$) and three internally-disjoint paths joining $\{a_i, b_i\}$ and $\{a_j, b_j\}$ avoiding the vertices a_k and b_k ($i < k < j$).*

Proof (Lemma 2.2). By symmetry, it is enough to prove only the first statement. So let $1 \leq i < j \leq n$. If $j - i$ is odd, then there are three internally-disjoint paths P_1, P_2, P_3 joining $\{a_i, b_i\}$ and $\{a_j, b_j\}$, where P_1 is the a -path joining a_i and a_j consisting of a_k ($i \leq k \leq j$), P_2 is formed by the b -path joining b_i and b_{j-1} consisting of b_k ($i \leq k \leq j$ and $k = i + 2t$) together with the edge $b_{j-1}a_j$, and P_3 is formed by the edge a_ib_{i+1} together with the b -path joining b_{i+1} and b_j consisting of b_k ($i \leq k \leq j$ and $k = i + 1 + 2t$).

If $j - i$ is even, then we have three internally-disjoint paths P_1, P_2, P_3 , joining $\{a_i, b_i\}$ and $\{a_j, b_j\}$, where P_1 is the a -path joining a_i and a_j consisting of a_k ($i \leq k \leq j$), P_2 is formed by the b -path joining b_i and b_j consisting of b_k ($i \leq k \leq j$ and $k = i + 2t$), and P_3 is formed by the edge a_ib_{i+1} together with the b -path joining b_{i+1} and b_{j-1} consisting of b_k ($i \leq k \leq j$ and $k = i + 1 + 2t$) and the edge $b_{j-1}a_j$. $\square$

Lemma 2.3. *Let T be any triangle in M_n ($n \geq 5$). Then $M_n - T$ is 3-connected.*

Proof (Lemma 2.3). Let T be any triangle in M_n . Then T is formed either by two a -vertices and one b -vertex or by one a -vertex and two b -vertices.

If triangle T is formed by two a -vertices and one b -vertex, then we have two possibilities, namely T is formed either by a_i, b_i and a_{i+1} or by a_{i-1}, b_i and a_i . If T is formed by one a -vertex and two b -vertices, then we have only one possibility for T , that is, T is formed by b_{i-1}, a_i and b_{i+1} .

In any of these cases, let P_a and P_b be respectively the a -path and b -path in $M_n - T$. We note that every a -vertex (b -vertex, respectively) in $M_n - T$ is adjacent to a vertex in P_b (P_a). Hence, deleting any two a -vertices (b -vertices) will not disconnect $M_n - T$. Also, note that each end of P_a (P_b , respectively) is adjacent to an inner vertex of P_b (P_a), and hence deleting an end of P_a or an end of P_b and any other vertex, will not disconnect $M_n - T$.

Let a_p and b_q be inner vertices of P_a and P_b respectively. Note that, for every inner vertex a_p of P_a , the vertex b_p is adjacent to three a -vertices. If $p = q$, then a_{p-1} and a_{p+1} are respectively adjacent to b_{p-1} and b_{p+1} , and thus the two remaining parts of P_a are connected by the path $a_{p-1}b_{p-1}b_{p+1}a_{p+1}$, whereas the two remaining parts of P_b are still attached to the remaining parts of P_a since their ends are adjacent. Therefore $(M_n - T) - \{a_p, b_q\}$ is connected.

Otherwise if $p \neq q$, then the remaining parts of P_a are joined by the path $a_{p-1}b_p a_{p+1}$, whereas the remaining parts of P_b are still attached to the parts of P_a since their ends are adjacent. Therefore $(M_n - T) - \{a_p, b_q\}$ is connected.

Thus deleting at most two vertices does not disconnect $M_n - T$. However, deleting the three neighbors of any vertex of degree three does disconnect $M_n - T$. Thus $M_n - T$ is 3-connected. $\square$ (*End of Proof Lemma 2.3*)

A direct consequence of Lemma 2.3 and the Menger's theorem is the following.

Corollary 2.4. *Let $\mathcal{D}$ be a drawing of M_n . Let T be any triangle in M_n ($n \geq 5$). Let $u \in V(M_n)$ drawn inside T and $v \in V(M_n)$ drawn outside T . Then there are at least three internally-disjoint paths joining u and v crossing T in $\mathcal{D}$. $\square$*

Lemma 2.5. *Let $n \geq 7$. Let T be any triangle in M_n and $V(M_n - T)$ be partitioned into S_1 and S_2 , with $|S_1| = 2$, $|S_2| \geq 2$. Then in $M_n - T$, there are at least five edges joining S_1 and S_2 .*

Proof (Lemma 2.5). Let $u, v \in S_1$, $u \neq v$. If S_1 is not connected in $M_n - T$, then each of the vertices in S_1 is adjacent to at least three vertices of S_2 , by Lemma 2.3. Hence in $M_n - T$, there are at least six edges joining S_1 and S_2 .

If S_1 is connected in $M_n - T$, then the number of outgoing edges from S_1 to S_2 is $\deg(u) + \deg(v) - 2$. Since the minimum degree of a vertex is three but the vertices of degree three are not adjacent (each of them is adjacent to a vertex of degree four). Therefore the minimum outgoing edges is $3 + 4 - 2 = 5$. $\square$

A direct consequence of Lemma 2.5 is the following.

Corollary 2.6. *Let $\mathcal{D}$ be a drawing of M_n . Let T be any triangle in M_n ($n \geq 5$). Let $V(M_n - T)$ be partitioned into S_1 and S_2 , with $|S_1| = 2$ and $|S_2| \geq 2$, where S_1 and S_2 are in different faces of T . Then in $\mathcal{D}$, there are at least five edges joining S_1 and S_2 crossing T . $\square$*

Lemma 2.7. *Let $n \geq 7$. Let T be any triangle in M_n . Let $V(M_n - T)$ be partitioned into S_1 and S_2 , with $|S_1| \geq 3$ and $|S_2| \geq 3$. Then in $M_n - T$ there are at least six edges joining S_1 and S_2 .*

Proof (Lemma 2.7). If S_1 (or S_2) is not connected in $M_n - T$, then for any two vertices u and v from distinct components in S_1 , there are three internally-disjoint paths joining any of these vertices to a vertex in S_2 , by Lemma 2.3. Hence, there are six edges joining S_1 and S_2 . Therefore we consider the case that both S_1 and S_2 are connected in $M_n - T$.

Since $M_n - T$ is 3-connected, there are three totally disjoint paths from distinct vertices in S_1 to distinct vertices in S_2 . Let P_1, P_2, P_3 be three such paths and let x_1y_1, x_2y_2, x_3y_3 be the edges in, respectively P_1, P_2, P_3 , joining S_1 and S_2 , where $x_1, x_2, x_3 \in S_1$ and $y_1, y_2, y_3 \in S_2$.

We consider two cases regarding the triangle T :

Case 1: T is formed by two a -vertices and one b -vertex. Then we have two possibilities, namely T is formed either by a_i, b_i, a_{i+1} or by a_{i-1}, b_i, a_i . We are going to consider the first possibility; the second possibility follows by reflective symmetry $a_{i+j} \rightarrow a_{i-j}$. So, let T be the triangle formed by a_i, b_i, a_{i+1} . We observe the following.

Observation 2.8.

- a) In $M_n - T$, every b -edge, except $b_{i-1}b_{i+1}$, is in one triangle with an a -vertex. Every a -edge is in two triangles; each with a b -vertex and every ab -edge of type a_kb_k , except $a_{i-1}b_{i-1}$ and $a_{i+2}b_{i+2}$, is in two triangles, each with an a -vertex. Any other ab -edge, except $a_{i-1}b_{i-2}$ and $a_{i+2}b_{i+1}$, is in two triangles; one with an a -vertex and the other with a b -vertex. Furthermore, each of the edges $a_{i-1}b_{i-2}, a_{i-1}b_{i-1}, a_{i+2}b_{i+1}, a_{i+2}b_{i+2}$ is in one triangle.
- b) In $M_n - T$, there is an a -path P_a containing all the a -vertices and joining a_{i+2} and

a_{i-1} . Also, there is a b -path P_b containing all the b -vertices and joining b_{i+2} and b_{i-2} with $b_{i-1}b_{i+1}$ as an internal edge. Let Q_{b-} (Q_{b+} respectively) be the path in P_b joining b_{i-1} and b_{i+2} (joining b_{i+1} and b_{i-2}).

Now, we consider the cases whether or not $b_{i-1}b_{i+1}$ joins S_1 and S_2 .

Case 1.1: Suppose that $b_{i-1}b_{i+1}$ does not join S_1 and S_2 .

Then by Observation 2.8 (a), for every edge $x_k y_k$, $1 \leq k \leq 3$, there is a vertex z_k that forms a triangle with x_k and y_k . If there are no two of such edges $x_k y_k$'s that form a rectangle with single diagonal, then we have six edges joining S_1 and S_2 . Otherwise, consider x_1, y_1, x_2 and y_2 . Without loss of generality, let z_1 coincide with x_2 and let z_2 coincide with y_1 . So we obtain a rectangle R formed by x_1, x_2, y_1, y_2 , with single diagonal $y_1 x_2$. So, from this rectangle with single diagonal, we have three edges joining S_1 and S_2 .

Let j be an index that is increasing in clockwise direction, that is, a_{j+1} is closer to a_{i-1} in P_a than a_j is. Now, up to symmetry, we have the following three possible configurations for the rectangle R : (i) R is formed by a_j, a_{j+1}, b_j and b_{j+2} ($i+2 \leq j \leq i-3$) with $y_1 x_2 = b_j a_{j+1}$; (ii) R is formed by a_j, a_{j+1}, a_{j+2} and b_{j+1} ($i+2 \leq j \leq i-3$) with $y_1 x_2 = a_{j+1} b_{j+1}$, or; (iii) R is formed by a_j, b_j, a_{j+1} and b_{j+1} ($i+2 \leq j \leq i-2$) with $y_1 x_2 = a_j a_{j+1}$. We investigate these possibilities:

i) R is formed by a_j, a_{j+1}, b_j , and b_{j+2} ($i+2 \leq j \leq i-3$) with $y_1 x_2 = b_j a_{j+1}$ as diagonal. The possible configurations for x_2 and y_1 in R are either $x_1 = a_j$ and $y_2 = b_{j+2}$ or $x_1 = b_{j+2}$ and $y_2 = a_j$. We look at these possibilities:

a) The case that $x_1 = a_j, y_2 = b_{j+2}$ ($i+2 \leq j \leq i-3$) with $y_1 x_2 = b_j a_{j+1}$.

For $i + 3 \leq j \leq i - 3$, $x_1y_1 = a_jb_j$ is in another triangle with a_{j-1} and $x_2y_2 = a_{j+1}b_{j+2}$ is in another triangle with a_{j+2} . So, together with x_3y_3 we have six edges joining S_1 and S_2 .

For $j = i + 2$, we have $x_1y_1 = a_jb_j = a_{i+2}b_{i+2}$ and $x_2y_2 = a_{i+3}b_{i+4}$. Now, $x_2y_2 = a_{i+3}b_{i+4}$ is in another triangle with a_{i+4} . If $a_{i+4} \in S_1$, then a_{i+4} and b_{i+4} forms another triangle with a_{i+5} . Otherwise, if $a_{i+4} \in S_2$, then a_{i+3} and a_{i+4} forms another triangle with b_{i+3} . So in either case, we have now five edges joining S_1 and S_2 . Note that each of these edges has an a -vertex as its end. Since $b_{i-1}b_{i+1}$ does not join S_1 and S_2 and b_{i+1} is adjacent to a_{i+2} , if $b_{i-1}b_{i+1}$ is in S_2 , then $a_{i+2}b_{i+1}$ joins S_1 and S_2 . Therefore we assume that $b_{i-1}b_{i+1}$ is in S_1 . Since both b_{i+2} and b_{i-1} are ends of b -path Q_{b-} (described in Observation 2.8 (b)) and none of the five edges we have so far is a b -edge, there must be a b -path from b_{i-1} to b_{i+2} . So there are six edges joining S_1 and S_2 .

- b) If $x_1 = b_{j+2}$ and $y_2 = a_j$ ($i + 2 \leq j \leq i - 3$) with $y_1x_2 = b_ja_{j+1}$, then we have $x_1y_1 = b_{j+2}b_j$ and $x_2y_2 = a_{j+1}a_j$. By Observation 2.8 (b), $x_2y_2 = a_{j+1}a_j$ is in another triangle with b_{j+1} .

If $b_{j+1} \in S_1$, then a_jb_{j+1} is in another triangle with b_{j-1} . Since $b_{i-1}b_{i+1}$ does not join S_1 and S_2 , $b_{i-1}b_{i+1}$ must be in one of the two. If $b_{i-1}b_{i+1}$ is in S_1 , then, since $b_{j-1}b_{j+1}$ and b_jb_{j+2} are in different Q_b -paths (described in Observation 2.8 (b)), then there is a b -path from b_{i+1} either to b_{j-1} not containing b_{j+1} or to b_j not containing b_{j+2} , because in the appropriate Q_b

-path, b_{i+1} is closer to b_{j-1} and b_j than b_{j+1} and b_{j+2} respectively. Otherwise, if $b_{i-1}b_{i+1}$ is in S_2 , then, since $b_{j-1}b_{j+1}$ and b_jb_{j+2} are in different Q_b -paths, either there is a path from b_{i-1} either to b_{j+2} not containing b_j or to b_{j+1} not containing b_{j-1} , again because in the appropriate Q_b -path b_{i-1} is closer to b_{j+1} and b_{j+2} than b_{j-1} and b_j respectively. So we obtain six edges joining S_1 and S_2 .

Otherwise, if $b_{j+1} \in S_2$, then $a_{j+1}b_{j+1}$ is in another triangle with a_{j+2} . So now we have five edges joining S_1 and S_2 . Note that in the five edges we have so far, the vertices adjacent to b_{j+1} are a -vertices. So, if $b_{i-1}b_{i+1}$ is in S_1 , then, since b_{j+1} and b_jb_{j+2} are in different Q_b -paths (described in Observation 2.8 (b)) and every b -edge is in either in path Q_{b-} with end b_{i-1} or in path Q_{b+} with end b_{i+1} and since the vertices adjacent to b_{j+1} are a -vertices, there is a b -path from either b_{i+1} or b_{i-1} to b_{j+1} . Otherwise, if $b_{i-1}b_{i+1}$ is in S_2 , then, since $b_{j-1}b_{j+1}$ and b_jb_{j+2} are in different Q_b -paths described in Observation 2.8 (b), then there is a path from b_{i-1} either to b_{j+2} not containing b_j or to b_{j+1} not containing b_{j-1} , because in the appropriate Q_b -path, b_{i-1} is closer to b_{j+1} and b_{j+2} than b_{j-1} and b_j respectively. So we obtain six edges joining S_1 and S_2 .

- ii) If R is formed by a_j, a_{j+1}, a_{j+2} and b_{j+1} ($i+2 \leq j \leq i-3$) with $y_1x_2 = a_{j+1}b_{j+1}$, then the possible configurations for x_1 and y_2 in R are: $x_1 = a_j$ and $y_2 = a_{j+2}$ or $x_1 = a_{j+2}$ and $y_2 = a_j$. We look at these possibilities:

- a) $x_1 = a_j$ and $y_2 = a_{j+2}$ ($i+2 \leq j \leq i-3$) with $y_1x_2 = a_{j+1}b_{j+1}$.

For $i + 2 \leq j \leq i - 4$, $x_1y_1 = a_ja_{j+1}$ is in another triangle with b_j . On the other side, $x_2y_2 = b_{j+1}a_{j+2}$ is in another triangle with b_{j+3} . So, together with x_3y_3 , we have six edges joining S_1 and S_2 .

For $j = i - 3$, $x_1y_1 = a_ja_{j+1} = a_{i-3}a_{i-2}$ is in with another triangle with b_{i-3} . If $b_{i-3} \in S_2$, then a_{i-3} and b_{i-3} forms another triangle with a_{i-4} . So now we have five edges joining S_1 and S_2 . On the other side, we have $x_2y_2 = b_{j+1}a_{j+2} = b_{i-2}a_{i-1}$. Since b_{i-1} is adjacent to a_{i-1} , whenever $b_{i-1}b_{i+1}$ is in S_1 , $a_{i-1}b_{i-1}$ joins S_1 and S_2 , hence we have six edges joining S_1 and S_2 . Therefore we assume that $b_{i-1}b_{i+1}$ is in S_2 . Since b_{i-2} and $b_{i-1}b_{i+1}$ are in different sides and none of these edges is a b -edge and the fact that all b -vertices are in P_b , there must be a b -edge joining b_{i-2} and $b_{i-1}b_{i+1}$. So there are six edges joining S_1 and S_2 .

Otherwise, if $b_{i-3} \in S_1$, then a_{i-2} and b_{i-3} forms another triangle with b_{i-1} which is adjacent to b_{i+1} in S_2 (because $b_{i-1}b_{i+1}$ does not join S_1 and S_2). Since b_{i-2} and b_{i+1} are ends of the b -path Q_{b+} (described in Observation 2.8 (b)) not containing b_{i-1} and b_{i-3} , there must be a b -edge of Q_{b+} joining S_1 and S_2 . So, there are six edges joining S_1 and S_2 .

- b) If $x_1 = a_{j+2}$ and $y_2 = a_j$ ($i + 2 \leq j \leq i - 3$) with $y_1x_2 = a_{j+1}b_{j+1}$, then for $i + 2 \leq j \leq i - 3$, $x_1y_1 = a_{j+2}a_{j+1}$ is in another triangle with b_{j+2} . On the other side, $x_2y_2 = b_{j+1}a_j$ is in another triangle with b_{j-1} . So, together with x_3y_3 , we have six edges joining S_1 and S_2 .

- iii) If the rectangle R is formed by a_j , b_j , a_{j+1} and b_{j+1} ($i + 2 \leq j \leq i - 2$) with y_1x_2

$= a_j a_{j+1}$, then the possible configurations for x_1 and y_2 in R are either $x_1 = b_j$ and $y_2 = b_{j+1}$ or $x_1 = b_{j+1}$ and $y_2 = b_j$. We discuss these possibilities:

a) $x_1 = b_j$ and $y_2 = b_{j+1}$ ($i + 2 \leq j \leq i - 2$).

For $i + 3 \leq j \leq i - 3$, $x_1 y_1 = b_j a_j$ is in another triangle with a_{j-1} . On the other side, $x_2 y_2 = a_{j+1} b_{j+1}$ is in another triangle with a_{j+2} . So, together with $x_3 y_3$, we have six edges joining S_1 and S_2 .

For $j = i + 2$, we have $x_1 y_1 = b_j a_j = b_{i+2} a_{i+2}$ and $x_2 y_2 = a_{j+1} b_{j+1} = a_{i+3} b_{i+3}$. The edge $a_{i+3} b_{i+3}$ is in another triangle with a_{i+4} . If $a_{i+4} \in S_1$, then b_{i+3} and a_{i+4} forms another triangle with b_{i+5} . Otherwise, if $a_{i+4} \in S_2$, then a_{i+3} and a_{i+4} forms another triangle with b_{i+4} . So now we have five edges joining S_1 and S_2 .

Now, since $b_{i+1} b_{i-1}$ is in one side (either S_1 or S_2) and a_{i+2} is adjacent to b_{i-1} , if $b_{i+1} b_{i-1}$ is in S_1 , then $a_{i+2} b_{i-1}$ joins S_1 and S_2 and hence we have six edges joining S_1 and S_2 .

Whereas if $b_{i+1} b_{i-1}$ is in S_2 , then, since b_{i+2} and b_{i-1} are ends of the b -path Q_{b-} (described in Observation 2.8 (b)) and none of the five edges we have so far is an b -edge, there must be a b -edge of Q_{b-} joining S_1 and S_2 which is our sixth edge.

For $j = i - 2$, we have $x_1 y_1 = b_j a_j = b_{i-2} a_{i-2}$ and $x_2 y_2 = a_{j+1} b_{j+1} = a_{i-1} b_{i-1}$. The edge $x_1 y_1 = a_{i-2} b_{i-2}$ is in another triangle with a_{i-3} . If $a_{i-3} \in S_1$, then a_{i-2} and a_{i-3} forms another triangle with b_{i-3} . Otherwise, if $a_{i-3} \in S_2$, then

a_{i-3} and b_{i-2} forms another triangle with b_{i-4} . So now we have five edges joining S_1 and S_2 .

Now, since $b_{i+1}b_{i-1}$ is in one side and $y_2 = b_{i-1}$, it must in S_2 . Since $x_1 = b_{i-2}$ and b_{i+1} are ends of Q_{b+} and b_{i-2} is adjacent to an a -vertex in the five edges joining S_1 and S_2 we have so far, there must be a b -edge of Q_{b+} joining S_1 and S_2 .

b) $x_1 = b_{j+1}$ and $y_2 = b_j$ ($i + 2 \leq j \leq i - 2$).

For $i + 2 \leq j \leq i - 3$, $x_1y_1 = b_{j+1}a_j$ is in another triangle with b_{j-1} and $x_2y_2 = a_{j+1}b_j$ is in another triangle with b_{j+2} . So, together with x_3y_3 , we have six edges joining S_1 and S_2 .

For $j = i - 2$, we have $x_1y_1 = b_{j+1}a_j = b_{i-1}a_{i-2}$ and $x_2y_2 = a_{j+1}b_j = a_{i-1}b_{i-2}$. The edges $b_{i-1}a_{i-2}$ is in another triangle with b_{i-3} . If $b_{i-3} \in S_1$, then $a_{i-2}b_{i-3}$ is in another triangle with a_{i-3} which in its turns, forms another triangle with b_{i-3} and a_{i-4} and hence we obtain six edges joining S_1 and S_2 . Otherwise, if $b_{i-3} \in S_2$, then, since $x_1 = b_{i-1} \in S_1$, $b_{i+1}b_{i-1}$ must be in S_1 . Since the edge incident with b_{i-2} is not a b -edge and the fact that b_{i-2} and b_{i+1} are ends of Q_{b+} , there must be a b -edge e_b of Q_{b+} joining S_1 and S_2 . If $x_3y_3 \neq e_b$, then, together with x_3y_3 (x_1, x_2, x_3 are different vertices), we have six edges joining S_1 and S_2 . Otherwise, if $x_3y_3 = e_b$, then, by Observation 2.8 (b) and the fact that the ends of e_b can not coincide with x_1, y_1, x_2 , and y_3 , the edge e_b is in another triangle, again by Observation 2.8 (b). Therefore there are six edges S_1 and S_2 .

Case 1.2: Suppose that $b_{i-1}b_{i+1}$ joins S_1 and S_2 .

In all cases in Case 1.1 above, except for the case that R is formed by $x_1 = b_{i-1}$, $x_2 = a_{i-1}$, $y_1 = a_{i-2}$, $y_2 = b_{i-2}$ with $a_{i-1}a_{i-2}$ on diagonal where $b_{i-1}a_{i-2}$ is involved in another triangle with $b_{i-3} \in S_2$ (the last part of case iii.b in case 1.1 above), either x_3y_3 as the sixth edge is free of choice or required to be a b -edge. In either case we can take $x_3y_3 = b_{i-1}b_{i+1}$.

So now, let us consider this exception, that is, the case that R is formed by $x_1 = b_{i-1}$, $x_2 = a_{i-1}$, $y_1 = a_{i-2}$, $y_2 = b_{i-2}$ with $a_{i-1}a_{i-2}$ on diagonal where $b_{i-1}a_{i-2}$ is in another triangle with $b_{i-3} \in S_2$. So far, we have $x_1y_1 = b_{i-1}a_{i-2}$, $x_2y_2 = a_{i-1}b_{i-2}$, $y_1x_2 = a_{i-2}a_{i-1}$ and $b_{i-1}b_{i-3}$ joining S_1 and S_2 . Since $x_1 = b_{i-1} \in S_1$, b_{i+1} must be in S_2 . So now we have five edges joining S_1 and S_2 . But x_1 , x_2 and x_3 are different vertices, hence x_3y_3 is the sixth edge joining S_1 and S_2 .

Case 2: Suppose that T is formed by one a -vertex and two b -vertices. Then we have only one possibility, that is, T is formed by b_{i-1} , a_i and b_{i+1} . We then observe the following:

Observation 2.9.

- a) Every edge in $M_n - T$ is in a triangle; every a -edge except $a_{i-1}a_{i-2}$ and $a_{i+1}a_{i+2}$ are in two triangles and every ab -edge except $a_{i-1}b_i$, $a_{i+1}b_i$, $a_{i-2}b_{i-3}$ and $a_{i+2}b_{i+3}$ are in two triangles.
- b) In $M_n - T$, there is an a -path P_a joining a_{i+1} and a_{i-1} that contains all the a -vertices. Also, there is a b -path P_b joining b_{i+3} and b_{i-3} that contains all the

b -vertices. Let Q_{b-} (Q_{b+} respectively) be the path in P_b joining b_i and b_{i-3} (joining b_i and b_{i+3}).

By Observation 2.9 (a), for every edge $x_k y_k$, $1 \leq k \leq 3$, there is a vertex z_k that forms a triangle with x_k and y_k . If there is no two of such edges $x_k y_k$'s that forms a rectangle with single diagonal, then we have six edges joining S_1 and S_2 . Otherwise, consider x_1, y_1, x_2 and y_2 . Without loss of generality, let z_1 coincide with x_2 and let z_2 coincide with y_1 . So we obtain a rectangle R formed by x_1, x_2, y_1, y_2 with single diagonal $y_1 x_2$. So, from this rectangle with single diagonal, we have three edges joining S_1 and S_2 . Let j be an index that is increasing in clockwise direction, that is, a_{j+1} is closer to a_{i-1} in P_a than a_j is. Now, up to symmetry, we have the following three possible configurations for the rectangle R : (i) R is formed by a_j, a_{j+1}, b_j , and b_{j+2} ($i+2 \leq j \leq i-4$) with $y_1 x_2 = b_j a_{j+1}$; (ii) R is formed by a_j, a_{j+1}, a_{j+2} and b_{j+1} ($i+1 \leq j \leq i-3$) with $y_1 x_2 = a_{j+1} b_{j+1}$; and (iii) R is formed by $b_{j+1}, a_{j+1}, a_j, b_j$ ($i+2 \leq j \leq i-2$) with $y_1 x_2 = a_j a_{j+1}$. We investigate these possibilities:

i) If R is formed by a_j, a_{j+1}, b_j and b_{j+2} ($i+2 \leq j \leq i-4$) with $y_1 x_2 = b_j a_{j+1}$, then the possible configurations for x_1 and y_2 in R are either $x_1 = a_j$ and $y_2 = b_{j+2}$ or $x_1 = b_{j+2}$ and $y_2 = a_j$. We discuss these possibilities:

a) Suppose that $x_1 = a_j, y_2 = b_{j+2}$ ($i+2 \leq j \leq i-4$) with $y_1 x_2 = b_j a_{j+1}$.

Then for $i+2 \leq j \leq i-4$, $x_1 y_1 = a_j b_j$ is in another triangle with a_{j-1} and $x_2 y_2 = a_{j+1} b_{j+2}$ is in another triangle with an a_{j+2} . So, together with $x_3 y_3$ we have six edges joining S_1 and S_2 .

b) $x_1 = b_{j+2}$ and $y_2 = a_j$ ($i + 2 \leq j \leq i - 4$) with $y_1x_2 = b_ja_{j+1}$, that is, $x_1y_1 = b_{j+2}b_j$ and $x_2y_2 = a_{j+1}a_j$. By Observation 2.9 (a), $x_2y_2 = a_{j+1}a_j$ is in another triangle with b_{j+1} . We investigate the possibilities where b_{j+1} is located.

If $b_{j+1} \in S_1$, then a_jb_{j+1} is in another triangle with b_{j-1} . Note that $b_{j-1}b_{j+1}$ and b_jb_{j+2} are in different Q_b -paths (described in Observation 2.9 (b)). Now we look at where b_i is located. If $b_i \in S_1$ (S_2), then by Observation 2.9 (b) again, there must be a path from b_i either to b_{j-1} (b_{j+1} respectively) not containing b_{j+1} and b_jb_{j+2} (b_{j-1} and b_jb_{j+2}) or to b_j (b_{j+2}) not containing b_{j+2} and $b_{j-1}b_{j+1}$ (b_j and $b_{j-1}b_{j+1}$). So there are six edges joining S_1 and S_2 .

Otherwise, if $b_{j+1} \in S_2$, then $a_{j+1}b_{j+1}$ is in another triangle with a_{j+2} . So now we have five edges joining S_1 and S_2 . Note that in the five edges we have so far, the vertices adjacent to b_{j+1} are a -vertices. Therefore if $b_i \in S_1$ (S_2 respectively), then, by Observation 2.9 (a) and the fact that b_{j+1} (b_j) and b_jb_{j+2} ($b_{j-1}b_{j+1}$) are in different Q_b -paths described in Observation 2.8 (b), there must be a path from b_i to b_{j+1} (b_{j-1}) not containing b_jb_{j+2} ($b_{j-1}b_{j+1}$) and b_{j-1} (b_{j+2}). So there are six edges joining S_1 and S_2 .

ii) If R is formed by a_j, a_{j+1}, a_{j+2} and b_{j+1} ($i + 1 \leq j \leq i - 3$) with $y_1x_2 = a_{j+1}b_{j+1}$, then the possible configurations for x_1 and y_2 in R are either $x_1 = a_j$ and $y_2 = a_{j+2}$ or $x_1 = a_{j+2}$ and $y_2 = a_j$. We investigate these possibilities.

a) $x_1 = a_j$ and $y_2 = a_{j+2}$ ($i + 1 \leq j \leq i - 3$) with $y_1x_2 = a_{j+1}b_{j+1}$.

For $i + 2 \leq j \leq i - 3$, $x_1y_1 = a_ja_{j+1}$ is in another triangle with b_j and

$x_2y_2 = b_{j+1}a_{j+2}$ is in another triangle with b_{j+3} . So, together with x_3y_3 , we have six edges joining S_1 and S_2 .

For $j = i + 1$, we have $x_1y_1 = a_ja_{j+1} = a_{i+1}a_{i+2}$ and $x_2y_2 = b_{j+1}a_{j+2} = b_{i+2}a_{i+3}$. The edge $b_{i+2}a_{i+3}$ is in another triangle with b_{i+4} . Now we look at where b_{i+4} is located.

If $b_{i+4} \in S_1$, then b_{i+4} and a_{i+3} form another triangle with a_{i+4} , which in its turn forms another triangle together with b_{i+4} and a_{i+5} . So we have six edges joining S_1 and S_2 . Otherwise, if $b_{i+4} \in S_2$, then we consider the vertex b_i . If $b_i \in S_1$, then we have the path P_1 formed by Q_{b+} (described in Observation 2.8 (b)) together with the edge $b_{i+3}a_{i+2}$ joining b_i and a_{i+2} not containing b_{i+2} and a_{i+1} , and the path P_2 disjoint from P_1 (except for b_i) formed by the edge b_ia_{i-1} together with the part of P_a joining a_{i-1} and a_{i+3} not containing a_{i+1} . So there are six edges joining S_1 and S_2 . Otherwise, if $b_i \in S_2$, then there are six edges joining S_1 and S_2 , because b_i is adjacent to a_{i+1} and b_{i+2} .

b) $x_1 = a_{j+2}$ and $y_2 = a_j$ ($i + 1 \leq j \leq i - 3$) with $y_1x_2 = a_{j+1}b_{j+1}$.

For $i + 1 \leq j \leq i - 4$, $x_1y_1 = a_{j+2}a_{j+1}$ is in another triangle with b_{j+2} , and $x_2y_2 = b_{j+1}a_j$ is in another triangle with b_{j-1} . So, together with x_3y_3 , we have six edges joining S_1 and S_2 .

For $j = i - 3$, by using the mappings $a_{i+k} \rightarrow a_{i-k}$ and $b_{i+k} \rightarrow b_{i-k}$, we have the same situation as for $j + 1$ in case (a) above. So there are six edges joining S_1 and S_2 .

iii) If R is formed by $b_{j+1}, a_{j+1}, a_j, b_j$, ($i + 2 \leq j \leq i - 3$) with $y_1x_2 = a_ja_{j+1}$, then the possible configurations for x_1 and y_2 in R are either $x_1 = b_j$ and $y_2 = b_{j+1}$ or $x_1 = b_{j+1}$ and $y_2 = b_j$. We investigate these possibilities.

a) If $x_1 = b_j$ and $y_2 = b_{j+1}$ ($i + 2 \leq j \leq i - 3$) with $y_1x_2 = a_ja_{j+1}$, then for $i + 2 \leq j \leq i - 3$, $x_1y_1 = b_ja_j$ is in another triangle with a_{j-1} and $x_2y_2 = a_{j+1}b_{j+1}$ is in another triangle with a_{j+2} . So, together with x_3y_3 , we have six edges joining S_1 and S_2 .

b) $x_1 = b_{j+1}$ and $y_2 = b_j$ ($i + 2 \leq j \leq i - 3$) with $y_1x_2 = a_ja_{j+1}$.

For $i + 3 \leq j \leq i - 4$, $x_1y_1 = b_{j+1}a_j$ is in another triangle with b_{j-1} and $x_2y_2 = a_{j+1}b_j$ is in another triangle with b_{j+2} . So, together with x_3y_3 , we have six edges joining S_1 and S_2 .

We are going to consider only the case for $j = i - 3$ (the case for $j = i + 2$ follows, by reflective symmetry $a_{i-k} \rightarrow a_{i+k}$ and $b_{i-k} \rightarrow b_{i+k}$). For $j = i - 3$, we have $x_1y_1 = b_{j+1}a_j = b_{i-2}a_{i-3}$ and $x_2y_2 = a_{j+1}b_j = a_{i-2}b_{i-3}$. The edges $b_{i-2}a_{i-3}$ is involved in another triangle with b_{i-4} . We look at where b_{i-4} is located.

If $b_{i-4} \in S_1$, then $a_{i-3}b_{i-4}$ is in another triangle with a_{i-4} , which in its turn forms another triangle with b_{i-4} and a_{i-5} and hence we have six edges joining S_1 and S_2 .

Otherwise, if $b_{i-4} \in S_2$, then we look at the vertex b_i . If $b_i \in S_1$, then we have the path P_1 formed by the b -path Q_{b-} joining b_i and b_{i-3} not containing

b_{i-2} , and the path P_2 disjoint from P_1 (except for b_i) formed by the edge $b_i a_{i+1}$ together with the part of P_a joining a_{i+1} and a_{i-3} not containing a_{i-1} . So there are six edges joining S_1 and S_2 . Otherwise, if $b_i \in S_2$, then there are six edges joining S_1 and S_2 , because b_i is adjacent to b_{i-2} and the path b_i, a_{i-1}, a_{i-2} joins b_i and a_{i-2} . $\square$ (*End of Proof Lemma 2.7*)

As a direct consequence of Lemma 2.7 is the following.

Corollary 2.10. *Let $\mathcal{D}$ be a drawing of M_n . Let T be any triangle in M_n ($n \geq 7$). Let $V(M_n - T)$ be partitioned into S_1 and S_2 with $|S_1| \geq 3$ and $|S_2| \geq 3$, where S_1 and S_2 are in different faces of T . Then in $\mathcal{D}$, there are at least six edges joining S_1 and S_2 crossing T . $\square$*

Lemma 2.11. *Let $H = \{a_i, a_{i+1}, b_i, b_{i+1}\}$. Then $M_n - H$ is 3-connected.*

Proof (Lemma 2.11). Let P_a be the a -path in $M_n - H$ joining a_{i-1} and a_{i+2} . Let P_{b_1} be the b -path joining b_{i-1} and b_{i+2} , and let P_{b_2} be the b -path joining b_{i-2} and b_{i+3} in $M_n - H$. Note that the edges $a_{i-1}b_{i-1}$, $a_{i-1}b_{i-2}$, $a_{i+2}b_{i+2}$, $a_{i+2}b_{i+3}$ are in $M_n - H$, that is, the ends of each of P_{b_1} and P_{b_2} are adjacent to the ends of P_a . Furthermore, every a_k ($1 \leq k \leq n$ and $k \neq i, i+1$) is adjacent to two b -vertices, one of which is b_k , and every b_k ($1 \leq k \leq n$ and $k \neq i, i+1$) is adjacent to two a -vertices, one of which is a_k . It follows that deleting any two a -vertices or any two b -vertices will not disconnect $M_n - H$. Also, deleting any end of the paths and any other vertex will not disconnect $M_n - H$.

Let a_p be an inner vertex of P_a and b_q be an inner vertex of P_{b_1} or P_{b_2} . Note that every vertex a_p is adjacent to three b -vertices of which either one is in P_{b_1} and two are P_{b_2} or two are in P_{b_1} and one is in P_{b_2} .

If $p = q$, then the two remaining parts of P_a are joined by the path $a_{p-1}, b_{p-1}, b_{p+1}, a_{p+1}$ and the remaining parts of P_{b_1} and P_{b_2} are still attached to the parts of P_a , because their ends on each side are adjacent. Thus $(M_n - H) - \{a_p, b_q\}$ is connected.

Otherwise, if $p \neq q$ then the two remaining parts of P_a are connected by the path a_{p-1}, b_p, a_{p+1} . Since on each side, the ends of P_{b_1} and P_{b_2} are adjacent to the ends of P_a , deleting any internal vertex different from b_p in P_{b_1} (P_{b_2} respectively), the remaining parts of P_{b_1} (P_{b_2}) are still attached to the remaining parts of P_a . Therefore $(M_n - H) - \{a_p, b_q\}$ is connected.

Thus deleting at most two vertices will not disconnect $M_n - H$. However, deleting the three neighbors of any vertex of degree three will disconnect $M_n - H$. Thus $M_n - H$ is 3-connected. $\square$

A direct consequence of Lemma 2.11 is the following.

Corollary 2.12. *Let $\mathcal{D}$ be a drawing of M_n . Let C be the four-cycle formed by $a_i, b_i, a_{i+1}, b_{i+1}$ in M_n ($n \geq 5$). Let $u \in V(M_n)$ be a vertex drawn inside C and let $v \in V(M_n)$ be a vertex drawn outside C . Then there are at least three internally-disjoint paths joining u and v crossing C in $\mathcal{D}$. $\square$*

Lemma 2.13. *Let C be the four-cycle formed by $a_i, a_{i+1}, b_i, b_{i+1}$ in M_n ($n \geq 5$). Let $V(M_n - C)$ be partitioned into S_1 and S_2 with $|S_1| = 2$ and $|S_2| \geq 2$, where S_1 and*

S_2 are in different faces of C . Then there are at least four edges joining S_1 and S_2 crossing C .

Proof (Lemma 2.13). Let $u, v \in C$. If S_1 is not connected in $M_n - C$, then from each vertex in S_1 , there are three edges to a vertex in S_2 , by Lemma 2.11. Hence, there are at least six edges joining S_1 and S_2 .

If S_1 is connected in $M_n - C$, then the number of edges joining S_1 and S_2 is $\deg(u) + \deg(v) - 2$. Since the minimum degree of a vertex is three, the minimal number of edges joining S_1 and S_2 is $3 + 3 - 2 = 4$. $\square$

Lemma 2.14. Let $\mathcal{D}$ be a drawing of M_n . Let C be the four-cycle formed by $a_i, a_{i+1}, b_i, b_{i+1}$ (the edge $a_i a_{i+1}$ is drawn outside the C) in M_n ($n \geq 5$). Let $V(M_n - C)$ be partitioned into S_1 and S_2 with $|S_1| \geq 3$ and $|S_2| \geq 3$, where S_1 and S_2 are in different faces of C . Then there are at least five edges joining S_1 and S_2 crossing C in $\mathcal{D}$.

Proof (Lemma 2.14). We note that in $M_n - C$, every edge is in a triangle. Since $M_n - C$ is 3-connected by Lemma 2.11, there are three edges joining S_1 and S_2 . Since each edge is in a triangle, there are five edges joining S_1 and S_2 . $\square$

2.4 Crossing Number of M_n

In this section, we discuss the crossing number of M_n (odd $n \geq 5$). We assume that $cr(M_5) = 6$; this was computed using a computer program [Bu99]. First, we show that for all odd $n \geq 5$, $cr(M_n) \leq 6$. Then, we show that in a drawing of M_n , odd $n \geq 7$, there are some edges without crossings so that when we contract these edges and remove some vertices, we obtain M_{n-2} with $cr(M_{n-2}) \leq cr(M_n)$.

Lemma 2.15. *For all odd $n \geq 5$, $cr(M_n) \leq 6$.*

Proof (Lemma 2.15). Let $n \geq 5$ be odd. To prove this, we only need to show a drawing of M_n with 6 crossings. To get an optimal drawing, consider the regular drawing of M_n . We flip all the b_i 's where i is even, to outside the n -gon to obtain $(n-1)/2$ pentagons (each with a_i , i is even, in the middle of the pentagon) without crossings. Then we draw the edges $b_i b_{i+2}$'s (i is even and $2 \leq i \leq n-3$) in the outer face without crossing. Furthermore, draw the edge $a_1 b_n$ crossing the edge $b_1 a_n$, the edge $b_1 b_{n-1}$ crossing the edges $a_1 a_n$ and $a_1 b_n$, and the edge $b_n b_2$ crossing the edges $a_1 a_n$, $b_1 a_n$, and $b_1 b_{n-1}$. So we have six crossings. Figure 2.2 gives an example of an optimal drawing of M_7 . Therefore, for all odd $n \geq 5$, we have $cr(M_n) \leq 6$. $\square$

The rest of this section is to prove $cr(M_n) \geq 6$.

Lemma 2.16. *Let $n \geq 7$ be odd. Let $\mathcal{D}_n$ be an optimal drawing of M_n and let L_i be the subgraph of M_n induced by $\{a_i, a_{i+1}, a_{i-1}, b_i, b_{i-1}, b_{i+1}, b_{i-2}\}$. If the number of crossings involving edges of L_i is at most two, then L_i has to be drawn in the way shown in Figure 2.3, that is, there is no crossing among its edges and no vertex of M_n is inside any triangle of L_i .*

Figure 2.2: *Optimal drawing of M_7 .*

Figure 2.3: *The drawing of L_i*

Proof (Lemma 2.16). Suppose that the number of crossings involving edges of L_i is at most 2. Let us consider the subgraph L' of L_i induced by $\{a_i, a_{i+1}, a_{i-1}, b_i, b_{i-1}, b_{i+1}\}$. Then by the simplicity of the drawing (edges with a common end do not cross), the edges incident with a_i must be drawn in the “star” form.

Let $v \neq a_i$ be any vertex of L' . Then by simplicity of drawing, va_i can only cross an edge xy in L' , where x and y are not adjacent to v . Let $u \in L'$ be a vertex adjacent to v different from a_i . If va_i crosses xy , then u is either inside or outside the triangle T_1 formed by x, y and a_i . If u is inside T_1 , then, in $\mathcal{D}_n$, there are three paths from v to u crossing the boundary of T_1 , by Lemma 2.4, which is a contradiction. So u must be outside T_1 . Now the edge uv must not cross xy , for otherwise it must also cross a_ix or a_iy where we obtain three crossings which is a contradiction. But this implies that the triangle T_2 formed by v, u and a_i separates x from y , and hence there are three crossings on the boundary of T_2 which is also a contradiction. Thus, in L' , an edge incident with a_i does not cross an edge not incident with a_i .

If two non adjacent edges e_1 and e_2 not incident with a_i cross each other, then the ends of one edge must be separated by the triangle T formed by the ends of the other edge and a_i , and therefore we have three crossings on the boundary of T which is a contradiction. Thus non adjacent edges not involving a_i do not cross each other. Thus L' must be drawn as a pentagon without crossing with a_i inside the pentagon.

Next, the vertex b_{i-2} can not be drawn inside any triangle of L' , for otherwise there are three paths from b_{i-2} to a vertex of L' not involved in the triangle, which is a contradiction. So, b_{i-2} must be drawn in the outer face of L' . If the edge $b_{i-2}b_i$ or $b_{i-2}a_{i-1}$ crosses the pentagon, it must cross two different edges on the boundary

of the pentagon and thus it must also cross an edge incident with a_i where we have three crossings, which contradicts the assumption.

Thus, L_i has to be drawn in the way shown in Figure 2.4 without crossings among its edges and none of its vertices is inside any of its triangles.

Furthermore, it is clear that if any other vertex of M_n is inside any triangle of L_i , then the triangle will separate that vertex from a vertex of L_i not in that triangle and hence we have three crossings on the boundary of the triangle, by Lemma 2.4. Thus a contradiction. Therefore there can not be any other vertex of M_n inside L_i .

□ (end of Proof Lemma 2.16)

Lemma 2.17. *Let $n \geq 7$ be odd. Suppose that $cr(\mathcal{D}_n) \leq 5$ (Assumption 0). Then, there is an optimal drawing of M_n for which there exists i such that the subgraph L_i of M_n induced by $\{a_i, a_{i+1}, a_{i-1}, b_i, b_{i-1}, b_{i+1}, b_{i-2}\}$, has no crossings on the “red edges” $a_i a_{i+1}$, $a_i a_{i-1}$, $b_{i-1} b_{i+1}$, and $b_i b_{i-2}$.*

Proof (Lemma 2.17). Let $n \geq 7$ be odd. Consider the subgraph L_i ($1 \leq i \leq n$) of M_n induced by $\{a_i, a_{i+1}, a_{i-1}, b_i, b_{i-1}, b_{i+1}, b_{i-2}\}$. Then we have the following cases regarding the number of crossings over L_i :

Case 1: Suppose that for all i ($1 \leq i \leq n$), the number of crossings over L_i is at least 3. Since each edge of M_n is in two or three of such L_i 's, we have

$$cr(\mathcal{D}_n) \geq \frac{1}{3} \sum_{i=1}^n cr(L_i) \geq \frac{1}{3} \sum_{i=1}^n 3 = \frac{1}{3}(7 \times 3) = 7,$$

which contradicts Assumption 0.

Case 2: Suppose that there is some i ($1 \leq i \leq n$) such that the number of crossings over L_i is at most 2. Then by Lemma 2.16, L_i must be drawn in the way shown in

Figure 2.3 (the “red edges” are indicated with bold lines). Note that every red edge is a common edge of exactly two L_i 's. If $cr(L_i) = 0$, then we are done. So let us consider the case that the number of crossings of L_i is one or two.

Subcase 2.1: Suppose that the number of crossings of L_i is 1 and suppose that the crossing involves a red edge of L_i .

Since the edges of L_i can not cross each other (by Lemma 2.16), the single crossing on L_i must involve an edge e of M_n which is not in L_i . By Lemma 2.16, there can not be a vertex inside any triangle of L_i , and the fact that L_i consists of triangles, e must have an end in L_i and crosses only one edge of L_i on the outer boundary which is not incident with its end. This implies that the edge e must have either b_i as its end and crosses $a_{i-1}b_{i-2}$ or a_{i-1} as its end and crosses $b_{i-2}b_i$. Since each of b_i and a_{i-1} has degree four in L_i , we have only two possibilities left, namely the edge $a_{i-2}a_{i-1}$ crosses b_ib_{i-2} or the edge b_ib_{i+2} crosses the edge $a_{i-1}b_{i-2}$. But the second possibility does not involve any red edge of L_i , hence the only one possible crossing is when $a_{i-2}a_{i-1}$ crosses b_ib_{i-2} .

Let c be the crossing point between $a_{i-2}a_{i-1}$ and b_ib_{i-2} . Let T be the triangle formed by a_{i-1} , a_{i-2} , and b_{i-2} , and let f be the interior face with boundary a_{i-2} , b_{i-2} and the segment of b_ib_{i-2} between b_{i-2} and the crossing point c . Furthermore, let R be the cycle formed by a_{i-1} , b_{i-1} , a_{i-2} , b_{i-2} .

Assumption 1: Let us assume that $a_{i-1}a_{i-2}$ has only one crossing (in case that $a_{i-1}a_{i-2}$ has more crossings we can apply a similar argument).

Since this is an optimal drawing, there must be at least two edge-disjoint paths P_1 and P_2 preventing $a_{i-2}a_{i-1}$ to take the straight route (we mean the “straight route”

is putting the vertices a_{i-1} and a_{i-2} as close as possible and draw the edge $a_{i-2}a_{i-1}$ as a straight line). If there is only one path, then we could have chosen this route to obtain another optimal drawing with the same number of crossings, and make the edge b_ib_{i-2} free of crossings to obtain L_i with free of crossings. So we have the following possibilities for P_1 and P_2 .

- 1) P_1 has b_{i-2} as its end and crosses $a_{i-2}b_{i-1}$; P_2 has b_{i-1} as its end and crosses $a_{i-2}b_{i-2}$.
- 2) None of P_1 and P_2 has b_{i-2} or b_{i-1} as its end but they cross $a_{i-2}b_{i-1}$ and $a_{i-2}b_{i-2}$.
- 3) P_1 has b_{i-2} as its end and crosses $a_{i-2}b_{i-1}$; P_2 does not have b_{i-2} or b_{i-1} as its end but it crosses $a_{i-2}b_{i-1}$ and $a_{i-2}b_{i-2}$.
- 4) P_2 has b_{i-1} as its end and crosses $a_{i-2}b_{i-2}$; P_1 does not have b_{i-2} or b_{i-1} as its end but it crosses $a_{i-2}b_{i-1}$ and $a_{i-2}b_{i-2}$.
- 5) P_1 has both b_{i-1} and b_{i-2} as its ends; P_2 is either a path with b_{i-2} as one of its end crossing $a_{i-2}b_{i-1}$ or a path not involving b_{i-1} or b_{i-2} crossing $a_{i-2}b_{i-1}$ and $a_{i-2}b_{i-2}$.
- 6) Both paths are from b_{i-2} and cross $a_{i-2}b_{i-1}$.

For the first possibility, we now have the crossings c , P_1 crosses P_2 , P_1 crosses $a_{i-2}b_{i-1}$, and P_2 crosses $a_{i-2}b_{i-2}$. By Assumption 1, one end w of P_2 must be inside the face f and so inside the triangle T . By Lemma 2.4, there are two more paths from w crossing the boundary of T to another vertex outside the triangle. And so we have six crossings, which contradicts Assumption 0.

For the second possibility, we now have five crossings based on the fact that P_1 and P_2 both cross $a_{i-2}b_{i-1}$ and $a_{i-2}b_{i-2}$ and the crossing c . By Assumption 1, each of P_1 and P_2 must have one end inside the face f and so inside T . Therefore by Corollary 2.6 and Corollary 2.10, there must be at least three more crossings of the boundary of the triangle. So we have eight crossings, which violates Assumption 0.

For the third possibility, we now have the crossings c , P_1 crosses $a_{i-2}b_{i-1}$, P_2 crosses $a_{i-2}b_{i-1}$, and P_2 crosses $a_{i-2}b_{i-2}$. By Assumption 1, one end w of P_2 must be inside the face f and thus inside the triangle T . By Lemma 2.4, there must be two other paths from w crossing the boundary of T to another vertex outside T . So we have six crossings, which contradicts Assumption 0.

For the fourth possibility, we now have the crossings c , P_1 crosses $a_{i-2}b_{i-1}$, P_1 crosses $a_{i-2}b_{i-2}$, and P_2 crosses $a_{i-2}b_{i-2}$. By Assumption 1, one end w of P_2 must be inside the face f . Also, each of P_1 and P_2 must have one end inside T . By Corollary 2.6 and Corollary 2.10, there must be three more crossings of the boundary of the triangle to another vertex outside T . So we have seven crossings, which violates Assumption 0.

For the fifth possibility, since $b_{i-2}a_{i-2}b_{i-1}$ and $b_{i-2}a_{i-1}b_{i-1}$ are the only paths joining b_{i-2} and b_{i-1} with single inner vertex, P_1 consists of at least two inner-vertices. Thus the cycle R formed by a_{i-1} , b_{i-2} , a_{i-2} , b_{i-1} separates these vertices from the other vertices of L_i , and hence there are at least four edges joining these two groups of vertices crossing R , by Lemma 2.13. If P_2 has a vertex in R , then by Lemma 2.14 we have five edges crossing R and hence together with c we have six crossings, which is a contradiction. Otherwise, if P_2 has no vertex in R , then in the interior face of R ,

P_1 and P_2 are disjoint and hence the crossing made by P_2 is different from the four crossings made by the paths from the inner vertices of P_1 to vertices outside R . So, there are at least five crossings on the boundary of R and thus together with c we have six crossings, which violates Assumption 0.

So we have only the sixth possibility left which we are going to consider now. We recall that R is the rectangle formed by $a_{i-1}, b_{i-1}, a_{i-2}$, and b_{i-2} . Note that $a_{i-2}b_{i-1}$ is involved in another triangle with b_{i-3} . Suppose that the vertex b_{i-3} is inside R . By Corollary 2.12 there must be three internally-disjoint paths Q_1, Q_2, Q_3 from b_{i-3} to a vertex outside R crossing the boundary of R . Since b_{i-2} is not adjacent to b_{i-3} , we have the following possibilities:

- i) If there are at least three vertices inside R , then by Lemma 2.14 we have at least five crossings of R , which together with c make the total six crossings.
- ii) Suppose that there are two vertices inside R , that is, besides b_{i-3} , there is one more vertex x inside R which can be a vertex of P_1 or P_2 (adjacent to b_{i-2}). If x is not adjacent to b_{i-2} , then we have at least six crossings, namely, the crossing c , three crossings made by the Q 's and two crossings made by the P 's. Otherwise, suppose that x is adjacent to b_{i-2} , i.e., it is a vertex of one of the P 's, say of P_2 . Then P_1 is disjoint from the Q 's, which means that P_1 crosses the path $a_{i-2}, b_{i-3}, b_{i-1}$ and R . Together with the four crossings of R , by Lemma 2.13, and c , we have at least seven crossings.
- iii) If b_{i-3} is the only vertex inside R , then the P 's and Q 's have no common vertex inside R . This implies that both P_1 and P_2 will cross the path $a_{i-2}, b_{i-3}, b_{i-1}$ and

R , which gives us four crossings. Together with the three crossings made by the Q 's, gives us too many crossings.

Each of these cases leads to at least six crossings which is a contradiction. So we conclude that the vertex b_{i-3} must be outside R (see Figure 2.4).

Let T_1 be the triangle formed by b_{i-1} , a_{i-2} , and b_{i-3} . Note that the vertex b_{i-2} is adjacent to a_{i-3} and b_{i-4} . Furthermore, let P_1 be the path containing a_{i-3} and P_2 be the path containing b_{i-4} described earlier.

If a_{i-3} and b_{i-4} are both in R , then by Lemma 2.13 there are four paths in total from a_{i-3} and b_{i-4} crossing R to any two vertices outside R . Now we have five crossings together with c . Furthermore, since both P_1 and P_2 cross $a_{i-2}b_{i-1}$, either they must cross T_1 at least twice or the next vertices of these paths are located in T_1 . In either case, there must be more crossings of T_1 and so we have more than five crossings, which contradicts Assumption 0.

If one of the vertices a_{i-3} and b_{i-4} is inside R and the other is inside T_1 , then by Corollaries 2.4 and 2.12 each of them has three paths crossing each boundary that bounds them. But a_{i-3} and b_{i-4} are adjacent, so we have four crossings. Together with the crossing c , we have five crossings. Now, a_{i-3} and b_{i-4} must be the only vertices in R and T_1 respectively, for otherwise, we must have had more than five crossings, by Corollary 2.6, Corollary 2.10, Lemma 2.13, and Lemma 2.14. Thus, in this case we have L_{i+1} with no crossings on its red edges.

If both a_{i-3} and b_{i-4} are inside T_1 , then by Corollaries 2.6 and 2.10, there must be five crossings on the boundary of T_1 , and so, together with c , we have six crossings, which violates Assumption 0.

Figure 2.4: *The possible one crossing on L_i and the position of b_{i-3}*

If one of the vertices a_{i-3} and b_{i-4} is inside T_1 and the other is outside T_1 (and also outside R and f), then the one inside T_1 has two more paths to any vertex outside T_1 . Whereas the one outside T_1 (and outside R and f), has crossed T_1 twice. Thus, together with c , we have five crossings. Now, there can be only one vertex in T_1 , for otherwise we must have had more crossings, by Corollaries 2.6 and 2.10. In this case we have L_{i+1} with no crossings on its red edges.

If both a_{i-3} and b_{i-4} are outside T_1 (and also outside R and f), then we have four crossings on the boundary of T_1 and together with the crossing c , we have five crossings in total. So we have L_{i+1} with no crossings on its red edges.

Thus we conclude that if there is one crossing on L_i , then there exists L_j for some $j \neq i$, with no crossings of its red edges.

Subcase 2.2: Suppose that the number of crossings of L_i is 2.

Let c_1 and c_2 be the points where the crossings happen. Since the edges of L_i can not cross each other (by Lemma 2.16), the two crossings on L_i must involve edges not in L_i . It follows that each of these edges must cross the outer boundary of L_i . Therefore these two crossings must be formed either by a combination of two edges not in L_i each with a single crossing on L_i or by an edge not in L_i with two crossings on different edges of L_i . In Subcase 2.1 above we saw that there are only two possibilities for the single crossing on L_i , namely $a_{i-2}a_{i-1}$ crosses $b_i b_{i-2}$ and $b_i b_{i+2}$ crosses $a_{i-1} b_{i-2}$. But a combination of these two possibilities will result in three crossings because they also cross each other inside the triangle b_i, a_{i-1}, b_{i-2} , and hence we have a contradiction. Thus these two crossings must be made by exactly one edge $e = uv$ of M_n which is not in L_i .

The edge e must cross the outer boundary of L_i anyway. If e crosses $b_{i-2}b_i$, then by Lemma 2.16 (no vertex inside the triangle of L_i) and the simplicity of the drawing, it can not cross b_ia_{i-1} (the edge of pentagon which is not on the outer boundary of L_i), and hence it must cross $b_{i-2}a_{i-1}$ (and vice versa). This also means that either $u, v \notin L_i$ or one of the ends of e , say $u \in \{b_{i-1}, b_{i+1}, a_{i+1}\}$.

Otherwise, if e crosses an edge $e_1 = xy$ of the pentagon on the outer boundary of L_i , then by the simplicity of the drawing, it can not cross e_1 twice, and hence it must cross either a_ix or a_iy , but these must be all the crossings on L_i . It follows that e can never cross b_ia_{i-1} . This also implies that if e crosses a_ix (a_iy respectively), one end of e , say u , must be the other neighbor of x (y respectively) in L_i different from a_i and y (a_i and x). Consequently, the other end v of e can not be in L_i .

Summarizing, we have the following possibilities:

- a) $u, v \notin L_i$.
- b) $u \in L_i$ and $v \notin L_i$.

We investigate these possibilities:

Subcase 2.2.1: the edge uv ($u, v \notin L_i$) crosses $a_{i-1}b_{i-2}$ and b_ib_{i-2} (see Figure 2.5(a)). Since we consider an optimal drawing, there must be at least three edge-disjoint paths preventing uv from taking other route, that is, there are at least three edge-disjoint paths not crossing uv and not including the edges $a_{i-1}b_{i-2}$ and b_ib_{i-2} from b_{i-2} to the set vertices of the pentagon. This implies that the other three edges of b_{i-2} not in L_i must be part of these paths. Let u be the end of e that is in the side of $a_{i-1}b_{i-2}$ and v be in the other side (the side of b_ib_{i-2}). We consider the location of u .

If u is in the triangle b_{i-2}, a_{i-1} and a_{i-2} , then by Corollary 2.4 there must be two other internally-disjoint paths joining u and v crossing the boundary of the triangle. So we have now four crossings. Suppose that a fifth crossing is in L_{i+1} . Then by Subcase 2.1 above the only possible crossing that involves a red edge of L_{i+1} is the crossing performed by $a_{i-1}a_i$ on $b_{i-1}b_{i+1}$. But these two edges are free of crossing, and hence the fifth crossing can not be on L_{i+1} . Thus we have L_{i+1} with no crossings on its red edges.

If u is in the triangle $b_{i-1}, a_{i-1}, a_{i-2}$, then we have now three crossings, and by Corollary 2.4, there must be two other internally-disjoint paths crossing the boundary of the triangle $b_{i-1}, a_{i-1}, a_{i-2}$. Now u must be the only vertex inside the triangle $b_{i-1}, a_{i-1}, a_{i-2}$, for otherwise, by Corollaries 2.6 and 2.10, we have more than five crossings, which is a contradiction of Assumption 0. Thus we have L_{i+1} with no crossings on its red edges.

If u is outside the triangles $b_{i-1}, a_{i-1}, a_{i-2}$ and $b_{i-2}, a_{i-1}, a_{i-2}$, then we have now four crossings, namely c_1, c_2, e crosses both $a_{i-1}a_{i-2}$ and $b_{i-1}a_{i-2}$. Suppose that the fifth crossing is in L_{i+1} . Then by Subcase 2.1 of this proof, the only possible crossing that involves a red edge of L_{i+1} is the crossing by $a_{i-1}a_i$ of $b_{i-1}b_{i+1}$. But these two edges are free of crossing, and hence the fifth crossing can not be on L_{i+1} . Thus we have L_{i+1} with no crossings on its red edges.

(a). Case 2.2.1

(b). Case 2.2.1.a

(c). Case 2.2.2.b.i

(d). Case 2.2.2.b.ii

(e). Case 2.2.2.c

Figure 2.5: The possible configurations used in Case 2.2 of Proof Lemma 2.17

Subcase 2.2.2: $u \in L_i$ and $v \notin L_i$.

From the analysis above, we see that the edge $e = uv$ must cross $a_i x$ and another edge $e_1 = xy \neq a_{i-1}b_{i-2}$ of the pentagon whose one end, say x , is a neighbor of u . Let z be the other neighbor of $u \in L_i$ different from x and a_i . Then we have the cases that v is adjacent to x , or v is adjacent to z , or v is adjacent to neither x nor z . We investigate these cases.

a) v is adjacent to x . Let T be the triangle formed by u , v and x . Without loss of generality, we assume a_i is in the outer face of T . With the condition that v is adjacent to x and by considering only the crossings that involve the red edges, we have the following possibilities:

- $a_{i+1}a_{i+2}$ crosses $a_i b_{i+1}$ and $b_{i-1}b_{i+1}$ ($uv = a_{i+1}a_{i+2}$ and $x = b_{i+1}$);
- $b_{i+1}a_{i+2}$ crosses $a_i a_{i+1}$ and $b_i a_{i+1}$ ($uv = b_{i+1}a_{i+2}$ and $x = a_{i+1}$);
- $b_i b_{i+2}$ crosses $a_i a_{i+1}$ and $b_{i+1}a_{i+1}$ ($uv = b_i b_{i+2}$ and $x = a_{i+1}$); and
- $a_{i-1}a_{i-2}$ crosses $a_i b_{i-1}$ and $b_{i-1}b_{i+1}$ ($uv = a_{i-1}a_{i-2}$ and $x = b_{i-1}$).

Without loss of generality, we consider the crossings made by $a_{i+1}a_{i+2}$ on $a_i b_{i+1}$ and $b_{i-1}b_{i+1}$ (we apply a similar argument to the other possibilities). So T is formed by a_{i+1} , a_{i+2} and b_{i+1} (see Figure 2.5 (b)). To prevent $a_{i+1}a_{i+2}$ from taking the straight route (indicated with dashed line in Figure 2.5 (b)), there must be three edge-disjoint paths not containing the edges of T and the edges of L_i , from any of the vertices of L_i (except a_{i+1} and b_{i+1}) crossing $b_{i+1}a_{i+2}$. This implies that the other ends of these paths must be inside T . From Corollaries 2.4, 2.6 and 2.10, and Assumption 0, it follows that there can only one vertex w inside T and thus

w must be the common end of these paths. Moreover, since every vertex in M_n is of degree five and we have already two crossings on L_i , the crossings on $b_{i+1}a_{i+2}$ must be exactly three. Therefore, w must be adjacent to both b_{i+1} and a_{i+2} , that is, $w = b_{i+3}$. Thus the crossings involve the edges $a_{i+1}a_{i+2}$, a_ib_{i+1} , $b_{i-1}b_{i+1}$, and three edges incident with b_{i+3} . But none of these edges are in L_{i-1} . Therefore, L_{i-1} has no crossings on its red edges.

b) v is adjacent to z .

Let T be the triangle formed by u , v , and z . Without loss of generality, we assume a_i is at the outer face of T . By considering only the crossings that involve the red edges, we have the two possibilities, namely:

- $a_{i+1}b_{i+2}$ crosses a_ib_{i+1} and $b_{i-1}b_{i+1}$ ($uv = a_{i+1}b_{i+2}$ and $z = b_i$); and
- b_ib_{i+2} crosses a_ia_{i-1} and $a_{i-1}b_{i-1}$ ($uv = b_ib_{i+2}$ and $z = a_{i+1}$).

We investigate these possibilities.

i) $a_{i+1}b_{i+2}$ crosses a_ib_{i+1} and $b_{i-1}b_{i+1}$ ($uv = a_{i+1}b_{i+2}$ and $z = b_i$).

So T is formed by a_{i+1} , b_{i+2} and b_i (see Figure 2.5(c)).

To prevent $a_{i+1}b_{i+2}$ from taking the straight route (indicated with dashed line in Figure 2.5 (c)), there must be three edge-disjoint paths not containing the edges of T and the edges of L_i , from b_{i+1} to any of the vertices of L_i (except a_{i+1} , b_{i+1} and b_i) crossing b_ib_{i+2} . But this is impossible, because b_{i+1} has already three edges in L_i . Thus b_ib_{i+2} has an alternative route with the same number of crossings as the current route. So we can choose the straight route and obtain L_i with no crossings on its edges.

ii) $b_i b_{i+2}$ crosses $a_i a_{i-1}$ and $a_{i-1} b_{i-1}$ ($uv = b_i b_{i+2}$ and $z = a_{i+1}$)

So T is formed by a_{i+1} , b_{i+2} and b_i (see Figure 2.5(d)).

To prevent $b_i b_{i+2}$ from taking the straight route (indicated with dashed line in Figure 2.5(d)), there must be three edge-disjoint paths not containing the edges of T and the edges of L_i , from a_{i-1} and b_{i-2} crossing $b_i b_{i+2}$. So now we have two vertices inside T . By Corollary 2.10 and the fact that there are three vertices of L_i outside T , there can not be more vertices inside T than these two vertices a_{i-1} and b_{i-2} , for otherwise we would have six crossings. But since all the other edges of a_{i-1} and b_{i-2} not in L_i must cross $b_i b_{i+2}$, this is a contradiction, because there are in total four edges of a_{i-1} and b_{i-2} not in L_i , and thus we have six crossings, which violates Assumption 0.

c) v is adjacent neither to x nor z .

By considering only the crossings that involve the red edges, we have only one possibility left, namely the crossings made by $b_{i+1} b_{i+3}$ crosses $a_i a_{i+1}$ and $a_{i+1} b_i$.

Note that b_{i+1} and b_{i+3} forms a triangle T with a_{i+2} . If the straight route is inside this triangle, there must be three edge-disjoint paths from a_{i+1} crossing the path $b_{i+1}, a_{i+2}, b_{i+3}$. But since a_{i+1} has already three edges in L_i , this is impossible. So the straight route must be outside T .

By Corollary 2.10, there can not be more than two vertices inside T . If there is another vertex w inside T except a_{i+1} , then by Corollary 2.6 we have five crossings only when w is adjacent to a vertex of T . But then we have L_{i-1} free of crossings.

Otherwise if a_{i+1} is the only vertex inside T (see Figure 2.5(e)), then we have an

alternative route which is cheaper, that is, $b_{i+1}b_{i+3}$ crosses $a_{i+1}a_{i+2}$. But this is a contradiction with the fact that $\mathcal{D}_n$ is an optimal drawing.

Thus we conclude that if there are two crossings on L_i , then there must exist three other crossings in $\mathcal{D}_n$ and that we are able to find L_j , for some $j \neq i$, with no crossings on its red edges. $\square$ (*End of Proof Lemma 2.17*)

Theorem 2.18. *For all odd $n \geq 5$, $cr(M_n) = 6$.*

Proof (*Theorem 2.18*). We prove that for all odd $n \geq 5$, $cr(M_n) \geq 6$ and do this by induction to n . Recall that we have assumed that $cr(M_5) = 6$; this is the base case. As the induction hypothesis, we assume that for odd $k < n$, $cr(M_k) \geq 6$.

Suppose that $cr(M_n) \leq 5$. Then by Lemma 2.17, there is an optimal drawing of M_n for which there exists some i ($1 \leq i \leq n$) such that the subgraph L_i of M_n , induced by $\{a_i, a_{i+1}, a_{i-1}, b_i, b_{i-1}, b_{i+1}, b_{i-2}\}$, has no crossings on the “red edges” $a_i a_{i+1}$, $a_i a_{i-1}$, $b_{i-1} b_{i+1}$, and $b_i b_{i-2}$. Now we contract these edges and delete other edges of L_i to get M_{n-2} with $cr(M_{n-2}) \leq cr(M_n)$. Hence, by induction hypothesis, we have $6 \leq cr(M_{n-2}) \leq cr(M_n)$, which contradicts the assumption that $cr(M_n) \leq 5$. Thus $cr(M_n) \geq 6$ and together with Lemma 2.15, we have $cr(M_n) = 6$. $\square$

Chapter 3

Sequences of Graphs with Constant Crossing Number

3.1 Introduction

The main point of this thesis is to consider the crossing numbers of graphs made up by gluing side by side many copies of the same small graph with the last copy glued to the first. As mentioned before, most, if not all, examples of known infinite families of graphs having the same crossing number are constructed this way.

We start in Section 3.2 with the definition of a *tile* (the basic building block) and related notions. Section 3.3 provides some operations on tiles and Section 3.4 gives notation for graphs built from these basic blocks.

The main work of this chapter is in Section 3.5, which determines the crossing

number of the “circular” graph described above in terms of properties of the building block, when the building block is planar. Thus, for a large number of copies of the tile at least, the crossing number of the large graph constructed by gluing can be determined (and is constant from some point on). In Section 3.6 we give a simple sufficient condition to compute the crossing number of the circular graphs satisfying the condition and in Section 3.7 we look at some variants of circular graphs.

3.2 Definition of Tile

In this section we give the definition of *tile* and related notions. A tile, being a graph with additional conditions, will be the basic building element of the circular graphs.

First, a *list* L over a set S is the sequence $L[1], L[2], \dots$, where $L[i] \in S$, for all $1 \leq i$. For a list L , we denote the length of L by $|L|$, i.e., if L has length t , then $|L| = t$. A list is *finite* if $|L| < \infty$. If L is a finite list with length t over a set S , then the *reverse list* of L , denoted by $L^\leftarrow$, is the list of length t over S where $L^\leftarrow[i] = L[t - i + 1]$, for all $1 \leq i \leq t$.

Definition 3.1 (*Tile*).

A *tile* is a 3-tuple $T = (G, L, R)$, where

- G is a connected graph;
- L is a finite list over $V(G)$, called the *left-wall*;
- R is a finite list over $V(G)$, called the *right-wall*.
- The vertices in L and R are all distinct. $\square$

For a tile $T = (G, L, R)$, we call a vertex in L (R , respectively) a *left- (right-) wall vertex* and an edge whose ends are both in L (R), a *left- (right-) wall edge*.

Remark: We use the “[.]” notation for the indices of list and the subscript notation for the indices of different tiles.

Definition 3.2 (Tile Crossing Number).

The *tile crossing number* $tcr(T)$ of a tile $T = (G, L, R)$ is the smallest number of crossings in any drawing of T in the unit square $S = \{(x, y) : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}$ such that the only vertices on the boundary $\{(x, y) \in S : \text{either } x \in \{-1, 1\} \text{ or } y \in \{-1, 1\}\}$ are in $L \cup R$, and the vertices in L occur in the line $x = -1$, with the y -coordinates of $L[1], L[2], \dots, L[|L|]$ decreasing and the vertices in R occur in the line $x = 1$, with the y -coordinates of $R[1], R[2], \dots, R[|R|]$ also decreasing. $\square$

Definition 3.3 (Planar tile, Perfect tile).

A tile $T = (G, L, R)$ is *planar* if $tcr(T) = 0$. A tile is *perfect* if

- (i) $G - L$ and $G - R$ are both connected;
- (ii) for every left- (right-, respectively) wall vertex v , there is a path to the right- (left-) wall disjoint from the left- (right-) wall except for v ;
- (iii) for every pair i and j , $i \neq j$, there are two disjoint paths, one joins $L[i]$ and $R[i]$, and the other joins $L[j]$ and $R[j]$. $\square$

Definition 3.4 (*Tile compatibility*).

A tile $T_1 = (G_1, L_1, R_1)$ is *compatible* with a tile $T_2 = (G_2, L_2, R_2)$ if the function $f : R_1 \rightarrow L_2$ defined by $f(R_1[i]) = L_2[i]$ is a bijection and

$$\forall v_1, v_2 \in R_1, v_1 v_2 \in E(G_1) \Leftrightarrow f(v_1) f(v_2) \in E(G_2).$$

A tile is *self-compatible* if it is compatible to itself. $\square$

Note that the compatibility refers to ordered pairs of tiles, that is, if T_1 is compatible with T_2 , T_2 does not need to be compatible with T_1 .

Example 3.5. Consider the graphs in Figure 3.1.

- $\mathbb{L} = (G_1, (a, b), (e, f))$ is a planar, perfect and self-compatible tile; this is the *Ladder tile*.
- $\mathbb{B} = (G_2, (a, b), (d, e))$ is a planar, perfect and self-compatible tile; this is the *Boutie tile*.
- $\mathbb{M} = (G_3, (a, b, c), (d, e, f))$ is a planar, but neither perfect ($G_3 - \{a, b, c\}$ is disconnected) nor self-compatible ($ab \notin E(G_3)$, $de \in E(G_3)$); this is the *M-tile*.
- $\mathbb{Z} = (G_4, (a, b, c), (d, e, f))$ is a planar and self-compatible tile, but not perfect ($G_4 - \{a, b, c\}$ is disconnected); this is the *Z-tile*.
- $\mathbb{K} = (G_5, (a, b, c), (i, j, k))$ is a planar, perfect, and self-compatible tile; this is the *Kochol tile*.
- $\mathbb{H} = (G_6, (a, b), (c, d))$ is self-compatible, but neither perfect ($G_6 - \{a, b\}$ is disconnected) nor planar (it has a crossing); this is the *Hour Glass tile*. $\square$

Figure 3.1: *Examples of (non) planar, (non) perfect, (non) self-compatible tiles.*

3.3 Operations on tiles

In this section we define some operations on tiles and discuss properties of the results obtained by applying these operations on tiles. We start with two auxiliary operations on general graphs.

Notation 3.6 ($G\{v_1, v_2\}$).

Let G be a graph. Let $v_1, v_2 \in V(G)$ and $v \notin V(G)$. We define $G\{v_1, v_2\}$ to be the graph whose vertex set is

$$(V(G) \setminus \{v_1, v_2\}) \cup \{v\}$$

and whose edge set is

$$(E(G) \setminus (\{v_1w \in E(G)\} \cup \{v_2w \in E(G)\})) \cup$$

$$\{vw : \text{either } v_1w \in E(G) \text{ or } v_2w \in E(G)\}. \quad \square$$

Intuitively, the operation $G\{v_1, v_2\}$ identifies the vertices v_1 and v_2 of G into a new vertex v . Note that if $v_1v_2 \in E(G)$, then $G\{v_1, v_2\}$ is simply the graph obtained by the contraction of that edge. The following lemma indicates that the order we identify the vertices is not important.

Lemma 3.7. *Let G be a graph and u_1, v_1, u_2, v_2 be distinct vertices in G . Let $G_1 = G\{u_1, v_1\}$ and $G_2 = G\{u_2, v_2\}$. Then $G_1\{u_2, v_2\} = G_2\{u_1, v_1\}$.*

Proof (Lemma 3.7). We add the edges u_1v_1 and u_2v_2 to G (if they are not present) and then contract these two edges. Note that contracting edges is independent of the order in which we proceed. $\square$

Notation 3.8 $((G \cup H)\{X, Y\})$.

Let G and H be (disjoint) graphs and let X and Y be lists of distinct vertices of G and H , respectively, having the same length t . Then $(G \cup H)\{X, Y\}$ is the graph $(\dots(((G \cup H)\{X[1], Y[1]\})\{X[2], Y[2]\})\dots)\{X[t], Y[t]\}$. $\square$

Definition 3.9 (*Glue*).

Let $S = (G, L, Q)$ and $T = (H, M, R)$ be compatible tiles. Then the *glue* of S and T , denoted by ST , is the tile $((G \cup H)\{Q, M\}, L, R)$. If $T_1, T_2, \dots, T_k$ are tiles such that, for each $i = 2, 3, \dots, k$, T_{i-1} is compatible with T_i , then $T_1 T_2 \dots T_k$ is the tile $((\dots(T_1 T_2) T_3) \dots) T_k$. $\square$

Example 3.10. Figure 3.2 shows the result $ST = (G, L, R)$ of gluing the tiles $S = (G_1, L, (u_1, u_2, u_3))$ and $T = (G_2, (v_1, v_2, v_3), R)$. $\square$

Definition 3.11 (*Close*).

Let $T = (G, L, R)$ be a self-compatible tile. The *close* of T is the graph $\circ(T) = G\{L, R\}$. $\square$

If $T = T_1 T_2 \dots T_k$ is a tile, where $T_i = (G_i, L_i, R_i)$, $1 \leq i \leq k$, are such that T_j is compatible with T_{j+1} , $1 \leq j \leq k-1$, then $\circ(T)$ is the graph obtained from $T_1 T_2 \dots T_k$ by identifying L_1 and R_k . Note that $\circ(T)$ is a graph and not a tile.

Lemma 3.12. *If T_1 is compatible with T_2 , then $\text{tcr}(T_1 T_2) \leq \text{tcr}(T_1) + \text{tcr}(T_2)$ and if T is self-compatible, then $\text{cr}(\circ(T)) \leq \text{tcr}(T)$.*

Proof (Lemma 3.12). Let $T_1 = (G_1, L_1, R_1)$ and $T_2 = (G_2, L_2, R_2)$. By the definition of tile crossing number, the vertices of R_1 and L_2 are on the outer boundary of T_1

Figure 3.2: *Gluing tiles S and T .*

and T_2 respectively. Since T_1 is compatible with T_2 , for every $1 \leq i \leq |R_1|$, we can add the edge $R_1[i]L_2[i]$ drawn on the outer face of T_1 and T_2 to obtain the tile T' . Since the order of the vertices R_1 corresponds to the order of the vertices of L_2 , these edges are free of crossings. Therefore $tc_r(T') \leq tc_r(T_1) + tc_r(T_2)$. By contracting these edges we obtain the desired tile T with number of crossing no more than T' has and thus $tc_r(T) \leq tc_r(T_1) + tc_r(T_2)$.

Using a similar argument as above with R and L instead of R_1 and L_2 respectively, we have $cr(\circ(T)) \leq tc_r(T)$. $\square$

3.4 Construction of the Sequences

In addition to $\circ(.)$ from Definition 3.11, we define explicitly some notation for the “circular” graphs built by tiles and state some consequences of Lemma 3.12 with regard to these notions.

Definition 3.13 ($\tilde{T}$, T^n , $T^\circ(n)$).

Let $T = (G, L, R)$ be a tile. We define

- $\tilde{T} = (G, L, R^-)$, the *twist* of T .
- $T^n = TT\dots T$, where there are n ($1 \leq n < \infty$) copies of T and T is a self-compatible tile, is the *block-tile of T of length n* .
- $T^\circ(n) = \circ(T^n\tilde{T})$, if T and $\tilde{T}$ are compatible to each other. $\square$

Example 3.14. Consider the tile *Bowtie* $\mathbb{B}$ in Example 3.5. Then Figure 3.3 shows $\mathbb{B}^{\otimes}(5)$. $\square$

Corollary 3.15. *Let T and $T_1, T_2, \dots, T_n$ be planar tiles.*

a) If T_k is compatible with T_{k+1} , $1 \leq k \leq n-1$, then $T_1 T_2 \dots T_n$ is planar.

b) If T is a self-compatible, then T^n is planar.

c) If T is a self-compatible, then $cr(T^{\otimes}(n)) \leq tcr(\tilde{T})$.

Proof (Corollary 3.15). These follow immediately from Lemma 3.12. Since $tcr(T) = tcr(T_k) = 0$, $1 \leq k \leq n$, we have $tcr(T_1 T_2 \dots T_n) \leq tcr(T_1) + tcr(T_2) + \dots + tcr(T_n) = 0$ and $cr(T^{\otimes}(n)) \leq tcr(T^n \tilde{T}) \leq tcr(T^n) + tcr(\tilde{T}) = tcr(\tilde{T})$. $\square$

Figure 3.3: $\mathbb{B}^{\odot}(5)$.

3.5 The Crossing Number of $T^\otimes(n)$.

We consider in this section $cr(T^\otimes(n))$. Let $T^\heartsuit(\ell, m)$ denote the tile $T^\ell \tilde{T} T^m$, $0 \leq \ell, m$. We show that these tiles determine $cr(T^\otimes(n))$. More precisely, the main result in this section is the following theorem:

Theorem 3.16. *Let T be a perfect planar tile. Then there are integers N_1, N_2 such that if $n \geq N_1$, then $cr(T^\otimes(n)) = tcr(T^\heartsuit(N_2, N_2))$.*

We start with the following result.

Theorem 3.17. *The sequence $tcr(T^\heartsuit(n, n))$ is non-increasing and therefore eventually constant.*

Proof (Theorem 3.17). First, we show by induction that, for all $n \geq 0$, $tcr(T^\heartsuit(n, n)) \leq tcr(\tilde{T})$. For $n = 0$, it is obvious. So let $n \geq 1$. Then by Lemma 3.12, we have $tcr(T^\heartsuit(n+1, n+1)) \leq tcr(T^\heartsuit(n, n+1)) \leq tcr(T^\heartsuit(n, n))$ which is at most $tcr(\tilde{T})$ by the induction hypothesis. So, the sequence $tcr(T^\heartsuit(n, n))$ is non-increasing. Moreover, since the tile crossing number can not be negative, at most $tcr(\tilde{T})$ times can there be strict inequality in the relation $tcr(T^\heartsuit(n+1, n+1)) \leq tcr(T^\heartsuit(n, n))$, so eventually there is an N such that $n \geq N$ implies $tcr(T^\heartsuit(n, n)) = tcr(T^\heartsuit(N, N))$. $\square$

The hard part of the proof of Theorem 3.16 is the inequality $tcr(T^\heartsuit(N_2, N_2)) \leq cr(T^\otimes(n))$. One ingredient in the argument that can be extracted is the following.

Lemma 3.18. *Let $T = (G, L, R)$ be a perfect tile. If $n \geq 3$, then, for any copy of $T_i = (G_i, L_i, R_i)$ in $T^\otimes(n)$: (i) if u, v are neighbors in $T_{i-1} - R_{i-1}$ of vertices in*

$R_{i-1} = L_i$, then there is a path in T_{i-1} joining u and v that is disjoint from T_i ; and
(ii) if u, v are neighbors in $T_{i+1} - L_{i+1}$ of vertices in $L_{i+1} = R_i$, then there is a path in T_{i+1} joining u and v that is disjoint from T_i .

Proof (Lemma 3.18). To prove (i), let u, v be neighbors in $T_{i-1} - R_{i-1}$ of vertices in $R_{i-1} = L_i$. Since T_{i-1} is perfect, so $T_{i-1} - R_{i-1}$ is connected and hence there is a path joining u and v in $T_{i-1} - R_{i-1}$. In an analogue way, we prove (ii). $\square$

Theorem 3.19. Let $T = (G, L, R)$ be a perfect planar tile and let c be a fixed positive integer. If $n \geq 8c + 5$ and $\mathcal{D}$ is a drawing of $T^\otimes(n)$ having at most c crossings, then there is a tile drawing of some $T^\otimes(\ell, m)$, with $\ell + m = n - 1$, having exactly $cr(\mathcal{D})$ crossings.

Proof (Theorem 3.19). Let $c = tcr(\tilde{T})$. Take $n \geq 8c + 5$. Ignore $\tilde{T}$ and the copies of T at each side of $\tilde{T}$; there are $8c + 3$ copies left. Since each crossing can involve two edges from two non-adjacent copies of T and the edges can be wall edges, at most $4c$ copies are involved in the crossings. Viewing the tiles involving in the crossing as separators (if the edge involved in the crossing is a wall edge, then the corresponding separator consists of the two consecutive copies containing that edge), we have most $2c$ separators (involving at most $4c$ copies) and at most $2c + 1$ intervals in the remaining $4c + 3$ copies. Hence one interval has at least $(4c + 3)/(2c + 1) = 2 + (1/(2c + 1)) > 2$ copies; that is, one interval contains at least 3 copies of T without crossings. Let $T_L = (G_L, L_L, R_L)$, $T_M = (G_M, L_M, R_M)$, $T_R = (G_R, L_R, R_R)$ be those three consecutive tiles.

Now consider the middle tile T_M of these three consecutive tiles without crossings.

Then there is a face of $\mathcal{D}(T_M)$ that contains the tile $\tilde{T}$. Since $\mathcal{D}(T_M)$ has no crossing and every vertex of $T^\circ(n)$ outside of T_M is joined to $\tilde{T}$ by a path disjoint from T_M , all of $T^\circ(n)$ other than T_M is drawn in that same face.

Furthermore, every wall vertex of T_M must be incident with that face. To see this, let $v \in L_M \cup R_M$. Without loss of generality, we assume that $v \in R_M$. By way of contradiction, suppose that v is not incident with that face. Since T is a perfect tile, condition (ii) of perfect tile guarantees that there is a path p in T_R from v to R_R which is disjoint from other wall vertices of $L_R (= R_M)$ and hence disjoint from the boundary of that face. So p must cross the boundary of that face which is a contradiction, since T_L, T_M, T_R are free of crossing. So every wall vertex of T_M must be incident with that face.

Let $W = (v_0, e_1, v_1, e_2, \dots, e_r, v_r)$ denote the boundary walk of this face. We have the following three claims:

Claim 1. *There do not exist $a < b < c < d$ and $p \neq q$ such that $v_a = L_M[p]$, $v_b = L_M[q]$, $v_c = R_M[p]$ and $v_d = R_M[q]$.*

Claim 2. *There do not exist $a < b < c < d < e < f$ and $p < q < r$ such that $v_a = L_M[p]$, $v_b = L_M[r]$, $v_c = L_M[q]$ and $v_d = R_M[q]$, $v_e = R_M[r]$, $v_f = R_M[p]$.*

Claim 3. *Let $i_1 < i_2 < \dots < i_s$ be the indices i such that v_i (in this position in W) is incident with an edge of $T^\circ(n)$ not in T_M . There do not exist $a < b < c < d$ such that v_{i_a}, v_{i_c} ($v_{i_a} \neq v_{i_c}$) are in L_M and v_{i_b}, v_{i_d} ($v_{i_b} \neq v_{i_d}$) are in R_M .*

Proof Claim 1. Suppose otherwise. Then by condition (iii) of perfect tile, there exist two disjoint paths, one joining $L_M[p]$ with $R_M[p]$ and the other joining $L_M[q]$

with $R_M[q]$. Since these paths are both in T_M , so they must be drawn inside W and cross each other which is a contradiction.

Proof Claim 2. Suppose otherwise. Then condition (iii) of perfect tile implies that there are two disjoint paths P_q and P_r , where P_q joins $L_M[q]$ and $R_M[q]$, and P_r joins $L_M[r]$ and $R_M[r]$. By the connectivity (and also condition (i)), there is a path P' joining P_q and either $L_M[p]$ or $R_M[p]$. By the planarity of the tile, P' does not cross P_q and so is disjoint from P_r . Note that both P_r and P' are paths of T_M and so must be drawn inside W . As a consequence, P' must cross P_r which is a contradiction.

Proof Claim 3. Suppose otherwise. Then by Lemma 3.18, there is a path in T_L joining v_{i_a} and v_{i_c} edge-disjoint from R_L and another path in T_R joining v_{i_b} and v_{i_d} edge-disjoint from L_R . Since these disjoint paths are not in T_M , they must be drawn outside W and cross each other which is a contradiction.

Note that it may happen that the same L_M vertex occurs twice as such v_{i_j} , but then they are consecutive and we may reroute the edges of $T^\circ(n)$ going to this vertex to all go to the same copy, just by running them along the boundary of the face. This gives a new drawing of $T^\circ(n)$ with the same number of crossings. Let γ be a curve obtained by going around the perimeter of the face and just going through a vertex in the boundary of the face when that vertex is incident with edges of $T^\circ(n)$ not in T_M .

Now γ has $\mathcal{D}(T_M)$ on one side, giving a planar drawing of T_M , and the rest of $T^\circ(n)$ on the other side, we delete the inner vertices and the inner edges (i.e., other than wall-vertices and wall-edges) of T_M from $\mathcal{D}(T^\circ(n))$ and obtain the tile drawing

of $T^\heartsuit(\ell, m)$, with $\ell + m = n - 1$. Note that all the crossing are located in $\mathcal{D}(T^\heartsuit(\ell, m))$ and so $tc_r(T^\heartsuit(\ell, m)) = cr(\mathcal{D})$. $\square$ (end proof of Theorem 3.19)

Now we prove the main theorem in this section.

Proof (Theorem 3.16). By Theorem 3.17, there is an integer N_0 such that if $n \geq N_0$, then $tc_r(T^\heartsuit(n, n)) = tc_r(T^\heartsuit(N_0, N_0))$. Let $c = tc_r(\tilde{T})$ and suppose $n \geq N_1 = \max\{2N_0, 8c + 5\}$. Let $\mathcal{D}$ be an optimal drawing of $T^\heartsuit(n)$. By Corollary 3.15 $cr(\mathcal{D}) \leq c$, and by Lemma 3.12 $cr(\mathcal{D}) \leq tc_r(T^\heartsuit(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)) \leq tc_r(T^\heartsuit(N_0, N_0))$. By Theorem 3.19, there is a tile drawing $\mathcal{D}'$ of $T^\heartsuit(\ell, m)$, for some positive integers ℓ, m such that $\ell + m = n - 1$, such that $cr(\mathcal{D}') = cr(\mathcal{D})$. Letting $a = \max\{\ell, m\}$, we see that $tc_r(T^\heartsuit(N_0, N_0)) \leq tc_r(T^\heartsuit(a, a)) \leq tc_r(T^\heartsuit(\ell, m)) \leq cr(\mathcal{D}') \leq cr(\mathcal{D}) \leq tc_r(T^\heartsuit(N_0, N_0))$. Therefore, we must have equality throughout, so if $n \geq N_1$, then $cr(T^\heartsuit(n)) = tc_r(T^\heartsuit(N_0, N_0))$. $\square$ (end proof of Theorem 3.16)

3.6 More on $cr(T^\heartsuit(n))$.

In this section we give a simple sufficient condition and show that if $T^\heartsuit(n)$ satisfies that condition, then $cr(T^\heartsuit(n)) = tc_r(\tilde{T})$.

Definition 3.20 (pair-disjoint, $pd(T)$).

Let $T = (G, L, R)$ be a tile. A set $\mathcal{P}$ of pairs of paths in $T\tilde{T}T = (G', L', R')$ is *pair-disjoint* if:

1. for $P \in \{P, Q\} \in \mathcal{P}$, there exist $i_P, j_P \in \{1, 2, \dots, |L|\}$ and paths P_1, P_2 , and P_3 in T such that: (i) P_1 joins $L[i_P]$ to $R[i_P]$, P_2 joins $L[i_P]$ to $R[j_P]$, and P_3

- joins $L[j_P]$ to $R[j_P]$; and (ii) $P = P_1 P_2 P_3$ joins $L'[i_P]$ to $R'[j_P]$;
2. for $\{P, Q\} \in \mathcal{P}$, P and Q are totally disjoint, and $i_P < i_Q$ if and only if $j_P > j_Q$;
and
3. if $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$ are distinct pairs in $\mathcal{P}$, then either $E(P_1) \cap E(P_2) = \emptyset$
or $E(Q_1) \cap E(Q_2) = \emptyset$.

For a tile T , $pd(T)$ denotes the maximum size of a pair-disjoint set. $\square$

Example 3.21. Let T be the *Bowtie* tile $\mathbb{B} = (G_2, (a, b), (d, e))$ from Example 3.5. Consider $T\tilde{T}T$ and let $\tilde{c}$ be the copy of vertex c in $\tilde{T}$ and let a', b', c', d' and e' be the copies of a, b, c, d and e respectively in the second T . Furthermore, let $p_1 = a d b' e'$, $p_2 = a c d \tilde{c} b' c' e'$, $p_3 = b c e \tilde{c} a' c' d'$, and $p_4 = b e a' d'$ be paths in $T\tilde{T}T$. Then the set $\mathcal{P} = \{\{p_1, p_3\}, \{p_1, p_4\}, \{p_2, p_4\}\}$ is pair-disjoint. Thus $pd(T) \geq 3$. $\square$

Lemma 3.22. *Let T be a perfect planar tile. For all $m, n \geq 1$, $tcr(T^\infty(m, n)) \geq pd(T)$.*

Proof (Lemma 3.22). Let $\mathcal{P}$ be a set of pair-disjoint paths in $T\tilde{T}T$. For $P \in \{P, Q\} \in \mathcal{P}$, let $P = P_1 P_2 P_3$ as in (1) of the definition of pair-disjoint. Let P^* be the path in $T^m \tilde{T} T^n$ consisting of all the copies of P_1 in the first m copies of T , P_2 in $\tilde{T}$, and all the copies of P_3 in the last n copies of T . Then P^* joins the i_P^{th} vertex of the left wall of $T^\infty(m, n)$ to the j_P^{th} vertex of the right wall of $T^\infty(m, n)$. By (2), if $\{P, Q\} \in \mathcal{P}$, then P^* and Q^* must cross in any tile drawing of $T^\infty(m, n)$. By (3), no two of these crossings are the same. $\square$

As a consequence of this Lemma 3.22, is the following:

Theorem 3.23. *Let T be a perfect planar tile. If $pd(T) = tcr(\tilde{T})$, then for n sufficiently large, $cr(T^\otimes(n)) = tcr(\tilde{T})$.*

Proof (Theorem 3.23). Assume that $pd(T) = tcr(\tilde{T})$. By Theorem 3.16, there are integers N_0, N such that if $n \geq N_0$, $tcr(T^\otimes(N, N)) = cr(T^\otimes(n))$. By Lemma 3.22, we have $tcr(\tilde{T}) = pd(T) \leq tcr(T^\otimes(N, N))$. By Lemma 3.12, $tcr(\tilde{T}) \geq tcr(T^\otimes(N, N))$. Hence $tcr(\tilde{T}) = tcr(T^\otimes(N, N)) = cr(T^\otimes(n))$. $\square$

Example 3.24. To prove that there is an infinite sequence of 4-regular, 4-connected, crossing-critical graphs having crossing number 3 (Richter and Thomassen [RT93]), we consider the tile $\mathbb{B} = (G_2, (a, b), (d, e))$ from Example 3.5. Example 3.21 shows that $3 \leq pd(\mathbb{B})$. An easy figure shows $tcr(\tilde{\mathbb{B}}) \leq 3$. Hence, by Lemma 3.22, $pd(\mathbb{B}) = tcr(\tilde{\mathbb{B}}) = 3$. Therefore, for large enough n , $cr(\mathbb{B}^\otimes(n)) = 3$, by Theorem 3.23. Deleting any edge in $\mathbb{B}$ will reduce $pd(\mathbb{B}) = tcr(\tilde{\mathbb{B}})$ and so reduce $cr(\mathbb{B}^\otimes(n))$. Therefore, $\mathbb{B}^\otimes(n)$ is crossing-critical. $\square$

The following application improves the result established by Salazar [Sal00] mentioned in Chapter 1.

Theorem 3.25. *Let $h \in \mathbb{N}$, $r \in \mathbb{Q} \cap (3.5, 4)$. Then there is an infinite family $G(h, r)$ of $\binom{2h+3}{2}$ -crossing critical simple graphs with average degree r .*

Proof (Theorem 3.25). Let $\mathbb{S} = (G, (a, b), (d, e))$ be the tile obtained from the tile $\mathbb{B}$ in Example 3.5 by splitting the middle vertex c into vertices f and g and

adding an edge joining f and g (see Figure 3.4 (i)) and let $G' = G - \{ad\}$ and $\mathbb{S}' = (G', (a, b), (d, e))$. Let $T_{(s,m)} = (G_T, L_T, R_T) = \mathbb{S}^s \mathbb{B}^m$. Furthermore, let $\mathbb{X}_{(h,s,m)} = (G_{(h,s,m)}, L, R)$, $h \geq 0$ and $s + m \geq 1$, be the tile consisting of $T_{(s,m)} = \mathbb{S}^s \mathbb{B}^m$ and h copies $\mathbb{S}'_i = (G'_i, (a_i, b_i), (d_i, e_i))$, $1 \leq i \leq h$, of $\mathbb{S}'$ “stacked” on each other with the first copy stacked under $T_{(s,m)}$, that is,

$$G_{(h,s,m)} = \left(\left(\dots \left(((G_T \cup G'_1) \{(L_T[2], R_T[2]), (a_1, d_1)\}) \cup G'_2 \} \{(b_1, e_1), (a_2, d_2)\} \dots \right) \cup G'_h \} \{(b_{h-1}, e_{h-1}), (a_h, d_h)\} \right),$$

$L = (L_T[1], a_1, a_2, \dots, a_h)$ and $R = (R_T[1], d_1, d_2, \dots, d_h)$, where a_i 's and d_i 's are vertices of $\mathbb{S}'_i$ (see Figure 3.4 (ii)).

Consider $\mathbb{X}_{(h,s,m)}$. Let p_i , $1 \leq i \leq h + 2$, be the horizontal paths joining $L[i]$ and $R[i]$ and let q_i , $1 \leq i \leq h + 1$, be the paths joining $L[i]$ and $R[i]$ disjoint from p_{i+1} and edge-disjoint from p_i . Similarly, let r_i , $1 \leq i \leq h + 1$, be the paths joining $L[i + 1]$ and $R[i + 1]$ disjoint from p_i and edge-disjoint from p_{i+1} .

Consider $\mathbb{X}_{(h,s,m)} \tilde{\mathbb{X}}_{(h,s,m)} \mathbb{X}_{(h,s,m)}$. Let P_i be the path in $\mathbb{X}_{(h,s,m)} \tilde{\mathbb{X}}_{(h,s,m)} \mathbb{X}_{(h,s,m)}$ consisting of p_i of the first $\mathbb{X}_{(h,s,m)}$ together with the “corresponding” path of p_i in the twist, and p_{h+3-i} of the second $\mathbb{X}_{(h,s,m)}$. Let P'_i be the path in $\mathbb{X}_{(h,s,m)} \tilde{\mathbb{X}}_{(h,s,m)} \mathbb{X}_{(h,s,m)}$ consisting of q_i of the first $\mathbb{X}_{(h,s,m)}$ together with the “corresponding” path of q_i in the twist, and r_{h+2-i} of the second $\mathbb{X}_{(h,s,m)}$ and let P''_i be the path in $\mathbb{X}_{(h,s,m)} \tilde{\mathbb{X}}_{(h,s,m)} \mathbb{X}_{(h,s,m)}$ consisting of r_i of the first $\mathbb{X}_{(h,s,m)}$ together with the “corresponding” path of r_i in the twist, and q_{h+2-i} of the second $\mathbb{X}_{(h,s,m)}$.

Then the following set is pair-disjoint:

$$\mathcal{P}_h = \left(\bigcup_{i=1}^{h+1} \bigcup_{j=i}^{h+1} \{ \{P_i, P_{j+1}\}, \{P_i, P''_j\} \} \right) \cup$$

(i): *Tile S*.

(ii): *Tile $X_{(h,s,m)}$* .

Figure 3.4: *Tiles S and $X_{(h,s,m)}$* .

$$\left(\bigcup_{i=1}^h \bigcup_{j=i+1}^{h+1} \{ \{P'_i, P_j\}, \{P'_i, P'_j\} \} \right) \cup \left(\bigcup_{i=1}^{h+1} \{ \{P'_i, P_{h+2}\} \} \right)$$

and

$$|\mathcal{P}_h| = h + 1 + \sum_{i=1}^{h+1} (2h - 2i + 4) + \sum_{i=1}^h (2h - 2i + 2).$$

We note that for all $h \geq 0$, $|\mathcal{P}_h| = \binom{2h+3}{2}$. Therefore, $pd(\mathbb{X}_{(h,s,m)}) \geq \binom{2h+3}{2}$. By drawing $\tilde{\mathbb{X}}_{(h,s,m)}$ such that the crossings only occur among the horizontal paths, we have $tcr(\tilde{\mathbb{X}}_{(h,s,m)}) \leq \binom{2h+3}{2}$. Hence, by Lemma 3.22, $pd(\mathbb{X}_{(h,s,m)}) = tcr(\tilde{\mathbb{X}}_{(h,s,m)}) = \binom{2h+3}{2}$. Therefore, for large enough n , $cr(\mathbb{X}_{(h,s,m)}^\circ(n)) = \binom{2h+3}{2}$, by Theorem 3.23.

It is easy to see that deleting any edge in any path p or any copy of the edge fg will save a crossing. Also, deleting any copy of edges af , ac , gd or cd (bf , bc , ge or ce respectively) in $T_{(s,m)}$ will save the crossing made by q_1 and p_2 (r_1 and p_1 respectively) in $\mathbb{X}_{(h,s,m)}^\circ(n)$. Similarly, deleting any copy of edge af or gd (bf or ge respectively) in the j -th copy $\mathbb{S}'_j$ of $\mathbb{S}'$ will save the crossing made by q_j and p_{j+1} (r_j and p_j respectively) in $\mathbb{X}_{(h,s,m)}^\circ(n)$. Therefore, $\mathbb{X}_{(h,s,m)}^\circ(n)$ is crossing-critical.

In $\mathbb{X}_{(h,s,m)}^\circ(n)$, we have $(2s + 2h)n$ vertices of degree 3, $(2(s + m) + m)n$ vertices of degree 4 and hn vertices of degree 6. Hence the average degree of $\mathbb{X}_{(h,s,m)}^\circ(n)$ is

$$\frac{(3(2s + 2h) + 4(2(s + m) + m) + 6h)n}{(2s + 2h + 2(s + m) + m + h)n} = \frac{14s + 12h + 12m}{4s + 3h + 3m} = 4 - \frac{2s}{4s + 3h + 3m}.$$

Obviously, $0 \leq \frac{2s}{4s + 3h + 3m} \leq 0.5$.

Given $r \in \mathbb{Q} \cap (3.5, 4)$, we want, for all positive integers h , infinitely many pairs (s, m) of positive integers so that $r = \frac{14s + 12h + 12m}{4s + 3h + 3m}$. This equation is the same as $s = \frac{12-3r}{4r-14}(m + h)$. For each h , there are infinitely many choices of positive integers m so that $\frac{12-3r}{4r-14}(m + h)$ is a positive integer. $\square$

3.7 Variants of $T^\otimes(n)$.

We consider in this section some variants of $T^\otimes(n)$ and state the corresponding variants of Theorem 3.16 and Theorem 3.23.

Definition 3.26. $(T^\leftarrow, T^\updownarrow, T^{\otimes h}(n), T^{\otimes v}(n))$

Let $T = (G, L, R)$ be a tile. We define

- $T^\leftarrow = (G, R, L)$, the *horizontal flip* of T ;
- $T^\updownarrow = (G, L^-, R^-)$, the *vertical flip* of T ;
- $T^{\otimes h}(n) = \circ((TT^\leftarrow)^n \tilde{T})$; and
- $T^{\otimes v}(n) = \circ((TT^\updownarrow)^n \tilde{T})$. $\square$

Observation 3.27. If $T = (G, L, R)$ is a planar tile, then both $T^\leftarrow$ and $T^\updownarrow$ are planar. $\square$

Example 3.28. Consider the tile $\mathbb{M} = (G_3, (a, b, c), (d, e, f))$ from Example 3.5. Then Figure 3.5 shows $\mathbb{M}^{\otimes h}(6)$ and $\mathbb{K}^{\otimes h}(3)$. $\square$

Let $T^{\otimes h}(m, n) = (TT^\leftarrow)^m \tilde{T} (TT^\leftarrow)^n$ and $T^{\otimes v}(m, n) = (TT^\updownarrow)^m \tilde{T} (TT^\updownarrow)^n$, with $m, n \geq 0$. We state the variants of Theorem 3.16 saying that these tiles determine $cr(T^{\otimes h}(n))$ and $cr(T^{\otimes v}(n))$, respectively. The proofs are completely analogous to that of Theorem 3.16.

 $M^{\otimes h}(6).$

 $K^{\otimes v}(3).$

Figure 3.5: $M^{\otimes h}(6)$ and $K^{\otimes v}(3)$.

Theorem 3.29.

Let T be a planar tile. If $TT^{\leftarrow}$ and $TT^{\downarrow}$ are perfect, then:

- i) $\exists N_1, N_2 \forall n \geq N_1 : cr(T^{\otimes h}(n)) = tcr(T^{\otimes h}(N_2, N_2));$ and
- ii) $\exists N_1, N_2 \forall n \geq N_1 : cr(T^{\otimes v}(n)) = tcr(T^{\otimes v}(N_2, N_2)). \square$

Next, we state the variants of Theorem 3.23 for $T^{\otimes h}(n)$ and $T^{\otimes v}(n)$, respectively, saying that if the given condition is satisfied, then $cr(T^{\otimes h}(n))$ and $cr(T^{\otimes v}(n))$ can be computed by just looking at $\tilde{T}$. Let T^* denote either $T^{\leftarrow}$ or $T^{\downarrow}$. We compute $pd(TT^*)$ in this context by considering paths in $TT^*\tilde{T}TT^*$, rather than $T\tilde{T}T$.

Theorem 3.30. Let T be a planar tile and let n be sufficiently large.

- i) If $TT^{\leftarrow}$ is perfect and $pd(TT^{\leftarrow}) = tcr(\tilde{T})$, then $cr(T^{\otimes h}(n)) = tcr(\tilde{T})$.
- ii) If $TT^{\downarrow}$ is perfect and $pd(TT^{\downarrow}) = tcr(\tilde{T})$, then $cr(T^{\otimes v}(n)) = tcr(\tilde{T}). \square$

Example 3.31. To show that the sequence of 5-regular simple graphs $\{M_{2n+1}\}_{n=N}^{\infty}$ from Chapter 2 has crossing number 6, consider the M -tile $\mathbb{M} = (G_3, (a, b, c), (d, e, f))$ from Example 3.5. Let $(\bar{a}, \bar{b}, \bar{c})$ be the right-wall of $\mathbb{M}\mathbb{M}^{\leftarrow}$. Consider $\mathbb{M}\mathbb{M}^{\leftarrow}\tilde{\mathbb{M}}\mathbb{M}\mathbb{M}^{\leftarrow}$ and let v' be the copy of vertex $v \in \mathbb{M}\mathbb{M}^{\leftarrow}$ in the second $\mathbb{M}\mathbb{M}^{\leftarrow}$. Furthermore, let $p_1 = a d \bar{a} c' f' \bar{c}'$, $p_2 = b d \bar{b} c' e' \bar{c}'$, $p_3 = b e \bar{b} b' e' \bar{b}'$, $p_4 = c e \bar{c} b' d' \bar{b}'$, and $p_5 = c f \bar{c} a' d' \bar{a}'$ be paths in $\mathbb{M}\mathbb{M}^{\leftarrow}\tilde{\mathbb{M}}\mathbb{M}\mathbb{M}^{\leftarrow}$. Then $\mathcal{P} = \{\{p_1, p_3\}, \{p_1, p_4\}, \{p_1, p_5\}, \{p_2, p_4\}, \{p_2, p_5\}, \{p_3, p_5\}\}$ is a pair-disjoint set and so $pd(\mathbb{M}\mathbb{M}^{\leftarrow}) \geq 6$. An easy figure shows $tcr(\tilde{\mathbb{M}}) \leq 6$. Hence, $pd(\mathbb{M}\mathbb{M}^{\leftarrow}) = tcr(\tilde{\mathbb{M}}) = 6$. Therefore, for large enough n , $cr(T^{\otimes}(n)) = 6$, by

Theorem 3.30. Now contract each copy of ad and cf in each copy of $\mathbb{M}$ (not in $\mathbb{M}^{\leftarrow}$) to obtain the graph M_{2n+1} . Since $\mathbb{M}^{\otimes h}(n)$ is a subdivision of M_{2n+1} , $cr(M_{2n+1}) = cr(\mathbb{M}^{\otimes h}(n)) = 6$. Note that by deleting the copies of edges bc and de of $\mathbb{M}$ in $\mathbb{M}^{\otimes h}(n)$, we obtain $\mathbb{Z}^{\otimes h}(n)$ with the same number of crossings, where $\mathbb{Z} = (G_4, (a, b, c), (d, e, f))$ is the $\mathbb{Z}$ -tile from Example 3.5. This proves that M_{2n+1} from Chapter 2 is not crossing-critical. $\square$

Example 3.32. To show that there is an infinite sequence $\{H_n\}_{n=N}^{\infty}$ of 3-connected crossing-critical simple graphs having crossing number 2, consider the *Kochol*-tile $\mathbb{K} = (G_3, (a, b, c), (i, j, k))$ from Example 3.5. Note that there is no set of pair-disjoint paths of size 2 in $\mathbb{K}\mathbb{K}^{\uparrow}$ (and hence not in $\mathbb{K}\mathbb{K}^{\uparrow}\tilde{\mathbb{K}}\mathbb{K}\mathbb{K}^{\uparrow}$). There are other pairs of paths, but the two that are used with the top horizontal path are not edge-disjoint. They have the “bases” (copies of the edge fh of $\mathbb{K}$ in Figure 3.1) of upright pentagons in common. Similarly, the two pairs with the bottom horizontal path have the bases of upsidedown pentagons in common. Note that the upright and the upsidedown pentagons are disjoint cycles. If $cr(\mathbb{K}^{\otimes v}(n)) = 1$, then it is because one each of these bases cross. But then the disjoint cycles of the pentagons cross again. Hence $cr(\mathbb{K}^{\otimes v}(n)) \geq 2$. A drawing with two crossings can be achieved by drawing $\tilde{\mathbb{K}}$ such that the pentagon is drawn in a “regular” way and the copy of the edge di in $\tilde{\mathbb{K}}$ crosses the copy of fh , and the copy of the edge gj crosses the copy of hk in $\tilde{\mathbb{K}}$ (see the bottom figure in Figure 3.5, for $\mathbb{K}^{\otimes v}(3)$). Therefore, $cr(\mathbb{K}^{\otimes v}(n)) = 2$.

It is an easy exercise to verify that deleting any edge in $\tilde{\mathbb{K}}$ reduces the number of crossings (one will involve the direct neighbors of $\tilde{\mathbb{K}}$ to the crossings). So $\mathbb{K}^{\otimes v}(n)$ is crossing-critical. Now contract each copy of ad, be, cf, di, gj, hk in each copy of

$\mathbb{K}$ (not in $\mathbb{K}^1$) to obtain the graph H_n without vertices of degree 2, with the same crossing number and crossing-critical. $\square$

Remark. The theory we have developed with respect to the crossing number of $T^\circ(n)$ generalizes in a very simple way. Let T and $T_1, T_2, T_3, \dots$ be compatible perfect planar tiles. Let $\mathcal{T}^\bowtie(j, k) = T_j T_{j-1} \dots T_1 \tilde{T} T_1 T_2 \dots T_k$. Then the sequence $tcr(\mathcal{T}^\bowtie(j, j))$ is eventually a constant and determines $cr(\circ(\mathcal{T}^\bowtie(j, j)))$. The analogue of $pd(T)$ is a little harder, since we need the crossing paths in $\mathcal{T}^\bowtie(j, j)$ and not just in T . $\square$

Chapter 4

Sequences of Graphs with Non-Constant Crossing Number

4.1 Introduction

In this chapter, we analyze $cr(o(T^n))$ for general tiles T . We do not attempt to be as precise as in Chapter 3 dealing with planar tiles. However, we do provide asymptotic estimates for $cr(o(T^n))$ in terms of a number $acr(T)$, which we call *average crossing number of T* .

The outline of this chapter is as follows. We begin by proving in Section 4.2 that $\lim_{n \rightarrow \infty} n^{-1} tcr(T^n)$ exists; this is $acr(T)$. Furthermore, if we permute the left and right walls of T with the same permutation π , producing the tile T_π (so T^n and T_π^n are the same graph), then $acr(T_\pi) = acr(T)$.

In Section 4.3, we prove that there is a constant c_T so that if n is large enough,

there is a $\pi = \pi(n)$ so that $cr(o(T^\pi)) \leq tcr(T_\pi^n) \leq cr(o(T^\pi)) + c_T$. This implies that $\lim_{n \rightarrow \infty} n^{-1}cr(o(T^n)) = acr(T)$, or in other words, that $cr(o(T^n))$ is essentially $n \cdot acr(T)$.

Then we discuss applications of this result to some standard families of graphs, namely the Generalized Petersen Graphs $P(n, k)$, in particular $P(3n, 3)$ in Section 4.4 and finally, the Cartesian product $C_m \times C_n$ of two cycles in Section 4.5.

4.2 On Average Crossing Number

Before showing that $\lim_{n \rightarrow \infty} n^{-1}tcr(T^n)$ exists, we state the following standard result which is sometimes referred as *Fekete's Lemma* or *the subadditivity lemma* (cf. [MS93, p.9]).

Lemma 4.1. *Let $\{a_n\}_{n \geq 1}$ be a sequence of positive real numbers which is subadditive, i.e., $a_{n+m} \leq a_n + a_m$. Then $\lim_{n \rightarrow \infty} n^{-1}a_n$ exists and is equal to $\inf_{n \geq 1} \{n^{-1}a_n\}$. $\square$*

By Lemma 3.12, $tcr(T^{m+n}) \leq tcr(T^m) + tcr(T^n)$, so an immediate consequence of Lemma 4.1 is the following.

Theorem 4.2. *Let T be a tile. Then $\lim_{n \rightarrow \infty} n^{-1}tcr(T^n)$ exists. $\square$*

Definition 4.3 (Average Crossing Number).

The *average crossing number* of a tile T is

$$acr(T) = \lim_{n \rightarrow \infty} \frac{tcr(T^n)}{n}. \quad \square$$

Let $T = (G, L, R)$ be a self-compatible tile. Let $t = |L| = |R|$. Let π be a permutation of $\{1, 2, \dots, t\}$. Let L_π be the list $L[\pi(1)], L[\pi(2)], \dots, L[\pi(t)]$ and similarly define R_π . Let $T_\pi = (G, L_\pi, R_\pi)$. Note that if $T^n = (G_1, L, R)$ and $T_\pi^n = (G_2, L_\pi, R_\pi)$, then G_1 and G_2 are isomorphic. We have the following result.

Theorem 4.4. *For any permutations π, π' ,*

$$acr(T_\pi) = acr(T_{\pi'})$$

Proof (Theorem 4.4). Let $\mathcal{D}_n$ be a tile drawing of T_π^n having $tcr(T_\pi^n)$ crossings. We can get a tile drawing $\mathcal{D}'_{n+2}$ of $T_{\pi'}^{n+2}$ by adding a tile at each end of T_π^n such that $cr(\mathcal{D}'_{n+2}) \leq cr(\mathcal{D}_n) + C(\pi, \pi')$, where $C(\pi, \pi')$ is the sum of the crossing numbers of these two added tiles (so $C(\pi, \pi')$ is independent of n). Therefore,

$$acr(T_{\pi'}) \leq \lim_{n \rightarrow \infty} \frac{tcr(\mathcal{D}'_{n+2})}{n+2} = \lim_{n \rightarrow \infty} \frac{tcr(\mathcal{D}'_{n+2})}{n} \leq \lim_{n \rightarrow \infty} \frac{tcr(\mathcal{D}_n) + C(\pi, \pi')}{n}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{tcr(\mathcal{D}_n) + C(\pi, \pi')}{n} = \lim_{n \rightarrow \infty} \frac{tcr(\mathcal{D}_n)}{n} + \lim_{n \rightarrow \infty} \frac{C(\pi, \pi')}{n} = acr(T_\pi),$$

it follows that, for any ordered pair (π, π') of permutations, $acr(T_\pi) \leq acr(T_{\pi'})$. This also holds when π and π' are interchanged and, therefore, $acr(T_\pi) = acr(T_{\pi'})$. $\square$

Note that, by taking the identity permutation for π' in Theorem 4.4, we have $acr(T_\pi) = acr(T)$.

Lemma 4.5. *Let T be any self-compatible tile. If $S = T^k$, then $acr(T) = k^{-1} acr(S)$.*

Proof (Lemma 4.5).

$$acr(T) = \lim_{n \rightarrow \infty} \frac{tcr(T^{nk})}{nk} = \lim_{n \rightarrow \infty} \frac{tcr(S^n)}{nk} = \frac{1}{k} \lim_{n \rightarrow \infty} \frac{tcr(S^n)}{n} = \frac{1}{k} acr(S). \quad \square$$

Lemma 4.6. *Let T be any self-compatible tile. Then, for each $n \geq 1$, $tcr(T^n) \geq n \cdot acr(T)$ and $cr(\circ(T^n)) \leq n \cdot tcr(T)$.*

Proof (Lemma 4.6). Let $n \geq 1$. Then by Lemma 4.1, $n^{-1}tcr(T^n) \geq acr(T)$. That $cr(\circ(T^n)) \leq n \cdot tcr(T)$ follows from Lemma 3.12. $\square$

4.3 The relationship between cr and acr .

We show in this section that $\lim_{n \rightarrow \infty} n^{-1}cr(\circ(T^n)) = acr(T)$.

Lemma 4.7. *Let G be a connected graph embedded without crossings in the plane. Let $W \subseteq V(G)$. Then there is a simple closed curve $\mathcal{C}$ in the plane such that $\mathcal{C} \cap V(G) = W$ and $\mathcal{C}$ meets each edge of G at most twice.*

Proof (Lemma 4.7). Let $\mathcal{T}$ be a tree in G containing all the vertices of W (this tree exists since G is connected). Consider the embedding of $\mathcal{T}$ induced by G . This has one face, and in traversing once the perimeter of this face, we can create a simple closed curve $\mathcal{C}$ that just goes through once each vertex in W . This closed curve will also cross each edge of G at most once for each end of the edge that is in $\mathcal{T}$, i.e., at most twice. $\square$

Theorem 4.8. *Let T be a tile. Then there is a permutation π and a constant c_T such that, for every $n \geq 1$,*

$$cr(\circ(T^n)) \leq tcr(T_\pi^n) \leq cr(\circ(T^n)) + c_T.$$

Proof (Theorem 4.8). Since $\circ(T^n) = \circ(T_\pi^n)$, the first inequality follows directly from Lemma 3.12. We only need to prove the second inequality. We do this by turning an optimal drawing of $\circ(T^n)$ into a tile drawing of T_π^n by adding relatively few crossings.

Consider an optimal drawing $\mathcal{D}$ of $\circ(T^n)$ with $cr(\circ(T^n))$ crossings. Let $\mathcal{T}$ be a tree in T containing all the right-wall vertices of T and let $\mathcal{T}_j$ be the copy of $\mathcal{T}$ in the j -th ($1 \leq j \leq n$) copy of T in $\circ(T^n)$. Let c_j the number of crossings of $\mathcal{T}_j$. Then

$$\sum_{j=1}^n c_j \leq 2cr(\circ(T^n))$$

and so there is some i such that

$$c_i \leq \frac{2cr(\circ(T^n))}{n}.$$

Therefore, by Lemma 4.6, $c_i \leq 2tcr(T)$.

Let W , with size $|W| = h$, be the set of the right-wall vertices of the i -th copy of T in $\circ(T^n)$ that are contained in the tree $\mathcal{T}_i$ (which has at most $2tcr(T)$ crossings). By Lemma 4.7, we can find a simple closed curve $\mathcal{C}$ along $\mathcal{T}_i$ in $\mathcal{D}$ containing all and only the vertices of W . Let $v_1, v_2, \dots, v_h$ be the cyclic order in which the vertices of W occur in $\mathcal{C}$. For every edge $e = uv$, $v \in W$, we call e a “blue edge” (“red edge”) if $u \in T_i - W$ ($u \in T_{i+1} - W$ respectively). Furthermore, let us call the edges crossing the curve as “green edges”. Now, we do the following splitting:

- we split each vertex v_j , $1 \leq j \leq h$, of W into v_j^L and v_j^R and view $\mathcal{C}$ as a rectangle, with $v_1, v_2, \dots, v_h$ in this order down the right-hand side. We now

relabel the v_j as v_j^R and add a v_j^L on the left-hand side, in the top-to-bottom order $v_1^L, v_2^L, \dots, v_h^L$;

- for every edge uv in $\circ(T^n)$, $v \in W$, if uv is a blue edge (red edge), we delete uv and add uv^L (uv^R respectively) to $E(\circ(T^n))$; for every right-wall edge uv ($u, v \in W$), add $u^L v^L$ (so $u^R v^R$ is still an edge).

After the splitting mechanism, we have each blue edge is incident with a v^L -vertex and a vertex of T_i and each red edge is adjacent to a v^R -vertex and a vertex of T_{i+1} . Now we make the curve $\mathcal{C}$ free of crossings by rerouting the blue, red and green edges such that:

- all the edges crossing the curve $\mathcal{C}$ are rerouted outside the rectangle, say passing the top side of the rectangle;
- the edges retain their positions as long as possible, that is, they reroute around the rectangle as close as possible to the rectangle;
- the additional crossings are only made with red, blue or green edges;

Figure 4.1 shows an example of splitting and rerouting. The curve is indicated with stipple lines, the blue edges b 's and the green edges g 's with narrow lines, and the red edges r 's with bold lines.

(a) before splitting and rerouting

(b) after splitting and rerouting

Figure 4.1: *Splitting and rerouting.*

Now there is no crossing on $\mathcal{C}$ and hence all the vertices v^L 's and v^R 's are in the same face. Note that the order of the vertices in both sides of the rectangles is a permutation of the the order of wall vertices of T . Say this permutation is π . So we have a drawing of T_π^n .

Rerouting the green edges eliminates the crossings on the wall edges, but each green edge might now cross every red edge and every blue edge. Thus, there are at most $c_i \sum_{j=1}^h (d_j^L + d_j^R)$ crossings of green edges with red or blue edges. Similarly, every red or blue edge might now cross all red edges and all blue edges, so there are at most $(\sum_{j=1}^h (d_j^L + d_j^R))^2$ crossings of red or blue with red or blue. Thus, we have added at most $c_i \sum_{j=1}^h (d_j^L + d_j^R) + (\sum_{j=1}^h (d_j^L + d_j^R))^2$ new crossings. Furthermore, we have a tile drawing of some T_π^n , since $\mathcal{C}$ is contained in one face of the drawing of the graph, and $\mathcal{C}$ goes through the left-wall vertices in the opposite order to how it goes through the right-wall vertices.

It follows that $tcr(T_\pi^n) \leq cr(\circ(T^n)) + c_i \sum_{j=1}^h (d_j^L + d_j^R) + (\sum_{j=1}^h (d_j^L + d_j^R))^2$. Letting $c_T = (2tcr(T)) \sum_{j=1}^h (d_j^L + d_j^R) + (\sum_{j=1}^h (d_j^L + d_j^R))^2$, the result follows. $\square$ (*end of proof Theorem 4.8*)

Corollary 4.9. *Let T be a tile. Then*

$$\lim_{n \rightarrow \infty} \frac{cr(\circ(T^n))}{n} = acr(T).$$

Proof (*Corollary 4.9*). Divide each side of the inequalities in Theorem 4.8 by n and take the limit as $n \rightarrow \infty$, to obtain

$$\lim_{n \rightarrow \infty} \frac{cr(\circ(T^n))}{n} \leq \lim_{n \rightarrow \infty} \frac{tcr(T_\pi^n)}{n} \leq \lim_{n \rightarrow \infty} \frac{cr(\circ(T^n)) + c_T}{n}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{cr(\circ(T^n)) + c_T}{n} = \lim_{n \rightarrow \infty} \frac{cr(\circ(T^n))}{n} + \lim_{n \rightarrow \infty} \frac{c_T}{n} = \lim_{n \rightarrow \infty} \frac{cr(\circ(T^n))}{n},$$

we have, by Theorem 4.4,

$$\lim_{n \rightarrow \infty} \frac{cr(\circ(T^n))}{n} = \lim_{n \rightarrow \infty} \frac{tcr(T_\pi^n)}{n} = acr(T_\pi) = acr(T). \quad \square$$

In the next example we show that average crossing number could be 0.5. However, it is not clear if average crossing number can be any arbitrary positive rational number.

Example 4.10. We recall that a graph is *outerplanar* if it can be embedded in the plane so that every vertex of that lies on the unbounded face. Also, we recall that a graph is outerplanar if and only if it contains no subgraph that is a subdivision of either K_4 or $K_{2,3}$.

Let $T = (G, (a, b, c), (d, e, f))$ be the tile in Figure 4.2. Consider T^n with L and R being the left- and right-wall respectively. Let T_i and T_{i+1} , $1 \leq i \leq n-1$, be two consecutive copies of T in T^n and v_i be the copy in T_i of vertex $v \in V(G)$. Then there is a subdivision $\mathcal{K}_i$ of $K_{2,3}$ induced by $\{a_i, b_i, d_i, e_i, f_i, x_{i+1}, d_{i+1}, e_{i+1}\}$ in $T_i T_{i+1}$ and so it is not outerplanar. There are only three possible drawings of $\mathcal{K}_i$ with no crossings, namely, drawing $\mathcal{D}_1$ with e_i and x_{i+1} inside the cycle $C_1 = a_i b_i f_i e_{i+1} d_{i+1} d_i a_i$, drawing $\mathcal{D}_2$ with a_i and b_i inside the cycle $C_2 = d_i e_i x_{i+1} f_i e_{i+1} d_{i+1} d_i$ and drawing $\mathcal{D}_3$ with d_{i+1} and e_{i+1} inside the cycle $C_3 = a_i b_i f_i x_{i+1} e_i d_i a_i$.

Figure 4.2: *Tile with $acr(T) = 0.5$.*

We claim that in each possible drawing, we can assign one crossing to $\mathcal{K}_i$. If any two edges of $\mathcal{K}_i$ cross, then we count the crossing as one and assign this crossing to $\mathcal{K}_i$. In $\mathcal{D}_1$, there is a path from e_i to L and a path from e_i to R . Each of these paths is disjoint from C_1 and so they both must cross that cycle. Count each of the crossings a half and so we have one crossing assigned for $\mathcal{K}_i$.

Note first that for $\mathcal{D}_2$ is only possible when $i \leq 2$. For this drawing, there are two disjoint paths from a_i and b_i to L . These paths are disjoint (except for a_i and b_i) from $\mathcal{K}_i$ and so each of them must cross C_2 . We count each of these crossings a half and so we have one full crossing to be assigned to $\mathcal{K}_i$.

Similarly, drawing $\mathcal{D}_3$ is only possible when $i \leq n - 2$. For this drawing, there are two disjoint paths from d_{i+1} and e_{i+1} to the right-wall. These paths are disjoint (except for d_{i+1} and e_{i+1}) from $\mathcal{K}_i$ and so each of them must cross C_3 . We count each of these crossings a half and so we have one full crossing to be assigned to $\mathcal{K}_i$.

So for each of the possible drawings of $\mathcal{K}_i$, we can assign one full crossing to it. Note that for each such $\mathcal{K}$, there is only one common edge with its adjacent neighbor, but an edge can not cross itself. Hence $tcr(T^n) \geq \frac{n}{2}$ and therefore,

$$acr(T) = \lim_{n \rightarrow \infty} \frac{tcr(T^n)}{n} \geq \lim_{n \rightarrow \infty} \frac{n/2}{n} = \frac{1}{2}. \quad \square$$

4.4 Application to Generalized Petersen Graphs

The Generalized Petersen Graphs, $P(n, k)$, is a family of graphs each consisting of an n -cycle $C = a_0 a_1 \dots a_{n-1} a_0$, and vertices $b_0, b_1, \dots, b_{n-1}$ all not in C and the edges $a_i b_i$ and $b_i b_{i+k}$ for every i modulo n . The (classical) Petersen Graph is $P(5, 2)$.

In [Sa97], Salazar shows that $cr(P(3k+h, 3)) = k+h$, if $h \in \{0, 2\}$, and $cr(P(3k+1, 3)) = k+3$, for each $k \geq 3$ with the single exception for $P(9, 3)$, whose crossing number is 2.

In this section, we demonstrate our technique, discussed in the previous sections, on the Generalized Petersen Graphs, in particular for $P(3n+k, 3)$, $k = 0, 1, 2$. We do $P(3n, 3)$ first by defining a corresponding tile T so that $\circ(T^n) = P(3n, 3)$ and show that $tcr(T^n) = n$, so that $acr(T) = 1$. This will imply that there is a constant c such that $cr(P(3n, 3)) \geq n - c$. Then we state the required modifications to get the results for $P(3n+1, 3)$ and $P(3n+2, 3)$.

Let $T = (G, L, R)$ be the tile in Figure 4.3 consisting of four disjoint paths, each joins $L[i]$ and $R[i]$, $1 \leq i \leq 4$; the *principle path* contains 3 inner vertices r, y, b and each of the red $\mathcal{R}$, yellow $\mathcal{Y}$ and blue $\mathcal{B}$ paths contains one inner vertex r', y' and b' respectively adjacent to r, y and b . We call these edges red, yellow, and blue *spokes* respectively. Note that $\circ(T^n)$ is a subdivision of $P(3n, 3)$.

We prove by induction that $tcr(T^n) \geq n - 2$. It is obvious for $n \leq 2$. So let $n > 2$. Let $T_i = (G_i, L_i, R_i)$, $1 \leq i \leq n$, be the i -th copy of T in the tile-block T^n . If a spoke in some copy T_i of T is crossed, then we delete all the spokes in T_i to obtain a subdivision of T^{n-1} . We apply the induction hypothesis to conclude $tcr(T^{n-1}) \geq n-3$ and together with the one crossing we removed we have $tcr(T^n) \geq n - 2$.

Figure 4.3: *Tile for $P(3n, 3)$.*

Now suppose otherwise that there is no spoke crossed. Then in every three consecutive copies $T_{i-1}T_iT_{i+1}$, $2 \leq i \leq n-1$, there is a subdivision $\mathcal{K}_i$ of $K_{3,3}$ formed by the path on $\mathcal{R}$ joining r'_{i-1} and r'_{i+1} , the path on $\mathcal{Y}$ joining y'_{i-1} and y'_i , the path on $\mathcal{B}$ joining b'_{i-1} and b'_i , the segment of the principle path joining r_{i-1} and r_{i+1} , the spoke $r_{i+1}r'_{i+1}$ and all the spokes in T_{i-1} and T_i where $\{y_{i-1}, r_i, b_i\}$ and $\{b_{i-1}, r'_i, y_i\}$ are the partition (in Figure 4.4, $\mathcal{K}_i$ is indicated with bold lines). By repeating this pattern, we have $n-2$ of them in T^n . Each of these gives a crossing. In two consecutive copies $\mathcal{K}_i$ and $\mathcal{K}_{i+1}$ of this subdivision of $K_{3,3}$, the common part is the cycle $r_i r'_i r'_{i+1} r_{i+1} b_i y_i r_i$, and the yellow and blue spokes in T_i . Since $\mathcal{K}_i$ and $\mathcal{K}_{i+1}$ are subdivisions of $K_{3,3}$ and recall that the spokes are assumed to be not crossed, the only possible common crossing is made by $r_i y_i$ and $b_i r_{i+1}$. But $r_i y_i$ is contained in a cycle C_1 formed by $y_{i-1}, y'_{i-1}, y'_i, y_i$ and $b_i r_{i+1}$ is contained in another cycle C_2 formed by $b_i, b'_i, b'_{i+1}, r_{i+1}$. If C_1 and C_2 cross, then they must cross an even number of times. So we have at least two crossings; one is assigned for $\mathcal{K}_i$ and the other for $\mathcal{K}_{i+1}$. Note that there is a common part for 3 consecutive $\mathcal{K}$'s but this is a spoke which is assumed not crossed. This proves $tcr(T^n) \geq n-2$.

Hence $acr(T) = \lim_{n \rightarrow \infty} n^{-1} tcr(T^n) \geq \lim_{n \rightarrow \infty} (n-2)/n = 1$. By Lemma 4.1,

$$\lim_{n \rightarrow \infty} \frac{tcr(T^n)}{n} = \inf_{n \geq 1} \frac{tcr(T^n)}{n},$$

that is, for all n , $n^{-1} tcr(T^n) \geq acr(T)$, or, in other words, $tcr(T^n) \geq n \cdot acr(T)$. But also, for some permutation π , $tcr(T_\pi^n) \geq n \cdot acr(T_\pi) = n \cdot acr(T) \geq n$, by Theorem 4.4. Hence $tcr(T_\pi^n) \geq n$ and by a simple picture $tcr(T_\pi^n) = n$. Therefore, by Theorem 4.8 and the fact that $\circ(T^n)$ is a subdivision of $P(3n, 3)$, $cr(P(3n, 3)) = cr(\circ(T^n)) \geq tcr(T_\pi^n) - c = n - c$, for some constant c .

Figure 4.4: Subdivision $\mathcal{K}_i$ of $K_{3,3}$ in $T_{i-1}T_iT_{i+1}$.

To get $P(3n+1, 3)$, we consider $\circ(T^n T^*)$, where $T = (G, L, R)$ is the same tile as that of $P(3n, 3)$ and T^* is obtained from T by applying the permutation $(1\ 2)$ to L and the permutation $(2\ 4)$ to R (see the top picture in Figure 4.5; the stippled edges are to be deleted and the white vertices are to be eliminated). After deleting the red and yellow spokes in T^* and eliminate all the vertices of degree two, we obtain $P(3n+1, 3)$.

To get $P(3n+2, 3)$, we consider $\circ(T^n T'')$, where $T = (G, L, R)$ is the same tile as that of $P(3n, 3)$ and T'' is obtained from T by applying the permutation $(1\ 2\ 4)$ to L (see the bottom picture in Figure 4.5; the stippled edge is to be deleted and the white vertices are to be eliminated). After deleting the red spoke in T'' and eliminating all the vertices of degree two, we obtain $P(3n+2, 3)$.

Remark. For the general $P(kn, k)$, $k \geq 4$, Salazar [personal communication] has suggested the following tile T_k . Draw the principle path P as a convex curve. Let s_i , $1 \leq i \leq k$, be the spoke incident with the i -th vertex from the left on P and let p_i be the non-principle path adjacent to spoke s_i . Then draw p_1 and p_2 above P with p_1 on the top and draw p_{k-1} and p_k under P with p_k at the bottom (so these paths do not cross P). Next, draw the remaining $k-2$ paths across P so that each of them crosses P twice and draw the spokes without crossing except for s_1 and s_k which cross p_2 and p_{k-1} respectively (see Figure 4.6). With this tile, we have $tcr(T_k^n) \leq 2n + 2n(k-4) = 2n(k-3)$ and by drawing all the copies of T_k this way, we have $cr(\circ(T_k^n)) = cr(P(kn, k)) \leq 2n(k-3)$.

$P(3n+1, 3).$

$P(3n+2, 3).$

Figure 4.5: $P(3n+1, 3)$ and $P(3n+2, 3)$.

Figure 4.6: *Tile for $P(kn, k)$.*

4.5 Application to $C_m \times C_n$

In this section, we show how our results apply to the Cartesian product $C_m \times C_n$ of an m -cycle with an n -cycle.

In 1973, Harary, Kainen and Schwenk [HKS73] conjectured that the crossing number of $C_m \times C_n$ is $(m-2)n$, assuming $m \leq n$. This has been proved for $n \leq 6$ and for $m = n = 7$. Recently, Adamsson and Richter [AD01] have proved the case $m = 7$ and any n . They approached the problem by introducing (m, n) -arrangements.

Definition 4.11 (*(m, n) -arrangement*).

An (m, n) -arrangement is a quintuple $(S, T, \mathcal{C}, \mathcal{P}, \psi)$, such that:

1. $\mathcal{C} = (C_1, C_2, \dots, C_n)$ is a sequence of cycles;
2. $\mathcal{P} = \{P_1, P_2, \dots, P_m\}$ is a set of paths;
3. S and T are cycles and $S[0, 1] \cup T[0, 1]$ is contained in one component of

$$\mathbb{R}^2 \setminus \bigcup_{i=1}^n C_i[0, 1];$$

4. $\psi: (\{S, T\} \cup \mathcal{C}) \times \mathcal{P} \rightarrow [0, 1] \times [0, 1]$ such that:

- for each $(C, P) \in (\{S, T\} \cup \mathcal{C}) \times \mathcal{P}$, if $\psi(C, P) = (s, t)$, then $C(s) = P(t)$;
- for each $P \in \mathcal{P}$, if $\psi(C_i, P) = (s_i, t_i)$, then $0 < t_1 < t_2 < \dots < t_n < 1$;
- for each $P \in \mathcal{P}$, there exists $s \in [0, 1]$ such that $\psi(S, P) = (s, 0)$; and
- for each $P \in \mathcal{P}$, there exists $t \in [0, 1]$ such that $\psi(T, P) = (t, 1)$. $\square$

An (m, n) -arrangement is basically putting two disjoint cycles labelled S and T on a plane, together with a sequence of cycles $(C_1, C_2, \dots, C_n)$ between them. Each path in $\{P_1, P_2, \dots, P_m\}$ starts at S , hits each of the C_i in turn, and ends at T .

Adamsson and Richter [AD01] proved the following result.

Theorem 4.12. *Let $\mathcal{A}$ be an (m, n) -arrangement. Then $cr(\mathcal{A}) \geq (m - 2)n$. $\square$*

Let $T_m = (G_m, L_m, R_m)$ be the tile in Figure 4.7, consisting of an m -cycle $C = v_1 v_2 \dots v_m v_1$, where $v_i \notin L_m \cup R_m$, and m disjoint paths $p_i = L_m[i] v_i R_m[i]$, $1 \leq i \leq m$.

Lemma 4.13. $acr(T_m) = m - 2$.

Proof (Lemma 4.13). Consider $T_m^n = (G, L, R)$. Let $\mathcal{D}$ be a drawing of T_m^n . Add the edges $L[1]L[m]$, $R[1]R[m]$ and also the edges $L[i]L[i+1]$ and $R[i]R[i+1]$, $1 \leq i \leq m-1$, without crossing, to $\mathcal{D}$ (so $L[1]L[m]$ and $R[1]R[m]$ are drawn on the outer face of T_m^n). Thus, we obtain a drawing $\mathcal{D}'$, of the new graph, with two new cycles S and T on the most left and most right respectively with n m -cycles between them, and m paths, each starts at S and hits the n cycles in turn and ends at T . Hence, $\mathcal{D}'$ is an (m, n) -arrangement $(S, T, \mathcal{C}, \mathcal{P}, \psi)$, where $\mathcal{C} = (C_1, C_2, \dots, C_n)$, C_k is the k -th copy of C , and $\mathcal{P}$ consists of the m paths, each being the union of the n copies of p_i . Note that $tcr(T_m^n) = cr(\mathcal{D}) = cr(\mathcal{D}')$. Thus, by Theorem 4.12, $tcr(T_m^n) = cr(\mathcal{D}') \geq (m - 2)n$. By the definition of acr , we have $acr(T(m)) \geq m - 2$. $\square$

As a consequence of Lemma 4.13 is the following result.

Theorem 4.14. $cr(C_m \times C_n) \geq (m - 2)n - c_m$, where c_m depends only on m . $\square$

Figure 4.7: *Tile for $C_m \times C_n$.*

Chapter 5

Future Work on Crossing Number

5.1 Introduction

The work in the previous chapters not only has solved some crossing number problems but also has established a new way of looking at the crossing number problems in particular for large graphs. Therefore it has a significant potential to open new directions for future research.

5.2 The crossing number of M_n

We computed in Chapter 2 the crossing number of the family of 5-regular graphs M_n having the same crossing number. In Chapter 3, it turns out that these graphs are not crossing-critical. The point is that in the tile building this graphs, there is always a “spoke” that is not involved in the pair-disjoint paths. This phenomenon seems to hold for any tile building a sequence of 5-regular graphs. This suggests the following

conjecture:

Conjecture 5.1. *There exists no sequence $\{G_n\}_{n \geq N}^\infty$ of 5-regular crossing-critical graphs having a constant crossing number. $\square$*

5.3 On Sequences with Constant cr .

The result in Theorem 3.16 applies to all $T^\otimes(n)$ made up from a perfect planar tile T . But unlike Theorem 3.23, it does not provide the actual procedure for computing the crossing number of $T^\otimes(n)$. However, the following question is interesting.

Question 5.2. *Let T be a perfect planar tile. Is there a polynomial time algorithm for computing $\lim_{n \rightarrow \infty} T^\otimes(n)$? $\square$*

5.4 On Sequences with non-Constant cr .

The results in Chapter 4 reduce the complexity of computing $cr(o(T^n))$ by just computing $acr(T)$. But for a large tile T , it is still hard to compute $tcr(T^n)$. The next natural step is to reduce the complexity furthermore by just looking at the tile itself, i.e., $tcr(T)$.

Let π, π' be permutations and $T = (G, L, R)$ be a tile. Let $\pi T \pi' = (G, \pi(L), \pi'(R))$, i.e., the tile obtained from T by applying the permutations π and π' on L and R respectively. A tile is called a *permutation-resistant*, if for pair of permutations π and π' , $tcr(T) \leq tcr(\pi T \pi')$. It is obvious, from Lemma 3.12, that $tcr(T^n) \leq n.tcr(T)$. The inequality to the other side still need to be proved:

Conjecture 5.3. *Let T be a permutation-resistant tile. Then*

$$\exists N \forall n : n \geq N \Rightarrow \lim_{n \rightarrow \infty} \frac{cr(\circ(T^n))}{n} = tcr(T),$$

or in other words, $tcr(T) = acr(T)$. $\square$

This result has direct applications to the Generalized Petersen Graphs $P(n, k)$ and the Cartesian Product $C_m \times C_n$ of two cycles.

It is natural to try this with small wall sizes. Wall size 1 seems easy, but even for wall size 2 we have only partial results.

We gave an example of tile T with $acr(T) = 0.5$. The following general properties still need to be proved:

Conjecture 5.4. *For any tile T , $acr(T)$ is rational. $\square$*

Conjecture 5.5. *For any positive rational number r , there is a tile T such that $acr(T) = r$. $\square$*

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