An Algorithm to Construct Representations of Finite Groups

by

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the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

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*To my wife Azadeh
for her love and her patience*

The undersigned hereby recommend to the Faculty of
Graduate Studies and Research acceptance of the thesis

**An Algorithm to Construct
Representations of Finite Groups**

Submitted by

Vahid Dabbaghian-Abdoly

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for the degree of Doctor of Philosophy.

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Abstract

Let G be a finite group. It is easy to compute the character of G corresponding to a given complex representation, but much more difficult to compute a representation affording a given character. In part this is due to the fact that a class of equivalent representations contains no natural canonical representation.

Although there is a large literature devoted to computing representations, and methods are known for particular classes of groups, no general method has been proposed which is practical for any but very small groups. For example, the function `IrreducibleRepresentationsDixon` which is supplied in the latest version 4.3 of the computer algebra system GAP [GAP], is very slow in computing representations for even moderately sized groups and fails to compute a representation in many cases.

We shall describe an algorithm to compute an irreducible matrix representation $\mathcal{R}$ which affords a given character χ of a given group G . The algorithm uses properties of the structure of G which can be computed efficiently by a program such as GAP, theoretical results from representation theory, theorems from group theory (including the classification of finite simple groups), and linear algebra. The algorithm has been implemented in GAP for a general group G when the character supplied has degree < 32 .

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Notations

$\lambda \vdash n$	A partition λ of n
$\mathbb{Z}$	Ring of integers
$\mathbb{C}$	Field of complex numbers
$\mathbb{F}$	A field
$\mathbb{F}_q$	Field with q elements
$ G : H $	Index of H in G
$Z(G)$	Centre of G
G'	Derived subgroup of G
$[x, y], [H, G]$	Commutator
$\text{Syl}(p)$	Sylow p -subgroup
$\mathcal{C}$	A conjugacy class
$C_G(H)$	Centralizer of H in G
$N_G(H)$	Normalizer of H in G
$\text{Fit}(G)$	Fitting subgroup of G
$\text{Aut}(G)$	Group of automorphisms of G
$\text{Inn}(G)$	Group of inner automorphisms of G
$\text{Out}(G)$	Group of outer automorphisms of G
$\text{Soc}(G)$	Socle of G
$\text{Irr}(G)$	Set of all irreducible characters of G
$\langle S \rangle$	Group generated by S
$\ll B \gg$	Subspace generated by B
$M_{n \times n}$	Algebra of $n \times n$ matrices

$M_{n \times n}(\mathbb{F})$	Algebra of $n \times n$ matrices on field $\mathbb{F}$
I_n	Identity matrix of dimension $n \times n$
$\text{tr}(M)$	Trace of a matrix M
$\det(M)$	Determinant of a matrix M
$\chi, \phi, \psi, \theta, \dots$	Characters
$\mathbf{1}$	Trivial character
$\mathcal{R}, \mathcal{S}, \mathcal{T}, \dots$	Representations
$\ker(\theta)$	Kernel of θ
$\text{Im}(f)$	Image of f
$I_G(\theta)$	Inertia group of θ in G
ϕ^G	Induced character
$\mathcal{R}^G$	Induced representation
χ_H	Restriction of χ to H
$\hat{\chi}, \hat{M}$	Inverse image
$\langle \phi, \psi \rangle$	Inner product of ϕ and ψ
$H \times K$	Direct product of H and K
$H \otimes K$	Tensor product of H and K

Introduction

The purpose of this thesis is presenting an efficient algorithm to construct representations of finite groups. The thesis is organized as follows:

Chapter 1 contains the basic definitions and concepts which we need throughout this thesis. In this chapter we introduce some specific notation while discussing a few preliminary results. Chapter 2 gives a brief description about the literature of the problem which has been discussed in this thesis. It contains a collection of results about constructing representations of finite groups. In Chapter 3 we introduce the methods which are used during the algorithm, while G is not a perfect group. Character restriction method is one of these methods which have been presented by J.D. Dixon [Dx93]. Also we introduce the definition of χ -subgroups when χ is an irreducible character of group G . In this chapter we present an algorithm which is efficient for constructing representations of solvable groups. In Chapter 4 we use the classification of finite simple groups to get χ -subgroups of simple groups and their central covers for each irreducible character χ of degree less than 32. In Chapter 5 we describe methods to construct representations of perfect groups. Finally Chapter 6 contains an implementation of our algorithm in GAP.

Chapter 1

Preliminaries

In this chapter we give a brief description of some concepts of general group theory and representation theory that we shall use latter. All groups which we consider will be finite.

Let G be a finite group and V be a vector space of dimension n over $\mathbb{C}$. The *general linear group* $\text{GL}(V)$ is the group of all invertible linear transformations of V into itself. If G is a subgroup of $\text{GL}(V)$, then a subspace W of V is an *invariant* subspace for G if $Wx = W$ for all $x \in G$. We say G is *irreducible* if it has no invariant subspace except V and 0 . Considering a basis for V , each linear transformation of V into itself corresponds uniquely to a $n \times n$ matrix with entries in $\mathbb{C}$. This correspondence defines an isomorphism from $\text{GL}(V)$ onto the group $\text{GL}(n, \mathbb{C})$ of all nonsingular $n \times n$ matrices over $\mathbb{C}$. A subgroup of $\text{GL}(n, \mathbb{C})$ is called a *matrix group*. If G is a matrix group then G is irreducible if its corresponding isomorphic subgroup of $\text{GL}(V)$ is irreducible.

Definition 1.0.1. A **representation** of G is a homomorphism $\mathcal{R}$ of G into $\text{GL}(n, \mathbb{C})$ for some $n \geq 1$. The number n is called the *degree* of $\mathcal{R}$ and is denoted by

$\deg \mathcal{R}$.

A representation $\mathcal{R}$ is called *irreducible* if $\mathcal{R}(G)$ is an irreducible matrix group.

Definition 1.0.2. Let $\mathcal{R}$ be a representation of G . Then the **character** χ of G afforded by $\mathcal{R}$ is a function of G into $\mathbb{C}$ given by $\chi(g) = \text{tr}(\mathcal{R}(g))$ for $g \in G$, which is the sum of the diagonal entries of $\mathcal{R}(g)$.

In this case the degree of χ is $\chi(1) = \deg \mathcal{R}$. A character χ afforded by a representation $\mathcal{R}$ is an irreducible character if $\mathcal{R}$ is irreducible. Characters of degree 1 are called *linear characters*. In particular the function $1(g) = 1$ for all $g \in G$ is a linear character and is called the *principal* (trivial) character. In this thesis the principal character is denoted by $\mathbf{1}$. We denote the set of all irreducible characters of G by $\text{Irr}(G)$. The number of irreducible characters equals the number of conjugacy classes of G ([Is] Corollary 2.5).

Definition 1.0.3. There is a character ρ of G such that $\rho(1) = |G|$ and $\rho(g) = 0$ for $g \in G$ and $g \neq 1$. This character is called the **regular character** of G .

We have

$$\rho = \sum_{\chi \in \text{Irr}(G)} \chi(1)\chi$$

([Is] Lemma 2.11).

Definition 1.0.4. Let ϕ and ψ be characters of G . Then

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}$$

is the **inner product** of ϕ and ψ .

If χ_i and χ_j are two irreducible character of G then

$$\langle \chi_i, \chi_j \rangle = \delta_{ij}$$

where Kronecker δ_{ij} is 0 or 1 depending on whether i and j are unequal or equal, respectively (see [Is] Chapter 2).

Every character χ of G can be written in the form

$$\chi = \sum_{\chi_i \in \text{Irr}(G)} m_i \chi_i$$

where the m_i are integers given by $\langle \chi, \chi_i \rangle = m_i$.

Let χ be a character of G . Then $\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}$ and $Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}$. A character χ is called faithful if $\ker \chi = 1$. If N is a normal subgroup of G , then there is a one-to-one correspondence between characters of G/N and characters of G with kernel containing N . Therefore we can identify $\text{Irr}(G/N)$ with $\{\chi \in \text{Irr}(G) \mid N \subseteq \ker \chi\}$.

If H is a subgroup of G and $\mathcal{R}$ is a representation of G , then its restriction to H , denoted by $\mathcal{R}_H$, is a representation of H . Similarly, the restriction χ_H of a character χ of G to H is a character of H and we can write

$$\chi_H = \sum_{\psi \in \text{Irr}(H)} m_\psi \psi$$

for suitable integers m_ψ .

Definition 1.0.5. Let H be a subgroup of G and ϕ be a character of H . Then ϕ^G , the **induced character** on G , is given by

$$\phi^G(g) = \frac{1}{|H|} \sum_{x \in G} \check{\phi}(xgx^{-1}),$$

where $\check{\phi}$ is defined by $\check{\phi}(h) = \phi(h)$ if $h \in H$ and $\check{\phi}(y) = 0$ if $y \notin H$.

If H is a subgroup of G and χ and ϕ are characters of G and H , respectively, then the *Frobenius reciprocity* Theorem shows that

$$\langle \phi, \chi_H \rangle = \langle \phi^G, \chi \rangle$$

([Isch] Lemma 5.2).

Definition 1.0.6. Let χ be an irreducible character of G . Then χ is called a **primitive** character if $\chi \neq \phi^G$ for any character ϕ of a proper subgroup of G . It is called a **monomial** character if $\chi = \lambda^G$, where λ is a linear character of some (not necessary proper) subgroup of G .

Lemma 1.0.7 (The Frattini Argument). Let G be a group and N be a normal subgroup of G . If P is a Sylow p -subgroup of N then $G = N_G(P)N$.

Proof: Let $g \in G$. Since N is normal in G so $P^g \leq N$ and is a Sylow p -subgroup of N . Thus by the Sylow theorems there exists $x \in N$ such that $P^g = P^x$. Therefore $gx^{-1} \in N_G(P)$ which this implies $g \in N_G(P)N$ ■

Chapter 2

A Brief Survey of Earlier Work

Let G be a finite group. Given a complex irreducible representation of the group we can easily get the corresponding character. However, the converse problem is more difficult, because there is no natural canonical representation of G affording a given character. This seems to be the main reason why the construction of representations is so difficult.

The construction of representations is a classical problem which dates back over a century. One early example appears in ([Brs] page 302) where an irreducible representation of degree 3 for the alternating group on five elements is constructed.

Even earlier (1878) C. Jordan constructed the primitive linear groups of degree 2. His work was extended by H. F. Blichfeldt in his book [Bhf] who showed that there exist 6 primitive groups of degree 3 and 21 primitive groups of degree 4 up to projective equivalence. In each of these cases he gives a set of matrices generating the group. The program was continued by R. Brauer and D. Wales. They classified the primitive linear groups of degree 5 and 7 (see [Brr], [Wls1] and [Wls2]). Using the classification of finite simple groups, work of J. D. Dixon and A. Zalesskii [DxZk]

describes all primitive finite linear groups of prime degree. In these latter cases, however, the groups are not actually constructed.

In this chapter we briefly survey the methods and algorithms which have been proposed to construct representations of finite groups. We can divide these works into two main classes: the methods which deal with the problem theoretically and the methods and algorithms which deal with it in a practical sense. So far none of them has proved effective as a method of solving the problem for more than a very limited class of groups.

There are satisfactory methods to construct modular representations of groups over finite fields and Parker's *MEATAXE* is an effective tools for computing with modular representations (see [Prk] and [HtRs]). The *MEATAXE* works with representations over small finite fields, but so far has not be extended to infinite fields. Perhaps the first example of a computer program which was written to compute the complex irreducible representations of finite groups is the program written by S. Flodmark and E. Blokker [FdBk]. It was a program in *FORTRAN* to compute irreducible representations of groups of order up to 50.

A few years later C. Brott and J. Neubüser [BrNs] described a computer program to construct monomial representations of groups of orders up to 10^4 , by induction from subgroups.

H. Gollan [GlGb] describes an algorithm to construct irreducible representations of finite groups. His algorithm was implemented in the computer algebra system *SCRATCHPAD* and used to compute irreducible representations of the symmetric groups over $\mathbb{Z}$.

In the following sections we briefly describe the principal methods proposed to construct representations of finite groups.

2.1 Central Idempotent Elements

Suppose M is an irreducible module over the complex group algebra $\mathbb{C}G$ and χ is the irreducible character corresponding to M . It is a well known theorem that

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g \quad (2.1)$$

is a central idempotent in $\mathbb{C}G$ with the property that e_χ acts as the identity operator on M and annihilates all irreducible $\mathbb{C}G$ -modules not isomorphic to M ([CtRn] Sect. 33).

Janusz [Jnz] established a method to construct primitive idempotents as a product $e_\chi e_\varphi$ of central idempotents where e_χ corresponds to an irreducible character χ of group G and e_φ corresponds to a linear character φ of some subgroup H of G such that $\langle \varphi^G, \chi \rangle = 1$. He proved that if $G = H_0 \supseteq H_1 \supseteq \dots \supseteq H_r$ is a chain of subgroups of G and χ_i an irreducible character of H_i ($0 \leq i \leq r$) such that $\langle \chi_{i+1}, (\chi_i)_{H_{i+1}} \rangle = 1$ for each i , and $\chi_r(1) = 1$ then $e = e_{\chi_0} e_{\chi_1} \dots e_{\chi_r}$ is a primitive idempotent of $\mathbb{C}G$ and $M \cong \mathbb{C}Ge$. He showed that in particular this method always works for solvable groups.

N.Yelkenkaya [Yky] and E.Güzel [Gzl] used Janusz's method to construct the full system of primitive idempotents of the complex group algebras of $SL(2, q)$ and $SL(3, q)$, respectively. Using the character table of $G = SL(3, q)$ (determined by J.S. Frame and W.A. Simpson [SpFm]) where q is a power of prime p , Güzel [Gzl] showed that for each irreducible character χ of G there exists a linear character φ of a Sylow p -subgroup H of G such that $\langle \varphi^G, \chi \rangle = 1$ (see Section 4.3).

The summation in (2.1) runs over all the elements of G and when G is large it is very expensive to compute. Although the central idempotents are closely linked with the irreducible representations, direct use of the idempotents to compute rep-

representations appears limited.

2.2 Polynomial-Time Algorithms

A *polynomial-time algorithm* is an algorithm that is guaranteed to terminate within a number of steps which is a polynomial function of the size of the problem. In 1990 L. Babai and L. Rónyai [BaRy] established a polynomial-time algorithm to find a complete set of inequivalent irreducible representations of a finite group G over the field of complex numbers. The running time of this algorithm is bounded by a polynomial in $|G|$.

L. Baum [Bm] presents a polynomial-time algorithm to compute the irreducible representations of a finite supersolvable group G . Recall that a finite supersolvable group is a group whose chief factors are cyclic of prime order. Since each supersolvable group is an M -group, its irreducible representations are monomial (see [IsC] page 87). The running time of this algorithm was proved to be $O(|G|\log|G|)$ by L. Baum and M. Clausen [BmCs]. An implementation has been described in [MuCs].

Since the running times of the above algorithms grow at least as fast as the order of the group, they are feasible only for very small groups. From our point of view polynomial-time in terms of $|G|$ is the wrong measure.

2.3 Gröbner Basis and Resultant

Suppose $G = \langle s_1, \dots, s_n \mid r_1, \dots, r_m \rangle$ is a presentation of G where $G = \langle s_1, \dots, s_n \rangle$ and $r_1 := r_1(s_1, \dots, s_n) = 1, \dots, r_m := r_m(s_1, \dots, s_n) = 1$ are m relations in the generators of G . Let χ be an irreducible character of G of degree d and $\mathcal{R}$ be a representation of G affording χ . Let $\mathcal{R}(s_k) = (x_{ij}^{(k)})$ for each $k = 1, \dots, n$ where $x_{ij}^{(k)}$ are indeterminates

for $1 \leq i, j \leq d$. By considering each entry of

$$0 = \mathcal{R}(r_l) - I_d = r_l(\mathcal{R}(s_1), \dots, \mathcal{R}(s_n)) - I_d \text{ for } l = 1, \dots, m$$

as a equation, then we have a system of $m \times d^2$ equations in $n \times d^2$ indeterminates. There exist two classical methods to eliminate indeterminates of such a system of equations, namely the resultant method and the Gröbner base method (see [Gds] and [Cox]). When the degree of χ is a small number, the Gröbner basis and resultant methods may be useful, but otherwise it appears too complicated to use these methods. W. Plesken and B. Souvignier [PkSg97] use these methods for constructing representations of finite groups to analyse finitely presented groups.

2.4 Alternative Methods

Methods to construct representations of some particular groups are known. I. Piatetski-Shapiro [Pts] describes the structure of representations of $\mathrm{GL}(2, q)$ and M. Pergler [Prg] constructs the cuspidal representations of degree $q - 1$ of $\mathrm{GL}(2, q)$ where q is a power of a prime. In an unpublished paper J. D. Dixon and H. W. Gollan [DxGl] describe the construction of the irreducible representations of degree $(p \pm 1)/2$ of $\mathrm{PSL}(2, p)$. The representations of degree $p - 1$ of group $\mathrm{PSL}(2, p)$ for prime p have also been computed in [Bg] and [PkSg98]. These methods of constructing representations do not seem to be applicable in more general situations.

Some methods of constructing approximate representations of finite groups are proposed in [Dx70] and [BaFd]. If χ is an irreducible character of a group G such that χ_H is irreducible for a subgroup H of G , [Mkz] and [PkSg98] describe some methods to extend a representation of H affording χ_H to a representation of G affording χ (see Section 3.2).

W. Plesken and B. Souvignier [PkSg96] propose a method of constructing rational representations of finite groups by computing and analysing endomorphisms and homomorphisms of modules.

Chapter 3

Constructing Representations of Finite Groups

Let $G = \langle S \rangle$ be a finite group and χ be an irreducible character of G . In this chapter we give an overview of our method to construct a representation $\mathcal{R}$ affording χ . We proceed recursively, reducing the problem to smaller subgroups of G or characters of smaller degree until we obtain a problem which we can deal with directly.

The inputs of this algorithm are a given set S of generators of G , and the character χ . The output is a list of matrices $\mathcal{R}(x)$ ($x \in S$) corresponding to the generators of G . We will show that this algorithm works for all groups and all irreducible characters χ of degree less than 32 although in principle the same methods can be extended to characters of larger degree.

This chapter contains five sections. In the first four sections we introduce and explain the methods used. These methods are induction, extension, tensor product and character restriction. Finally, in the last section we explain how these methods of induction and extension enable us to reduce our problem to the case where G is

perfect.

3.1 Induction

Given a finite group G with a subgroup H . How can we get representations of G from those of H ? We can answer this by defining the *induced representation* and using the following theorems.

Definition 3.1.1. Let H be a subgroup of index n of group G and let $t_1, \dots, t_n$ be a left transversal of H in G . Let $\mathcal{R}$ be a representation of degree m of H . Then the *induced representation* $\mathcal{R}^G$ of G is defined

$$\mathcal{R}^G(g) = \begin{pmatrix} \check{\mathcal{R}}(t_1^{-1}gt_1) & \check{\mathcal{R}}(t_1^{-1}gt_2) & \dots & \check{\mathcal{R}}(t_1^{-1}gt_n) \\ \check{\mathcal{R}}(t_2^{-1}gt_1) & \check{\mathcal{R}}(t_2^{-1}gt_2) & \dots & \check{\mathcal{R}}(t_2^{-1}gt_n) \\ \vdots & \vdots & \ddots & \vdots \\ \check{\mathcal{R}}(t_n^{-1}gt_1) & \check{\mathcal{R}}(t_n^{-1}gt_2) & \dots & \check{\mathcal{R}}(t_n^{-1}gt_n) \end{pmatrix}$$

where $g \in G$ and

$$\check{\mathcal{R}}(x) = \begin{cases} \mathcal{R}(x) & \text{if } x \in H \\ 0 & \text{if } x \notin H. \end{cases}$$

Note: In general, $\mathcal{R}^G$ is not irreducible, even when $\mathcal{R}$ is.

Theorem 3.1.2. If H is a subgroup of index n of group G and $\mathcal{R}$ is a representation of degree m of H , then $\mathcal{R}^G$ is a representation of G of degree mn .

Proof: See ([CtRn] Section 12D) ■

Let θ be a character of a normal subgroup N of G and $g \in G$. Then θ^g is defined by $\theta^g(x) = \theta(gxg^{-1})$ for all $x \in N$. This is also a character of N and is called a *conjugate* of θ under G .

Theorem 3.1.3 (Clifford). *Let G be a finite group and χ be an irreducible character of G . If N is a normal subgroup of G then*

$$\chi_N = e \sum_{i=1}^t \theta_i$$

where $\theta = \theta_1, \dots, \theta_t$ are the distinct conjugates of an irreducible character θ of N and e is a positive integer.

Proof: [Is] Theorem 6.2 ■

Definition 3.1.4. *Let N be a normal subgroup of G and θ be an irreducible character of N . Then*

$$I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$$

*is the **inertia subgroup** of θ in G .*

Put $T := I_G(\theta)$. If we consider the action of G on $\text{Irr}(N)$ then T is the stabilizer of θ in this action so $N \subseteq T$ and the index $|G : T|$ is the size of the orbit of θ . Thus in the formula

$$\chi_N = e \sum_{i=1}^t \theta_i$$

of Theorem 3.1.3, we have $t = |G : T|$. Now if χ is an irreducible character of G then with the above assumptions we have

Theorem 3.1.5. *There exists an irreducible character ψ of T such that $\psi_N = \theta$ and $\psi^G = \chi$.*

Proof: See [Is] Theorem 6.11 ■

If $t > 1$ in Theorem 3.1.3, then T is a proper subgroup of G and given a representation $\mathcal{R}$ of T affording ψ , $\mathcal{R}^G$ is a representation of G affording $\psi^G = \chi$. This will enable us in many cases to reduce our problem of finding a representation affording χ to one of finding a representation affording a character on a proper subgroup.

3.2 Extension

Suppose G is a group and H is a subgroup of G . Let χ be an irreducible character of G . T. Minkwitz [Mkz] proposed a method to extend a representation affording χ_H of H to G , when the character χ_H is irreducible. In the following theorem we show another method to extend a representation.

Theorem 3.2.1. *Let G be a group with an irreducible character χ of degree n and H be a subgroup of G such that χ_H is irreducible. Let $\mathcal{R}_0$ be a representation of H affording χ_H . By a theorem of Burnside ([Wfz] Corollary 1.17) there exist $w_1, \dots, w_{n^2} \in H$ such that $\mathcal{B} = \{\mathcal{R}_0(w_1), \dots, \mathcal{R}_0(w_{n^2})\}$ is a basis for the full matrix algebra $M_{n \times n}$. Then $\mathcal{R}_0$ can be extended uniquely to a representation $\mathcal{R}$ of G affording χ and the entries of $\mathcal{R}(z)$ for $z \in G$ are determined by the equations*

$$\chi(w_k z) = \text{tr}(\mathcal{R}_0(w_k) \mathcal{R}(z)) \text{ for } k = 1, \dots, n^2.$$

Proof : Let $\mathcal{R}$ be a representation affording χ and extending $\mathcal{R}_0$. We find the entries of $\mathcal{R}(z) := [z_{ij}]$ for $z \in G$. Suppose $\mathcal{S} = \{E_{ij} \mid 1 \leq i, j \leq n\}$ is the standard basis of $M_{n \times n}$, where the entries of E_{ij} are 1 at position (i, j) and 0 elsewhere. Since $\mathcal{B}$ is a basis for $M_{n \times n}$ therefore for each $1 \leq i, j \leq n$ we have

$$E_{ij} = \sum_{k=1}^{n^2} \alpha_{ijk} \mathcal{R}_0(w_k) \text{ for some } \alpha_{ijk} \in \mathbb{C}$$

where the α_{ijk} are unique. Since

$$\chi(w_k z) = \text{tr} \mathcal{R}(w_k z) = \text{tr}(\mathcal{R}_0(w_k) \mathcal{R}(z))$$

therefore

$$\begin{aligned} \sum_{k=1}^{n^2} \alpha_{ijk} \chi(w_k z) &= \sum_{k=1}^{n^2} \alpha_{ijk} \text{tr}(\mathcal{R}_0(w_k) \mathcal{R}(z)) \\ &= \text{tr}(\sum_{k=1}^{n^2} \alpha_{ijk} \mathcal{R}_0(w_k) \mathcal{R}(z)) = \text{tr}(E_{ij} \mathcal{R}(z)) \end{aligned}$$

However $\text{tr}(E_{ij}\mathcal{R}(z)) = z_{ji}$ so

$$z_{ji} = \sum_{k=1}^{n^2} \alpha_{ijk} \chi(w_k z) \text{ for } 1 \leq i, j \leq n.$$

This completes the proof ■

3.3 Tensor Product

Suppose $G = HK$ is the central product of groups H and K . This means that H and K are normal subgroups of G and $[H, K] = 1$, so $H \cap K \leq Z(G)$. Suppose χ is an irreducible character of G and $\mathcal{R}$ is a representation of G affording χ . In this section we show that there exist representations $\mathcal{S}$ and $\mathcal{T}$ of H and K , respectively, such that $\mathcal{R}(hk) = \mathcal{S}(h) \otimes \mathcal{T}(k)$ for all $h \in H$ and $k \in K$.

Theorem 3.3.1. *Suppose $G = HK$ is the central product of groups H and K . Let $\chi \in \text{Irr}(G)$. Then $\chi_H = \psi(1)\phi$ and $\chi_K = \phi(1)\psi$ for some $\phi \in \text{Irr}(H)$ and $\psi \in \text{Irr}(K)$. Furthermore if $\mathcal{S}$ and $\mathcal{T}$ are representations of H and K affording ϕ and ψ , respectively, then $xy \mapsto \mathcal{S}(x) \otimes \mathcal{T}(y)$ is a representation of G affording χ .*

Proof : (Compare with [CtRn] Corollary 51.13.) Define $f : H \times K \rightarrow G$ by $f(x, y) = xy$ for $x \in H$ and $y \in K$. Since $[H, K] = 1$, f is a homomorphism onto G . By ([Isc] Lemma 2.22) each element of $\text{Irr}(G)$ corresponds to an element of $\text{Irr}(H \times K)$. Suppose $\hat{\chi} \in \text{Irr}(H \times K)$ is corresponding to χ . Then by ([Isc] Theorem 4.21) there exist irreducible characters $\phi \in \text{Irr}(H)$ and $\psi \in \text{Irr}(K)$ such that

$$\hat{\chi}((h, k)) = \phi(h)\psi(k) \text{ for all } h \in H, k \in K.$$

Since $(h, 1) \in H \times K$ for each $h \in H$ so $\chi(h) = \hat{\chi}(h, 1) = \phi(h)\psi(1)$ and this means $\chi_H = \psi(1)\phi$. Similarly we have $\chi_K = \phi(1)\psi$.

Suppose $\mathcal{S}$ and $\mathcal{T}$ are representations of H and K affording ϕ and ψ , respectively. We show that if $x, x' \in H$ and $y, y' \in K$ then $xy = x'y'$ implies $\mathcal{S}(x) \otimes \mathcal{T}(y) = \mathcal{S}(x') \otimes \mathcal{T}(y')$. Since $xy = x'y'$ and $[H, K] = 1$ so $z := x^{-1}x' = y(y')^{-1} \in H \cap K \subseteq Z(G)$. On the other hand $\mathcal{S}(z) = \mathcal{S}(1)\lambda(z)$ and $\mathcal{T}(z) = \mathcal{T}(1)\mu(z)$ for some linear characters λ and μ of $H \cap K$ (see [Iscl] Lemma 2.27). Since $z \in H \cap K$ so

$$\begin{aligned}\chi(z) &= \psi(1)\phi(z) = \psi(1)\phi(1)\lambda(z) \\ &= \phi(1)\psi(z) = \phi(1)\psi(1)\mu(z)\end{aligned}$$

and this means $\lambda(z) = \mu(z)$. Thus

$$\begin{aligned}\mathcal{S}(x') \otimes \mathcal{T}(y') &= \mathcal{S}(xz) \otimes \mathcal{T}(yz^{-1}) \\ &= \mathcal{S}(x)\lambda(z) \otimes \mathcal{T}(y)\mu(z^{-1}) \\ &= \mathcal{S}(x) \otimes \mathcal{T}(y)\end{aligned}$$

as required. This shows $xy \mapsto \mathcal{S}(x) \otimes \mathcal{T}(y)$ is well defined.

Let $h \in H$ and $k \in K$. Then

$$\text{tr}(\mathcal{S}(h) \otimes \mathcal{T}(k)) = \text{tr}(\mathcal{S}(h))\text{tr}(\mathcal{T}(k)) = \phi(h)\psi(k) = \hat{\chi}((h, k)) = \chi(hk).$$

This implies that $hk \mapsto \mathcal{S}(h) \otimes \mathcal{T}(k)$ is a representation of G affording χ ■

3.4 Character Restriction

An efficient and simple method to construct the representations of finite groups in certain cases has been presented by John D. Dixon [Dx93]. In this section we want to explain that method. We follow the original paper quite closely except for some misprints which have been corrected here.

Suppose G is a finite group of size m and χ is an irreducible character of G of degree d which is afforded over some field $\mathbb{F} \subseteq \mathbb{C}$. Then

$$e := (d/|G|) \sum_{x \in G} \chi(x^{-1})x$$

is a central idempotent in $\mathbb{F}G$ which corresponds to a representation ρ affording χ of $\mathbb{F}G$ with kernel $(1 - e)\mathbb{F}G$ and image $M_{d \times d}(\mathbb{F})$ (see Satz V.5.15 of [Hup]).

Lemma 3.4.1. *If $a \in \mathbb{F}G$ such that $\rho(a)$ has rank 1, then $ae\mathbb{F}G$ is a submodule of $e\mathbb{F}G$ of dimension d which affords the character χ .*

Proof: Since $\ker(\rho) = (1 - e)\mathbb{F}G$ so the restriction of ρ on $e\mathbb{F}G$ is an isomorphism. Hence $e\mathbb{F}G \cong \rho(e\mathbb{F}G) = M_{d \times d}(\mathbb{F})$ and has dimension d^2 . Now if $a \in \mathbb{F}G$ such that $\rho(a)$ has rank 1 then

$$ae\mathbb{F}G \cong \rho(ae\mathbb{F}G) = \rho(a)\rho(e\mathbb{F}G) = \rho(a)M_{d \times d}(\mathbb{F}).$$

Suppose E_{ij} is a $d \times d$ matrix with entries 1 at position (i, j) and 0 elsewhere. Since $\rho(a)$ has rank 1 so using the row and column operations we have $\rho(a) = CE_{11}D$ where C and D are two invertible matrices. Therefore

$$ae\mathbb{F}G \cong CE_{11}DM_{d \times d}(\mathbb{F}) = CE_{11}M_{d \times d}(\mathbb{F}).$$

Since $\{E_{ij} \mid 1 \leq i, j \leq d\}$ is a basis (standard basis) for $M_{d \times d}(\mathbb{F})$, $E_{11}E_{1j} = E_{1j}$ and $E_{11}E_{ij} = 0$ for $i \geq 2$ so $\{E_{1j} \mid 1 \leq j \leq d\}$ is a basis for $E_{11}M_{d \times d}(\mathbb{F})$. Thus $E_{11}M_{d \times d}(\mathbb{F})$ has dimension d and since C is invertible so $CE_{11}M_{d \times d}(\mathbb{F})$ has dimension d and this means that $ae\mathbb{F}G$ is a submodule of $e\mathbb{F}G$ of dimension d .

Now we prove $ae\mathbb{F}G$ affords χ . Suppose M is an irreducible submodule which corresponds to χ . Since e is an idempotent element corresponding to χ so e annihilates all submodules not isomorphic to M . Since dimension of $e\mathbb{F}G$ is d^2 so

$e\mathbb{F}G = dM$. On the other hand $ae\mathbb{F}G$ is a submodule of $e\mathbb{F}G$ of dimension d so $ae\mathbb{F}G \cong M$ ■

The construction of a suitable “rank-1” element is considerably simplified when we can find a subgroup H such that for some linear character θ of H , $\langle \chi_H, \theta \rangle = 1$. If H is such subgroup we define

$$a := \sum_{h \in H} \theta(h^{-1})h \in \mathbb{F}H$$

then $a^2 = |H|a$ and this means $a/|H|$ is an idempotent element and

$$\begin{aligned} \text{tr}(\rho(a)) &= \text{tr}(\rho(\sum_{h \in H} \theta(h^{-1})h)) \\ &= \sum_{h \in H} \theta(h^{-1})\text{tr}(\rho(h)) \\ &= \sum_{h \in H} \theta(h^{-1})\chi(h) \\ &= |H|\langle \chi_H, \theta \rangle = |H|. \end{aligned}$$

Therefore $\rho(a/|H|)$ is an idempotent with a single eigenvalue equal 1 and all other eigenvalues 0. In particular, $\rho(a)$ has rank 1.

Set $f := (|G|/d)ae$, where e is the central idempotent defined above. Then $f = \sum_{t \in G} \alpha(t)t$, where

$$\alpha(t) = \sum_{h \in H} \theta(h^{-1})\chi(ht^{-1}) \text{ for all } t \in G; \quad (3.1)$$

and from the lemma above $f\mathbb{F}G$ is a module of rank d which affords χ .

Suppose $A := [\alpha(yx^{-1})]_{x,y \in G}$ is a $m \times m$ matrix. Since $fx = \sum_{t \in G} \alpha(tx^{-1})t$, the x th row of A gives the components of fx in $\mathbb{F}G$. Therefore $\text{rank } A = \dim f\mathbb{F}G = d$ and $\alpha(t^{-1}) = \overline{\alpha(t)}$ implies A is hermitian. Hermitian matrices are diagonalizable,

and so if $x_1, \dots, x_d \in G$ index a set of d linearly independent rows of A then the $d \times d$ submatrix

$$[\alpha(x_j x_i^{-1})] \quad (3.2)$$

is non-singular.

The following theorem shows that if we define the matrices $A(x) := [\alpha(x_j x^{-1} x_i^{-1})]_{1 \leq i, j \leq d}$ for each $x \in G$ then the mapping

$$x \longmapsto A(x)A(1)^{-1} \quad (3.3)$$

defines a representation of degree d of G which affords the character χ .

Theorem 3.4.2. *If the indices $x_1, \dots, x_d$ correspond to a set of linearly independent rows for the matrix A , then the mapping (3.3) defines a representation of G of degree d affording χ .*

Proof: First recall that $f\mathbb{F}G$ is an $\mathbb{F}G$ -module of dimension d which affords the character χ , and that $fx_1, \dots, fx_d$ are linearly independent elements of $\mathbb{F}G$ by choice of $x_1, \dots, x_d$. Therefore $fx_1, \dots, fx_d$ is an $\mathbb{F}$ -basis for $f\mathbb{F}G$, and so the mapping $x \mapsto [\xi_{ij}]$ defined by

$$(fx_i)x = \sum_{j=1}^d \xi_{ij}(fx_j)$$

is a representation of G affording χ . This latter relation is equivalent to

$$\alpha(tx^{-1}x_i^{-1}) = \sum_{j=1}^d \xi_{ij}\alpha(tx_j^{-1}) \text{ for all } t \in G \text{ and } i = 1, \dots, d.$$

In particular, $A(x) = [\xi_{ij}]A(1)$, and so $[\xi_{ij}] = A(x)A(1)^{-1}$. Thus (3.3) defines the required representation ■

We shall call this algorithm the Character Restriction Method and introduce the following definition.

Definition 3.4.3. Let G be a finite group and χ be an irreducible character of G . The subgroup H of G is called a χ -subgroup if there exist a linear character θ of H such that $\langle \chi_H, \theta \rangle = 1$.

As we see there are two problems that we need to solve to use the character restriction method to construct a representation affording χ .

Problem 1. Find a χ -subgroup H of G .

Problem 2. Find a set of d linearly independent rows of the matrix A .

Neither of these problems is dealt with in any generality in [Dx93].

To solve Problem 2 the following theorem shows that on the average at most $2d$ random selection of rows are enough to yield a set with the desired property.

Theorem 3.4.4. Let $G = \langle T \rangle$ be a group where $T = \{t_1, \dots, t_n\}$ and let V be an irreducible G -module. Suppose U is a proper subspace of V . Put $X = \{t_1^{\epsilon_1} \dots t_n^{\epsilon_n} \mid \epsilon_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq n\}$. As the ϵ_i take values 0 or 1 at random, the probability that $Ux \neq U$ for $x \in X$, is at least $1/2$.

Proof: (Compare with [CoFi] Prop. 2.1.) For $g \in G$ we say g has $\mathcal{P}$ if $Ug \neq U$. At least one t_i has $\mathcal{P}$ since otherwise $Ug = U$ for all $g \in G$ contrary to hypothesis. Now every element of X has the form

$$x = \overbrace{t_1^{\epsilon_1} \dots t_{k-1}^{\epsilon_{k-1}}}^y t_k^{\epsilon_k} \overbrace{t_{k+1}^{\epsilon_{k+1}} \dots t_n^{\epsilon_n}}^z.$$

where k is the largest i such that t_i has $\mathcal{P}$.

To prove the theorem it is enough to show that for fixed y and z at least one of $yt_k z$ or yz has $\mathcal{P}$. Indeed $Uz = U$ by the choice of k . Hence if $Uyz = U$ then $Uy = U$. But then $Uyt_k = Ut_k \neq U$ and so $yt_k z$ has $\mathcal{P}$ as required ■

We can use the Theorem above to find d linearly independent rows of the matrix A defined in the character restriction method as follows. Let $R = \{r_1, \dots, r_m\}$ be the set of rows of A , $V = \langle\langle R \rangle\rangle$ be the vector space spanned by the rows of A , and

$$X = \{t_1^{\epsilon_1} \dots t_m^{\epsilon_m} \mid \epsilon_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq m\}$$

where $t_1, \dots, t_m$ is a generators set for G . Note that V is an irreducible module for G .

Choose $u_1 \in R$ and let $U_1 = \langle\langle u_1 \rangle\rangle$ so $0 \neq U_1 \leq V$. If $U_1 = V$ then we are finished, otherwise since the probability of choosing $x_1 \in X$ such that $(U_1)x_1 \neq U_1$ is $\geq 1/2$, an average of at most 2 selections is required to find $x_1 \in X$ such that $(U_1)x_1 \neq U_1$. Now suppose $u_2 := u_1x_1$ and $U_2 = \langle\langle u_1, u_2 \rangle\rangle$ so $0 \neq U_1 < U_2 \leq V$. If $U_2 < V$ then again after an average of at most 2 selections we find $x_2 \in X$ such that $(U_2)x_2 \neq U_2$. Hence we can add a vector $u_3 \in (U_2)x_2 \setminus U_2$ to a basis of U_2 to get U_3 such that $U_2 < U_3 \leq V$. Since the dimension of V is d , after an average of at most $2d$ selections we get a set $u_1, \dots, u_d$ of rows of A which are linearly independent.

For Problem 1 we may need to examine the full lattice of subgroups of G to find a χ -subgroup. Indeed there is no guarantee that any χ -subgroup exists for a given character χ . G. Glauberman has given examples of solvable groups where no such subgroups exist; see [Jnz]. Using GAP we have also found non-solvable examples. For example the covering group $6.A_7$ of the group A_7 has 3 characters of degree 36 and for two of them there is no such subgroup. In the next chapter we show that for all irreducible characters χ of degree less than 32 of simple groups and their covering groups there exists a χ -subgroup, and then in most cases this can be chosen as a p -subgroup.

3.5 Reduction to the Perfect Case

If G is not perfect then we can reduce our problem to constructing a representation for a proper subgroup. Suppose

$$G = G^{(0)} \triangleright G^{(1)} \triangleright \dots \triangleright G^{(\infty)}$$

is the derived series of G . If G is solvable then $G^{(\infty)} = 1$, otherwise $G^{(\infty)}$ is a non-trivial perfect subgroup of G . Since each $G^{(i)}$ is normal in G , Theorem 3.1.3 shows that for each $i \geq 1$ we have

$$\chi_{G^{(i)}} = e_i \sum_{j=1}^{t_i} \theta_{ij}. \quad (3.4)$$

where $\theta_i = \theta_{i1}, \dots, \theta_{it_i}$ are distinct G -conjugates of the irreducible character θ_i of $G^{(i)}$. Now we consider three different possibilities for the values of t_i and e_i in (3.4) for $i \geq 1$.

Case I. At least one $t_i > 1$:

Put $\theta := \theta_{i1}$ and $T := I_G(\theta)$, so $G^{(i)} \leq T < G$ and $|G : T| = t_i$. Then by Theorem 3.1.5 there exists an irreducible character ψ of T such that $\psi_{G^{(i)}} = \theta$ and $\psi^G = \chi$. Hence if $\mathcal{R}$ is a representation of T affording ψ then $\mathcal{R}^G$ is a representation of G affording χ .

Case II. All $t_i = 1$ and $e_1 > 1$:

Let

$$G = K_0 \triangleright K_1 \triangleright \dots \triangleright K_m = G^{(1)}$$

be a composition series of G through $G^{(1)}$. Then for $1 \leq u \leq m$, $|K_{u-1} : K_u|$ is a prime and K_u is a normal subgroup of G . Since $t_1 = 1$ and $e_1 > 1$, $\chi_{G^{(1)}}$ is reducible. This means there exists $1 \leq u \leq m$ such that $\chi_{K_{u-1}}$ is irreducible

and χ_{K_u} is reducible. Since $|K_{u-1} : K_u| = p$ is a prime and K_u is normal in G ([Isc] Corollary 6.19) shows that

$$\chi_{K_u} = \sum_{j=1}^p \varphi_j$$

where $\varphi = \varphi_1, \dots, \varphi_p \in \text{Irr}(K_u)$ are the distinct conjugates of φ in G . Hence if we consider $T := I_G(\varphi)$ then $K_u \leq T < G$ and $|G : T| = p$, and by Theorem 3.1.5 there exists $\psi \in \text{Irr}(T)$ such that $\psi_{K_u} = \varphi$ and $\psi^G = \chi$. Now if $\mathcal{R}$ is a representation of T affording ψ then $\mathcal{R}^G$ is a representation of G affording χ .

Case III. All $t_i = 1$ and $e_1 = 1$:

If for some $j \geq 1$, $e_j = 1$ and $e_{j+1} > 1$ then $\chi_{G^{(j)}}$ is an irreducible character of $G^{(j)}$. Let $\mathcal{R}$ be a representation affording $\chi_{G^{(j)}}$ for $G^{(j)}$. Then Theorem 3.2.1 shows how to construct an extension of $\mathcal{R}$ to G which is a representation of G affording χ .

If all $e_i = 1$ then put $H := G^{(\infty)}$. If G is solvable then $H = 1$ and this will never happen except in the trivial case when $\chi(1) = 1$. If G is not solvable then χ_H is an irreducible character of $H \neq 1$. Since H is perfect, we have to deal with the irreducible characters of perfect groups. This is the topic which we discuss in chapter 5.

Chapter 4

Simple Groups and Covers

From the previous chapter recall that if χ is an irreducible character of G then a subgroup H of G is called a χ -subgroup if there exists a linear character θ of H such that $\langle \chi_H, \theta \rangle = 1$. If H is a χ -subgroup then using the character restriction method we can construct a representation of G affording χ .

If G is a large group then it is very complicated to check all subgroups of G to find a χ -subgroup. In the following sections we use the classification of finite simple groups and the Atlas of Finite Groups [Atlas] to show that for all characters χ of degree less than 32 for simple groups and their covering groups there always exist χ -subgroups, and in most cases these can be chosen as p -subgroups.

This chapter has four sections. The first one deals with alternating groups and their covering groups. The second and third sections discuss the groups $\text{PSL}(2, q)$ and $\text{PSL}(3, q)$, $\text{PSU}(3, q)$ and their covers, respectively. Finally the last section deals with other simple groups and some exceptional cases. We summarize our results with in the table (Table 1.8).

4.1 Irreducible Characters of Symmetric Groups and Alternating Groups

In this section we study the structure of irreducible characters of symmetric groups and alternating groups. We use theorems from the theory of symmetric groups and alternating groups to find χ -subgroups for the alternating groups and their covers when χ is a non-linear character of degree less than 32. The notation used in this section can be found in [Sgn].

Definition 4.1.1. A **partition** $\lambda = (\lambda_1, \dots, \lambda_l)$ of a natural number n is a weakly decreasing sequence $\lambda_1 \geq \dots \geq \lambda_l > 0$ of integers with $|\lambda| = \sum_{i=1}^l \lambda_i = n$, for short we write $\lambda \vdash n$. The integer $l = l(\lambda)$ is the length of λ , the numbers λ_i are the parts of λ . We also write the partition exponentially as $\lambda = (l_1^{a_1}, \dots, l_m^{a_m})$, $l_1 > \dots > l_m > 0$, when we have a_i parts of size l_i .

Definition 4.1.2. Suppose $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$. The **Ferrers diagram** of λ is an array of n dots having l left-justified rows with row i containing λ_i dots for $1 \leq i \leq l$.

The dot in row i and column j has coordinates (i, j) , as in a matrix. As an example, the partition $\lambda = (4, 4, 2, 1) = (4^2, 2, 1)$ has Ferrers diagram

Given a partition λ , one obtains another partition λ' , the conjugate partition of λ , by transposing the Ferrers diagram about its main diagonal. So for $\lambda = (4^2, 2, 1)$ the Ferrers diagram for λ' is

and $\lambda' = (4, 3, 2^2)$.

Definition 4.1.3. *If λ is a partition of n , then each inner corner dot of the Ferrers diagram of λ is a **node** whose removal leaves a diagram, the Ferrers diagram of a partition of $n - 1$.*

Note that the inner corners of λ are exactly those nodes at the end of a row and column of the diagram of λ . Thus the nodes at the end of rows of $\lambda = (4^2, 2, 1)$ have coordinates $(1, 4)$, $(2, 4)$, $(3, 2)$ and $(4, 1)$.

We know that the number of irreducible characters of a group is equal to the number of conjugacy classes, which in the case of S_n is the number of partitions of n . Therefore we can label the irreducible characters of S_n by partitions of n . We denote the irreducible character labelled by the partition λ by $[\lambda]$, so $\text{Irr}(S_n) = \{[\lambda] \mid \lambda \vdash n\}$. Now for our purpose we introduce the following definition (this is not a standard definition).

Definition 4.1.4. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition of n . We say the partition has **level** k if $k = \lambda_2 + \dots + \lambda_l$ ($= n - \lambda_1$). Similarly we say that the corresponding irreducible character $[\lambda]$ of S_n has level k .*

The principal character $\mathbf{1} = [n]$ is a character of level 0. By the definition above the number of irreducible characters of level k of S_n is equal to the number of partitions of k . For example the characters $[6, 2, 1]$, $[6, 3]$ and $[6, 1^3]$ of S_9 have level 3.

The following theorem shows the behavior of irreducible characters of S_n when we restrict them to the subgroup S_{n-1} .

Theorem 4.1.5 (Branching Theorem). *If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \vdash n$, then*

$$[\lambda]_{S_{n-1}} = \sum_{i=1}^l \epsilon_i [\lambda_1, \dots, \lambda'_i, \dots, \lambda_l] \text{ where } \lambda'_i = \begin{cases} \lambda_i - 1 & \text{if } \lambda_i > \lambda_{i+1} \text{ or } i = l \\ \lambda_i & \text{if } \lambda_i = \lambda_{i+1} \end{cases} \quad (4.1)$$

and $\epsilon_i = 0$ if $\lambda'_i = \lambda_i$ otherwise $\epsilon_i = 1$.

Proof: See ([Sgn], Theorem 2.8.3). ■

Lemma 4.1.6. *If $r \geq 1$ and $[\lambda]$ is an irreducible character of S_n of level $k \geq r$ then the level of each constituent of $[\lambda]_{S_{n-r}}$ is at most k . Furthermore $[\lambda]_{S_{n-r}}$ has at least one constituent of level $k - r$. In particular for $r = k$ the principal character is a constituent of $[\lambda]_{S_{n-k}}$.*

Proof: By the Definition 4.1.4 and the Branching Theorem the constituents of $[\lambda]_{S_{n-1}}$ have level k , in case $\lambda'_1 = \lambda_1 - 1$, and level $k - 1$ in the other cases. On the other hand, if $k > 0$ then $[\lambda_1, \dots, \lambda_{l-1}, \lambda_l - 1]$ is a constituent of level $k - 1$ of $[\lambda]_{S_{n-1}}$. Now by induction on r the constituents of $[\lambda]_{S_{n-r}}$ have level at most k and $[\lambda]_{S_{n-r}}$ has a constituent of level $k - r$. If $r = k$ then $[\lambda]_{S_{n-r}}$ has a constituent of level 0 which is the principal character ■

Definition 4.1.7. *If $\nu = (i, j)$ is a node in the diagram of λ , then its **hook** is*

$$H_\nu = H_{i,j} := \{(i, j') \mid j' \geq j\} \cup \{(i', j) \mid i' \geq i\}$$

with corresponding **hooklength**

$$h_\nu = h_{i,j} := |H_{i,j}|.$$

To illustrate, if $\lambda = (4^2, 2, 1)$, then the circle dots in

are the hook $H_{2,2}$ with hooklength $h_{2,2} = 4$.

The following theorem gives the degrees of irreducible characters of S_n .

Theorem 4.1.8 (Hook Formula). *If $\lambda \vdash n$, then*

$$[\lambda](1) = \frac{n!}{\prod_{(i,j) \in \lambda} h_{i,j}}.$$

Proof: See ([Sgn], Theorem 3.10.2) ■

Theorem 4.1.9. *Let $k \geq 0$ be fixed, and suppose $(\lambda_2, \dots, \lambda_l) \vdash k$. Consider the irreducible character $[\lambda] = [n - k, \lambda_2, \dots, \lambda_l]$ of S_n of level k . Then $[\lambda](1)$ is a polynomial in n of degree k .*

Proof: Let H_{ij} be the hooks of the diagram of $[\lambda]$ corresponding to the nodes (i, j) . Using Definition 4.1.7 we have $|H_{ij}| = h_{ij} \leq k$ for $i \geq 2$. Also there exist $n - k$ hooks, H_{1j} , such that $|H_{1j}| = h_{1j}$ has a value of the form $(n - m_j)$ with $m_1 < m_2 < \dots < m_{n-k}$ for $1 \leq j \leq n - k$. Simplifying the Hook Formula,

$$[\lambda](1) = \frac{n!}{\prod_{(i,j) \in \lambda} h_{i,j}} = \frac{n!}{(\prod_{(1,j) \in \lambda} h_{1,j})(\prod_{(i \geq 2, j) \in \lambda} h_{i,j})}$$

only k factors remain in the numerator. This means $[\lambda](1)$ is a polynomial in n of degree k ■

Theorem 4.1.10. *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of n of level k . If $0 \leq r \leq \lambda_1 - \lambda_2$, then $[\lambda_1 - r, \lambda_2, \dots, \lambda_l]$ is a constituent with multiplicity one in $[\lambda]_{S_{n-r}}$ and all other constituents have level $< k$*

Proof: If $\lambda_1 - \lambda_2 = 0$ then $r = 0$ and there is nothing to prove. Now suppose $\lambda_1 - \lambda_2 > 0$. We prove the theorem by induction on r .

If $r = 1$ the Branching Theorem shows

$$[\lambda]_{S_{n-1}} = [\lambda_1 - 1, \lambda_2, \dots, \lambda_l] + \sum_{i=2}^l \epsilon_i [\lambda_1, \lambda_2, \dots, \lambda'_i, \dots, \lambda_l]$$

where

$$\lambda'_i = \begin{cases} \lambda_i - 1 & \text{if } \lambda_i > \lambda_{i+1} \text{ or } i = l \\ \lambda_i & \text{if } \lambda_i = \lambda_{i+1}. \end{cases}$$

and $\epsilon_i = 1$ if $\lambda'_i = \lambda_i - 1$ otherwise $\epsilon_i = 0$. The constituent $[\lambda_1 - 1, \lambda_2, \dots, \lambda_l]$ has level k and has multiplicity one in $[\lambda]_{S_{n-1}}$ and the other constituents have level $k - 1$.

Now suppose the theorem is true for $r \geq 1$ and we have

$$[\lambda]_{S_{n-r}} = [\lambda_1 - r, \lambda_2, \dots, \lambda_l] + \sum_j e_j \psi_j$$

where $[\lambda_1 - r, \lambda_2, \dots, \lambda_l]$ is the constituent of level k and $\psi_j \in \text{Irr}(S_{n-r})$ have level $< k$. We prove the theorem for $r + 1$.

If we restrict $[\lambda]$ to $S_{n-(r+1)}$ then we get

$$[\lambda]_{S_{n-(r+1)}} = [\lambda_1 - r, \lambda_2, \dots, \lambda_l]_{S_{n-(r+1)}} + \sum_j e_j (\psi_j)_{S_{n-(r+1)}}.$$

Using the Branching theorem we have

$$[\lambda_1 - r, \lambda_2, \dots, \lambda_l]_{S_{n-(r+1)}} = [\lambda_1 - (r + 1), \lambda_2, \dots, \lambda_l] + \sum_{i=2}^l \epsilon_i [\lambda_1 - r, \lambda_2, \dots, \lambda'_i, \dots, \lambda_l]$$

where $[\lambda_1 - (r + 1), \lambda_2, \dots, \lambda_l]$ has level k and $[\lambda_1 - r, \lambda_2, \dots, \lambda'_i, \dots, \lambda_l]$ have level $k - 1$. On the other hand since the characters ψ_j have level $< k$ so by Lemma 4.1.6 the constituents of $(\psi_j)_{S_{n-(r+1)}}$ have level $< k$. This means none of the constituents of $(\psi_j)_{S_{n-(r+1)}}$ are equal $[\lambda_1 - (r + 1), \lambda_2, \dots, \lambda_l]$. Therefore $[\lambda_1 - (r + 1), \lambda_2, \dots, \lambda_l]$ is the only constituent of $[\lambda]_{S_{n-(r+1)}}$ of level k and has multiplicity one. This completes the proof ■

Theorem 4.1.11. *Let $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ have level k . Suppose $\lambda_1 = n - k \geq k + \lambda_2$, then for all $m \geq n$ the characters*

$$[m - k, \lambda_2, \dots, \lambda_l]_{S_{n-k}} \text{ and } [n - k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$$

have the same constituents (although the multiplicities may be different). Furthermore $[n - 2k, \lambda_2, \dots, \lambda_l]$ is a constituent with multiplicity one for both of these characters.

Proof: We use induction on m . If $m = n$ then there is nothing to prove. Suppose $m > n$ and

$$[m - k, \lambda_2, \dots, \lambda_l]_{S_{n-k}} \text{ and } [n - k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$$

have the same constituents and $[n - 2k, \lambda_2, \dots, \lambda_l]$ is the only constituent of level k with multiplicity one for both of these characters. We prove the Theorem for $m + 1$.

Using the Branching Theorem we have

$$[m + 1 - k, \lambda_2, \dots, \lambda_l]_{S_m} = [m - k, \lambda_2, \dots, \lambda_l] + \sum_{i=2}^l \epsilon_i [m + 1 - k, \lambda_2, \dots, \lambda'_i, \dots, \lambda_l] \quad (4.2)$$

where

$$\lambda'_i = \begin{cases} \lambda_i - 1 & \text{if } \lambda_i > \lambda_{i+1} \text{ or } i = l \\ \lambda_i & \text{if } \lambda_i = \lambda_{i+1} \end{cases}$$

Since $[m - k, \lambda_2, \dots, \lambda_l]$ is a constituent of $[m + 1 - k, \lambda_2, \dots, \lambda_l]_{S_m}$ and by assumption

$$[m - k, \lambda_2, \dots, \lambda_l]_{S_{n-k}} \text{ and } [n - k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$$

have the same constituents so each constituent of $[n - k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$ is a constituent of $[m + 1 - k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$.

Now we show each constituent of $[m + 1 - k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$ is a constituent of $[n - k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$.

Let $1 \leq r \leq k$. We claim each constituent of $[m+1-k, \lambda_2, \dots, \lambda_l]_{S_{(m+1)-r}}$ is either a constituent of $[m-k, \lambda_2, \dots, \lambda_l]_{S_{m-(r-1)}}$ or a character of level $k-r$ (note that $(m+1)-r = m-(r-1)$).

Using the Branching Theorem, if $r = 1$ then $[m+1-k, \lambda_2, \dots, \lambda_l]_{S_{(m+1)-1}}$ has two different constituents, $[m-k, \lambda_2, \dots, \lambda_l] = [m-k, \lambda_2, \dots, \lambda_l]_{S_{m-(1-1)}}$ and $[m+1-k, \mu_2, \dots, \mu_l]$ where $\mu_2 + \dots + \mu_l = k-1$ and this proves our claim for $r = 1$. If $r = 2$ then the constituents of $[m+1-k, \mu_2, \dots, \mu_l]_{S_{(m+1)-2}}$ are either $[m-k, \mu_2, \dots, \mu_l]$ which is a constituent of $[m-k, \lambda_2, \dots, \lambda_l]_{S_{m-(2-1)}}$ or $[m+1-k, \eta_2, \dots, \eta_l]$ where $\eta_2 + \dots + \eta_l = k-2$. This means each constituent of $[m+1-k, \lambda_2, \dots, \lambda_l]_{S_{(m+1)-2}}$ is either a constituent of $[m-k, \lambda_2, \dots, \lambda_l]_{S_{m-(2-1)}}$ or a character of level $k-2$. Therefore for $1 \leq r \leq k$ if $[m+1-k, \tau_2, \dots, \tau_l]$ is a constituent of $[m+1-k, \lambda_2, \dots, \lambda_l]_{S_{(m+1)-r}}$ of level $r-1$ then each constituent of $[m+1-k, \tau_2, \dots, \tau_l]_{S_{(m+1)-r}}$ is either $[m-k, \tau_2, \dots, \tau_l]$ which is a constituent of $[m-k, \lambda_2, \dots, \lambda_l]_{S_{m-(r-1)}}$ or $[m+1-k, \zeta_2, \dots, \zeta_l]$ where $\zeta_2 + \dots + \zeta_l = k-r$ and this proves our claim. Now if $r = k$ then each constituent of $[m+1-k, \lambda_2, \dots, \lambda_l]_{S_{(m+1)-k}}$ is either a constituent of $[m-k, \lambda_2, \dots, \lambda_l]_{S_{m-(k-1)}}$ or a character of level 0 which is the trivial character **1**. Therefore by restricting the character $[m+1-k, \lambda_2, \dots, \lambda_l]$ on S_{n-k} we have each constituent of $[m+1-k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$ is either a constituent of $[m-k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$ or the trivial character **1**. Now by assumption

$$[m-k, \lambda_2, \dots, \lambda_l]_{S_{n-k}} \text{ and } [n-k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$$

have the same constituents and on the other hand by Lemma 4.1.6 the principal character is a constituent of $[n-k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$. Thus each constituent of $[m+1-k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$ is a constituent of $[n-k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$. Using Theorem 4.1.10 the character $[n-2k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$ is a constituent of $[m+1-k, \lambda_2, \dots, \lambda_l]_{S_{n-k}}$ with multiplicity one. This completes the proof ■

Suppose $n - k \geq k + \lambda_2$ in $[\lambda] = [n - k, \lambda_2, \dots, \lambda_l]$ for $k = \lambda_2 + \dots + \lambda_l$ then using the Branching Theorem we have

$$[\lambda]_{S_{n-r}} = [n - k - 1, \lambda_2, \dots, \lambda_l]_{S_{n-r}} + \sum_{i=2}^l \epsilon_i [n - k, \lambda_2, \dots, \lambda'_i, \dots, \lambda_l]_{S_{n-r}}$$

for $r \geq 1$. Now if $n - r \geq k + \lambda_2$ then the constituent of level k has multiplicity one and if we want to know the structure of $[\lambda]_{S_{n-r}}$ we need to know the structure of $[n - k, \lambda_2, \dots, \lambda'_i, \dots, \lambda_l]_{S_{n-r}}$. It means that if we want to know the structure of characters of level k , we should know the structure of characters of level $k - 1$. Then by this recursive method we can get the structure of all irreducible characters of S_n when we restrict them on S_{n-r} .

In the following theorem we describe the restriction of the irreducible characters $[\lambda]$ of S_n of level $k = 1, 2, 3$.

Theorem 4.1.12. *Let $r \geq 0$ then,*

1. *If $[\lambda] = [n - 1, 1]$ then for $n - r \geq 2$*

$$[\lambda]_{S_{n-r}} = [n - (r + 1), 1] + r.1$$

2. *If $[\lambda] = [n - 2, 2]$ then for $n - r \geq 4$*

$$[\lambda]_{S_{n-r}} = [n - (r + 2), 2] + r[n - (r + 1), 1] + \frac{r(r - 1)}{2}.1$$

3. *If $[\lambda] = [n - 2, 1^2]$ then for $n - r \geq 3$*

$$[\lambda]_{S_{n-r}} = [n - (r + 2), 1^2] + r[n - (r + 1), 1] + \frac{r(r - 1)}{2}.1$$

4. *If $[\lambda] = [n - 3, 3]$ then for $n - r \geq 6$*

$$[\lambda]_{S_{n-r}} = [n - (r + 3), 3] + r[n - (r + 2), 2] + \frac{r(r - 1)}{2}[n - (r + 1), 1] + \frac{r(r - 1)(r - 2)}{6}.1$$

5. If $[\lambda] = [n-3, 1^3]$ then for $n-r \geq 4$

$$[\lambda]_{S_{n-r}} = [n-(r+3), 1^3] + r[n-(r+2), 1^2] + \frac{r(r-1)}{2}[n-(r+1), 1] + \frac{r(r-1)(r-2)}{6}.1$$

6. If $[\lambda] = [n-3, 2, 1]$ then for $n-r \geq 5$

$$\begin{aligned} [\lambda]_{S_{n-r}} &= [n-(r+3), 2, 1] + r([n-(r+2), 2] + [n-(r+2), 1^2]) \\ &\quad + r(r-1)[n-(r+1), 1] + \frac{r(r-1)(r-2)}{3}.1 \end{aligned}$$

Proof: It is clear by induction on r ■

As a further example if we consider the irreducible character $[n-4, 3, 1]$ of level 4 of S_n then by induction on $r \geq 1$ we can show

$$[n-4, 3, 1]_{S_{n-r}} = [n-(r+4), 3, 1] + \sum_{i=1}^r ([n-(r+4-i), 2, 1]_{S_{n-r}} + [n-(r+4-i), 3]_{S_{n-r}})$$

which by using parts (4) and (6) of the theorem above gives us all constituents and their multiplicities.

Similarly for the other characters of level 4 of S_n we have

$$[n-4, 4]_{S_{n-r}} = [n-(r+4), 4] + \sum_{i=1}^r [n-(r+4-i), 3]_{S_{n-r}}$$

$$[n-4, 2, 1^2]_{S_{n-r}} = [n-(r+4), 2, 1^2] + \sum_{i=1}^r ([n-(r+4-i), 1^3]_{S_{n-r}} + [n-(r+4-i), 2, 1]_{S_{n-r}})$$

$$[n-4, 2^2]_{S_{n-r}} = [n-(r+4), 2^2] + \sum_{i=1}^r [n-(r+4-i), 2, 1]_{S_{n-r}}$$

and

$$[n-4, 1^4]_{S_{n-r}} = [n-(r+4), 1^4] + \sum_{i=1}^r [n-(r+4-i), 1^3]_{S_{n-r}}.$$

Since the alternating group A_n is a normal subgroup of index 2 in S_n , so by Clifford's Theorem the restriction of each irreducible character of S_n to A_n is either

irreducible or splits into two conjugate irreducible characters of A_n . Let $\lambda \vdash n$ and λ' be the conjugate partition of λ .

If $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ with $\lambda_1 \neq l$, then clearly $\lambda \neq \lambda'$. Under the condition $\lambda \neq \lambda'$, $[\lambda]_{A_n}$ is irreducible. Therefore the following characters are irreducible $[n-1, 1]_{A_n}$ for $n \geq 4$, $[n-2, 2]_{A_n}$ for $n \geq 5$, $[n-2, 1^2]_{A_n}$ and $[n-3, 3]_{A_n}$ for $n \geq 6$, $[n-3, 1^3]_{A_n}$ for $n \geq 8$ and $[n-3, 2, 1]_{A_n}$ for $n \geq 7$. This proves the following corollary.

Corollary 4.1.13. *Let $r \geq 0$. Then the restrictions of irreducible characters of A_n decompose as follows:*

1. For $n-r \geq 4$

$$[n-1, 1]_{A_{n-r}} = [n-(r+1), 1]_{A_{n-r}} + r \cdot \mathbf{1}$$

2. For $n-r \geq 4$

$$[n-2, 2]_{A_{n-r}} = [n-(r+2), 2]_{A_{n-r}} + r[n-(r+1), 1]_{A_{n-r}} + \frac{r(r-1)}{2} \cdot \mathbf{1}$$

3. For $n-r \geq 6$

$$[n-2, 1^2]_{A_{n-r}} = [n-(r+2), 1^2]_{A_{n-r}} + r[n-(r+1), 1]_{A_{n-r}} + \frac{r(r-1)}{2} \cdot \mathbf{1}$$

4. For $n-r \geq 6$

$$\begin{aligned} [n-3, 3]_{A_{n-r}} &= [n-(r+3), 3]_{A_{n-r}} + r[n-(r+2), 2]_{A_{n-r}} \\ &\quad + \frac{r(r-1)}{2} [n-(r+1), 1]_{A_{n-r}} + \frac{r(r-1)(r-2)}{6} \cdot \mathbf{1} \end{aligned}$$

5. For $n-r \geq 8$

$$\begin{aligned} [n-3, 1^3]_{A_{n-r}} &= [n-(r+3), 1^3]_{A_{n-r}} + r[n-(r+2), 1^2]_{A_{n-r}} \\ &\quad + \frac{r(r-1)}{2} [n-(r+1), 1]_{A_{n-r}} + \frac{r(r-1)(r-2)}{6} \cdot \mathbf{1} \end{aligned}$$

6. For $n - r \geq 7$

$$\begin{aligned} [n-3, 2, 1]_{A_{n-r}} &= [n-(r+3), 2, 1]_{A_{n-r}} + r([n-(r+2), 2]_{A_{n-r}} + [n-(r+2), 1^2]_{A_{n-r}}) \\ &\quad + r(r-1)[n-(r+1), 1]_{A_{n-r}} + \frac{r(r-1)(r-2)}{3} \cdot \mathbf{1} \end{aligned}$$

Now we find χ -subgroups for all irreducible characters χ of alternating groups and their covers when χ has degree less than 32. First consider the characters $\chi := [n-1, 1]$, $[n-2, 2]$ and $[n-2, 1^2]$ of degrees $n-1$, $(n^2-3n)/2$ and $(n^2-3n+2)/2$, respectively.

Theorem 4.1.14.

1. If $n \geq 4$ and $\chi = [n-1, 1]_{A_n}$, then A_3 is a χ -subgroup.
2. If $n \geq 6$ and $\chi = [n-2, 2]_{A_n}$ or $[n-2, 1^2]_{A_n}$, then $\text{Syl}_{A_6}(3)$, a Sylow 3-subgroup of A_6 , is a χ -subgroup.

Proof: Suppose $\chi = [n-1, 1]_{A_n}$. By Corollary 4.1.13 with $n-r=4$

$$\chi_{A_4} = [3, 1]_{A_4} + (n-4) \cdot \mathbf{1}.$$

On the other hand a simple calculation shows that $[3, 1]_{A_3} = \mathbf{1} + \varphi_1 + \varphi_2$ where φ_1 and φ_2 are distinct non-trivial linear characters of A_3 . Therefore

$$\chi_{A_3} = (n-3) \cdot \mathbf{1} + \varphi_1 + \varphi_2.$$

This means $\langle \chi_{A_3}, \varphi_1 \rangle = \langle \chi_{A_3}, \varphi_2 \rangle = 1$ and proves (1).

Let $H := \text{Syl}_{A_6}(3)$ be a Sylow 3-subgroup of A_6 . Then H is an abelian subgroup of order 9 and $\text{Irr}(H) = \{\varphi_1, \dots, \varphi_9\}$ where $\varphi_1 = \mathbf{1}$ and $\varphi_2, \dots, \varphi_9$ are non-trivial linear characters of H . If $\chi = [n-2, 2]_{A_n}$, then Corollary 4.1.13 with $n-r=6$ shows

$$\chi_{A_6} = [4, 2]_{A_6} + (n-6)[5, 1]_{A_6} + \frac{(n-6)(n-7)}{2} \cdot \mathbf{1}.$$

On the other hand a simple calculation shows that $[4, 2]_H = \sum_{i=1}^9 \varphi_i$ and $[5, 1]_H = \sum_{i=1}^5 \varphi_i$ for a suitable ordering of the characters φ_i . Therefore

$$\begin{aligned}\chi_H &= \sum_{i=1}^9 \varphi_i + (n-6) \sum_{i=1}^5 \varphi_i + \frac{(n-6)(n-7)}{2} \cdot 1 \\ &= \sum_{i=6}^9 \varphi_i + (n-5) \sum_{i=2}^5 \varphi_i + \frac{n^2 - 11n + 32}{2} \cdot 1\end{aligned}$$

and this implies $\langle \chi_H, \varphi_i \rangle = 1$ for $6 \leq i \leq 9$ as required.

Now consider $\chi = [n-2, 1^2]_{A_n}$. Then by Corollary 4.1.13 with $n-r=6$,

$$\chi_{A_6} = [4, 1^2]_{A_6} + (n-6)[5, 1]_{A_6} + \frac{(n-6)(n-7)}{2} \cdot 1$$

where $[4, 1^2]_H = 2\varphi_1 + \sum_{i=2}^9 \varphi_i$ and $[5, 1]_H = \sum_{i=1}^5 \varphi_i$. Therefore

$$\begin{aligned}\chi_H &= 2\varphi_1 + \sum_{i=2}^9 \varphi_i + (n-6) \sum_{i=1}^5 \varphi_i + \frac{(n-6)(n-7)}{2} \cdot 1 \\ &= \sum_{i=6}^9 \varphi_i + (n-5) \sum_{i=2}^5 \varphi_i + \frac{n^2 - 11n + 34}{2} \cdot 1.\end{aligned}$$

This implies $\langle \chi_H, \varphi_i \rangle = 1$ for $6 \leq i \leq 9$, and so the proof of (2) is complete ■

Now consider the other characters, if we look at the character tables of A_n in [Atlas], we see that for $n \geq 10$ the other characters have degrees ≥ 32 . Also for $n \geq 12$, the covering group $2.A_n$ has no character of degree < 32 .

Therefore the only remaining cases that we should check are the characters of A_n for $n < 10$ and the characters of covering groups for $n < 12$. Using computations in GAP we have found that for most of these characters χ there is at least one p -subgroup which is a χ -subgroup. These are listed at Table 1.8. For the other characters there are χ -subgroups which are not p -subgroups. These characters have been considered at section 4.4.

4.2 $\text{PSL}(2, q)$ and its cover

Our main reference in this section is ([Dhf] Chapter.38).

By ([Kpk] Theorem 7.1.1) the group $\text{SL}(2, q)$ is the unique covering group of the simple group $\text{PSL}(2, q)$, except for $q = 9$. In the later case $\text{PSL}(2, 9) \cong A_6$ and this has been dealt with in the previous section. Also $\text{PSL}(2, q)$ is the factor group of $\text{SL}(2, q)$ by its centre so its characters correspond to the characters of $\text{SL}(2, q)$ whose kernels contains the centre. Thus it is enough to find χ -subgroups for the irreducible characters χ of $\text{SL}(2, q)$.

Let $G = \text{SL}(2, q)$ where $q = p^n$ for some prime p and let

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \mid \beta \in \mathbb{F}_q \right\}.$$

Then H is an abelian Sylow p -subgroup of G of order q . The following tables are the tables of values of characters of G on elements 1 and $1 \neq h \in H$, when q is odd and when q is even (see [Dhf] pages 228, 235).

	1	h
1	1	1
η_1	$(q-1)/2$	$(-1 \mp \sqrt{\epsilon q})/2$
η_2	$(q-1)/2$	$(-1 \pm \sqrt{\epsilon q})/2$
ξ_1	$(q+1)/2$	$(1 \mp \sqrt{\epsilon q})/2$
ξ_2	$(q+1)/2$	$(1 \pm \sqrt{\epsilon q})/2$
ρ	q	0
ψ_i	$q-1$	-1
θ_j	$q+1$	1

Table 1.1: VALUES OF CHARACTERS OF $\text{SL}(2, q)$
ON ELEMENTS OF H WHEN q IS ODD

where $\epsilon = (-1)^{(q-1)/2}$, $1 \leq i \leq (q-1)/2$ and $1 \leq j \leq (q-3)/2$. Note that $\eta_1(h) + \eta_2(h) = -1$ and $\xi_1(h) + \xi_2(h) = 1$ for all $1 \neq h \in H$.

	1	h
1	1	1
ρ	q	0
ψ_i	$q-1$	-1
θ_j	$q+1$	1

Table 1.2: VALUES OF CHARACTERS OF
 $\text{SL}(2, q)$ ON ELEMENTS OF H WHEN q IS EVEN.

where $1 \leq i \leq q/2$ and $1 \leq j \leq (q-2)/2$.

Now we show H is a χ -subgroup for all irreducible characters of G . In this case there is no restriction on the degrees of characters.

We shall need the following lemma.

Lemma 4.2.1. *Let χ be an irreducible character of group G and suppose $p \nmid (|G|/\chi(1))$ for some prime p . Then $\chi(g) = 0$ whenever $p \mid o(g)$. In particular if G has a Sylow subgroup H and an irreducible character χ such that $|H| = \chi(1)$, then χ_H is the regular character of H and so $\langle \chi_H, \varphi \rangle = 1$ for each linear character φ of H .*

Proof: See [Isa] Theorem 8.17 ■

Theorem 4.2.2. *Let $G = \text{SL}(2, q)$ for $q = p^n \geq 4$ and H be a Sylow p -subgroup of G . Then for all characters χ of G , H is a χ -subgroup.*

Proof: By Lemma 4.2.1 the character ρ_H of degree q is the regular character of H . Since H is abelian, all irreducible characters $\varphi_1 := \mathbf{1}, \varphi_2, \dots, \varphi_q$ of H are linear. On the other hand $\psi_j(h) = -1$ and $\theta_i(h) = 1$ for all $1 \neq h \in H$ so

$$(\psi_j)_H = \rho_H - \mathbf{1}$$

and

$$(\theta_i)_H = \rho_H + \mathbf{1}.$$

Also when q is odd we have $\eta_1(h) + \eta_2(h) = -1$ and $\xi_1(h) + \xi_2(h) = 1$ for all $1 \neq h \in H$ so

$$(\eta_1)_H + (\eta_2)_H = \rho_H - \mathbf{1}$$

and

$$(\xi_1)_H + (\xi_2)_H = \rho_H + \mathbf{1}.$$

Now since $\rho_H = \sum_{i=1}^q \varphi_i$ and $q \geq 4$, therefore the restriction of each irreducible character of G to H has at least one linear constituent with multiplicity 1 ■

4.3 $\text{PSL}(3, q)$ and $\text{PSU}(3, q)$ and their covers

In this section there is no restriction on the degree of the characters.

By ([Kpk] Theorem 7.1.1) the group $\text{SL}(3, q)$ where $q = p^n > 2$ and p is a prime, is the unique covering group of the simple group $\text{PSL}(3, q)$ except when $q = 4$ ($\text{PSL}(3, 4)$ has 7 different covering groups, see Table 1.8) . Also $\text{PSU}(3, q)$ is a Lie group of type ${}^2A_2(q)$ and by ([Kpk] Table 8.5) the group $\text{SU}(3, q)$ is the unique covering group of the simple group $\text{PSU}(3, q)$ (see [Gsn] Corollary 5.1.3).

On the other hand $\text{PSL}(3, q)$ and $\text{PSU}(3, q)$ are factor groups of $\text{SL}(3, q)$ and $\text{SU}(3, q)$ by their centres, respectively. Thus their characters correspond to the characters of $\text{SL}(3, q)$ and $\text{SU}(3, q)$, respectively, whose kernels contain the centre. Thus it is enough to find a χ -subgroup for each irreducible character χ of $\text{SL}(3, q)$ and $\text{SU}(3, q)$.

Let $G = \text{SL}(3, q)$ or $\text{SU}(3, q)$. We shall show that for each irreducible character χ of G either a Sylow subgroup H or a p -subgroup of order q^2 of G is a χ -subgroup. The character tables of these groups are known by the work of J. S. Frame and W. A. Simpson [SpFm] which is our main reference for this section. We shall use those tables to get the values of characters on the different conjugacy classes of G which contain the elements of the Sylow subgroup H defined below.

Define

$$H := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_q \right\}.$$

Then the order of H is q^3 and H is a Sylow p -subgroup for both $\text{SL}(3, q)$ and $\text{SU}(3, q)$.

The following table is a part of Table 1a of [SpFm] that shows the structure of conjugacy classes of G which contain some elements of the Sylow p -subgroup H . In this section $d = \gcd(3, q - 1)$, $\epsilon \in \text{GF}(q)$ and $\epsilon^3 \neq 1$ in case $G = \text{SL}(3, q)$ and $d = \gcd(3, q + 1)$, $\epsilon \in \text{GF}(q^2)$ and $\epsilon^3 \neq 1$ in case $G = \text{SU}(3, q)$. In the following table ω is a cube root of unity.

Conjugacy class	Canonical representative	Parameters
$\mathcal{C}_1^{(k)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^k \end{pmatrix}$	$0 \leq k \leq (d - 1)$
$\mathcal{C}_2^{(k)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ 1 & \omega^k & 0 \\ 0 & 0 & \omega^k \end{pmatrix}$	$0 \leq k \leq (d - 1)$
$\mathcal{C}_3^{(k,l)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ \epsilon^l & \omega^k & 0 \\ 0 & \epsilon^l & \omega^k \end{pmatrix}$	$0 \leq k, l \leq (d - 1)$

Table 1.3: CONJUGACY CLASSES OF $\text{SL}(3, q)$ AND $\text{SU}(3, q)$
WHICH CONTAIN ELEMENTS OF THE SYLOW p -SUBGROUP H FOR $d = 1, 3$.

Each element of H is contained in one of the conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,l)}$ of G .

The centre

$$Z(H) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z & 0 & 1 \end{pmatrix} \mid z \in \mathbb{F}_q \right\}$$

is an elementary abelian p -group of order q . By using the canonical representative elements of conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,l)}$ we see that the minimal polynomials of elements of these conjugacy classes have degrees 1, 2 and 3, respectively and the minimal polynomials of nontrivial elements of $Z(H)$ have degree 2 so nontrivial elements of $Z(H)$ are contained in the conjugacy class $\mathcal{C}_2^{(0)}$.

The following lemma gives us some properties of H .

Lemma 4.3.1. *Suppose G is one of the groups $\text{SL}(3, q)$ or $\text{SU}(3, q)$ where q is a power of the prime p . If H is a Sylow p -subgroup of G then we have:*

1. H has $q^2 + q - 1$ conjugacy classes.
2. H has q^2 linear characters and $q - 1$ non-linear characters of degree q such that their values on nontrivial elements of $Z(H)$ are 1 and $q\omega^i$ for some $1 \leq i \leq p$ respectively, where ω is a primitive p^{th} root of unity.
3. If τ is an irreducible character of degree q of H then $\tau(x) = 0$ for $x \notin Z(H)$, and $\sum_{1 \neq z \in Z(H)} \tau(z) = -q$.

Proof : First of all we show $H/Z(H)$ is abelian. Let $x, y \in H$ so it is enough to show $x^{-1}y^{-1}xy \in Z(H)$. Let

$$x = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \text{ and } y = \begin{pmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ e & f & 1 \end{pmatrix}$$

then

$$x^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{pmatrix} \text{ and } y^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -d & 1 & 0 \\ df-e & -f & 1 \end{pmatrix}$$

and so

$$x^{-1}y^{-1}xy = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ af-dc & 0 & 1 \end{pmatrix} \in Z(H).$$

Hence $H/Z(H)$ is abelian and $H' \subseteq Z(H)$. Conversely if

$$z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & 0 & 1 \end{pmatrix} \in Z(H)$$

then $z = x^{-1}y^{-1}xy \in H'$ where

$$x = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ b & c & 1 \end{pmatrix} \text{ and } y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e & 1 & 1 \end{pmatrix}$$

for $b, c, e \in \mathbb{F}_q$. Therefore $H' = Z(H)$. Now suppose

$$h = \begin{pmatrix} 1 & 0 & 0 \\ h_1 & 1 & 0 \\ h_2 & h_3 & 1 \end{pmatrix} \in H \setminus Z(H)$$

so at least one of h_1, h_3 is not 0. Then

$$x^{-1}hx = h^x = \begin{pmatrix} 1 & 0 & 0 \\ h_1 & 1 & 0 \\ h_1c - ah_3 - h_2 & h_3 & 1 \end{pmatrix}$$

As x runs over H , $h_1c - ah_3 - h_2$ runs over $\mathbb{F}_q$. Thus the conjugacy class $\{h^x \mid x \in H\}$ has order q . Therefore each conjugacy class of H has order 1 or q and H has q single element conjugacy classes, since $|Z(H)| = q$. If n is the number of conjugacy class of order q then $|H| = (q \times 1) + (n \times q)$ and so $n = q^2 - 1$. Thus H has $q^2 + q - 1$ conjugacy classes.

Since $|H : H'| = q^2$ therefore H has q^2 linear characters and since the number of conjugacy classes of H is $q^2 + q - 1$ so H has $q - 1$ non-linear characters. Let τ be a non-linear irreducible character of H . Since $Z(H) \subseteq Z(\tau)$ and by ([Is] Corollary

2.30)

$$\tau^2(1) \leq |H : Z(\tau)| \leq |H : Z(H)| = q^2 \quad (4.3)$$

so $\tau(1) \leq q$. On the other hand the number of conjugacy classes of H is $q^2 + q - 1$ and the order of H is q^3 so

$$q^3 = |H| = \sum_{i=1}^{q^2} \varphi_i(1)^2 + \sum_{j=1}^{q-1} \tau_j(1)^2$$

where φ_i and τ_j are linear and non-linear irreducible characters of H , respectively. Since $\tau_j(1) \leq q$ so $\tau_j(1) = q$ and (4.3) implies $Z(H) = Z(\tau)$. Since $H' = Z(H)$ so the value of all linear characters of H on $Z(H)$ is 1. Also for an irreducible character τ of degree q if ρ is the representation which affords τ then $\rho(z)$ is an scalar for all $1 \neq z \in Z(H)$ thus $\tau(z) = q\omega^j$ for some $1 \leq j \leq p$ where ω is a primitive p^{th} -root of unity.

Since $\tau^2(1) = q^2 = |H : Z(H)|$, ([Is] Corollary 2.30) shows that $\tau(x) = 0$ for all $x \notin Z(H)$. Now by using the first orthogonality relation we get

$$\frac{1}{|H|} \sum_{x \in H} \tau(x) 1(x^{-1}) = \frac{1}{|H|} \sum_{x \in H} \tau(x) = \frac{1}{|H|} \sum_{z \in Z(H)} \tau(z) = 0.$$

Therefore $\tau(1) = q$ implies

$$\sum_{1 \neq z \in Z(H)} \tau(z) = -q \quad (4.4)$$

and this completes the proof ■

The following table shows the values of the restriction of the irreducible characters of group $SL(3, q)$ on the elements of Sylow subgroup H when $d = 1$ (see [SpFm] Table 1b). Since $d = 1, l = 0$.

	$C_1^{(0)}$	$C_2^{(0)}$	$C_3^{(0,0)}$
$\mathbf{1}$	1	1	1
ψ	$q^2 + q$	q	0
ρ	q^3	0	0
ζ_i	$q^2 + q + 1$	$q + 1$	1
η_j	$q^3 + q^2 + q$	q	0
ε_r	$q^3 + 2q^2 + 2q + 1$	$2q + 1$	1
μ_s	$q^3 - 1$	-1	-1
ν_t	$q^3 - q^2 - q + 1$	$1 - q$	1

Table 1.4: VALUES OF CHARACTERS OF $\text{SL}(3, q)$

ON ELEMENTS OF H WHEN $d = 1$.

where $1 \leq i, j \leq q - 2$, $1 \leq r \leq (q^2 - 5q + 6)/6$, $1 \leq s \leq (q^2 - q)/2$ and $1 \leq t \leq (q^2 + q)/3$.

Lemma 4.3.2. *Let $G = \text{SL}(3, q)$ where $q > 2$ is a power of the prime p and let H be the Sylow p -subgroup of G and ψ be the irreducible character of degree $q^2 + q$ of G . Then*

1. $\langle \psi_H, \mathbf{1} \rangle = 2$.
2. $\langle \psi_H, \tau \rangle = 1$ for each irreducible character τ of degree q of H .
3. There exist some non-principal linear characters φ and ϕ of H such that $\langle \psi_H, \varphi \rangle = 0$ and $\langle \psi_H, \phi \rangle = 1$.

Proof: Suppose that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \in H$$

is contained in the conjugacy class $\mathcal{C}_2^{(0)}$ of G . Since each element in $\mathcal{C}_2^{(0)}$ has a minimal polynomial of degree 2,

$$(x - 1)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ac & 0 & 0 \end{pmatrix} = 0.$$

This, together with $x \notin Z(H)$ implies $a = 0$ or $c = 0$ but not both. Therefore the number of possibilities for the elements x with above properties is $2q(q - 1)$. The elements of $Z(H)$ are also contained in $\mathcal{C}_2^{(0)}$ and the values of ψ on $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,0)}$ are $q^2 + q$, q and 0, respectively. Thus we have

$$\begin{aligned} \langle \psi_H, \mathbf{1} \rangle &= \frac{1}{|H|} \sum_{x \in H} \psi_H(x) \mathbf{1}(x) \\ &= \frac{1}{q^3} (\psi_H(1) + \sum_{1 \neq z \in Z(H)} \psi_H(z) + \sum_{z \notin Z(H)} \psi_H(z)) \\ &= \frac{1}{q^3} ((q^2 + q) + (q - 1)q + 2q(q - 1)q) = 2. \end{aligned}$$

This proves the first assertion.

Now suppose τ is an irreducible character of degree q of H . By using Table 1.4 for the value of ψ on the conjugacy class $\mathcal{C}_2^{(0)}$ of G which contains the elements of $Z(H)$, Lemma 4.3.1 and (4.4) we have

$$\langle \psi_H, \tau \rangle = \frac{1}{|H|} \sum_{x \in H} \psi_H(x) \overline{\tau(x)}$$

$$\begin{aligned}
&= \frac{1}{q^3}(\psi_H(1)\tau(1) + \sum_{1 \neq z \in Z(H)} \psi_H(z)\overline{\tau(z)} + \sum_{z \notin Z(H)} \psi_H(z)\overline{\tau(z)}) \\
&= \frac{1}{q^3}((q^2 + q)q - q^2 + 0) = 1.
\end{aligned}$$

Therefore for each irreducible character τ of degree q of H

$$\langle \psi_H, \tau \rangle = 1$$

as claimed.

Now since $\langle \psi_H, \tau \rangle = 1$ for each irreducible character τ of degree q of H , hence $\psi_H = \sum_{i=1}^{q-1} \tau_i + \sum_{j=1}^t m_j \phi_j$ where ϕ_j are linear characters of H with multiplicity m_j . Since $\psi(1) = q^2 + q$ and $\sum_{i=1}^{q-1} \tau_i(1) = q^2 - q$ we have $\sum_{j=1}^t m_j \phi_j(1) = 2q$. Since H possesses $q^2 - 1$ non-principal linear characters, there exists at least one non-principal linear character φ such that $\langle \psi_H, \varphi \rangle = 0$.

Put $\langle \psi_H, \mathbf{1} \rangle = m$. Since $(\nu_t)_H = \chi_H - \psi_H + \mathbf{1}$ we have $\langle (\nu_t)_H, \mathbf{1} \rangle = 2 - m$. Hence $m \leq 2$ and so $\sum_{j=1}^t m_j \phi_j(1) \geq 2q - 2$ where $\sum'$ runs over $\phi_j \neq \mathbf{1}$. This means there exists some non-principal linear character ϕ of H such that $\langle \psi_H, \phi \rangle \neq 0$. If we suppose $\langle \psi_H, \phi \rangle = n$, then $\langle (\nu_t)_H, \phi \rangle = 1 - n$. Since $n \neq 0$ this shows that $n = 1$. Therefore there exists a non-principal linear character ϕ such that $\langle \psi_H, \phi \rangle = 1$ and this completes the proof ■

Theorem 4.3.3. *Let $G = \text{SL}(3, q)$ where $q > 2$ is a power of prime p and $d = 1$ and let H be a Sylow p -subgroup of G . Then for all irreducible characters χ of G , H is a χ -subgroup.*

Proof: By Table (1.4) the characters ρ and ψ have degrees q^3 and $q^2 + q$ respectively. Now if we restrict them to H we see that for all nontrivial $x \in H$ we have $\rho_H(x) = 0$ and $\psi_H(x) = q$ or 0 . Thus from the values of the other characters of G on H we get:

$$(\zeta_i)_H = \psi_H + \mathbf{1},$$

$$(\eta_j)_H = \rho_H + \psi_H,$$

$$(\varepsilon_r)_H = \rho_H + 2\psi_H + \mathbf{1},$$

$$(\mu_s)_H = \rho_H - \mathbf{1}$$

and

$$(\nu_t)_H = \rho_H - \psi_H + \mathbf{1}.$$

Since $\rho(1) = q^3$ is the order of H , Lemma 4.2.1 shows that ρ_H is the regular character of H and $\rho_H = \sum_{v \in \text{Irr}(H)} v(1)v$. On the other hand by the Lemma 4.3.2 there exists a non-principal linear character φ of H such that $\langle \psi_H, \varphi \rangle = 0$ then, since $\langle \rho_H, \varphi \rangle = 1$, we have

$$\langle \rho_H + \psi_H, \varphi \rangle = 1,$$

$$\langle \rho_H + 2\psi_H + \mathbf{1}, \varphi \rangle = 1,$$

$$\langle \rho_H - \mathbf{1}, \varphi \rangle = 1$$

and

$$\langle \rho_H - \psi_H + \mathbf{1}, \varphi \rangle = 1.$$

Also by the Lemma 4.3.2 there exists a non-principal linear character ϕ of H such that $\langle \psi_H, \phi \rangle = 1$. Thus

$$\langle \psi_H + \mathbf{1}, \phi \rangle = 1.$$

This completes the proof ■

Now we consider the case that $d = 3$. We have the following table of the values of irreducible characters of $\text{SL}(3, q)$ on the conjugacy classes which contain the elements of H (see [SpFm] Table 1b).

	$\mathcal{C}_1^{(0)}$	$\mathcal{C}_2^{(0)}$	$\mathcal{C}_3^{(0,l)}$
1	1	1	1
ψ	$q^2 + q$	q	0
ρ	q^3	0	0
ζ_i	$q^2 + q + 1$	$q + 1$	1
η_j	$q^3 + q^2 + q$	q	0
θ_k	$(q^3 + 2q^2 + 2q + 1)/3$	$(2q + 1)/3$ or $(1 - q)/3$	$(2q + 1)/3$ or $(1 - q)/3$
ε_r	$q^3 + 2q^2 + 2q + 1$	$2q + 1$	1
μ_s	$q^3 - 1$	-1	-1
ν_t	$q^3 - q^2 - q + 1$	$1 - q$	1
ω_m	$(q^3 - q^2 - q + 1)/3$	$(1 - q)/3$ or $(2q + 1)/3$	$(1 - q)/3$ or $(2q + 1)/3$
γ_n	$(q^3 - q^2 - q + 1)/3$	$(1 - q)/3$ or $(2q + 1)/3$	$(1 - q)/3$ or $(2q + 1)/3$

Table 1.5: VALUES OF CHARACTERS OF $\text{SL}(3, q)$ ON ELEMENTS OF H WHEN $d = 3$.

where $1 \leq i, j \leq q - 2$, $1 \leq r \leq (q^2 - 5q + 4)/6$, $1 \leq s \leq (q^2 - q)/2$, $1 \leq t \leq (q^2 + q - 2)/3$ and $1 \leq k, m, n \leq 3$.

By the values of characters ω_m and γ_n on the conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,l)}$ in the Table 1b of [SpFm], we have

$$\{(\omega_1)_H, (\omega_2)_H, (\omega_3)_H\} = \{(\gamma_1)_H, (\gamma_2)_H, (\gamma_3)_H\}. \quad (4.5)$$

Definition 4.3.4. Let H be a normal subgroup of group G and α and β be characters of H . Then β is called a conjugate of α in G if there exists $g \in G$ such that $\beta = \alpha^g$, where $\alpha^g(h) = \alpha(ghg^{-1}) = \alpha(h^{g^{-1}})$ for $h \in H$.

Theorem 4.3.5. *Let $G := \text{GL}(3, q)$, then the elements of each set of characters $\{\theta_1, \theta_2, \theta_3\}$, $\{\omega_1, \omega_2, \omega_3\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ of $\text{SL}(3, q)$, are conjugate in G .*

Proof: See [SpFm] ■

Lemma 4.3.6. *Let H be a subgroup of G , $x \in N_G(H)$ and ϑ be a character of H . Then ϑ^x is a character of H and $\vartheta^x(1) = \vartheta(1)$. Furthermore $\langle \vartheta^x, \vartheta^x \rangle = \langle \vartheta, \vartheta \rangle$ and in particular ϑ^x is irreducible if and only if ϑ is irreducible.*

Proof: Since $x \in N_G(H)$ so $xhx^{-1} \in H$ for $h \in H$. On the other hand by Definition 4.3.4 $\vartheta^x(h) = \vartheta(xhx^{-1}) = \vartheta(h^{x^{-1}})$ so ϑ^x is a character of H . Also $\vartheta^x(1) = \vartheta(xx^{-1}) = \vartheta(1)$.

$$\begin{aligned} \langle \vartheta^x, \vartheta^x \rangle &= \frac{1}{|H|} \sum_{h \in H} \vartheta^x(h) \overline{\vartheta^x(h)} = \frac{1}{|H|} \sum_{h \in H} \vartheta(h^{x^{-1}}) \overline{\vartheta(h^{x^{-1}})} \\ &= \frac{1}{|H|} \sum_{z=h^{x^{-1}} \in H} \vartheta(z) \overline{\vartheta(z)} = \langle \vartheta, \vartheta \rangle. \end{aligned}$$

Therefore $\vartheta^x \in \text{Irr}(H)$ if and only if $\vartheta \in \text{Irr}(H)$ ■

Lemma 4.3.7. *Let G be a normal subgroup of group L and H be a Sylow subgroup of G . Let χ and ϑ be irreducible characters of G and H , respectively. Let $l \in L$ then*

$$\langle \chi_H, \vartheta \rangle = \langle \chi_H^l, \vartheta^x \rangle \text{ for some } x \in N_L(H).$$

In particular $\langle \chi_H, \mathbf{1} \rangle = \langle \chi_H^l, \mathbf{1} \rangle$.

Proof: The Frattini argument (Lemma 1.0.7) shows $L = GN_L(H)$. If $l \in L$ then $l = gx$ where $g \in G$ and $x \in N_L(H)$. Thus

$$\langle \chi_H, \vartheta \rangle = \frac{1}{|H|} \sum_{h \in H} \chi_H(h) \vartheta(h) = \frac{1}{|H|} \sum_{h^{x^{-1}} \in H} \chi_H(h^{x^{-1}}) \vartheta(h^{x^{-1}})$$

$$= \frac{1}{|H|} \sum_{h \in H^x = H} \chi_H^x(h) \vartheta^x(h) = \langle \chi_H^x, \vartheta^x \rangle.$$

Since $\chi \in \text{Irr}(G)$ so $\chi^g = \chi$ thus $\chi_H^l = \chi_H^{gx} = \chi_H^x$ and this implies $\langle \chi_H, \vartheta \rangle = \langle \chi_H^l, \vartheta^x \rangle$.
In particular if $\vartheta = \mathbf{1}$ then since $\mathbf{1}^x = \mathbf{1}$ so $\langle \chi_H, \mathbf{1} \rangle = \langle \chi_H^l, \mathbf{1} \rangle$ ■

Lemma 4.3.8. *Let $G = \text{SL}(3, q)$ where $q > 5$ is a power of prime p , $d = 3$ and H be a Sylow p -subgroup of G . Suppose θ_k , ω_m and γ_n are the irreducible characters of G in Table 1.5. Then for $1 \leq k, m, n \leq 3$*

$$\langle (\theta_k)_H, \mathbf{1} \rangle = 2 \text{ and } \langle (\omega_m)_H, \mathbf{1} \rangle = \langle (\gamma_n)_H, \mathbf{1} \rangle = 0.$$

Proof: Since ρ_H is the regular character of H so $\langle \rho_H, \mathbf{1} \rangle = 1$. Also $\langle \psi_H, \mathbf{1} \rangle = 2$ by Lemma 4.3.2. On the other hand

$$(\zeta_i)_H = \psi_H + \mathbf{1},$$

$$(\eta_j)_H = \rho_H + \psi_H,$$

$$(\varepsilon_r)_H = \rho_H + 2\psi_H + \mathbf{1},$$

$$(\mu_s)_H = \rho_H - \mathbf{1}$$

and

$$(\nu_t)_H = \rho_H - \psi_H + \mathbf{1}.$$

Therefore by Frobenius reciprocity

$$\langle \zeta_i, \mathbf{1}^G \rangle = \langle (\zeta_i)_H, \mathbf{1} \rangle = 3,$$

$$\langle \eta_j, \mathbf{1}^G \rangle = \langle (\eta_j)_H, \mathbf{1} \rangle = 3,$$

$$\langle \varepsilon_r, \mathbf{1}^G \rangle = \langle (\varepsilon_r)_H, \mathbf{1} \rangle = 6,$$

$$\langle \mu_s, \mathbf{1}^G \rangle = \langle (\mu_s)_H, \mathbf{1} \rangle = 0,$$

$$\langle \nu_t, \mathbf{1}^G \rangle = \langle (\nu_t)_H, \mathbf{1} \rangle = 0,$$

Let

$$\langle \theta_k, \mathbf{1}^G \rangle = \langle (\theta_k)_H, \mathbf{1} \rangle = K_k,$$

$$\langle \omega_m, \mathbf{1}^G \rangle = \langle (\omega_m)_H, \mathbf{1} \rangle = M_m,$$

$$\langle \gamma_n, \mathbf{1}^G \rangle = \langle (\gamma_n)_H, \mathbf{1} \rangle = N_n.$$

for $1 \leq k, m, n \leq 3$.

If we induce the principal character $\mathbf{1}$ to G we get

$$\begin{aligned} \mathbf{1}^G &= \mathbf{1} + \psi + \rho + 3(q-2)\zeta_i + 3(q-2)\eta_j + 6((q^2 - 5q + 4)/6)\varepsilon_r \\ &\quad + \sum_{k=1}^3 K_k \theta_k + \sum_{m=1}^3 M_m \omega_m + \sum_{n=1}^3 N_n \gamma_n \end{aligned}$$

but $\mathbf{1}^G(1) = |G : H|$ so if we calculate the value at 1 and simplify the above equation we have

$$|G : H| = -1 - 4q - 5q^2 - 3q^3 + q^5 + \sum_{k=1}^3 K_k \theta_k(1) + \sum_{m=1}^3 M_m \omega_m(1) + \sum_{n=1}^3 N_n \gamma_n(1).$$

Since $|G : H| = q^5 - q^3 - q^2 + 1$ we get

$$\sum_{k=1}^3 K_k \theta_k(1) + \sum_{m=1}^3 M_m \omega_m(1) + \sum_{n=1}^3 N_n \gamma_n(1) = 2q^3 + 4q^2 + 4q + 2.$$

Since

$$\theta_k(1) = (q^3 + 2q^2 + 2q + 1)/3 \quad \text{and} \quad \omega_m(1) = \gamma_n(1) = (q^3 - q^2 - q + 1)/3$$

so we have

$$\begin{aligned} & \left(\sum_{k=1}^3 K_k \right) ((q^3 + 2q^2 + 2q + 1)/3) + \left(\sum_{m=1}^3 M_m + \sum_{n=1}^3 N_n \right) ((q^3 - q^2 - q + 1)/3) \\ &= 2q^3 + 4q^2 + 4q + 2. \end{aligned}$$

Hence by considering $K = \sum_{k=1}^3 K_k$, $M = \sum_{m=1}^3 M_m$ and $N = \sum_{n=1}^3 N_n$ we get

$$K((q^3 + 2q^2 + 2q + 1)/3) + (M + N)((q^3 - q^2 - q + 1)/3) = 2q^3 + 4q^2 + 4q + 2$$

so

$$\begin{aligned} (K + M + N)q^3 + (2K - (M + N))q^2 + ((2K - (M + N))q + (K + M + N)) \\ = 6q^3 + 12q^2 + 12q + 6. \end{aligned}$$

Thus

$$(A - 6)(q^3 + 1) = (12 - B)(q^2 + q) \quad (4.6)$$

where $A = K + M + N$ and $B = 2K - (M + N)$. Since K , M and N are non negative integers and are not all equal 0, so A is a positive integer. Since $q \mid (12 - B)(q^2 + q)$ so $q \mid A - 6$ and this means that $A - 6 = tq$ for some integer t . Hence simplifying equation (4.6) implies $12 - B = t(q^2 - q + 1)$. Therefore

$$0 \leq 3K = A + B = 18 - t(q - 1)^2.$$

Since $q > 5$ so $A = 6 + tq > 0$ implies $t \geq 0$ and $A + B = 18 - t(q - 1)^2 \geq 0$ shows $t \leq 0$. Therefore $t = 0$, $A = 6$ and $B = 12$ which these imply $K = 6$ and $M + N = 0$. Thus $\sum_{k=1}^3 K_k = 6$ and $\sum_{m=1}^3 M_m + \sum_{n=1}^3 N_n = 0$.

Theorem 4.3.5 shows the characters θ_1, θ_2 and θ_3 are conjugate in $L = \text{GL}(3, q)$ so there exist $a, b \in L$ such that $\theta_2 = \theta_1^a$ and $\theta_3 = \theta_1^b$. Using the Lemma 4.3.7 we have

$$\langle (\theta_1)_H, \mathbf{1} \rangle = \langle (\theta_2)_H, \mathbf{1} \rangle = \langle (\theta_3)_H, \mathbf{1} \rangle$$

which $\sum_{k=1}^3 K_k = 6$ implies $K_k = 2$ for $1 \leq k \leq 3$. Also $\sum_{m=1}^3 M_m + \sum_{n=1}^3 N_n = 0$ implies $M_m = N_n = 0$ for $1 \leq m, n \leq 3$. This proves the lemma ■

Since in the lemma above $d = \gcd(3, q - 1) = 3$ so the only case that we should consider is $q = 4$. Fortunately the group $\text{SL}(3, 4)$ is available in the GAP library and using some simple computation we see that the lemma above is also correct for this group. Therefore we prove the following theorem for $q > 2$.

Theorem 4.3.9. *Let $G = \text{SL}(3, q)$ where $q > 2$ is a power of the prime p and $d = 3$. Let H be a Sylow p -subgroup of G . If χ is an irreducible character of G then H is a χ -subgroup.*

Proof: (See [Gzl]). We use the same method as we used to prove Theorem 4.3.3. Thus ρ_H is the regular character of H and ψ_H is the character of degree $q^2 + q$ of H . Therefore for characters $\zeta_i, \eta_j, \varepsilon_r, \mu_s$ and ν_t we have:

$$(\zeta_i)_H = \psi_H + \mathbf{1},$$

$$(\eta_j)_H = \rho_H + \psi_H,$$

$$(\varepsilon_r)_H = \rho_H + 2\psi_H + \mathbf{1},$$

$$(\mu_s)_H = \rho_H - \mathbf{1}$$

and

$$(\nu_t)_H = \rho_H - \psi_H + \mathbf{1}.$$

Lemma 4.3.2 shows H has some non-principal linear characters φ and ϕ such that $\langle \psi_H, \varphi \rangle = 0$ and $\langle \psi_H, \phi \rangle = 1$. Since ρ_H is the regular character of H so $\langle \rho_H, \varphi \rangle = \langle \rho_H, \phi \rangle = 1$. Hence

$$\langle (\eta_j)_H, \varphi \rangle = \langle (\varepsilon_r)_H, \varphi \rangle = \langle (\mu_s)_H, \varphi \rangle = \langle (\nu_t)_H, \varphi \rangle = 1 \text{ and } \langle (\zeta_i)_H, \varphi \rangle = 0.$$

Also $\langle \psi_H, \phi \rangle = 1$ implies $\langle (\zeta_i)_H, \phi \rangle = 1$. Therefore the restriction of characters $\zeta_i, \eta_j, \varepsilon_r, \mu_s$ and ν_t on H have at least one linear constituent with multiplicity one. So the only remaining characters to consider are θ_k, ω_m and γ_n for $1 \leq k, m, n \leq 3$.

Using the Frobenius reciprocity we have,

$$\langle \eta_j, \varphi^G \rangle = \langle \varepsilon_r, \varphi^G \rangle = \langle \mu_s, \varphi^G \rangle = \langle \nu_t, \varphi^G \rangle = 1 \text{ and } \langle \zeta_i, \varphi^G \rangle = 0.$$

Also if

$$\langle (\theta_k)_H, \varphi \rangle = K_k, \langle (\omega_m)_H, \varphi \rangle = M_m \text{ and } \langle (\gamma_n)_H, \varphi \rangle = N_n,$$

then

$$\langle \theta_k, \varphi^G \rangle = K_k, \langle \omega_m, \varphi^G \rangle = M_m \text{ and } \langle \gamma_n, \varphi^G \rangle = N_n$$

for $1 \leq k, m, n \leq 3$. Therefore if we induce φ to G we get

$$\begin{aligned} \varphi^G &= \rho + (q-2)\eta_j + ((q^2-5q+4)/6)\varepsilon_r + ((q^2-q)/2)\mu_s \\ &+ ((q^2+q-2)/3)\nu_t + \sum_{k=1}^3 K_k \theta_k + \sum_{m=1}^3 M_m \omega_m + \sum_{n=1}^3 N_n \gamma_n. \end{aligned}$$

Using the fact that $\varphi^G(1) = |G : H|\varphi(1)$ we calculate the value at 1 and simplify the above equation we have

$$|G : H| = -q^2 - q^3 + q^5 + \sum_{k=1}^3 K_k \theta_k(1) + \sum_{m=1}^3 M_m \omega_m(1) + \sum_{n=1}^3 N_n \gamma_n(1).$$

Since $|G : H| = q^5 - q^3 - q^2 + 1$ we get

$$\sum_{k=1}^3 K_k \theta_k(1) + \sum_{m=1}^3 M_m \omega_m(1) + \sum_{n=1}^3 N_n \gamma_n(1) = q^3 + 1.$$

Since

$$\theta_k(1) = (q^3 + 2q^2 + 2q + 1)/3$$

and

$$\omega_m(1) = \gamma_n(1) = (q^3 - q^2 - q + 1)/3$$

so we have

$$\left(\sum_{k=1}^3 K_k\right)((q^3 + 2q^2 + 2q + 1)/3) + \left(\sum_{m=1}^3 M_m + \sum_{n=1}^3 N_n\right)((q^3 - q^2 - q + 1)/3) = q^3 + 1.$$

Hence by considering $K = \sum_{k=1}^3 K_k$, $M = \sum_{m=1}^3 M_m$ and $N = \sum_{n=1}^3 N_n$ we get

$$K((q^3 + 2q^2 + 2q + 1)/3) + (M + N)((q^3 - q^2 - q + 1)/3) = q^3 + 1$$

so

$$(K + M + N)q^3 + (2K - (M + N))q^2 + ((2K - (M + N))q + (K + M + N)) = 3(q^3 + 1).$$

Thus

$$(A - 3)(q^3 + 1) = -B(q^2 + q) \quad (4.7)$$

where $A = K + M + N$ and $B = 2K - (M + N)$. Since K , M and N are non negative integers and are not all equal 0, so A is a positive integer. Since $q \mid -B(q^2 + q)$ so $q \mid A - 3$ and this means that $A - 3 = tq$ for some integer t . Hence simplifying equation (4.7) implies $-B = t(q^2 - q + 1)$. Thus

$$0 \leq 3K = A + B = 3 - t(q - 1)^2.$$

Since $d = \gcd(3, q - 1) = 3$ so we can consider $q > 3$, which in this case $A = 3 + tq > 0$ implies $t \geq 0$ and $A + B = 3 - t(q - 1)^2 \geq 0$ shows $t \leq 0$. Thus $t = 0$, $A = 3$ and $B = 0$ which these conclude $K = 1$ and $M + N = 2$. Therefore $\sum_{k=1}^3 K_k = 1$ and $\sum_{m=1}^3 M_m + \sum_{n=1}^3 N_n = 2$. So for some k , $K_k = 1$ and $\langle (\theta_k)_H, \varphi \rangle = 1$.

Let $\langle (\theta_1)_H, \varphi \rangle = 1$ then by Theorem 4.3.5 the characters θ_1, θ_2 and θ_3 are conjugate in $L = \text{GL}(3, q)$. Hence by Lemma 4.3.7 we have

$$\langle (\theta_1)_H, \varphi \rangle = \langle (\theta_2)_H, \varphi^x \rangle = \langle (\theta_3)_H, \varphi^y \rangle = 1$$

for some $x, y \in N_L(H)$. On the other hand by Lemma 4.3.6, φ^x and φ^y are linear characters of H so the restriction of characters θ_1, θ_2 and θ_3 to H have at least a constituent of degree one with multiplicity one.

Also by (4.5),

$$\{(\omega_1)_H, (\omega_2)_H, (\omega_3)_H\} = \{(\gamma_1)_H, (\gamma_2)_H, (\gamma_3)_H\}$$

so $\sum_{m=1}^3 M_m = 1$ and $\sum_{n=1}^3 N_n = 1$. Therefore for some m and n we have $N_n = 1$ and $M_m = 1$ which this means $\langle (\omega_m)_H, \varphi \rangle = \langle (\gamma_n)_H, \varphi \rangle = 1$. Without any ambiguity we can suppose $\langle (\omega_1)_H, \varphi \rangle = \langle (\gamma_1)_H, \varphi \rangle = 1$. Theorem 4.3.5 shows the elements of each set of characters $\{\omega_1, \omega_2, \omega_3\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ are conjugate in $L = \text{GL}(3, q)$. Therefore by Lemma 4.3.7 and Lemma 4.3.6 there exist $r, s, t, u \in N_L(G)$ such that $\varphi^r, \varphi^s, \varphi^t$ and φ^u are linear characters of H and

$$\langle (\omega_2)_H, \varphi^r \rangle = \langle (\omega_3)_H, \varphi^s \rangle = \langle (\gamma_2)_H, \varphi^t \rangle = \langle (\gamma_3)_H, \varphi^u \rangle = 1.$$

Hence for $1 \leq m, n \leq 3$ the characters $(\omega_m)_H$ and $(\gamma_n)_H$ have a linear constituent with multiplicity 1. This completes the proof ■

Now we prove corresponding theorems for the groups $\text{SU}(3, q)$. The following table shows the values of the restriction of the irreducible characters of the groups $\text{SU}(3, q)$ on the elements of Sylow subgroup H when $d = 1$. Since $d = 1, l = 0$.

	$C_1^{(0)}$	$C_2^{(0)}$	$C_3^{(0,0)}$
1	1	1	1
ψ	$q^2 - q$	$-q$	0
ρ	q^3	0	0
ζ_i	$q^2 - q + 1$	$-q + 1$	1
η_j	$q^3 - q^2 + q$	q	0
ε_r	$q^3 - 2q^2 + 2q - 1$	$2q - 1$	-1
μ_s	$q^3 + 1$	1	1
ν_t	$q^3 + q^2 - q - 1$	$-q - 1$	-1

Table 1.6: VALUES OF CHARACTERS OF $\text{SU}(3, q)$

ON ELEMENTS OF H WHEN $d = 1$.

where $1 \leq i, j \leq q$, $1 \leq r \leq (q^2 - q)/6$, $1 \leq s \leq (q^2 - q - 2)/2$ and $1 \leq t \leq (q^2 - q)/3$.

Theorem 4.3.10. *Let $G = \text{SU}(3, q)$ where $q > 2$ is a power of the prime p and $d = 1$. Let H be a Sylow p -subgroup of G . Then H is a χ -subgroup for all irreducible characters χ of G such that $\chi(1) \neq q^2 - q$. If $\chi(1) = q^2 - q$ then there is an abelian subgroup of order q^2 in H which is a χ -subgroup.*

Proof: Let ψ is the irreducible character of degree $q^2 - q$ of G and τ be an irreducible character of degree q of H . Then using Table 1.6 for the value of ψ on the conjugacy class $C_2^{(0)}$ containing the nontrivial elements of $Z(H)$, and Lemma 4.3.1 we have

$$\begin{aligned} \langle \psi_H, \tau \rangle &= \frac{1}{|H|} \sum_{x \in H} \psi_H(x) \overline{\tau(x)} \\ &= \frac{1}{q^3} (\psi_H(1) \tau(1) + \sum_{1 \neq z \in Z(H)} \psi_H(z) \overline{\tau(z)} + \sum_{z \notin Z(H)} \psi_H(z) \overline{\tau(z)}) \\ &= \frac{1}{q^3} ((q^2 - q)q + (-q)(-q) + 0) = 1. \end{aligned}$$

Since H has $q - 1$ irreducible characters of degree q , we have

$$\psi_H = \sum_{i=1}^{q-1} \tau_i \tag{4.8}$$

Let ρ be the irreducible character of degree q^3 of G . Since $\rho(1) = |H|$, Lemma 4.2.1 shows that ρ_H is the regular character of H . But H has q^2 linear characters so for each linear character φ of H we have $\langle \rho_H, \varphi \rangle = 1$ and by (4.8) we have $\langle \psi_H, \varphi \rangle = 0$.

On the other hand Table 1.6 shows that:

$$(\zeta_i)_H = \psi_H + \mathbf{1},$$

$$(\eta_j)_H = \rho_H - \psi_H,$$

$$(\varepsilon_r)_H = \rho_H - 2\psi_H - \mathbf{1},$$

$$(\mu_s)_H = \rho_H + \mathbf{1}$$

and

$$(\nu_t)_H = \rho_H + \psi_H - \mathbf{1}.$$

Therefore if φ is a non-principal linear character of H then, since $\langle \rho_H, \varphi \rangle = 1$ and $\langle \psi_H, \varphi \rangle = 0$, we get

$$\langle (\eta_j)_H, \varphi \rangle = \langle (\varepsilon_r)_H, \varphi \rangle = \langle (\mu_s)_H, \varphi \rangle = \langle (\nu_t)_H, \varphi \rangle = 1$$

and

$$\langle (\zeta_i)_H, \mathbf{1} \rangle = 1.$$

So we have proved the theorem for all characters except the character ψ of degree $q^2 - q$. In the later case we proved as follows.

Define

$$K := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 1 \end{pmatrix} \mid \text{for } a, b \in \mathbb{F} \right\}.$$

Then K is an abelian subgroup of H of order q^2 and $Z(H) \subset K$. Let

$$k = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 1 \end{pmatrix} \in K \setminus \{1\}.$$

Then

$$(k - 1)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a^2 & 0 & 0 \end{pmatrix}$$

Thus, if $(k-1)^2 = 0$, then $a = 0$ and $k \in Z(H)$. Otherwise the minimal polynomial for k has degree 3. Since the minimal polynomials of elements in the conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,0)}$ have degrees 1, 2 and 3, respectively, we have $k \in \mathcal{C}_2^{(0)}$ when $1 \neq k \in Z(H)$ and $k \in \mathcal{C}_3^{(0,0)}$ when $k \notin Z(K)$. Let ϕ be a non-principal linear character of K . Then using the values of ψ on $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,0)}$ we have

$$\begin{aligned} \langle \psi_K, \phi \rangle &= \frac{1}{|K|} \sum_{k \in K} \psi_K(k) \overline{\phi(k)} \\ &= \frac{1}{|K|} (\psi_K(1)\phi(1) + \sum_{1 \neq k \in Z(H)} \psi_K(k) \overline{\phi(k)} + \sum_{k \notin Z(H)} \psi_K(k) \overline{\phi(k)}) \\ &= \frac{1}{q^2} ((q^2 - q) + (-q) \sum_{1 \neq k \in Z(H)} \overline{\phi(k)} + 0). \end{aligned}$$

Put $Z := Z(H)$ then $Z \subset K$ and ϕ_Z is a linear character of Z . Using the first orthogonality relation we have $\sum_{k \in Z} \overline{\phi(k)} = \sum_{k \in Z} \overline{\phi_Z(k)} = 0$. Therefore

$$\sum_{1 \neq k \in Z(H)} \overline{\phi(k)} = -1.$$

This shows $\langle \psi_K, \phi \rangle = 1$ as required ■

Now we consider the case that $d = 3$. Then we have the following table of the values of irreducible characters of the groups $\text{SU}(3, q)$ on the conjugacy classes which contain the elements of H .

	$\mathcal{C}_1^{(0)}$	$\mathcal{C}_2^{(0)}$	$\mathcal{C}_3^{(0,l)}$
1	1	1	1
ψ	$q^2 - q$	$-q$	0
ρ	q^3	0	0
ζ_i	$q^2 - q + 1$	$-q + 1$	1
η_j	$q^3 - q^2 + q$	q	0
θ_k	$(q^3 - 2q^2 + 2q - 1)/3$	$(2q - 1)/3$ or $(-q - 1)/3$	$(2q - 1)/3$ or $(-q - 1)/3$
ε_r	$q^3 - 2q^2 + 2q - 1$	$2q - 1$	-1
μ_s	$q^3 + 1$	1	1
ν_t	$q^3 + q^2 - q - 1$	$-q - 1$	-1
ω_m	$(q^3 + q^2 - q - 1)/3$	$(-q - 1)/3$ or $(2q - 1)/3$	$(-q - 1)/3$ or $(2q - 1)/3$
γ_n	$(q^3 + q^2 - q - 1)/3$	$(-q - 1)/3$ or $(2q - 1)/3$	$(-q - 1)/3$ or $(2q - 1)/3$

Table 1.7: VALUES OF CHARACTERS OF $\text{SU}(3, q)$ ON ELEMENTS OF H WHEN $d = 3$.

where $1 \leq i, j \leq q$, $1 \leq r \leq (q^2 - q - 2)/6$, $1 \leq s \leq (q^2 - q - 2)/2$, $1 \leq t \leq (q^2 - q - 2)/3$ and $1 \leq k, m, n \leq 3$.

Theorem 4.3.11. *Let $G := \text{GU}(3, q)$, then the elements of each set of characters $\{\theta_1, \theta_2, \theta_3\}$, $\{\omega_1, \omega_2, \omega_3\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ of $\text{SU}(3, q)$, are conjugate in G .*

Proof: See [SpFm] ■

Theorem 4.3.12. *Let $G = \text{SU}(3, q)$ where $q > 2$ is a power of the prime p and $d = 3$ and let H be a Sylow p -subgroup of G . Then for all irreducible characters χ of G such that $\chi(1) \neq q^2 - q$, H is a χ -subgroup. If $\chi(1) = q^2 - q$ then H contains an abelian subgroup of order q^2 which is a χ -subgroup.*

Proof: We use the same method as was used to prove Theorem 4.3.10. Thus ρ_H is the regular character of H , ψ_H is the character of degree $q^2 - q$ of H , and $\psi_H = \sum_{i=1}^{q-1} \tau_i$ where the τ_i are the irreducible characters of degree q of H . Also for the characters $\zeta_i, \eta_j, \varepsilon_r, \mu_s$ and ν_t we have:

$$(\zeta_i)_H = \psi_H + \mathbf{1},$$

$$(\eta_j)_H = \rho_H + \psi_H,$$

$$(\varepsilon_r)_H = \rho_H + 2\psi_H + \mathbf{1},$$

$$(\mu_s)_H = \rho_H - \mathbf{1}$$

and

$$(\nu_t)_H = \rho_H - \psi_H + \mathbf{1}.$$

The structure of ρ_H and ψ_H shows $\langle \rho_H, \varphi \rangle = 1$ and $\langle \psi_H, \varphi \rangle = 0$ for all non-principal linear characters φ of H . Hence

$$\langle (\eta_j)_H, \varphi \rangle = \langle (\varepsilon_r)_H, \varphi \rangle = \langle (\mu_s)_H, \varphi \rangle = \langle (\nu_t)_H, \varphi \rangle = 1$$

and

$$\langle (\zeta_i)_H, \mathbf{1} \rangle = 1.$$

As the same as Theorem 4.3.10 we define

$$K := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 1 \end{pmatrix} \mid \text{for } a, b \in \mathbb{F} \right\}.$$

Then K is an abelian subgroup of H of order q^2 and $Z(H) \subset K$. Suppose

$$k = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 1 \end{pmatrix} \in K \setminus \{1\}.$$

Then $(k - 1)^2 = 0$ if and only if $a = 0$ and $k \in Z(H)$. Otherwise the minimal polynomial for k has degree 3. Since the minimal polynomials of elements in the conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,l)}$ have degrees 1, 2 and 3, respectively, we have $k \in \mathcal{C}_2^{(0)}$ when $1 \neq k \in Z(H)$ and $k \in \mathcal{C}_3^{(0,l)}$ for some $0 \leq l \leq 2$ when $k \notin Z(K)$. Now if ϕ is a non-principal linear character of K . Then using the values of ψ on $\mathcal{C}_2^{(0)}$ and $\mathcal{C}_3^{(0,l)}$ we have

$$\begin{aligned} \langle \psi_K, \phi \rangle &= \frac{1}{|K|} \sum_{k \in K} \psi_K(k) \overline{\phi(k)} \\ &= \frac{1}{|K|} (\psi_K(1)\phi(1) + \sum_{1 \neq k \in Z(H)} \psi_K(k) \overline{\phi(k)} + \sum_{k \notin Z(H)} \psi_K(k) \overline{\phi(k)}) \\ &= \frac{1}{q^2} ((q^2 - q) + (-q) \sum_{1 \neq k \in Z(H)} \overline{\phi(k)} + 0). \end{aligned}$$

Put $Z := Z(H)$ then $Z \subset K$ and ϕ_Z is a linear character of Z . Using the first orthogonality relation we have

$$\sum_{k \in Z} \overline{\phi(k)} = \sum_{k \in Z} \overline{\phi_Z(k)} = 0.$$

Therefore

$$\sum_{1 \neq k \in Z(H)} \overline{\phi(k)} = -1.$$

This shows $\langle \psi_K, \phi \rangle = 1$.

The only remaining characters to consider are θ_k , ω_m and γ_n for $1 \leq k, m, n \leq 3$.

Suppose φ is a non-principal linear character of H then $\langle \psi_H, \varphi \rangle = 0$ and $\langle \rho_H, \varphi \rangle = 1$ so

$$\langle (\eta_j)_H, \varphi \rangle = \langle (\varepsilon_r)_H, \varphi \rangle = \langle (\mu_s)_H, \varphi \rangle = \langle (\nu_t)_H, \varphi \rangle = 1 \text{ and } \langle (\zeta_i)_H, \varphi \rangle = 0.$$

Using the Frobenius reciprocity we have,

$$\langle \eta_j, \varphi^G \rangle = \langle \varepsilon_r, \varphi^G \rangle = \langle \mu_s, \varphi^G \rangle = \langle \nu_t, \varphi^G \rangle = 1 \text{ and } \langle \zeta_i, \varphi^G \rangle = 0.$$

Also if

$$\langle (\theta_k)_H, \varphi \rangle = K_k, \langle (\omega_m)_H, \varphi \rangle = M_m \text{ and } \langle (\gamma_n)_H, \varphi \rangle = N_n,$$

then

$$\langle \theta_k, \varphi^G \rangle = K_k, \langle \omega_m, \varphi^G \rangle = M_m \text{ and } \langle \gamma_n, \varphi^G \rangle = N_n$$

for $1 \leq k, m, n \leq 3$.

Now if we induce φ to G we get

$$\begin{aligned} \varphi^G &= \rho + q\eta_j + ((q^2 - q - 2)/6)\varepsilon_r + ((q^2 - q - 2)/2)\mu_s \\ &+ ((q^2 - q - 2)/3)\nu_t + \sum_{k=1}^3 K_k \theta_k + \sum_{m=1}^3 M_m \omega_m + \sum_{n=1}^3 N_n \gamma_n \end{aligned}$$

but $\varphi^G(1) = |G : H|\varphi(1)$ so if we calculate the value at 1 and simplify the above equation we have

$$|G : H| = q^2 - 2q^3 + q^5 + \sum_{k=1}^3 K_k \theta_k(1) + \sum_{m=1}^3 M_m \omega_m(1) + \sum_{n=1}^3 N_n \gamma_n(1).$$

Since $|G : H| = q^5 - q^3 + q^2 - 1$ and we get

$$\sum_{k=1}^3 K_k \theta_k(1) + \sum_{m=1}^3 M_m \omega_m(1) + \sum_{n=1}^3 N_n \gamma_n(1) = q^3 - 1.$$

Since $\theta_k(1) = (q^3 - 2q^2 + 2q - 1)/3$ and $\omega_m(1) = \gamma_n(1) = (q^3 + q^2 - q - 1)/3$ so we have

$$\left(\sum_{k=1}^3 K_k\right)((q^3 - 2q^2 + 2q - 1)/3) + \left(\sum_{m=1}^3 M_m + \sum_{n=1}^3 N_n\right)((q^3 + q^2 - q - 1)/3) = q^3 - 1.$$

Hence by considering $K = \sum_{k=1}^3 K_k$, $M = \sum_{m=1}^3 M_m$ and $N = \sum_{n=1}^3 N_n$ we get

$$K((q^3 - 2q^2 + 2q - 1)/3) + (M + N)((q^3 + q^2 - q - 1)/3) = q^3 - 1$$

so

$$(K + M + N)q^3 - (2K - (M + N))q^2 + ((2K - (M + N))q - (K + M + N)) = 3(q^3 - 1).$$

Thus

$$(A - 3)(q^3 - 1) = B(q^2 - q) \quad (4.9)$$

where $A = K + M + N$ and $B = 2K - (M + N)$. Since $q \mid B(q^2 - q)$ so $q \mid A - 3$ and this means that $A - 3 = tq$ for some integer t . Hence simplifying equation (4.9) implies $B = t(q^2 + q + 1)$. Therefore

$$0 \leq 3K = A + B = 3 + t(q + 1)^2$$

and

$$0 \leq 3(M + N) = 2A - B = 6 - t(q^2 - q + 1).$$

If $q > 2$ then the first inequality shows that $t \geq 0$ and the second shows that $t \leq 0$. So $t = 0$, $A = 3$ and $B = 0$ which this concludes $K = 1$ and $M + N = 2$. Hence $\sum_{k=1}^3 K_k = 1$ and $\sum_{m=1}^3 M_m + \sum_{n=1}^3 N_n = 2$. Therefore, for some k , $K_k = 1$ and $\langle (\theta_k)_H, \varphi \rangle = 1$.

Let $\langle (\theta_1)_H, \varphi \rangle = 1$ then by Theorem 4.3.11 the characters θ_1, θ_2 and θ_3 are conjugate in $L = \text{GU}(3, q)$. Hence by Lemma 4.3.7 we have

$$\langle (\theta_1)_H, \varphi \rangle = \langle (\theta_2)_H, \varphi^x \rangle = \langle (\theta_3)_H, \varphi^y \rangle = 1$$

for some $x, y \in N_L(H)$. On the other hand by Lemma 4.3.6, φ^x and φ^y are linear characters of H so the restriction of characters θ_1, θ_2 and θ_3 to H have at least a constituent of degree one with multiplicity one.

Also equation (4.5) shows $\sum_{m=1}^3 M_m = \sum_{n=1}^3 N_n$ and so both sums equal 1. Therefore for some m and n we have $N_n = 1$ and $M_m = 1$ which this means $\langle (\omega_m)_H, \varphi \rangle = \langle (\gamma_n)_H, \varphi \rangle = 1$. Without any ambiguity we can suppose $\langle (\omega_1)_H, \varphi \rangle = \langle (\gamma_1)_H, \varphi \rangle = 1$. Theorem 4.3.11 shows the elements of each set of characters $\{\omega_1, \omega_2, \omega_3\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ are conjugate in $L = \text{GU}(3, q)$. Therefore by Lemma 4.3.7 and Lemma 4.3.6 there exist $r, s, t, u \in N_L(G)$ such that $\varphi^r, \varphi^s, \varphi^t$ and φ^u are linear characters of H and

$$\langle (\omega_2)_H, \varphi^r \rangle = \langle (\omega_3)_H, \varphi^s \rangle = \langle (\gamma_2)_H, \varphi^t \rangle = \langle (\gamma_3)_H, \varphi^u \rangle = 1.$$

Hence for $1 \leq m, n \leq 3$ the characters $(\omega_m)_H$ and $(\gamma_n)_H$ have a linear constituent with multiplicity 1. This completes the proof ■

4.4 Other simple groups and covers

In the first section of this chapter we showed that (with a few exceptions), for each irreducible character χ of degree less than 32 of the alternating groups and their covers there exists a Sylow p -subgroup which is a χ -subgroup, for some prime p . Also without any restriction on the degree of characters, if G is one of the groups $\text{PSL}(2, q)$, $\text{PSL}(3, q)$, $\text{PSU}(3, q)$ or their covers and χ is an irreducible character of G , then there exists a Sylow subgroup or a p -subgroup of G , which is a χ -subgroup.

Lemma 4.2.1 shows that if a group G has a Sylow subgroup P and an irreducible character χ such that $|P| = \chi(1)$, then χ_P is the regular character of P . In this case $\langle \chi_P, \varphi \rangle = 1$ for each linear character φ of P (i.e., P is a χ -subgroup) and we can

use the character restriction method to construct a representation of G affording χ . Using this Lemma and some computations in GAP, we found all cases where G is a simple group or a cover of a simple group listed in [Atlas] and χ an irreducible character of degree less than 32 of G , for which there exists a Sylow subgroup which is a χ -subgroup. We have summarized our results in the following table.

The groups which have not been listed in this table and have irreducible characters of degree less than 32 are $3.O_7(3)$, $3.U_6(2)$ and covering groups of $U_4(3)$. Suppose $\tilde{G}$ is one of these groups and χ is an irreducible character of $\tilde{G}$ of degree less than 32. We shall use [Atlas] to find a maximal subgroup $\tilde{M}$ of $\tilde{G}$ such that $\chi_{\tilde{M}}$ is irreducible. It is enough to find a maximal subgroup $\tilde{M}$ such that

$$\langle \chi_{\tilde{M}}, \chi_{\tilde{M}} \rangle = \langle \chi_{\tilde{M}} \bar{\chi}_{\tilde{M}}, \mathbf{1} \rangle = 1.$$

Since χ is irreducible, $\langle \chi, \chi \rangle = \langle \chi \bar{\chi}, \mathbf{1} \rangle = 1$. Note that the kernel of $\chi \bar{\chi}$ contains the centre of $\tilde{G}$ and so we can consider $\chi \bar{\chi}$ as a character of $G = \tilde{G}/Z(\tilde{G})$. So

$$\chi_M \bar{\chi}_M = \mathbf{1} + \sum_{\mathbf{1} \neq \psi_i \in \text{Irr}(G)} m_i (\psi_i)_M. \quad (4.10)$$

where $M = \tilde{M}/Z(\tilde{G})$. Now if we find a maximal subgroup M of G such that for each constituent $(\psi_i)_M$ of equation (4.10), $\langle (\psi_i)_M, \mathbf{1} \rangle = 0$ then $\langle \chi_M \bar{\chi}_M, \mathbf{1} \rangle = 1$. This means that the restriction $\chi_{\tilde{M}}$ of χ to the inverse image $\tilde{M}$ of M in the centre of $\tilde{G}$ is irreducible.

Covering Groups of $U_4(3)$: Suppose $G = U_4(3)$. The covering groups $2.G$ and $4.G$ have one and two characters of degree 21, respectively. The covering group $3_1.G$ has two characters of degree 15 and two characters of degree 21. Finally the covering group $6_1.G$ has two characters of degree 6. If χ is an

irreducible character of degree less than 32 of one of these covers such that $\chi(1) \neq 21$, then for the maximal subgroup $M \cong 3^4:A_6$ of index 112 of G we have $\langle \chi_M \bar{\chi}_M, \mathbf{1} \rangle = 1$ which this means $\chi_{\tilde{M}}$ is irreducible. On the other hand consider a Sylow 3-subgroup H of G and let C be the centralizer H' in H . Then the normalizer M of C in G (i.e., $M = N_G(C_H(H'))$) is a maximal subgroup of index 112 of G such that $\chi_{\tilde{M}}$ is irreducible. If $\chi(1) = 21$ then G has a maximal subgroup M isomorphic to $\text{PSL}(3, 4)$ of index 162 such that $\langle \chi_M \bar{\chi}_M, \mathbf{1} \rangle = 1$ and so $\chi_{\tilde{M}}$ is irreducible.

3.O₇(3) : Suppose $G = O_7(3)$. The covering group $3.G$ has two characters of degree 27, and G has a maximal subgroup M of index 364 such that $M \cong 3^5:U_4(2):2$. For each character χ of degree 27, $\chi_{\tilde{M}}$ is irreducible. If we take H as a Sylow 3-subgroup of G and U_6 as the sixth term of the upper central series of H then $|U_6| = 81$. Suppose C is the centralizer of U_6 in H then M is the normalizer of C in G (i.e., $M = N_G(C_H(U_6))$).

3.U₆(2) : Suppose $G = U_6(2)$. The covering group $3.G$ has two irreducible characters of degree 21, and G has a maximal subgroup M of index 891 such that $M \cong 2^9:L_3(4)$. We find that $\chi_{\tilde{M}}$ is irreducible where χ is either of the characters of degree 21. If we take H as a Sylow 2-subgroup of G and U_3 as the third term of the upper central series of H then $|U_3| = 64$. Suppose C is the centralizer of U_3 in H then M is the normalizer of C in G (i.e., $M = N_G(C_H(U_3))$).

According to section 4.1 if G is one of the covering groups $6.A_6$, $2.A_7$, $3.A_7$, $6.A_7$ and $2.A_8$ then G has some irreducible characters of degree less than 32 which

their restriction on all p -subgroups of G do not have any linear constituent with multiplicity one. In other words, if χ is a character of this type then G does not have any p -subgroup as a χ -subgroup. Although for computations it may be less convenient, we have to look elsewhere for a χ -subgroup. Computation in GAP enabled us to find the following χ -subgroups of G .

6.A₆ : Suppose $G = 6.A_6$ then G has four irreducible characters of degree 12, say χ , such that none of the p -subgroups of G is a χ -subgroup. However GAP shows that G has a subgroup of order 60 which contains the centre of G and is a χ -subgroup.

2.A₇ : Suppose $G = 2.A_7$ then G has two irreducible characters of degree 20, say χ , such that none of the p -subgroups of G is a χ -subgroup. However GAP shows that G has a subgroup of order 40 which contains the centre of G and is a χ -subgroup.

3.A₇ : Suppose $G = 3.A_7$ then G has two irreducible characters of degree 21 and four irreducible characters of degree 24, say χ , such that none of the p -subgroups of G is a χ -subgroup. However GAP shows that, if $\chi(1) = 21$ then G has an abelian subgroup of order 36 which contains the centre of G and is a χ -subgroup. If $\chi(1) = 24$ then G has a subgroup of order 60 which contains the centre of G and is a χ -subgroup.

6.A₇ : Suppose $G = 6.A_7$ then G has two irreducible characters of degree 20, two irreducible characters of degree 21 and eight irreducible characters of degree 24, say χ , such that none of the p -subgroups of G is a χ -subgroup. However GAP shows that if $\chi(1) = 20, 21$ and 24 then G has a subgroup of order 120, 72 and 120, respectively, which contains the centre of G and is a χ -subgroup.

2.A₈ : Suppose $G = 2.A_8$ then G has two irreducible characters of degree 24, say χ , such that none of the p -subgroups of G is a χ -subgroup. However GAP shows that G has an abelian subgroup of order 30 which contains the centre of G and is a χ -subgroup.

4.4.1 Summary

Table 1.8: χ -subgroups of simple groups and covering groups
which are p -subgroups

G	$ G $	$1 < \chi(1) < 32$	χ -subgroups
A_5	$2^2.3.5$	3, 4, 5	Syl(3)
$2.A_5$	$2^3.3.5$	2, 3, 4, 5, 6	Syl(5)
A_6	$2^3.3^2.5$	5, 8, 9, 10	Syl(2)
$2.A_6$	$2^4.3^2.5$	4, 5, 8, 9, 10	Syl(3)
$3.A_6$	$2^3.3^3.5$	3, 5, 6, 8, 9, 10, 15	Syl(2)
$(6.A_6)^*$	$2^4.3^3.5$	3, 4, 5, 6, 8, 9, 10, 12, 15	Syl(5) for $\chi(1) < 10$ Syl(3) for $\chi(1) = 10$ Syl(2) for $\chi(1) = 15$
A_7	$2^3.3^2.5.7$	6, 10, 14, 15, 21	Syl(3)
$(2.A_7)^*$	$2^4.3^2.5.7$	4, 6, 10, 14, 15, 20, 21	Syl(3) for $\chi(1) \neq 20$
$(3.A_7)^*$	$2^3.3^3.5.7$	4, 6, 10, 14, 15, 21, 24	Syl(2) for $\chi(1) \neq 21, 24$
$(6.A_7)^*$	$2^4.3^3.5.7$	4, 6, 10, 14, 15, 20, 21, 24	Syl(7) for $\chi(1) < 14$ Syl(3) for $\chi(1) = 14$ Syl(2) for $\chi(1) = 15$
M_{11}	$2^4.3^2.5.11$	10, 11, 16	Syl(11)
A_8	$2^6.3^2.5.7$	7, 14, 20, 21, 28	Syl(3) for $\chi(1) < 28$ Syl(2) for $\chi(1) = 28$

(Continued over)...

Table 1.8:(Continued)

G	$ G $	$1 < \chi(1) < 32$	χ -subgroups
$(2.A_8)^*$	$2^7.3^2.5.7$	7, 8, 14, 20, 21, 24, 28	Syl(7) for $\chi(1) < 14$ Syl(3) for $\chi(1) = 14, 20, 21$ Syl(2) for $\chi(1) = 28$
$2.L_3(4)$	$2^7.3^2.5.7$	10, 20, 28	Syl(2)
$3.L_3(4)$	$2^6.3^3.5.7$	15, 20, 21	Syl(2)
$4_1.L_3(4)$	$2^8.3^2.5.7$	8, 10, 20, 28	Syl(2)
$4_2.L_3(4)$	$2^8.3^2.5.7$	10, 20, 28	Syl(2)
$6.L_3(4)$	$2^7.3^3.5.7$	6, 10, 15, 20, 21, 28	Syl(2)
$12_1.L_3(4)$	$2^8.3^3.5.7$	6, 10, 15, 20, 21, 24, 28	Syl(2)
$12_2.L_3(4)$	$2^8.3^3.5.7$	6, 10, 15, 20, 21, 28	Syl(2)
$U_4(2)$	$2^6.3^4.5$	5, 6, 10, 15, 20, 24, 30	Syl(5) for $\chi(1) = 6$ Syl(3) for $\chi(1) \neq 6$
$Sp_4(3)$	$2^7.3^4.5$	4, 5, 6, 10, 15, 20, 24, 30	Syl(5) for $\chi(1) = 6$ Syl(3) for $\chi(1) \neq 6$
$Sz(8)$	$2^6.5.7.13$	14	Syl(13)
M_{12}	$2^6.3^3.5.11$	11, 16	Syl(2)
$2.M_{12}$	$2^6.3^3.5.11$	10, 11, 12, 16	Syl(2)
A_9	$2^6.3^4.5.7$	8, 21, 27, 28	A_3 for $\chi(1) = 8$ Syl(2) for $\chi(1) \neq 8$
$2.A_9$	$2^7.3^4.5.7$	8, 21, 27, 28	Syl(7) for $\chi(1) = 8$ Syl(3) for $\chi(1) \neq 8$

(Continued over)...

Table 1.8:(Continued)

G	$ G $	$1 < \chi(1) < 32$	χ -subgroups
M_{22}	$2^7.3^2.5.7.11$	21	Syl(2)
$2.M_{22}$	$2^8.3^2.5.7.11$	10, 21	Syl(3) for $\chi(1) = 10$ Syl(2) for $\chi(1) = 21$
$3.M_{22}$	$2^7.3^3.5.7.11$	21	Syl(2)
J_2	$2^7.3^3.5^2.7$	14, 21	Syl(5)
$2.J_2$	$2^8.3^3.5^2.7$	6, 14, 21	Syl(5)
$S_4(4)$	$2^8.3^2.5^2.17$	18	Syl(5)
$S_6(2)$	$2^9.3^4.5.7$	7, 15, 21, 27	Syl(7) for $\chi(1) = 7$ Syl(3) for $\chi(1) \neq 7$
$2.S_6(2)$	$2^{10}.3^4.5.7$	7, 8, 15, 21, 27	Syl(7) for $\chi(1) = 7, 8$ Syl(3) for $\chi(1) \neq 7, 8$
A_{10}	$2^7.3^4.5^2.7$	9	A_3
$2.A_{10}$	$2^8.3^4.5^2.7$	9, 16	Syl(5)
$U_4(3)$	$2^7.3^6.5.7$	21	Syl(2)
$G_2(3)$	$2^6.3^6.7.13$	14	Syl(13)
$3.G_2(3)$	$2^6.3^7.7.13$	14, 27	Syl(2)
$S_4(5)$	$2^6.3^2.5^4.13$	13	Syl(13)
$Sp_4(5)$	$2^7.3^2.5^4.13$	12, 13	Syl(3)
$L_4(3)$	$2^7.3^6.5.13$	26	Syl(3)
$L_5(2)$	$2^{10}.3^2.5.7.31$	30	Syl(2)
M_{23}	$2^7.3^2.5.7.11.23$	22	Syl(23)

(Continued over)...

Table 1.8:(Continued)

G	$ G $	$1 < \chi(1) < 32$	χ -subgroups
$U_5(2)$	$2^{10}.3^5.5.11$	10, 11	Syl(11)
${}^2F_4(2)'$	$2^{11}.3^3.5^2.13$	26, 27	Syl(3)
A_{11}	$2^7.3^4.5^2.7.11$	10	A_3
$2.A_{11}$	$2^8.3^4.5^2.7.11$	10, 16	Syl(11)
HS	$2^9.3^2.5^3.7.11$	22	Syl(2)
$3.J_3$	$2^7.3^6.5.17.19$	18	Syl(17)
$O_8^+(2)$	$2^{12}.3^5.5^2.7$	28	Syl(5)
$2.O_8^+(2)$	$2^{13}.3^5.5^2.7$	8, 28	Syl(5)
${}^3D_4(2)$	$2^{12}.3^4.7^2.13$	26	Syl(7)
A_{12}	$2^9.3^5.5^2.7.11$	11	A_3
M_{24}	$2^{10}.3^3.5.7.11.23$	23	Syl(23)
$2.G_2(4)$	$2^{13}.3^3.5^2.7.13$	12	Syl(13)
M^cL	$2^7.3^6.5^3.7.11$	22	Syl(2)
A_{13}	$2^9.3^5.5^2.7.11.13$	12	A_3
$S_6(3)$	$2^9.3^9.5.7.13$	13	Syl(13)
$2.S_6(3)$	$2^{10}.3^9.5.7.13$	13, 14	Syl(13)
$U_6(2)$	$2^{15}.3^6.5.7.11$	22	Syl(3)
$2.Ru$	$2^{15}.3^3.5^3.7.13.29$	28	Syl(29)
$6.Suz$	$2^{14}.3^8.5^2.7.11.13$	12	Syl(13)
Co_3	$2^{10}.3^7.5^3.7.11.23$	23	Syl(23)
Co_2	$2^{18}.3^6.5^3.7.11.23$	23	Syl(23)
$2.Co_1$	$2^{22}.3^9.5^4.7^2.11.13.23$	24	Syl(23)

Note: Groups marked by * in the table above have characters of degree < 32 which have no p -subgroups which are χ -subgroups. We have described χ -subgroups for these exceptional cases in the previous section.

Chapter 5

Representation of Perfect Groups

In this chapter we present methods to construct representations of perfect groups. The main references in this chapter are [HtPk89] and libraries from [GAP].

An important class of perfect groups is the class of simple groups and their central covers. As we saw in Chapter 4, there exists a χ -subgroup for each irreducible character of degree less than 32 for a simple group or a covering group of a simple group (except a few exceptions which have been mentioned on page 67). So for those groups we can use the character restriction method. In the present chapter we shall be looking at the other perfect groups.

Let G be a group. Then $\text{Soc}(G)$, the *socle* of G , is defined to be the subgroup generated by the set of all minimal normal subgroups of G . Every nontrivial finite group has at least one minimal normal subgroup and so has a nontrivial socle. If G is a finite group then $\text{Soc}(G)$ is a direct product of some minimal normal subgroups of G (see [DxMr] Theorem 4.3A).

In the next two sections we classify the class of perfect groups G into two sub-classes depending on whether $\text{Soc}(G/Z(G))$ is abelian or non-abelian.

5.1 Perfect Groups with $\text{Soc}(G/Z(G))$ Non-Abelian

Lemma 5.1.1. *If H is a perfect subgroup of a group G and K is normal in G such that G/K is solvable then $H \leq K$.*

Proof: Since $H/(H \cap K) \cong HK/K \leq G/K$ so $H/(H \cap K)$ is solvable. But H is perfect, so $H/(H \cap K) = 1$, and this implies $H \leq K$ ■

Lemma 5.1.2. *Let $T_1, T_2, \dots, T_k$ be k non-abelian simple groups. Then there exists a homomorphism*

$$\sigma : \text{Aut}(T_1 \times T_2 \times \dots \times T_k) \rightarrow S_k$$

with $\ker(\sigma) = \prod_{i=1}^k \text{Aut}(T_i)$.

Proof: Let $G := T_1 \times T_2 \times \dots \times T_k$, $A := \text{Aut}(G)$. By assumption for $1 \leq i \leq k$ the group T_i is a non-abelian simple group so it is a minimal normal subgroup of G and $T_1, T_2, \dots, T_k$ are the only minimal normal subgroups of G (see [Ros] Lemma 8.9). Hence if we consider $\Omega = \{T_1, T_2, \dots, T_k\}$ then A acts on Ω by mapping each minimal normal subgroup of G to a minimal normal subgroup of G . Therefore for each $\psi \in A$ there exist a permutation $\pi \in S_k$ such that $\psi(T_i) \cong T_{\pi(i)}$ for $1 \leq i \leq k$, where the mapping

$$\psi \mapsto \pi$$

defines a homomorphism from A to S_k .

Now consider $\psi \in \ker(\sigma)$. Then $\psi(T_i) = T_i$ and

$$\psi|_{T_i} : T_i \rightarrow T_i$$

is an automorphism of T_i for each i , so $\psi|_{T_i} \in \text{Aut}(T_i)$. Hence $\psi \in \prod_{i=1}^k \text{Aut}(T_i)$. Conversely each $\psi \in \prod_{i=1}^k \text{Aut}(T_i)$ is an automorphism of G which maps each T_i into itself. Thus $\ker(\sigma) = \prod_{i=1}^k \text{Aut}(T_i)$ ■

According to the lemma above if $A = \text{Aut}(T_1 \times T_2 \times \dots \times T_k)$ and $A^* = \prod_{i=1}^k \text{Aut}(T_i)$ then A/A^* is isomorphic to a subgroup of S_k .

Notation 5.1.3. Let $Z = Z(G)$ be the centre of the group G . For each subgroup H of G which contains Z denote H/Z by $\bar{H}$.

Theorem 5.1.4. Let G be a finite perfect group and $\bar{S} = \text{Soc}(\bar{G}) = \bar{T}_0 \times \bar{T}_1 \times \dots \times \bar{T}_n$ where $\bar{T}_0$ is the abelian part of $\bar{S}$ and for $1 \leq i \leq n$, $\bar{T}_i$ are non-abelian simple groups. Let $\bar{S}^* = \bar{T}_1 \times \bar{T}_2 \times \dots \times \bar{T}_n$ and $\bar{C} = C_{\bar{G}}(\bar{S}^*)$. If $n \leq 4$, then $\bar{G} = \bar{C} \times \bar{S}^*$ and G is the central product of $C, T_1, T_2, \dots, T_n$. In particular if $\bar{T}_0 = 1$ then $\bar{C} = 1$ and so G is the central product of $T_1, T_2, \dots, T_n$.

Proof: By assumption $\bar{C} = C_{\bar{G}}(\bar{S}^*)$ so $\bar{S}^* \triangleleft \bar{G}$ implies $\bar{C} \triangleleft \bar{G}$ and $\bar{C} \cap \bar{S}^* = 1$ since $Z(\bar{S}^*) = 1$. Since $\bar{G}$ acts by conjugation on $\bar{S}^*$ so we define a homomorphism

$$\phi : \bar{G} \longrightarrow \text{Aut}(\bar{S}^*)$$

with $\ker(\phi) = \bar{C}$ and $\bar{G}/\bar{C}$ is isomorphic to a subgroup $B = \phi(\bar{G})$ of $A := \text{Aut}(\bar{S}^*)$. On the other hand

$$A^* = \text{Aut}(\bar{T}_1) \times \text{Aut}(\bar{T}_2) \times \dots \times \text{Aut}(\bar{T}_n) \triangleleft A.$$

Since $n \leq 4$ Lemma 5.1.2 shows we can embed A/A^* in S_4 .

Since $\bar{C} \cap \bar{S}^* = 1$ so the restriction of ϕ on $\bar{S}^*$ is one to one and this means $\bar{S}^* \cong \phi(\bar{S}^*)$. Since $\bar{T}_i$ are non-abelian simple for $i \neq 0$, $Z(\bar{T}_i) = 1$ and thus

$$\bar{S}^* \cong \text{Inn}(\bar{T}_1) \times \text{Inn}(\bar{T}_2) \times \dots \times \text{Inn}(\bar{T}_n) = \text{Inn}(\bar{S}^*) = \phi(\bar{S}^*) = I^*, \text{ say.}$$

Now by ‘‘Schreier’s Conjecture’’ the group of outer automorphisms of each finite simple group is solvable (see [DxMr] page 133). Thus $\text{Out}(\bar{T}_i) = \text{Aut}(\bar{T}_i)/\text{Inn}(\bar{T}_i)$ are solvable and

$$A^*/I^* \cong \text{Out}(\bar{T}_1) \times \text{Out}(\bar{T}_2) \times \dots \times \text{Out}(\bar{T}_n) = O^*$$

is solvable. The following diagram shows the situation above.

$$\begin{array}{ccc}
A & & \\
| & \} & \hookrightarrow S_4 \\
A^* & & \\
| & \} & \cong O^* \\
I^* & &
\end{array}$$

Since A/A^* is isomorphic to a subgroup of S_4 , it is solvable. On the other hand B is a perfect subgroup of A , so by Lemma 5.1.1 we have B is a perfect subgroup of A^* . I^* is a normal subgroup of A^* and $A^*/I^* \cong O^*$ is solvable thus by Lemma 5.1.1 we get B is isomorphic to a subgroup of I^* . However we have $\bar{S}^* \cong I^*$ embedded in B . Hence $\bar{S}^* \cong B \cong \bar{G}/\bar{C}$. Since $\bar{C} \cap \bar{S}^* = 1$, this implies $\bar{G} \cong \bar{C} \times \bar{S}^*$ which is $\bar{G} \cong \bar{C} \times \bar{T}_1 \times \bar{T}_2 \times \dots \times \bar{T}_n$. Now we show that G is a central product of $C, T_1, T_2, \dots, T_n$. Suppose U equals C or some T_i and V is the product of the remaining factors. It means if $U = C$ then $V = T_1 \times \dots \times T_n$ and if $U = T_i$ for some $1 \leq i \leq n$ then $V = C \times T_1 \times \dots \times T_{i-1} \times T_{i+1} \times \dots \times T_n$. We show that $[U, V] = 1$. Let $u \in U$ and consider the mapping $f_u : V \longrightarrow Z(G)$ defined by $f_u(y) = [u, y]$. Since $[u, y] \in Z(G)$

$$\begin{aligned}
f_u(yy') &= [u, yy'] = u^{-1}(y')^{-1}y^{-1}uyy' = u^{-1}(y')^{-1}uu^{-1}y^{-1}uyu^{-1}yy' \\
&= [u, y]u^{-1}(y')^{-1}uu^{-1}yy' = [u, y][u, y'] = f_u(y)f_u(y')
\end{aligned}$$

shows f_u is a homomorphism. Since $\text{Im}(f_u) \leq Z(G)$ is abelian, $\ker(f_u)$ contains V' . Since G is perfect, $V = V'$ and so $\ker(f_u) = V$. Thus $[u, y] = 1$ for all $y \in V$. Therefore $[U, V] = 1$ and G is a central product of U and V . Now the general result follows by induction on the number of factors in G .

If $\bar{T}_0 = 1$ and $\bar{C} \neq 1$ then $\bar{C}$ contains a minimal normal subgroup of $\bar{G}$ and we get $\bar{C} \cap \bar{S}^* \neq 1$ which contradicts $Z(\bar{S}^*) = 1$. Therefore $\bar{C} = 1$ and G is a central product of $T_1, T_2, \dots, T_n$ ■

Since G is perfect, $T_1, \dots, T_n$ are non-abelian perfect groups, so the degree of a nontrivial character of each T_i is at least 2. Therefore if $n \geq 5$ and χ is a faithful character of G then $\chi(1) \geq 2^5 = 32$ (Theorem 3.3.1). Since we have restricted the degrees of our characters to less than 32 we can assume $n \leq 4$ and so the theorem above applies.

Suppose G is a group satisfying the conditions of the theorem above. If $\bar{G}$ has more than one factor then we can use the tensor product method to construct representations of G (Theorem 3.3.1). Otherwise $\bar{S} = \bar{T}_1 = \bar{G}$ or $\bar{S} = \bar{T}_0$. If $\bar{G} = \bar{T}_1$ is a non-abelian simple group then G is a covering group of a simple group and we can use the character restriction method (see Section 3.4).

The case where $\bar{S} = \bar{T}_0$ is abelian is the topic of the next section.

5.2 Perfect Groups with $\text{Soc}(G/Z(G))$ Abelian

Suppose G is a perfect group with $\text{Soc}(G/Z(G))$ abelian and χ is a faithful irreducible character of G . If there exists an abelian normal subgroup A of G such that $A \not\leq Z(G)$ then using Clifford's Theorem (Theorem 3.1.3)

$$\chi_A = e \sum_{i=1}^t \phi_i$$

where $\phi = \phi_1, \dots, \phi_t$ are distinct irreducible characters of A conjugate to ϕ in G . Since $A \not\leq Z(G)$, $t > 1$ and we can use the induction method.

Thus suppose $Z(G)$ is the maximal normal abelian subgroup of G . The following theorem is due to Isaacs [Is75].

Theorem 5.2.1 ([Sht] page 31). *Let G be a group with centre $Z := Z(G)$ and H be a subgroup of G such that*

1. $Z(H) = Z$,
2. $[H, G] \leq Z$ and
3. $|\text{Hom}(H/Z, Z)| \leq |H/Z|$.

Then $G/Z = H/Z \times C_G(H)/Z$.

Theorem 5.2.2 ([Dx71] Theorem 4.4). *Let χ be a faithful irreducible character of degree n of a group G and suppose $Z := Z(G)$ is the unique maximal normal abelian subgroup of G . Let $F := \text{Fit}(G)$ be the Fitting subgroup of G . Then:*

1. F/Z is the unique maximal normal abelian subgroup of G/Z .
2. $|F : Z| = d^2$ for some divisor d of n and in particular $d = n$ if χ_F is irreducible.
3. The Sylow subgroups of F/Z are elementary abelian.
4. Let $d = p_1^{l_1} \dots p_s^{l_s}$ be the prime decomposition of d . Then there exists a homomorphism θ of G into $\prod_{i=1}^s \text{Sp}(2l_i, p_i)$ with

$$\ker(\theta) = \{g \in G \mid [g, x] \in Z \text{ for all } x \in F\} = C_G(F/Z).$$

Suppose G is a finite group, $Z(G)$ is the maximal normal abelian subgroup and $\text{Soc}(G/Z(G)) = T_0/Z(G)$ is abelian. We shall show that in this case the $\text{Fit}(G) = C_G(\text{Fit}(G)/Z(G))$ by proving the following lemma, which is a generalization of ([Dx71] Theorem 4.5).

Lemma 5.2.3. *Let G be an irreducible subgroup of $GL(V)$ where V is a vector space over $\mathbb{C}$ and let $Z(G)$ be the maximal normal abelian subgroup of G . Suppose $\text{Soc}(G/Z(G))$ is abelian. Then $C_G(\text{Fit}(G)/Z(G)) = \text{Fit}(G)$.*

Proof: Define $Z := Z(G)$, $F := \text{Fit}(G)$ and $C := C_G(F/Z)$. Then by Theorem 5.2.2, F/Z is abelian and we get $F \leq C$. Now we check the conditions of Theorem 5.2.1 for the group C and subgroup F . Since Z is the maximal normal abelian subgroup of G and $Z(F)$ is a characteristic subgroup of F therefore $Z(F) \subseteq Z$ and $Z(C) \subseteq Z$. Since $Z \subseteq C$ therefore $Z \subseteq Z(C)$, and since $Z \subseteq F$ therefore $Z \subseteq Z(F)$. Hence $Z(F) = Z(C) = Z$. This proves condition (1) of Theorem 5.2.1. For (2) the definition of C shows $[F, C] \leq Z(C)$. Finally since Z is a group of scalar, Z is isomorphic to a subgroup of $\mathbb{C}^*$ and this means each homomorphism from F/Z to Z is a representation of degree one. By Theorem 5.2.2, F/Z is abelian so there are exactly $|F/Z|$ representations of degree 1 for F/Z which implies condition (3). Therefore by Theorem 5.2.1, $C/Z = F/Z \times C_C(F)/Z$.

Finally we show $C_C(F)/Z = 1$. If $C_C(F)/Z \neq 1$ then G/Z has a minimal normal subgroup contained in $C_C(F)/Z$. Since $C_C(F)/Z \cap F/Z = 1$ and $\text{Soc}(G/Z) \subseteq F/Z$ by hypothesis we have a contradiction. Hence $C_C(F)/Z = 1$ and so $F/Z = C/Z$ and therefore $F = C$ ■

Theorem 5.2.4. *Let G be a group such that $Z(G)$ is the unique maximal normal abelian subgroup of G and suppose $\text{Soc}(G/Z(G))$ is abelian. If χ is a faithful irreducible character of G of degree n then $|\text{Fit}(G) : Z(G)| = d^2$ for some divisor $d = p_1^{l_1} \dots p_s^{l_s}$ of n and the factor group $G/\text{Fit}(G)$ is isomorphic to a subgroup of $\prod_{i=1}^s \text{Sp}(2l_i, p_i)$.*

Proof: This follows at once from Theorem 5.2.2 and Lemma 5.2.3 ■

Put $F := \text{Fit}(G)$. If χ_F contains at least two distinct irreducible constituents, then χ is imprimitive and we can reduce our problem to construction of a representation for a proper subgroup of G (see Section 3.1). Thus we consider the case where $\chi_F = e\theta$ for some $e \geq 1$ and $\theta \in \text{Irr}(F)$. In this case θ is G -invariant (i.e., $I_G(\theta) = G$).

If $e = 1$ then χ_F is an irreducible character of F , so the Theorem 3.2.1 shows how we can extend a representation of F affording χ_F to a representation of G affording χ .

Now consider the case that $e > 1$. We consider the different possible values of e and $\theta(1)$. First of all we show that $\theta(1) > 1$. If $\theta(1) = 1$ then the elements of F are scalar and so $F \subseteq Z(G)$ and this means that $F/Z(G) = 1$. On the other hand $T_0/Z(T_0)$ is a homomorphic image of $\bar{T}_0$ and so is abelian. Therefore T_0 is nilpotent and this implies $T_0 \subseteq F$. Hence $\bar{T}_0 = \text{Soc}(G/Z(G)) = 1$ and this happens when $G = Z(G)$ which is a contradiction with $\chi(1) = e$.

Definition 5.2.5. *Let H be a subgroup of group G and μ is an irreducible character of H . Then μ is called **extendible** to G if there exists an irreducible character ρ of G such that $\rho_H = \mu$.*

Theorem 5.2.6 (Gallagher). *Suppose G is a group and χ is an irreducible character of G . Let N be a normal subgroup of G and μ be an irreducible character of N such that $\langle \chi_N, \mu \rangle \neq 0$. If μ is extendible to G (i.e., $\rho_N = \mu$ for some $\rho \in \text{Irr}(G)$), then there exists $\alpha \in \text{Irr}(G/N)$ such that $\chi = \alpha\rho$.*

Proof: See ([Isc] Corollary 6.17) ■

Thus in the case where $\chi_F = e\theta$ ($\theta \in \text{Irr}(F)$ and $e > 1$) we can solve our problem if we can find $\rho \in \text{Irr}(G)$ such that $\rho_F = \theta$ and $\alpha \in \text{Irr}(G/F)$ such that $\chi = \alpha\rho$. The theorem below gives a condition for θ to be extendible to G .

Theorem 5.2.7. *Let $N \triangleleft G$ and let $\theta \in \text{Irr}(N)$ be invariant in G . Suppose that for each prime p there is a Sylow p -subgroup P/N of G/N such that θ is extendible to P . Then θ is extendible to G .*

Proof: See ([Is] Corollary 11.31) ■

By the theorem above if we show that for each Sylow subgroup P/F of G/F the irreducible character θ of F is extendible to P , then θ is extendible to G . Then we know that there exists an irreducible character α of G with $F \subseteq \ker \alpha$ such that $\alpha(1) = e$ and $\chi = \alpha\rho$. If $\mathcal{R}$ and $\mathcal{S}$ are representations of G affording α and ρ , respectively, then Theorem 3.3.1 shows $x \mapsto \mathcal{R}(x) \otimes \mathcal{S}(x)$ is a representation of G affording χ .

Theorem 5.2.8. *Suppose G is a perfect group, $Z(G)$ is a maximal normal abelian subgroup and $\text{Soc}(G/Z(G))$ is abelian. Let χ be a faithful irreducible character of G , $F := \text{Fit}(G)$, and $\chi_F = e\theta$ for $\theta \in \text{Irr}(F)$ and $e > 1$. If $e = p$ is a prime then either θ is extendible to G or χ_P is irreducible, where P/F is a Sylow p -subgroup of G/F .*

Proof: Suppose Q/F is a Sylow q -subgroup of G/F and $\chi_Q = \sum e_i \psi_i$ for $\psi_i \in \text{Irr}(Q)$. Since θ is invariant in G , $(\psi_i)_F = f_i \theta$ and so $\chi_F = e\theta$ implies $e = \sum e_i f_i$. By Clifford's Theorem $f_i \mid |Q : F|$ so $f_i = q_i^{\beta_i}$ for some $\beta_i \geq 0$. If $\beta_i = 0$ for some i then $f_i = 1$ and we have $(\psi_i)_F = \theta$ and θ is extendible to Q . Therefore using Theorem 5.2.7 we have θ is extendible to G unless for some q we have $\beta_i > 0$ for all i .

Suppose $\beta_i > 0$ for all i . Then $q \mid f_i$ for each i and so $q \mid e$. Since $e = p$ is prime $q = p$. Since $e = \sum e_i f_i$ we conclude that $e_1 = 1$, $f_1 = p$ and $e = e_1 f_1$. This implies $\chi_Q = \psi_1$ is an irreducible character of Q and Q/F is a Sylow p -subgroup of G/F ■

Theorem 5.2.9. *If χ is a faithful irreducible character of G of degree n and $G/Z(G)$ is abelian then $|G : Z(G)| = n^2$.*

Proof: See ([Dx71] Theorem 4.3) or ([Is] Theorem 2.31) ■

Corollary 5.2.10. *Suppose G is a perfect group for which $Z(G)$ is a maximal normal abelian subgroup and $F := \text{Fit}(G)$. If χ is a faithful irreducible character of G such that $\chi_F = e\theta$ for $\theta \in \text{Irr}(F)$, then $|F : Z(G)| = \theta(1)^2$.*

Proof: Since $Z(G)$ is the maximal normal abelian subgroup of G and $Z(F)$ is a characteristic subgroup of F so $Z(F) \subseteq Z(G)$. On the other hand $Z(G) \subseteq F$ implies $Z(G) \subseteq Z(F)$. Hence $Z(F) = Z(G)$. Now since χ is faithful for G and $\chi_F = e\theta$, therefore θ is faithful for F . Hence the Theorem shows $|F : Z(G)| = \theta(1)^2$ ■

Theorem 5.2.11. *Suppose G is a perfect group with $Z(G)$ a maximal normal abelian subgroup and $\text{Soc}(G/Z(G))$ abelian. Put $F := \text{Fit}(G)$. If χ is a faithful irreducible character of G with $\chi_F = e\theta$ for $\theta \in \text{Irr}(F)$, then $\theta(1) \notin \{2, 3, 6\}$.*

Proof: If $\theta(1) = d = p_1^{l_1} \dots p_s^{l_s}$ then Theorem 5.2.4 and Corollary 5.2.10 implies G/F is isomorphic to a subgroup of $\prod_{i=1}^s \text{Sp}(2l_i, p_i)$. Since G is perfect, G/F is perfect. If $\theta(1) = 2, 3$ or 6 then G/F is isomorphic to a subgroup of $\text{Sp}(2, 2), \text{Sp}(2, 3)$ or $\text{Sp}(2, 2) \times \text{Sp}(2, 3)$, respectively. But none of these groups contains any perfect subgroup so $\theta(1) \notin \{2, 3, 6\}$ ■

Theorem 5.2.8 shows how to deal with the case where e is a prime and Theorem 5.2.11 shows that the case where $\theta(1) \mid 6$ cannot occur. Thus if we restrict the degree of the character χ to less than 32 then the only remaining cases to consider are:

$$(\chi(1), \theta(1), e) = (16, 4, 4), (20, 5, 4), (24, 4, 6), (28, 7, 4), (30, 5, 6).$$

Lemma 5.2.12. *If G is a perfect group and χ is a faithful irreducible character of degree n for G , then $|Z(G)| \mid n$.*

Proof: Let $\chi(1) = n$ and $\mathcal{R}$ be a representation of G affording χ . Since $Z(G) \subset G'$ so for each $z \in Z(G)$ there exist $x, y \in G$ such that $z = x^{-1}y^{-1}xy$. Thus

$$\det(\mathcal{R}(z)) = \det(\mathcal{R}(x^{-1}))\det(\mathcal{R}(y^{-1}))\det(\mathcal{R}(x))\det(\mathcal{R}(y)) = 1.$$

On the other hand since χ is faithful so $Z(G)$ is cyclic (see [Is] Theorem 2.32). Suppose $Z(G) = \langle z \rangle$ has order m , say. Since $\mathcal{R}(z)$ is an scalar, there is an m^{th} root of unity ω such that

$$\mathcal{R}(z) = \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega \end{pmatrix}$$

Since $\det(\mathcal{R}(z)) = 1$, $\omega^n = 1$ and this implies $m \mid n$ ■

We now deal with the exceptional cases listed above.

5.2.1 Characters of Degree 16

Consider the case where χ is of degree 16 and $\chi_F = 4\theta$ where θ is an irreducible character of F of degree 4. Theorem 5.2.4 shows that the perfect group G/F is isomorphic to a subgroup of $\text{Sp}(4, 2)$. Since A_5 and A_6 are the only perfect subgroups of $\text{Sp}(4, 2)$ so $G/F \cong A_5$ or $G/F \cong A_6$. Using [HtPk89], [GAP] and Lemma 5.2.12, for the case $G/F \cong A_5$ we have 11 different perfect groups G with an irreducible character of degree 16. Computations in GAP shows that these perfect groups all contain χ -subgroups, for each irreducible character χ of degree 16. Therefore using the character restriction method, we can construct a representation of G affording χ . The details are as follows (the notation is from [GAP]).

1. $G = A_5 2^{4'} \mathbf{C} \mathbf{N} 2^1 :$

Then $|G| = 1920$, $|Z(G)| = 2$ and G has one irreducible character χ of degree 16. We find that G contains a 2-subgroup H of size 32 with $Z(G) \subset H$ which is a χ -subgroup.

2. $G = A_5 2^{4'} \mathbf{C} 2^1 :$

Then $|G| = 1920$, $|Z(G)| = 2$ and G has one irreducible character χ of degree 16. We find that G contains a subgroup H of size 48 with $Z(G) \subset H$ which is a χ -subgroup.

3. $G = A_5 (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} 2^1 \mathbf{I} :$

Then $|G| = 3840$, $|Z(G)| = 4$ and G has two irreducible characters χ of degree 16. We find that for each of these characters G contains a subgroup H of size 48 with $Z(G) \subset H$ which is a χ -subgroup.

4. $G = A_5 (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} 2^1 \mathbf{II} :$

Then $|G| = 3840$, $|Z(G)| = 4$ and G has two irreducible characters χ of degree 16. We find that for each of these characters G contains an abelian 2-subgroup H of size 64 with $Z(G) \subset H$ which is a χ -subgroup.

5. $G = A_5 2^1 \times (2^{4'} \mathbf{C} 2^1) :$

Then $|G| = 3840$, $|Z(G)| = 4$ and G has two irreducible characters χ of degree 16. We find that for each of these characters G contains a subgroup H of size 48 with $Z(G) \subset H$ which is a χ -subgroup.

6. $G = A_5 (2^4 \mathbf{E} (2^1 \mathbf{A} \times 2^1)) \mathbf{C} 2^1 :$

Then $|G| = 7680$, $|Z(G)| = 8$ and G has four irreducible characters χ of degree 16. We find that for each of these characters G contains a subgroup H of size 96 with $Z(G) \subset H$ which is a χ -subgroup.

7. $G = A_5 (2^4 \mathbf{E} (2^1 \mathbf{A} \times \mathbf{N} 2^1)) \mathbf{C} 2^1 \mathbf{I} :$

Then $|G| = 7680$, $|Z(G)| = 8$ and G has four irreducible characters χ of degree 16. We find that for each of these characters G contains a subgroup H of size 96 with $Z(G) \subset H$ which is a χ -subgroup.

8. $G = A_5 (2^4 \mathbf{E} (2^1 \mathbf{A} \times \mathbf{N} 2^1)) \mathbf{C} 2^1 \mathbf{II} :$

Then $|G| = 7680$, $|Z(G)| = 8$ and G has four irreducible characters χ of degree 16. We find that for each of these characters G contains an abelian 2-subgroup H of size 128 with $Z(G) \subset H$ which is a χ -subgroup.

9. $G = A_5 2^1 \times (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} 2^1 :$

Then $|G| = 7680$, $|Z(G)| = 8$ and G has four irreducible characters χ of degree 16. We find that for each of these characters G contains a subgroup H of size 96 with $Z(G) \subset H$ which is a χ -subgroup.

10. $G = A_5 (2^4 \mathbf{E} (2^1 \mathbf{A} \times 2^1 \mathbf{A})) \mathbf{C} (2^1 \times 2^1) :$

Then $|G| = 15360$, $|Z(G)| = 16$ and G has 12 irreducible characters χ of degree 16. We find that for each of these characters G contains a subgroup H of size 192 with $Z(G) \subset H$ which is a χ -subgroup.

11. $G = A_5 2^1 \times (2^4 \mathbf{E} (2^1 \mathbf{A} \times 2^1)) \mathbf{C} 2^1 :$

Then $|G| = 15360$, $|Z(G)| = 16$ and G has eight irreducible characters χ of degree 16. We find that for each of these characters G contains a subgroup H of size 192 with $Z(G) \subset H$ which is a χ -subgroup.

When $G/F \cong A_6$ there exist two perfect groups which contain an irreducible character of degree 16.

1. $G = A_6 (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} 2^1 :$

Then $|G| = 23040$, $|Z(G)| = 4$ and G has four irreducible characters χ of degree 16. We find that for each of these characters G contains a 2-subgroup H of size 64 with $Z(G) \subset H$ which is a χ -subgroup.

2. $G = A_6 2^1 \times (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} 2^1 :$

Then $|G| = 46080$, $|Z(G)| = 8$ and G has four irreducible characters χ of degree 16. We find that for each of these characters G contains a 2-subgroup H of size 128 with $Z(G) \subset H$ which is a χ -subgroup.

When $G/F \cong A_6$ we have an alternative method of constructing representations which is described in section 5.2.6.

5.2.2 Characters of Degree 20

Suppose χ is of degree 20 such that $\chi_F = 4\theta$ where θ is an irreducible character of F of degree 5. Theorem 5.2.4 shows the perfect group G/F is isomorphic to a subgroup of $\mathrm{Sp}(2, 5) \cong \mathrm{SL}(2, 5)$. Since $\mathrm{SL}(2, 5)$ is the only perfect subgroup of $\mathrm{Sp}(2, 5)$ therefore $G/F \cong \mathrm{SL}(2, 5)$. Using [HtPk89], [GAP] and Lemma 5.2.12 we have only one perfect group which has an irreducible character of degree 20 and $G/F \cong \mathrm{SL}(2, 5)$.

1. $G = A_5 2^1 5^2 \mathbf{C} 5^1 :$

Then $|G| = 15000$, $|Z(G)| = 5$ and G has eight irreducible characters χ of degree 20. We find that for each of these characters G contains an abelian subgroup H of size 50 with $Z(G) \subset H$ which is a χ -subgroup.

5.2.3 Characters of Degree 24

Suppose χ is of degree 24 such that $\chi_F = 6\theta$ where θ is an irreducible character of F of degree 4. Theorem 5.2.4 shows that the perfect group G/F is isomorphic to a subgroup of $\text{Sp}(4, 2)$. Since A_5 and A_6 are the only perfect subgroups of $\text{Sp}(4, 2)$ therefore $G/F \cong A_5$ or $G/F \cong A_6$. Using [HtPk89], [GAP] and Lemma 5.2.12, for the case $G/F \cong A_5$ we have six different perfect groups G with an irreducible character of degree 24.

1. $G = A_5 \ 2^{4'} \ \mathbf{C} \ 2^1 :$

Then $|G| = 1920$, $|Z(G)| = 2$ and G has one irreducible character χ of degree 24. We find that G contains a subgroup H of size 48 which $Z(G) \subset H$ and is a χ -subgroup.

2. $G = A_5 \ (2^4 \ \mathbf{E} \ 2^1 \ \mathbf{A}) \ \mathbf{C} \ 2^1 \ \mathbf{I} :$

Then $|G| = 3840$, $|Z(G)| = 4$ and G has two irreducible characters χ of degree 24. We find that for each of these characters G contains a subgroup H of size 96 with $Z(G) \subset H$ which is a χ -subgroup.

3. $G = A_5 \ 2^1 \times (2^{4'} \ \mathbf{C} \ 2^1) :$

Then $|G| = 3840$, $|Z(G)| = 4$ and G has one irreducible character χ of degree 24. We find that G contains a subgroup H of size 96 with $Z(G) \subset H$ which is a χ -subgroup.

4. $G = A_5 \ (2^4 \ \mathbf{E} \ (2^1 \ \mathbf{A} \times 2^1)) \ \mathbf{C} \ 2^1 :$

Then $|G| = 7680$, $|Z(G)| = 8$ and G has two irreducible characters χ of degree 24. We find that for each of these characters G contains a subgroup H of size 192 with $Z(G) \subset H$ which is a χ -subgroup.

5. $G = A_5 (2^4 \mathbf{E} (2^1 \mathbf{A} \times \mathbf{N} 2^1)) \mathbf{C} 2^1 \mathbf{I} :$

Then $|G| = 7680$, $|Z(G)| = 8$ and G has four irreducible characters χ of degree 24. We find that for each of these characters G contains a subgroup H of size 192 with $Z(G) \subset H$ which is a χ -subgroup.

6. $G = A_5 2^1 \times (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} 2^1 :$

Then $|G| = 7680$, $|Z(G)| = 8$ and G has two irreducible characters χ of degree 24. We find that for each of these characters G contains a subgroup H of size 192 with $Z(G) \subset H$ which is a χ -subgroup.

When $G/F \cong A_6$ there exist 12 perfect groups which contain an irreducible character of degree 24.

1. $G = A_6 2^4 \mathbf{E} 2^1 :$

Then $|G| = 11520$, $|Z(G)| = 2$ and G has one irreducible character χ of degree 24. We find that for G contains an abelian subgroup H of size 24 with $Z(G) \subset H$ which is a χ -subgroup.

2. $G = A_6 2^4 \mathbf{E} \mathbf{N} 2^1 :$

Then $|G| = 11520$, $|Z(G)| = 2$ and G has one irreducible character χ of degree 24. We find that for G contains an abelian subgroup H of size 24 with $Z(G) \subset H$ which is a χ -subgroup.

3. $G = A_6 (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} 2^1 :$

Then $|G| = 23040$, $|Z(G)| = 4$ and G has one irreducible character χ of degree 24. We find that for G contains a subgroup H of size 48 with $Z(G) \subset H$ which is a χ -subgroup.

4. $G = A_6 (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} \mathbf{N} 2^1 :$

Then $|G| = 23040$, $|Z(G)| = 4$ and G has one irreducible character χ of degree 24. We find that for G contains a subgroup H of size 48 with $Z(G) \subset H$ which is a χ -subgroup.

5. $G = A_6 2^1 \times (2^4 \mathbf{E} 2^1) :$

Then $|G| = 23040$, $|Z(G)| = 4$ and G has two irreducible characters χ of degree 24. We find that for each of these characters G contains an abelian subgroup H of size 48 with $Z(G) \subset H$ which is a χ -subgroup.

6. $G = A_6 3^1 \times 2^4 \mathbf{E} 2^1 :$

Then $|G| = 34560$, $|Z(G)| = 6$ and G has three irreducible characters χ of degree 24. We find that for each of these characters G contains an abelian subgroup H of size 72 with $Z(G) \subset H$ which is a χ -subgroup.

7. $G = A_6 3^1 \times 2^4 \mathbf{E} \mathbf{N} 2^1 :$

Then $|G| = 34560$, $|Z(G)| = 6$ and G has three irreducible characters χ of degree 24. We find that for each of these characters G contains an abelian subgroup H of size 72 with $Z(G) \subset H$ which is a χ -subgroup.

8. $G = A_6 2^1 \times (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} 2^1 :$

Then $|G| = 46080$, $|Z(G)| = 8$ and G has two irreducible characters χ of degree 24. We find that for each of these characters G contains a subgroup H of size 96 with $Z(G) \subset H$ which is a χ -subgroup.

9. $G = A_6 3^1 \times (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} 2^1 :$

Then $|G| = 69120$, $|Z(G)| = 12$ and G has 11 irreducible characters χ of degree 24. We find that for each of these characters G contains a subgroup H of size 192 with $Z(G) \subset H$ which is a χ -subgroup.

10. $G = A_6 3^1 \times (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} \mathbf{N} 2^1 :$

Then $|G| = 69120$, $|Z(G)| = 12$ and G has seven irreducible characters χ of degree 24. We find that for each of these characters G contains a subgroup H of size 192 with $Z(G) \subset H$ which is a χ -subgroup.

11. $G = A_6 3^1 \times 2^1 \times (2^4 \mathbf{E} 2^1) :$

Then $|G| = 69120$, $|Z(G)| = 12$ and G has six irreducible characters χ of degree 24. We find that for each of these characters G contains an abelian subgroup H of size 144 with $Z(G) \subset H$ which is a χ -subgroup.

12. $G = A_6 3^1 \times 2^1 \times (2^4 \mathbf{E} 2^1 \mathbf{A}) \mathbf{C} 2^1 :$

Then $|G| = 138240$, $|Z(G)| = 24$ and G has 18 irreducible characters χ of degree 24. We find that for each of these characters then a subgroup H of size 288 or subgroup H of size 384 with $Z(G) \subset H$ which is a χ -subgroup.

5.2.4 Characters of Degree 28

Suppose χ is of degree 28 such that $\chi_F = 4\theta$ where θ is an irreducible character of F of degree 7. Theorem 5.2.4 shows that the perfect group G/F is isomorphic to a subgroup of $\mathrm{Sp}(2, 7) \cong \mathrm{SL}(2, 7)$. Since $\mathrm{SL}(2, 7)$ is the only perfect subgroup of $\mathrm{Sp}(2, 7)$ therefore $G/F \cong \mathrm{SL}(2, 7)$. Using [HtPk89], [GAP] and Lemma 5.2.12 we have only one perfect group which has an irreducible character of degree 28 and $G/F \cong \mathrm{SL}(2, 7)$.

1. $G = \mathrm{PSL}(3, 2) 2^1 7^2 \mathbf{C} 7^1 :$

Then $|G| = 115248$, $|Z(G)| = 7$ and G has 12 irreducible characters χ of degree 28. We find that for each of these characters G contains an abelian subgroup H of size 98 with $Z(G) \subset H$ which is a χ -subgroup.

The perfect group G also contains two maximal subgroups M_1 and M_2 such that at least one of the restrictions χ_{M_i} is irreducible, for all irreducible characters χ of degree 28 (see section 5.2.6).

5.2.5 Characters of Degree 30

Suppose χ is of degree 30 such that $\chi_F = 6\theta$ where θ is an irreducible character of F of degree 5. Theorem 5.2.4 shows that the perfect group G/F is isomorphic to a subgroup of $\text{Sp}(2, 5) \cong \text{SL}(2, 5)$. Since $\text{SL}(2, 5)$ is the only perfect subgroup of $\text{Sp}(2, 5)$ therefore $G/F \cong \text{SL}(2, 5)$. Using [HtPk89], [GAP] and Lemma 5.2.12 we have only one perfect group which has an irreducible character of degree 30 and $G/F \cong \text{SL}(2, 5)$.

1. $G = A_5 \ 2^1 \ 5^2 \ \mathbf{C} \ 5^1$:

Then $|G| = 15000$, $|Z(G)| = 5$ and G has four irreducible characters χ of degree 30. We find that for each of these characters G contains an abelian 5-subgroup H of size 125 with $Z(G) \subset H$ which is a χ -subgroup.

5.2.6 Alternative method

The following theorem gives an alternative method to construct representations of G affording characters of degree 16 in case $G/F \cong A_6$ and characters of degree 28.

Theorem 5.2.13. *Suppose G is a perfect group in which $Z(G)$ is a maximal normal abelian subgroup and $\text{Soc}(G/Z(G))$ is abelian. Put $F := \text{Fit}(G)$. Let χ be a faithful irreducible character of G such that $\chi_F = e\theta$ for $\theta \in \text{Irr}(F)$. Write $\chi\bar{\chi}$ in terms of the irreducible characters of G :*

$$\chi\bar{\chi} = \sum_{F \subseteq \ker(\psi_i)} m_i \psi_i + \sum_{F \not\subseteq \ker(\psi_j)} m_j \psi_j \quad (5.1)$$

If $\rho = \sum_{F \subseteq \ker(\psi_i)} m_i \psi_i$ then $\rho(1) = e^2$ and $\langle \rho, \mathbf{1} \rangle = 1$.

Proof: Since $\chi_F = e\theta$ we have $\chi_F \bar{\chi}_F = e^2(\theta \bar{\theta})$ and Theorem 5.2.10 shows $|F : Z| = \theta(1)\bar{\theta}(1) = \theta^2(1)$. Since χ is faithful, θ is faithful, so using ([Is] Corollary 2.30) we have $(\theta \bar{\theta})(x) = 0$ for $x \in F \setminus Z$. Since $\theta \bar{\theta}$ has degree $|F/Z|$, $\theta \bar{\theta}$ is the regular character of F/Z and so

$$\theta \bar{\theta} = \mathbf{1} + \sum_{\mathbf{1} \neq \phi_i} \phi_i(1) \phi_i$$

where $\phi_i \in \text{Irr}(F/Z)$. Thus

$$\chi_F \bar{\chi}_F = e^2(\theta \bar{\theta}) = e^2 \mathbf{1} + e^2 \sum_{\mathbf{1} \neq \phi_i} \phi_i(1) \phi_i. \quad (5.2)$$

Suppose $\psi \in \text{Irr}(G)$, then $F \subseteq \ker(\psi)$ if and only if $\psi_F = \psi(1)\mathbf{1}$. Since $\rho = \sum_{F \subseteq \ker(\psi_i)} m_i \psi_i$ so $\rho_F = \rho(1)\mathbf{1}$. Comparing the equations (5.1) and (5.2) implies $\rho(1) = e^2$.

Since χ is irreducible, $1 = \langle \chi, \chi \rangle = \langle \chi \bar{\chi}, \mathbf{1} \rangle = \langle \rho, \mathbf{1} \rangle$. This completes the proof ■

Characters of Degree 16: Suppose $\chi_F = 4\theta$ and $G/F \cong A_6$. Using Theorem 5.2.13 we have $\rho(1) = 16$ and $\rho = \mathbf{1} + \sum m_i \psi_i$ where $\mathbf{1}, \psi_1, \psi_2, \dots$ are distinct irreducible characters of $\text{Irr}(A_6)$. The group A_6 has six nontrivial irreducible characters $\psi_2, \dots, \psi_7$ of degrees 5, 5, 8, 8, 9 and 10, respectively. Therefore we can write

$$\rho(1) = 16 = 1 + (m_2 + m_3)5 + (m_4 + m_5)8 + m_6 9 + m_7 10.$$

If we consider all the possibilities for m_i we have

$$\begin{aligned} \rho &= \mathbf{1} + m_2 \psi_2 + m_3 \psi_3 + \psi_7 & \text{with } m_2 + m_3 &= 1 \\ \rho &= \mathbf{1} + m_2 \psi_2 + m_3 \psi_3 & \text{with } m_2 + m_3 &= 3 \end{aligned}$$

Suppose M/F is the maximal subgroup of $G/F \cong A_6$ of order 36 then

$$\langle (\psi_2)_{M/F}, \mathbf{1} \rangle = \langle (\psi_3)_{M/F}, \mathbf{1} \rangle = \langle (\psi_7)_{M/F}, \mathbf{1} \rangle = 0$$

and this implies $\langle \rho_M, \mathbf{1} \rangle = 1$. Since $\langle \chi_M, \chi_M \rangle = \langle \rho_M, \mathbf{1} \rangle$ so χ_M is irreducible. Therefore using Theorem 3.2.1 we can extend a representation of M affording χ_M to a representation of G affording χ .

Characters of Degree 28: Suppose $\chi_F = 4\theta$ and $G/F \cong \text{SL}(2, 7)$. Using Theorem 5.2.13 we have $\rho(1) = 28$ and $\rho = \mathbf{1} + \sum m_i \psi_i$ where $\mathbf{1}, \psi_1, \psi_2, \dots$ are distinct irreducible characters in $\text{Irr}(\text{SL}(2, 7))$. The group $\text{SL}(2, 7)$ has 10 nontrivial irreducible characters $\psi_2, \dots, \psi_{11}$ of degrees 3, 3, 4, 4, 6, 6, 6, 7, 8 and 8, respectively. Therefore we can write

$$\rho(1) = 16 = 1 + (m_2 + m_3)3 + (m_4 + m_5)4 + (m_6 + m_7 + m_8)6 + m_9 7 + (m_{10} + m_{11})8$$

If we consider all the possibilities for m_i we have

1. $\rho = \mathbf{1} + \psi_9 + m_{10}\psi_{10} + m_{11}\psi_{11}$ with $m_{10} + m_{11} = 1$.
2. $\rho = \mathbf{1} + m_4\psi_4 + m_5\psi_5 + \psi_9$ with $m_4 + m_5 = 2$.
3. $\rho = \mathbf{1} + m_2\psi_2 + m_3\psi_3 + m_6\psi_6 + m_7\psi_7 + m_8\psi_8$
with $m_2 + m_3 = 1$ and $m_6 + m_7 + m_8 = 2$.
4. $\rho = \mathbf{1} + m_2\psi_2 + m_3\psi_3 + m_4\psi_4 + m_5\psi_5 + m_{10}\psi_{10} + m_{11}\psi_{11}$
with $m_2 + m_3 = m_4 + m_5 = m_{10} + m_{11} = 1$.
5. $\rho = \mathbf{1} + m_2\psi_2 + m_3\psi_3 + m_6\psi_6 + m_7\psi_7 + m_8\psi_8$
with $m_2 + m_3 = 3$ and $m_6 + m_7 + m_8 = 1$.

6. $\rho = \mathbf{1} + m_2\psi_2 + m_3\psi_3$ with $m_2 + m_3 = 5$.

7. $\rho = \mathbf{1} + m_2\psi_2 + m_3\psi_3 + m_4\psi_4 + m_5\psi_5$ with $m_2 + m_3 = 1$ and $m_4 + m_5 = 3$.

The group $G/F \cong \mathrm{SL}(2, 7)$ has two maximal subgroups M_1/F and M_2/F of orders 42 and 48, respectively. We have $\langle (\psi_i)_{M_1/F}, \mathbf{1} \rangle = 0$ for $i \neq 9$ and $\langle (\psi_i)_{M_2/F}, \mathbf{1} \rangle = 0$ for $i \neq 6$. Since for each of the possibilities for ρ , at least one of the characters ψ_6 and ψ_9 does not appear, either $\langle \rho_{M_1}, \mathbf{1} \rangle = 1$ or $\langle \rho_{M_2}, \mathbf{1} \rangle = 1$. Since $\langle \chi_{M_i}, \chi_{M_i} \rangle = \langle \rho_{M_i}, \mathbf{1} \rangle$, either χ_{M_1} or χ_{M_2} is irreducible. Therefore using Theorem 3.2.1 we can extend a representation of M_i affording χ_{M_i} to a representation of G affording χ .

Chapter 6

Implementation

This chapter contains an implementation in GAP of the algorithms presented in this thesis. The first three sections deal with the methods: Character Restriction, Induction, Extension. The fourth section deals with representations of perfect groups. The last section presents the main program which incorporates these methods.

6.1 Character Restriction Method

Suppose G is a finite group and χ is an irreducible character of G . In this section we present a program to implement the character restriction method (see chapter 3.4).

IrrRepsOfGroup:

```
IrrRepsOfGroup( G, chi )  
IrrRepsOfGroup( G, chi, chisub )
```


computes an irreducible complex representation for the finite group G affording the irreducible character chi according to the character restriction method.

chi is assumed to be an irreducible character. In the first case the program finds a chi -subgroup and a representation affording chi is returned. If given, $chisub$ must be a chi -subgroup. In this case a representation is computed using this chi -subgroup.

The method will fail if for the given irreducible character chi there is no chi -subgroup.

Note: A similar function `IrreducibleRepresentationsDixon` is available as a function in the standard distribution of GAP4.3. However that function does not seem to be very effective and sometimes the output is not correct. For example, when $G := \text{SL}(2, 5)$ and χ is the irreducible character of degree 5 of G , the output is a mapping from G to a linear group of size infinity.

```
IrrRepsOfGroup:=function ( arg )
  local  r, E, H, f, m1, Y, S, s, U, M, gp, chi;
  gp := arg[1];
  U := GeneratorsOfGroup( gp );
  if arg[2][1]=1 then r:=List([1..Length(U)],i->[[U[i]^arg[2]]]);
  return GroupHomomorphismByImagesNC( gp, Group( r ), U, r );
fi;
if Length( arg ) = 2 then
  if IsClassFunction( arg[2] ) and IsCharacter( arg[2] ) then
    chi := arg[2];
    S := PSubgroupsRepsn( gp );
    s := SubgroupRepsn( gp, chi, S );
    if s = fail then
```

```

    Info( InfoWarning, 1, "need to compute subgroup lattice" );
    S := List( ConjugacyClassesSubgroups(gp), Representative);
    s := SubgroupRepsn( gp, chi, S );
    if s = fail then
        Error ("Dixon's method does not work for given character");
    fi;
fi;
H := s[1];
f := s[2];
else Error( "second argument must be ordinary character");
fi;
fi;
if Length( arg ) = 3 then
    if IsSubgroup (gp, arg[3] ) and IsClassFunction( arg[2] )
        and IsCharacter( arg[2] ) then
        chi := arg[2];
        H:= arg[3];
        S:=[H];
        s := SubgroupRepsn( gp, chi, S );
        if s=fail then Error("given subgroup is not a chi-subgroup");
        fi;
        f := s[2];
    else Error ("Second argument is not a character or
                third argument is not a subgroup of given group");
    fi;
fi;

```

```

fi;
E := Elements( H );
M := MatrixRepsn( chi, gp, H, f, E );
m1 := M[1];
Y := M[2];
r := RepsRepsn( U, Y, m1, f, chi, E );
return GroupHomomorphismByImagesNC( gp, Group( r ), U, r );
end;

```

IrrRepsOfGroup calls the following subprograms.

1. PSubgroupsRepsn:

This a short program to get all p -subgroups of G for each prime $p \mid |G|$. If none of these p -subgroups is a χ -subgroup then the main program searches to find a χ -subgroup among the lattice of subgroups.

```

PSubgroupsRepsn:=function(gp)
local s,S,l,L;
L:=[];
S:=List(Set(FactorsInt(Size(gp))),i->SylowSubgroup(gp,i));;
for s in S do
  L:=Union(L,List(ConjugacyClassesSubgroups(s),Representative));;
  l:=List(L,Size);
  SortParallel( l, L );
od;
return L;
end;

```

3. SubgroupRepsn:

This is a program to get a χ -subgroup from a list of subgroups of G .

```
SubgroupRepsn:=function ( gp, chi, S )
local   i, j, C, R, m;
for j   in [1..Length(S)] do
    R := RestrictedClassFunction( chi, S[j] );
    C := ConstituentsOfCharacter( R );
    m := MatScalarProducts( C, [ R ] );
    for i   in [ 1 .. Length( C ) ] do
        if m[1][i] * C[i][1] = 1 then return [ S[j] , C[i] ]; fi;
    od;
od;
return fail;
end;
```

3. AlphaRepsn:

The following program corresponds to equation (3.1) to get the entries of the submatrix of equation (3.2).

```
AlphaRepsn:=function ( x, y, f, chi, E )
local   z, sum;
sum := 0;
for z   in E do
    sum := sum + (z ^ -1) ^ f * (z * x * y ^ -1) ^ chi;
od;
return sum;
```

```
end;
```

4. MatrixRepsn:

This program constructs the submatrix of equation (3.2).

```
MatrixRepsn:=function ( chi, gp, H, f, E )
    local mat, matn, X, j, n, x, d, R, h;
    R := List( RightCosets( gp, H ), Representative );
    h:=Size(H);
    mat := [ [ h ] ];
    X := [ Identity( gp ) ];
    d := DegreeOfCharacter( chi );
    if d = 1 then
        return [ mat, X ];
    fi;
    for x in R do
        matn := [ ];
        n := Length( X );
        for j in [ 1 .. n ] do
            matn[j] := AlphaRepsn( x, X[j], f, chi, E );
            mat[j][n + 1] := ComplexConjugate( matn[j] );
        od;
        matn[n + 1] := h;
        mat[n + 1] := matn;
        if Determinant( mat ) <> 0 then
            X[n + 1] := x;
            if Length( X ) = d then
```

```

        return [ mat, X ];
    fi;
fi;
od;
return;
end;

```

5. RepsRepsn:

This program computes the matrices $A(x)A(1)^{-1}$ of the mapping (3.3).

```

RepsRepsn:=function ( U, Y, m1, f, chi, E )
    local u, i, j, m2, L;
    L := [ ];
    for u in U do
        m2 := [ [ ] ];
        for i in [ 1 .. Length( Y ) ] do
            for j in [ 1 .. Length( Y ) ] do
                m2[i][j] := AlphaRepsn( Y[i] * u, Y[j], f, chi, E );
            od;
            if i < Length( Y ) then
                Add( m2, [ ] );
            else
                Add( L, m2 * Inverse( m1 ) );
            fi;
        od;
    od;
    return L;
end;

```

```
end;
```

6.2 Induction Method

```
InducedReps( G, rep )
```

computes a representation of G induced from the representation rep of a subgroup of G . It is assumed that rep is a representation of a subgroup of G .

```
InducedReps := function( g, rep )
  local M,M1,mat,u,U,rt,i,j,k,LL,L,m,Ls,Lr,l,d,h;
  mat := [ ]; L := [ ];
  h := PreImagesRange(rep);
  d := DimensionsMat(GeneratorsOfGroup(Image(rep)) [1]) [1];
  U := GeneratorsOfGroup(g);
  rt := RightTransversal(g,h);
  for u in U do
    M := [ ];
    for i in [1..Length(rt)] do
      M1 := [ ];
      for j in [1..Length(rt)] do
        if rt[i]*u*rt[j]^(-1) in h then
          Add(M1,ImagesRepresentative(rep,rt[i]*u*rt[j]^(-1)));
        else Add(M1,NullMat(d,d));
        fi;
      od;
    od;
  od;
```

```

    Add(M,M1);
od;
Add(mat,M);
od;
for m in mat do
LL := [ ]; Lr := [ ]; Ls := [ ];
for i in [1..Index(g,h)] do
for j in [1..d] do
for k in [1..Index(g,h)] do
Add(LL,m[i][k][j]);
od;
Add(Ls,Concatenation(LL));
LL := [ ];
od;
Add(Lr,Ls);
od;
Add(L,Lr);
od;
l := List([1..Length(L)],i->L[i][1]);
return GroupHomomorphismByImagesNC( g, Group( l ), U, l );
end;

```


6.3 Extension Method

`ExtendedReps( chi, rep )`

computes the extension of the representation *rep* of a subgroup *H* of *G* to *G*. It is assumed that *chi* is an irreducible character of *G* and that *rep* is an irreducible representation of a subgroup *H* of *G*, affording the restriction of *chi* to *H*.

```
ExtendedReps := function( chi, rep )
local S,C,mat,B,i,j,s,u,U,L,n,M,l,h,g;
S := [ ]; C := [ ]; mat := [ ]; L := [ ]; M := [ ];
n := chi[1];
h := PreImagesRange(rep);
g := UnderlyingGroup(chi);
U := GeneratorsOfGroup(g);
B := ModuleBasis(h,rep);
for u in U do
  for i in [1..n^2] do
    for j in [1..n] do
      Append(L,TransposedMat(B[1][i])[j]);
    od;
    Add(mat,L);
    L := [ ];
    Add(C,(B[2][i]*u)^chi);
  od;
s := SolutionMat(TransposedMat(mat),C);
```

```

    for i in [0..n-1] do
        l:=List([i*n+1..(i+1)*n],j->s[j]);
        Add(M,l);
    od;
    Add(S,M);
    mat := [ ]; C := [ ]; M := [ ];
od;
return GroupHomomorphismByImagesNC( g, Group( S ), U, S );
end;

```

The following program is called during the `ExtendedReps`.

```

ModuleBasis := function( g, rep )
local V,f,B,u,x,L,l,d,i,N,P;
l := [ ]; L := [ ]; B := [ ];
d := DimensionsMat(GeneratorsOfGroup(Image(rep)))[1])[1];
for i in [1..d^2+d] do
    x := Random(g);
    g := Difference(g,[x]);
    u := ImagesRepresentative(rep,x);
    Add(l,u);
od;
f := FieldOfMatrixGroup(Image(rep));
V := VectorSpace(f,l);
while Dimension(V) < d^2 do
    x := Random(g);

```

```

g := Difference(g,[x]);
u := ImagesRepresentative(rep,x);
if u in V then;
else Add(l,u);
    V := VectorSpace(f,l);
fi;
od;
for i in [1..Length(l)] do
    Add(L,Concatenation(l[i]));
od;
N := LinearIndependentColumns(TransposedMat(L));
for i in N do
    Add(B,l[i]);
od;
P := List([1..d^2],i->PreImagesRepresentative(rep,B[i]));
return [B,P];
end;

```

6.4 Perfect Groups

PerfectReps(G, chi)

computes an irreducible representation affording the irreducible character *chi* for the finite perfect group *G*. It is assumed that *chi* is an irreducible character of *G*.

Before we define `PerfectReps` and its subprograms we need to define the following variable.

```

IrreducibleReps := "we define this variable for future use";

PerfectReps := function( G, chi )
local h,n,a,r,s,S,c,C,i,R,l,L,M,Co;
h := NaturalHomomorphismByNormalSubgroup( G, Centre( G ) );
M := [[6*360,6],[2*2520,2],[3*2520,3],[6*2520,6],[2*20160,2],
      [2*3265920,2],[3*3265920,3],[4*3265920,4],[6*3265920,6],
      [3*4585351680,3],[3*9196830720,3]];
if chi[1] in [6,12,15,20,21,24,27] then
  if [Size(G), Size(Centre(G))] in M and IsSimple( Image(h) ) then
    return SpecialCoversReps( G, chi );
  fi;
fi;
L := []; M := [];
n := MinimalNormalSubgroups( Image(h) );
a := Filtered( n, IsAbelian );
if a=n then return PerfectAbelianSocleReps( G, chi );
else S := Difference( n, a );
  for s in S do
    Append( M, GeneratorsOfGroup(s) );
  od;
c := Centralizer( Image(h), Group(M) );
if Size(c) > 1 then Add( S, c ); fi;
S := List( S, i->PreImage( h, i ) );

```

```

Co := [1];
for i in [1..Length(S)] do
  R := RestrictedClassFunction( chi, S[i] );
  C := ConstituentsOfCharacter( R );
  r := IrrRepsOfGroup( S[i], C[1] );
  Add( L, GeneratorsOfGroup( Image(r) ) );
  Co := Concatenation(List( Co,u->u*GeneratorsOfGroup(S[i])));
od;
if Length(L) = 1 then return r;
fi;
r := L[1];
for i in [2..Length(L)] do
  r := RepsKronecker( r, L[i] );
od;
fi;
return GroupHomomorphismByImagesNC( G, Group( r ), Co, r );
end;

```

PerfectReps calls the following subprograms during the program, respectively.

1. SpecialCoversReps:

```
SpecialCoversReps( G, chi )
```

computes irreducible representations of groups which are mentioned in Section 4.4.

```
SpecialCoversReps := function(G,chi)
```

```

local H,h,k,g,d,u,c,C,n,m,R,r,L;
L := [ ];
h := NaturalHomomorphismByNormalSubgroup( G, Centre( G ) );
g := Image(h);
if Size(g) in [360, 2520, 20160] then
  C := List( ConjugacyClassesSubgroups(Image(h) ),Representative);
  for c in C do
    if Size(c) in [10,12,15,20] then
      Add( L, PreImage( h, c ) );
    fi;
  od;
  H := SubgroupRepsn( chi, L )[1];
  return IrrRepsOfGroup( G, chi, H );
fi;
if Size(g) = 3265920 then
  if chi[1] in [6,15] then
    k := SylowSubgroup( g, 3 );
    d := DerivedSubgroup( k );
    c := Centralizer( k, d );
    n := Normalizer( g, c );
  fi;
  if chi[1] = 21 then
    r := IsomorphicSubgroups( g, PSL(3,4) );
    n := Image( r[1] );
  fi;

```

```

fi;
if Size(g) = 4585351680 then
    k := SylowSubgroup( g, 3 );
    u := CompositionSeries( k );
    c := Centralizer( k, u[6] );
    n := Normalizer( g, c );
fi;
if Size(g) = 9196830720 then
    k := SylowSubgroup( g, 2 );
    u := UpperCentralSeries( k );
    c := Centralizer( k, u[3] );
    n := Normalizer( g, c );
fi;
m := PreImage( h, n );
R := RestrictedClassFunction( chi, m );
return ExtendedReps( chi, IrreducibleReps( m, R ) );
end;

```

2. MinimalNormalSubgroups:

```
MinimalNormalSubgroups( G )
```

returns a list containing those proper normal subgroups of the group G that are minimal among the proper normal subgroups. I wish to thank Bettina Eick who provided this program for GAP users.

```
MinimalNormalSubgroups := function(G)
```

```

local cl, nt, c, U;
cl := ConjugacyClasses(G);
nt := [ ];
for c in cl do
    if IsPrime(Order(Representative(c))) then
        U := Subgroup( G, [Representative(c)] );
        U := NormalClosure( G, U );
        nt := AddToSubgroupList( nt, U );
    fi;
od;
return nt;
end;

```

The following subprogram is called during the MinimalNormalSubgroups.

```

AddToSubgroupList := function( nt, U )
    local i, N;
    for N in nt do
        if IsSubgroup(U, N) then
            return nt;
        fi;
    od;
    for i in [1..Length(nt)] do
        if IsSubgroup(nt[i], U) then
            nt[i] := false;
        fi;
    od;
end;

```



```

    nt := Filtered( nt, x -> not IsBool(x) );
    Add(nt, U);
    return nt;
end;

```

3. RepsKronecker:

The following program computes the Kronecker product of the elements of two lists of matrices (see Theorem 3.3.1).

```

RepsKronecker := function(r1,r2)
local u,v,L;
L := [ ];
for u in r1 do
  for v in r2 do
    Add(L,KroneckerProduct(u,v));
  od;
od;
return L;
end;

```

4. PerfectAbelianSocleReps:

```

PerfectAbelianSocleReps( G, chi )

```

returns a representation of the finite perfect group G affording the irreducible character chi , when the socle of $G/Z(G)$ is abelian.

```

PerfectAbelianSocleReps := function( G, chi )

```

```

local l,e1,e2,r,f,R,Rc,Cc,Ri,Rf,Ci,C,I,t,psi,i,h,im,r1,r2,U,P,m,w;
U := GeneratorsOfGroup( G );
f := FittingSubgroup( G );
R := RestrictedClassFunction( chi , f );
C := ConstituentsOfCharacter( R );
m := MatScalarProducts( C, [ R ] );
if IsIrreducibleCharacter( R ) then
    return ExtendedReps( chi, IrreducibleReps( f, R ) );
fi;
w := [[16,4],[20,4],[24,6],[28,4],[30,6]];
if [chi[1],m[1][1]] in w and Length( C )=1 then
    return SpecialCharactersReps( G, chi );
fi;
h := NaturalHomomorphismByNormalSubgroup( G, f );
if Length( C )=1 and IsPrime( m[1][1] ) then
    P := PreImage( h , SylowSubgroup( Image(h), m[1][1] ) );
    R := RestrictedClassFunction( chi , P );
    if IsIrreducibleCharacter( R ) then
        return ExtendedReps( chi, IrreducibleReps( P, R ) );
    fi;
fi;
if Length( C ) > 1 then
    I := InertiaSubgroup( G, C[1] );
    Ri := RestrictedClassFunction( chi, I );
    Ci := ConstituentsOfCharacter( Ri );

```

```

for t in Ci do
  Rc := RestrictedClassFunction( t, f );
  Cc := ConstituentsOfCharacter( Rc );
  if C[1] in Cc then
    return InducedReps( G, IrreducibleReps( I, t ) );;
  fi;
od;
fi;
for i in [1..Length(Irr(G))] do
  Rf := RestrictedClassFunction( Irr(G)[i], f );
  if Rf = C[1] then
    psi := Irr(G)[i];
    i := Length( Irr( G ) )+1;
    e1 := IrreducibleReps( G, psi );
    r1 := GeneratorsOfGroup(Image(e1));
  fi;
od;
im := Image( h );
for i in [1..Length( Irr(im) )] do
  if m[1][1] = Irr(im)[i][1] then
    R := RestrictedClassFunction( Irr(im)[i], h );
    if chi = psi*R then
      e2 := IrreducibleReps( im, Irr(im)[i] );
      l := List([1..Length(U)],i->ImagesRepresentative( h,U[i]));
      r2 := List([1..Length(l)],i->ImagesRepresentative(e2,l[i]));

```

```

        r := List([1..Length(r1)],i->KroneckerProduct(r1[i],r2[i]));
        i := Length( Irr(im) )+1;
    fi;
fi;
od;
return GroupHomomorphismByImagesNC( G, Group( r ), U, r );
end;

```

The following subprogram is called during the PerfectAbelianSocleReps (see Section 5.2.1 to Section 5.2.6).

```

SpecialCharactersReps:=function( G, chi )
local L, h, C, c, H, m, M, R, f;
L := [ ];
f := FittingSubgroup( G );
h := NaturalHomomorphismByNormalSubgroup( G, f );
if chi[1] in [16,28] and Size(Image(h)) in [360, 336] then
    M := MaximalSubgroupClassReps( Image(h) );
    for m in M do
        if Size(m) in [36,42,48] then
            H := PreImage( h, m );
            R := RestrictedClassFunction( chi, H );
            if IsIrreducibleCharacter( R ) then
                return ExtendedReps( chi, IrreducibleReps( H, R ) );
            fi;
        fi;
    od;
end;

```

```

else
  h := NaturalHomomorphismByNormalSubgroup( G, Centre( G ) );
  C := List(ConjugacyClassesSubgroups(Image(h)),Representative);
  for c in C do
    if Size( c ) in [10,12,16,24] then
      Add( L, PreImage( h, c ) );
    fi;
  od;
  H := SubgroupRepsn( chi, L )[1];
  return IrrRepsOfGroup( G, chi, H );
fi;
end;

```

6.5 Irreducible Representations

`IrreducibleReps( G, chi )`

This is a recursive program which returns a representation of the group G affording the irreducible character chi . It is assumed that chi is an irreducible character of G .

```

IrreducibleReps := function( G, chi )
local n,k,g,c,rep,U,r,l;
if chi[1] = 1 then return IrrRepsOfGroup( G, chi ); fi;
U := GeneratorsOfGroup( G );
k := NaturalHomomorphismByNormalSubgroup( G, Kernel( chi ) );

```

```

g := Image(k);
c := Filtered(Irr(Image(k)), x->chi=RestrictedClassFunction( x, k ));
if IsPerfect( g ) then rep := PerfectReps( g, c[1] );
else
  n := NormalChain( g, c[1] );
  if n[2][1] < c[1][1] then
    rep := InducedReps( g, IrreducibleReps( n[1], n[2] ) );
  fi;
  if n[2][1] = c[1][1] then
    Info( InfoWarning, 1,
      "Need to extend a representation. This may take a while." );
    rep := ExtendedReps( c[1], IrreducibleReps( n[1], n[2] ) );
  fi;
fi;
l := List( [1..Length(U)], i->ImagesRepresentative( k, U[i] ) );
r := List( [1..Length(l)], i->ImagesRepresentative( rep, l[i] ) );
return GroupHomomorphismByImagesNC( G, Group( r ), U, r );
end;

```

The following subprogram is called during the IrreducibleReps

```

NormalChain := function( G, chi )
local k,i,N,I,R,Ri,Rc,C,Ci,Cc,c,L,n;
N := DerivedSeries( G );
k := 0;
for i in [2..Length(N)] do
  R := RestrictedClassFunction( chi, N[i] );

```

```

C := ConstituentsOfCharacter(R);
if Length(C) > 1 then
  I := InertiaSubgroup( G, C[1] );
  Ri := RestrictedClassFunction( chi, I );
  Ci := ConstituentsOfCharacter( Ri );
  for c in Ci do
    Rc := RestrictedClassFunction( c, N[i] );
    Cc := ConstituentsOfCharacter( Rc );
    if C[1] in Cc then return [I,c];
  fi;
od;
elif IsIrreducibleCharacter( R ) then
  k := i; c := R;
fi;
od;
if k > 0 then return [ N[k], c ];
fi;
L := ChiefSeriesThrough( G, [N[2]] );
n := Length(L);
for i in [2..n] do
  R := RestrictedClassFunction( chi, L[i] );
  C := ConstituentsOfCharacter( R );
  if Length(C) > 1 then
    I := InertiaSubgroup( G, C[1] );
    Ri := RestrictedClassFunction( chi, I );

```

```

Ci := ConstituentsOfCharacter( Ri );
  for c in Ci do
    Rc := RestrictedClassFunction( c, L[i] );
    Cc := ConstituentsOfCharacter( Rc );
    if C[1] in Cc then return [I,c];
  fi;
od;
fi;
od;
end;

```


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