List of Figures

7.1	Triviality constraint as a function of cutoff	60
7.2	Range of α allowed by triviality	62
7.3	Bound state : $M = 1.998m_B$, $J_3 = 0$ (+ + & + -)	63
7.4	Bound state : $M = 1.998m_B$, $J_3 = 0$ (- + & - -)	64
7.5	Bound state : $M = 1.998m_B$, $J_3 = 1$ (+ + & + -)	66
7.6	Bound state : $M = 1.998m_B$, $J_3 = 1$ (- + & - -)	67
7.7	Bound state : $M = 1.940m_B$, $J_3 = 0$ (+ + & + -)	69
7.8	Bound state : $M = 1.940m_B$, $J_3 = 0$ (- + & - -)	70
7.9	Bound state : $M = 1.940m_B$, $J_3 = 1$ (+ + & + -)	71
7.10	Bound state : $M = 1.940m_B$, $J_3 = 1$ (- + & - -)	72
7.11	Bound state mass as a function of coupling ($m_a = m_H$)	73
7.12	Bound state mass as a function of coupling ($m_a = 0.1m_H$)	74
7.13	Bound state mass as a function of coupling ($m_a = 0.01m_H$)	75
7.14	Coupling as a function of quadrature order	76

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**TWO FERMION BOUND STATE EQUATION
USING LIGHT FRONT TAMM-DANCOFF FIELD
THEORY IN 3+1 DIMENSIONS**

by

PHILIP M. WORT, B.Sc., M.Sc.

**A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of**

Doctor of Philosophy

Department of Physics

**Carleton University
Ottawa-Carleton Institute of Physics**

Ottawa, Ontario

July 5, 1992

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USING LIGHT FRONT TAMM-DANCOFF FIELD
THEORY IN 3+1 DIMENSIONS"**

submitted by
PHILIP M. WORT, B.Sc., M.Sc.

in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

Chair, Department of Physics

Thesis Supervisor

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Carleton University

Abstract

Attention has recently been paid to Light Front Tamm-Dancoff field theory in 1+1 dimensions, including the proposition of a sector dependent renormalisation procedure. We extend this formalism to 3+1 dimensions, and examine a two fermion system using Light Front field theory in the lowest non-trivial Tamm-Dancoff approximation. It is found that the extension to 3+1 dimensions results in an integral equation with a kernel which does not decrease rapidly enough to formally admit solutions for $L_3 = 0$ states. Proceeding with a cutoff reveals evidence of physically reasonable bound states.

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Contents

Acceptance Sheet	ii
Abstract	iii
Acknowledgements	iv
Table of Contents	v
List of Figures	vii
1 Introduction	1
1.1 Motivation	1
1.2 Overview	5
1.3 The Tamm-Dancoff Method	8
1.4 Light Front Coordinates	10
2 First Tamm-Dancoff approximation for a single fermion	14
3 First Tamm-Dancoff approximation for two fermions	22
4 Coupling Constant Renormalisation	32
4.1 Explicit Mass Correction Calculation	37
4.2 Triviality	40
5 Asymptotic Behaviour	43
6 Normalisation and Consistency Considerations	51
7 Numerical Results	58
8 General Considerations	77

9 Discussion and Conclusion	81
9.1 Discussion	81
9.2 Conclusion	86
A Light Front Variables	88
B Light Front Hamiltonian	94
C Numerical Methods	107
C.1 The Lanczos Algorithm	112
Bibliography	116

List of Figures

7.1	Triviality constraint as a function of cutoff	60
7.2	Range of α allowed by triviality	62
7.3	Bound state : $M = 1.998m_B$, $J_3 = 0$ (+ + & + -)	63
7.4	Bound state : $M = 1.998m_B$, $J_3 = 0$ (- + & - -)	64
7.5	Bound state : $M = 1.998m_B$, $J_3 = 1$ (+ + & + -)	66
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7.9	Bound state : $M = 1.940m_B$, $J_3 = 1$ (+ + & + -)	71
7.10	Bound state : $M = 1.940m_B$, $J_3 = 1$ (- + & - -)	72
7.11	Bound state mass as a function of coupling ($m_a = m_H$)	73
7.12	Bound state mass as a function of coupling ($m_a = 0.1m_H$) . .	74
7.13	Bound state mass as a function of coupling ($m_a = 0.01m_H$) . .	75
7.14	Coupling as a function of quadrature order	76

Chapter 1

Introduction

1.1 Motivation

Many of the particles we encounter in nature are not truly elementary, but are instead composite, composed of more elementary constituents. For example the atomic nucleus is known to be composed of protons and neutrons, while the nucleons themselves are considered to be composed of partons, which are identified with quarks and gluons.

In general the substructure of a particle can be revealed through scattering experiments, since as the energy of the scattering process is increased the distance scales to which the process is sensitive decreases. Thus to reveal the substructure of a nucleon requires a more energetic probe than is needed for a nucleus.

To take a specific example of this process, and to illustrate the difficulty of describing composite, or bound, systems we will examine the nucleon case.

The substructure of the proton was first indicated by the onset of Bjorken scaling (see for example Ref [1]) in deep inelastic scattering experiments. This behaviour led to the introduction of the parton model [2, 3]. In this model the form factors needed to account for the extended nature of the proton, are essentially traded for parton distribution functions which describe the partonic content of the nucleon. This description of a nucleon as composed of ‘elementary’ constituents provides a physical explanation of the observed scaling behaviour, but the distribution functions must still be determined from experiment (although their general behaviour can be described through an approach due to Altarelli and Parisi [4]).

Prior to the introduction of the parton model the quark model had been introduced [5] essentially as a classification tool for the observed spectrum of hadronic states. This model hypothesised the existence of parton like particles, the quarks, from which the hadrons were constructed. At the time of its introduction these particles were considered to be only a group theoretical device, but with the introduction of the parton model the identification of the quarks with the partons was a logical step.

In order to achieve this identification the fact that the parton model indicated the existence of large numbers of partons, while the quark model proposed, for example, that the proton was a three quark state, lead to the

notion of valence and sea quarks. The valence quarks carried the quantum numbers of the proton, and the sea quarks were paired in flavour singlets and thus contributed no net flavour.

It should be noted from this brief description of the quark/parton model that no reference has been made to quantum field theory, which is usually the primary tool used for the description of elementary particles.

The currently accepted field theory for describing the strong nuclear forces which act on hadrons is Quantum Chromodynamics (QCD) (see for example Ref [6]). Prior to QCD, field theoretical treatments of hadronic systems had been largely Yukawa type theories, which considered the pion (a pseudoscalar meson) as the mediator of the strong force.

QCD describes fermions interacting through the exchange of massless vector gluons. It is a non-Abelian gauge theory and is asymptotically free. This asymptotic freedom (which essentially implies that the effective coupling constant goes to zero as interaction energies increase) implies that in high momentum exchange processes, such as the deep inelastic regime, a system of strongly interacting particles will behave as if they are essentially free particles. The smallness of the coupling has allowed perturbative QCD to be applied to deep inelastic processes. QCD also provides a mechanism, via fermion pair production for example, for the generation of the sea quarks

needed to reconcile the quark model and the parton model.

There are however many problems encountered when trying to describe a bound state using QCD. First the running of the coupling constant, responsible for asymptotic freedom, also implies that for low momentum processes the coupling is large. Thus in this regime perturbation theory cannot be expected to be a useful tool. A further problem lies in the masses of the quarks. There exist good theoretical reasons (from current algebra for example [6]) for assuming that chiral symmetry is an approximate symmetry of the strong interactions. The quark mass terms in the QCD lagrangian violate chiral symmetry, and thus we have constraints on the quark masses (chirally symmetric QCD has zero mass quarks).

In the constituent quark model however the valence quarks are responsible for the mass of the nucleon, and thus are quite massive. This has lead, for example, to the notion that the constituent quarks of the quark model are in fact themselves bound states involving the 'QCD' (current) quarks [6].

The purpose of the above discussion is to illustrate that there is a need to better understand relativistic bound states. It is not just for QCD that there are difficulties, in fact even for a simple system like the hydrogen atom there exists no exact treatment. Part of the problem in general lies in the fact that the description of a stable bound state must respect that stability,

and thus have the bound state existing for all time. Thus the asymptotic states in the theory must also be bound states. In conventional perturbation theory one considers that the asymptotic states are free particles. Here the constituent particles are always bound.

In what follows we will consider the case of a two fermion system, such as the deuteron. We will describe the interaction between the fermions as due to the exchange of a pseudoscalar meson. This examination is intended as a step toward the better understanding of the bound state problem, and thus we are not deterred by the fact that the pseudoscalar meson exchange model of the strong interactions is known to be only an approximation to the true interaction.

1.2 Overview

There has recently been a renewal of interest in Tamm-Dancoff field theory as a tool for the investigation of relativistic bound states. This approach involves an approximation based on a truncation of Fock space, and was investigated in the 1950's but ultimately abandoned, essentially due to difficulties involving the construction of the physical vacuum.

The renewal of interest stems from the observation [7] that since the vacuum in light front field theory is trivial [8] (for massive particles), application of the Tamm-Dancoff procedure within this framework should not be subject

to the same difficulties as were found in the original 'equal time' approach. In the following chapters we discuss the application of light front Tamm-Dancoff field theory to the case of fermions and mesons interacting through a Yukawa coupling, in $3+1$ dimensions. This work is intended to be seen as an extension of that performed by R.J. Perry, A. Harindranath and K.G. Wilson [7, 9] who considered the same problem in $1+1$ dimensions. In particular a non-perturbative, sector dependent renormalisation scheme was proposed by these authors, and the primary motivation for the work presented here is to determine whether or not this scheme is applicable in $3+1$ dimensions.

As a vehicle for this examination, a two fermion system is considered in the lowest non-trivial Tamm-Dancoff approximation, and a bound state equation for this system obtained. In a two fermion system there are two basic processes occurring in parallel. One is the exchange of mesons between the two fermions, and the other is the dressing of the individual fermions. The renormalisation scheme used requires that the self energy correction which arises from each individual fermion dressing itself be regulated by introducing the mass counterterm from the one fermion system (where this dressing is the sole process). The goal of this procedure is to ensure that above the two particle production threshold the individual fermions propagate with their physical masses. Once this has been accomplished attention can be focused on the exchange interactions responsible for producing the bound state.

The introduction is concluded with two sections discussing the nature and background of both the Tamm-Dancoff approximation and the use of light front coordinates.

Next the first Tamm-Dancoff approximation for a single fermion is examined. This examination is taken to the point at which a mass counterterm is available, and we then move directly to the two fermion case. It is found that (in contrast to the 1+1 case) it is necessary to renormalise the coupling constant in addition to the masses. A chapter is devoted to this renormalisation, and contains within it a description of the triviality constraint to which this theory is subject.

We then examine the asymptotic behaviour of the equation which results from the above procedure, and show that it is ill defined in the sense that it formally admits no solutions for states related to $L_3 = 0$. Following this we consider the question of the normalisation of the bound state, and then move on to numerical results. The penultimate chapter concerns itself with trying to identify, within the context of our bound state equation, the cause of the bad high momentum behaviour. Details of the conventions used, the quantisation procedure (commutation relations etc.), and details of the numerical techniques, are relegated to appendices A, B and C.

1.3 The Tamm-Dancoff Method

The Tamm-Dancoff method, named for I.Tamm [10] and S.M. Dancoff [11] who proposed it (in 1945 and 1950 respectively) is an approximation method for field theoretic problems based on a truncation of Fock space.

As an example of this method, suppose we have a Schrodinger equation for a state $|\Psi\rangle$, i.e.

$$H|\Psi\rangle = (H_0 + H_I)|\Psi\rangle = E|\Psi\rangle \quad (1.1)$$

This equation can, in principle, be solved by expanding the state $|\Psi\rangle$ in terms of the eigenstates of H_0 , or equivalently the eigenstates of the number operators of the (free) fields of the problem, which constitute the Fock space. For example if we suppose that the only free field is that of a meson with creation operator $a^\dagger$, we would have (in momentum space)

$$|\Psi(p)\rangle = C_1(p) a^\dagger(p) |0\rangle + \int dp_1 C_2(p_1, p - p_1) a^\dagger(p_1) a^\dagger(p - p_1) |0\rangle + \dots \quad (1.2)$$

The expansion coefficients $C_1(p)$, $C_2(p, p')$ etc. represent the probability amplitude for finding a particular Fock state within the state $|\Psi\rangle$. Performing this expansion and then contracting the equation with each member of the space in turn would reduce the Schrodinger equation to an (infinite dimensional) matrix-integral equation for the expansion amplitudes. This

matrix representation is entirely equivalent to the original equation, and the problem now becomes the diagonalisation of this matrix.

In the Tamm-Dancoff method (see Ref [12]) a subspace of the space of all Fock states is chosen comprising, for example, those states which would seem to be reasonable intermediate states in a given problem. Then it is decided that only states within this truncated Fock space will be considered. This has the effect of limiting the expansion of $|\Psi\rangle$ to a finite number of terms, and hence the resulting matrix is finite. Now the diagonalisation of this matrix will yield approximations to the true amplitudes, and hence to the true $|\Psi\rangle$.

This method was employed in the 1950's (in the context of the usual equal time quantisation) with some success (see for example Ref [13]) but problems were encountered when dealing even with a relatively small Fock space due to the fact that in general the interaction Hamiltonian contains terms which can create particles from the vacuum, or annihilate particles into the vacuum. In solving the coupled equations which result from the expansion of $|\Psi\rangle$ one needs to eliminate amplitudes. In this process vacuum creation and annihilation terms are brought together and disconnected, divergent vacuum diagrams result. This problem is related to the fact that in normal perturbation theory the perturbative and physical vacua are not the same; i.e. the perturbative vacuum is not an eigenstate of the full Hamiltonian.

When considering scattering problems in the usual perturbative framework (which essentially consists of a coupling constant expansion) the fact that disconnected vacuum terms do not contribute to transitions makes it clear that they are spurious. This fact also makes them easy to dispose of, since their contributions factor out (simply because they are disconnected). In the Tamm-Dancoff case, however, the disconnected vacuum terms appear within integral equations and it was not clear how they should be dealt with. Attempts were made to remove, or avoid, this problem [14, 15] but ultimately this was not found to be possible in the general case (i.e. with arbitrarily large Fock space), and the approach was abandoned.

1.4 Light Front Coordinates

The notion of using light front coordinates for discussing relativistic dynamics stems from a paper by Dirac [16], in 1949, in which the Poincare generators were examined for different choices of quantisation surface. The usual choice of quantisation surface is $t = 0$, and Poincare generators are classified as being kinematic or dynamic depending on whether their action on a coordinate point on this surface leaves it on the surface, or moves it off the surface (i.e. generates time displacement). This is referred to as the Equal Time (or Instant) case, for obvious reasons.

In the Light Front case one chooses the quantisation surface to be $z+t = 0$.

If we use coordinates

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3) \quad (1.3)$$

with x^i unchanged for $i = 1, 2$, we can write this condition as $x^+ = 0$. With this choice of coordinates the scalar product becomes

$$p^\mu x_\mu = p^+ x^- + p^- x^+ - \vec{p}_\perp \cdot \vec{x}_\perp \quad (1.4)$$

with $\vec{p}_\perp = (p^1, p^2)$ etc. Since we are selecting $x^+ = 0$ we call x^+ the light front time. This choice for the ‘time’ variable dictates that we denote p^- the light front energy, since it is the conjugate variable to x^+ .

Generalising the definition of dynamic (kinematic) operators to those which do (do not) generate displacements in light front time we find that a different division of the Poincare generators results (see Appendix A). In particular boosts, which were dynamic in the equal time case become kinematic in the light front case, with the converse being true for $\vec{J}_\perp$. In fact in Light Front coordinates we have one more kinematic operator than in the Equal Time case. The utility of the Light Front system stems from an observation made by Weinberg [8] when considering dynamics in the infinite momentum frame. He noticed that in the limit $p^3 \rightarrow \infty$ a large number of possible diagrams were suppressed, in particular those contributing to vacuum structure. The fact that in this limit the motion of the particles is essentially along $x^3 = x^0$ or $x^- = 0$ which is perpendicular to $x^+ = 0$ implies

that a reasonable choice of coordinates in this case would be the Light Front coordinates introduced above.

From the definitions of $p^\pm$ it can be seen that for positive p^0 both p^+ and p^- must be positive for massive particles. Now p^+ is a conserved momentum and since it is always positive it is not possible to create particles from the vacuum, or annihilate particle into the vacuum. It is this fact that makes this system of coordinates so useful, since the vacuum never gets dressed, and thus the perturbative vacuum is the same as the physical vacuum. In the context of the Schrodinger equation shown above (Equation 1.1) we can note that the parts of the Hamiltonian which would be responsible for the generation of disconnected vacuum terms do not appear due to this conservation of light front momentum.

For massless particles p^+ can be zero. This fact creates difficulties for theories with massless quanta, but since we are not considering such theories in what follows this point will not be addressed further.

It should be stressed that while the motivation for this coordinate system came initially from consideration of the infinite momentum frame, the triviality of the vacuum is ensured here by the choice of coordinate system, and is frame independent.

A great deal of work has been performed using light front variables [17] for the reasons outlined above. What is most significant from the point of

view of what follows, is the observation by Wilson [7] that the triviality of the vacuum should render the Tamm-Dancoff method viable once again.

Chapter 2

First Tamm-Dancoff approximation for a single fermion

As a prelude to the examination of the two fermion problem (and because we need some of the results from the single fermion case for that problem) we consider the case of a single fermion in the first Tamm-Dancoff approximation. This is an approximation to the case of a physical fermion, in the sense that the Tamm-Dancoff truncation limits the ‘dressing’ of the fermion to a maximum of one meson.

Our basic equation is the Einstein equation

$$(2P^+P^- - P_\perp^2)|\Psi\rangle = M^2|\Psi\rangle \quad (2.1)$$

We split the Hamiltonian P^- into $P_M^- + P_f^-$ (i.e. into terms diagonal and non-diagonal, respectively, in the creation and annihilation operators), after

which our equation reads

$$(M^2 - 2P^+ P_M^- + P_\perp^2)|\Psi\rangle = 2P^+ P_l^- |\Psi\rangle \quad (2.2)$$

The required truncation of Fock space is effected by the expansion for $|\Psi\rangle$

$$\begin{aligned} |\Psi(P)\rangle &= \sum_{\alpha} C_0(P; \alpha) b_{\alpha}^{\dagger}(P) |0\rangle \\ &+ \sum_{\alpha} \int da^+ d^2 a_{\perp} db^+ d^2 b_{\perp} \Delta_{\delta}(\Lambda, a, b) C_1(a, b; \alpha) b_{\alpha}^{\dagger}(a) a^{\dagger}(b) |0\rangle \end{aligned} \quad (2.3)$$

where ‘ α ’ is the fermion spin index, and

$$\begin{aligned} \Delta_{\delta}(\Lambda, a, b) &= \theta(a^+) \theta(b^+) \delta(P^+ - a^+ - b^+) \delta^2(\vec{P}_{\perp} - \vec{a}_{\perp} - \vec{b}_{\perp}) \\ &\times \theta\left(\Lambda - \frac{(m_a^2 + a_{\perp}^2)}{2a^+} - \frac{(m_b^2 + b_{\perp}^2)}{2b^+}\right) \end{aligned} \quad (2.4)$$

which ensures momentum conservation, the positivity of a^+ and b^+ , and regulates the integral via a cutoff, Λ . For the sake of convenience we sometimes include the momentum conserving δ -functions in the definition of the cutoff functions, and when this is the case a subscript, δ , is appended.

Upon substitution, we project the equation onto non-interacting one-fermion and one-fermion/one-boson states

$$|p_b; r\rangle = b_r^{\dagger}(p_b) |0\rangle \quad (2.5)$$

$$|p_b, p_a; r\rangle = b_r^{\dagger}(p_b) a^{\dagger}(p_a) |0\rangle \quad (2.6)$$

which, via orthogonality of the basis states, will produce coupled equations for the expansion coefficients C_0 and C_1 .

Using the Hamiltonian in Appendix B we obtain

$$\left[M^2 - \left(m_{b_0}^2 + \frac{\lambda^2 \beta(1)}{2(2\pi)^3} \right) \right] C_0(P; r) = \frac{\lambda m_b \sqrt{2P^+}}{(2\pi)^{\frac{3}{2}}} \\ \times \sum_{\alpha} \int d^2 q \int_0^1 dz \frac{\Delta(\Lambda, P - q, q)}{[(1-z)z]^{\frac{1}{2}}} \bar{u}_r(P) \Gamma u_{\alpha}(P - q) C_1(P - q, q; \alpha) \quad (2.7)$$

and

$$\left[M^2 - \frac{1}{x} \left(m_{b_1}^2 + \frac{\lambda^2 \beta(x)}{2(2\pi)^3} \right) - \frac{1}{(1-x)} \left(m_{a_1}^2 + \frac{\lambda^2 \alpha(1-x)}{2(2\pi)^3} \right) - \right. \\ \left. \frac{(1-x)}{x} p_{b\perp}^2 - \frac{x}{(1-x)} p_{a\perp}^2 + 2\vec{p}_{b\perp} \cdot \vec{p}_{a\perp} \right] \Delta(\Lambda, p_b, p_a) C_1(p_b, p_a; r) \\ = \frac{\lambda m_b}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{2}{P^+}} \sum_{\alpha} \frac{1}{[(1-x)x]^{\frac{1}{2}}} \bar{u}_r(p_b) \Gamma u_{\alpha}(P) C_0(P; \alpha) \quad (2.8)$$

respectively, where we have expressed the ‘+’ components of the momenta as fractions of P^+ ; explicitly

$$p_b^+ = x P^+$$

$$p_a^+ = (1-x) P^+$$

$$q^+ = z P^+$$

In deriving the above equations the sector dependent renormalisation scheme proposed by Perry and Harindranath [9] was employed. This involves assuming that the bare masses that appear in these coupled equations

depend on the Fock space sector from which they originate. Thus in the first equation (which arises from the contraction with $|p_b; r >$) we use m_{b_0} , and in the second (contracting with $|p_b, p_a; r >$) m_{b_1} . For masses which appear due to interaction terms we use the physical masses, since we are adopting the convention that "... a fermion which cannot dress itself propagates with the physical mass, and the proper mass for other cases is dependent on the degree to which the fermion can dress itself" [7]. If we were using a coupling constant expansion we could justify this by arguing that any corrections to these masses must be of a higher order in λ . Ostensibly we cannot use this argument here, since the Tamm-Dancoff truncation is not equivalent to an expansion in the coupling constant. We can note, however, that our choice of the truncated Fock state basis would be highly inappropriate if indeed the coupling constant were large, since we would have no reason in that case to expect only the lowest Fock states to be physically significant. While this observation lends support to the use of physical masses in the interaction terms in our case, we must essentially include this 'ansatz' in the description of the renormalisation procedure, as an assumption.

Another assumption made is that we can omit the contribution of the 'instantaneous' terms in the Hamiltonian to the equation which arises from the highest sector of Fock space under consideration. If we were performing

calculations in the Tamm-Dancoff limit we would find that all instantaneous terms arising from particular sectors were needed to cancel terms appearing in higher sectors. However since we are truncating Fock space we rule that these terms are artifacts of the truncation, and we omit them. The most obvious consequence of this omission is that the highest sector equation is algebraic rather than being an integral equation, since the inclusion of these ‘instantaneous’ terms would have yielded an additional integral term in C_1 on the right hand side of Equation 2.8.

Now in the highest sector we can use a threshold condition to determine the counterterms needed to renormalise the fermion and meson masses in this sector. At threshold we expect C_1 to contain a pole corresponding to the scattering states, and for this to remain a solution of Equation 2.8 its coefficient must vanish. Thus, setting $\vec{p}_{b\perp} = \vec{p}_{a\perp} = 0$ at threshold, and denoting the appropriate momentum fraction x_{th} , this condition becomes

$$M^2 - \frac{1}{x_{th}} \left(m_{b_1}^2 + \frac{\lambda^2 \beta(x_{th})}{2(2\pi)^3} \right) - \frac{1}{(1-x_{th})} \left(m_{a_1}^2 + \frac{\lambda^2 \alpha(1-x_{th})}{2(2\pi)^3} \right) = 0 \quad (2.9)$$

The momentum fractions can be recovered from the fact that at threshold $M = m_b + m_a$ and $P^+ = M/\sqrt{2}$. The use of these allows us to identify

$$m_b^2 = m_{b_1}^2 + \frac{\lambda^2 \beta(x_{th})}{2(2\pi)^3}$$

$$m_a^2 = m_{a_1}^2 + \frac{\lambda^2 \alpha (1 - x_{th})}{2(2\pi)^3} \quad (2.10)$$

which we accept as the form of the highest sector mass counterterms. These counterterms (which represent an infinite renormalisation) are used to eliminate the bare masses in favour of the physical masses in this sector.

The next step is to eliminate C_1 between Equations 2.7 and 2.8 in order to obtain an equation in C_0 only. In doing this we can drop the cutoff function, Δ , since its arguments are external momenta which we can choose to satisfy the cutoff condition. Denoting

$$D_1(p_b, p_a) = M^2 - \frac{m_b^2}{x} - \frac{m_a^2}{(1-x)} - \frac{(1-x)p_{b\perp}^2}{x} - \frac{x p_{a\perp}^2}{(1-x)} + 2\vec{p}_{b\perp} \cdot \vec{p}_{a\perp} \quad (2.11)$$

we find

$$C_1(p_b, p_a; r) = \frac{\lambda m_b}{(2\pi)^{\frac{5}{2}}} \sqrt{\frac{2}{P^+}} \sum_{\alpha} \frac{1}{D_1(p_b, p_a)} \frac{1}{[(1-x)x]^{\frac{1}{2}}} \bar{u}_r(p_b) \Gamma u_{\alpha}(P) C_0(P; \alpha) \quad (2.12)$$

Using this equation to provide an expression for $C_1(P - q, q; \alpha)$ and substituting into Equation 2.7 yields

$$\begin{aligned} & \left[M^2 - \left(m_{b_0}^2 + \frac{\lambda^2 \beta(1)}{2(2\pi)^3} \right) \right] C_0(P; r) = \\ & \lambda^2 \frac{2m_b^2}{2(2\pi)^3} \int d^2 q \int_0^1 dz \frac{\Delta(\Lambda, P - q, q)}{z(1-z)} \frac{1}{D_1(P - q, q)} \\ & \times \sum_{\alpha, \beta} \bar{u}_r(P) \Gamma u_{\alpha}(P - q) \bar{u}_{\alpha}(P - q) \Gamma u_{\beta}(P) C_0(P; \beta) \end{aligned} \quad (2.13)$$

The spinor sum can be evaluated using completeness relations to give

$$\sum_{\alpha} \bar{u}_r(P) \Gamma u_{\alpha}(P-q) \bar{u}_{\alpha}(P-q) \Gamma u_L(P) = \frac{1}{2} \delta_{r,\beta} \left(\frac{P.(P-q)_{m_b}}{m_b^2} - 1 \right) \quad (2.14)$$

for $\Gamma = i\gamma_5$, where the ‘ m_b ’ suffix is meant to imply that the vector $(P-q)$ is on shell, rather than either of its component vectors. Explicitly we have

$$P.(P-q)_{m_b} = \frac{m_b^2 + (\vec{P}_{\perp} - \vec{q}_{\perp})^2}{2(1-z)} + \frac{(1-z)(m_b^2 + \vec{P}_{\perp}^2)}{2} - \vec{P}_{\perp} \cdot (\vec{P}_{\perp} - \vec{q}_{\perp}) \quad (2.15)$$

Thus our equation becomes

$$\begin{aligned} & \left[M^2 - (m_{b_0}^2 + \frac{\lambda^2 \beta(1)}{2(2\pi)^3}) \right] C_0(P; r) \\ &= \lambda^2 \frac{2m_b^2}{2(2\pi)^3} \int d^2 q \int_0^1 dz \frac{\Delta(\Lambda, P-q, q)}{D_1(P-q, q)} \frac{1}{z(1-z)} \\ & \quad \times \frac{1}{2m_b^2} (P.(P-q)_{m_b} - m_b^2) C_0(P; r) \end{aligned} \quad (2.16)$$

which is purely algebraic in $C_0(P; r)$.

Below the two particle production threshold we naturally interpret $M = m_b$, and so the equation for C_0 provides us with the mass counterterm for m_{b_0} , namely

$$m_{b_0}^2 = m_b^2 - \frac{\lambda^2 \beta(1)}{2(2\pi)^3}$$

$$\begin{aligned}
& - \lambda^2 \frac{2m_b^2}{2(2\pi)^3} \int d^2 q \int_0^1 dz \frac{\Delta(\Lambda, P - q, q)}{D_1(P - q, q)} \frac{1}{z(1 - z)} \\
& \times \frac{1}{2m_b^2} (P \cdot (P - q)_{m_b} - m_b^2)
\end{aligned}
\tag{2.17}$$

This counterterm is all we require from the one fermion equation at present, since it represents the self energy correction due to the possibility of the fermion dressing itself with a single meson. The hope is that when dealing with more complicated configurations we will be able to use this counterterm to remove the divergence associated with this self energy. To what extent this hope is realised will be addressed in the following chapter. For now we can note that, unlike the 1+1 case, this renormalisation is an infinite one, diverging logarithmically as the cutoff is removed.

Chapter 3

First Tamm-Dancoff approximation for two fermions

We next consider the case of two fermions in the first Tamm-Dancoff approximation. Here we have, in addition to the self energy effects seen in the one fermion case, interactions between the two fermions, mediated by the single meson that is allowed in this approximation.

Our basic equation is, as before, the Einstein equation, which after the separation of the diagonal and non-diagonal terms, reads

$$(M^2 - 2P^+ P_M^- + P_\perp^2)|\Psi\rangle = 2P^+ P_I^- |\Psi\rangle \quad (3.1)$$

In this case the required truncation of Fock space is effected by the expansion for $|\Psi\rangle$

$$|\Psi(P)\rangle = \sum_{\alpha,\beta} \int da db \Delta_\delta^0(\Lambda, a, b) C_0(a, b; \alpha, \beta) B_\alpha^\dagger(a) b_\beta^\dagger(b) |0\rangle$$

$$+ \sum_{\alpha, \beta} \int da db dc \Delta_{\delta}^1(\Lambda, a, b, c) C_1(a, b, c; \alpha, \beta) B_{\alpha}^{\dagger}(a) b_{\beta}^{\dagger}(b) a^{\dagger}(c) |0 > \quad (3.2)$$

where $da \equiv da^+ d^2 a_{\perp}$ etc. and we have used ' $B^{\dagger}$ ' and ' $b^{\dagger}$ ' to denote the creation operators for the two (assumed distinct) types of fermions in the problem. The cutoff functions are defined, in exact analogy to the one fermion case, by

$$\begin{aligned} \Delta_{\delta}^0(\Lambda, a, b) &= \theta(a^+) \theta(b^+) \delta(P^+ - a^+ - b^+) \delta^2(\vec{P}_{\perp} - \vec{a}_{\perp} - \vec{b}_{\perp}) \\ &\times \theta \left(\Lambda - \frac{(m_a^2 + a_{\perp}^2)}{2a^+} - \frac{(m_b^2 + b_{\perp}^2)}{2b^+} \right) \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \Delta_{\delta}^1(\Lambda, a, b, c) &= \theta(a^+) \theta(b^+) \theta(c^+) \\ &\times \delta(P^+ - a^+ - b^+ - c^+) \delta^2(\vec{P}_{\perp} - \vec{a}_{\perp} - \vec{b}_{\perp} - \vec{c}_{\perp}) \\ &\times \theta \left(\Lambda - \frac{(m_a^2 + a_{\perp}^2)}{2a^+} - \frac{(m_b^2 + b_{\perp}^2)}{2b^+} - \frac{(m_c^2 + c_{\perp}^2)}{2c^+} \right) \end{aligned} \quad (3.4)$$

In this case, after substituting for $|\Psi >$, we project the equation onto non-interacting two-fermion and two-fermion/one-boson states

$$|p_B, p_b; r, s > = B_r^{\dagger}(p_B) b_s^{\dagger}(p_b) |0 > \quad (3.5)$$

$$|p_B, p_b, p_a; r, s > = B_r^{\dagger}(p_B) b_s^{\dagger}(p_b) a^{\dagger}(p_a) |0 > \quad (3.6)$$

As in the one fermion case, we expect the equations which arise from these projections to be coupled equations in C_0 and C_1 . Following, again, the

policy of discarding 'instantaneous' terms in the highest sector of Fock space we obtain

$$\begin{aligned}
& \left[M^2 - \frac{1}{x} \left(m_{B_0}^2 + \frac{\lambda^2 \beta(x)}{2(2\pi)^3} \right) - \frac{1}{(1-x)} \left(m_{b_0}^2 + \frac{\lambda^2 \beta(1-x)}{2(2\pi)^3} \right) - \right. \\
& \left. \frac{(1-x)}{x} p_{B\perp}^2 - \frac{x}{(1-x)} p_{b\perp}^2 + 2\vec{p}_{B\perp} \cdot \vec{p}_{b\perp} \right] \Delta^0(\Lambda, p_B, p_b) C_0(p_B, p_b; r, s) \\
& = \frac{\lambda m_B}{(2\pi)^{\frac{3}{2}}} \sqrt{2P^+} \sum_{\alpha} \int d^2 q \int_0^x dz \frac{1}{[x(x-z)z]^{\frac{1}{2}}} \bar{u}_r(p_B) \Gamma u_{\alpha}(p_B - q) \\
& \quad \times \Delta^1(\Lambda, p_B - q, p_b, q) C_1(p_B - q, p_b, q; \alpha, s) \\
& + \frac{\lambda m_b}{(2\pi)^{\frac{3}{2}}} \sqrt{2P^+} \sum_{\alpha} \int d^2 q \int_0^{1-x} dz \frac{1}{[(1-x)(1-x-z)z]^{\frac{1}{2}}} \bar{u}_s(p_b) \Gamma u_{\alpha}(p_b - q) \\
& \quad \times \Delta^1(\Lambda, p_B, p_b - q, q) C_1(p_B, p_b - q, q; r, \alpha) \tag{3.7}
\end{aligned}$$

and

$$\begin{aligned}
& \left[M^2 - \frac{1}{x} \left(m_{B_1}^2 + \frac{\lambda^2 \beta(x)}{2(2\pi)^3} \right) - \frac{1}{(1-x-y)} \left(m_{b_1}^2 + \frac{\lambda^2 \beta(1-x-y)}{2(2\pi)^3} \right) - \right. \\
& \frac{1}{y} \left(m_{a_1}^2 + \frac{\lambda^2 \alpha(y)}{2(2\pi)^3} \right) - \frac{(1-x)}{x} p_{B\perp}^2 - \frac{x+y}{(1-x-y)} p_{b\perp}^2 - \frac{(1-y)}{y} p_{a\perp}^2 + \\
& \left. 2(\vec{p}_{B\perp} \cdot \vec{p}_{b\perp} + \vec{p}_{B\perp} \cdot \vec{p}_{a\perp} + \vec{p}_{b\perp} \cdot \vec{p}_{a\perp}) \right] C_1(p_B, p_b, p_a, r, s) \\
& = \frac{\lambda m_B}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{2}{P^+}} \frac{1}{[x(x+y)y]^{\frac{1}{2}}} \sum_{\alpha} \bar{u}_r(p_B) \Gamma u_{\alpha}(p_B + p_a) \\
& \quad \times C_0(p_B + p_a, p_b; \alpha, s) \\
& + \frac{\lambda m_b}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{2}{P^+}} \frac{1}{[(1-x)(1-x-y)y]^{\frac{1}{2}}} \sum_{\alpha} \bar{u}_s(p_b) \Gamma u_{\alpha}(p_b + p_a) \\
& \quad \times C_0(p_B, p_b + p_a; r, \alpha) \tag{3.8}
\end{aligned}$$

where we have again expressed the ‘+’ components of the momenta as fractions of P^+ ; namely

$$p_B^+ = xP^+$$

$$p_a^+ = yP^+$$

$$q^+ = zP^+$$

the fraction for p_b^+ being fixed by the condition that $P^+ = p_B^+ + p_b^+ + p_a^+$

As in the one fermion case, we have used the physical values for masses appearing due to interaction terms, but otherwise have allowed the bare masses to depend on the Fock space sector from which they originate. Given the neglect of the ‘instantaneous’ terms in the highest sector we see that the equation arising from this sector is algebraic (i.e. is not an integral equation), and we can again use a threshold condition to determine the mass counterterms. Thus we consider the three particle threshold, where $M = m_B + m_b + m_a$. Immediately we can set $\vec{p}_B = \vec{p}_b = \vec{p}_a = 0$, and the threshold momentum fractions can be determined from the fact that $P^+ = M/\sqrt{2}$ and, for example, $p_B^+ = m_B/\sqrt{2}$. At threshold we expect C_1 to contain a pole corresponding to the scattering states, and for this to remain a solution of Equation 3.8 its coefficient must vanish. Thus we are led to identify

$$m_B^2 = m_{B_1}^2 + \frac{\lambda^2 \beta(x_{th})}{2(2\pi)^3}$$

$$\begin{aligned}
m_b^2 &= m_{b_1}^2 + \frac{\lambda^2 \beta (1 - x_{th} - y_{th})}{2(2\pi)^3} \\
m_a^2 &= m_{a_1}^2 + \frac{\lambda^2 \alpha (y_{th})}{2(2\pi)^3}
\end{aligned} \tag{3.9}$$

which we use to eliminate the bare masses in this sector.

As in the one fermion case we can now solve this equation for C_1 , and use the result to eliminate it from the lowest sector equation. Denoting

$$\begin{aligned}
D_1(p_B, p_b, p_a) &= M^2 - \frac{m_B^2}{x} - \frac{m_b^2}{(1-x-y)} - \frac{m_a^2}{y} - \frac{(1-x)}{x} p_{B\perp}^2 \\
&- \frac{(x+y)}{(1-x-y)} p_{b\perp}^2 - \frac{(1-y)}{y} p_{a\perp}^2 \\
&+ 2(\vec{p}_{B\perp} \cdot \vec{p}_{b\perp} + \vec{p}_{B\perp} \cdot \vec{p}_{a\perp} + \vec{p}_{b\perp} \cdot \vec{p}_{a\perp})
\end{aligned} \tag{3.10}$$

we find

$$\begin{aligned}
C_1(p_B, p_b, p_a, r, s) &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{2}{P^+}} \sum_{\alpha} \frac{1}{D_1(p_B, p_b, p_a)} \\
&\times \left(\frac{m_B}{[x(x+y)y]^{\frac{1}{2}}} \bar{u}_r(p_B) \Gamma u_{\alpha}(p_B + p_a) C_0(p_B + p_a, p_b; \alpha, s) + \right. \\
&\left. \frac{m_b}{[(1-x)(1-x-y)y]^{\frac{1}{2}}} \bar{u}_s(p_b) \Gamma u_{\alpha}(p_b + p_a) C_0(p_B, p_b + p_a; r, \alpha) \right) \tag{3.11}
\end{aligned}$$

In the lowest sector equation we need expressions for $C_1(p_B - q, p_b, q; \alpha, s)$ and $C_1(p_B, p_b - q, q; r, \alpha)$ thus we need to modify the arguments in the above expression in order to perform the substitutions required. This having been done, we eliminate C_1 to obtain

$$\begin{aligned}
& \left[M^2 - \frac{1}{x} \left(m_{B_0}^2 + \frac{\lambda^2 \beta(x)}{2(2\pi)^3} \right) - \frac{1}{(1-x)} \left(m_{b_0}^2 + \frac{\lambda^2 \beta(1-x)}{2(2\pi)^3} \right) - \right. \\
& \quad \left. \frac{(1-x)}{x} p_{B\perp}^2 - \frac{x}{(1-x)} p_{b\perp}^2 + 2\vec{p}_{B\perp} \cdot \vec{p}_{b\perp} \right] \Delta^0(\Lambda, p_B, p_b) C_0(p_B, p_b; \tau, s) \\
&= \frac{\lambda^2}{(2\pi)^3} \left[\frac{m_B^2}{x} \int d^2 q \int_0^x \frac{dz}{(x-z)z} \frac{\Delta^1(\Lambda, p_B - q, p_b, q)}{D_1(p_B - q, p_b, q)} \frac{(p_B \cdot (p_B - q)_{m_B} - m_B^2)}{m_B^2} \right. \\
& \quad \left. + \frac{m_b^2}{(1-x)} \int d^2 q \int_0^{1-x} \frac{dz}{(1-x-z)z} \frac{\Delta^1(\Lambda, p_B, p_b - q, q)}{D_1(p_B, p_b - q, q)} \frac{(p_b \cdot (p_b - q)_{m_b} - m_b^2)}{m_b^2} \right] \\
& \quad \times C_0(p_B, p_b, q; \tau, s) \\
&+ \frac{2\lambda^2}{(2\pi)^3} \sum_{\alpha, \beta} \left[m_B m_b \int d^2 q \int_0^x dz \frac{\bar{u}_r(p_B) \Gamma u_\alpha(p_B - q) \bar{u}_r(p_b) \Gamma u_\beta(p_b + q)}{[x(x-z)(1-x)(1-x+z)]^{\frac{1}{2}} z} \right. \\
& \quad \times \frac{\Delta^1(\Lambda, p_B - q, p_b, q)}{D_1(p_B - q, p_b, q)} C_0(p_B - q, p_b + q; \alpha, \beta) \\
& \quad + m_b m_B \int d^2 q \int_0^{1-x} dz \frac{\bar{u}_s(p_b) \Gamma u_\alpha(p_b - q) \bar{u}_r(p_B) \Gamma u_\beta(p_B + q)}{[(1-x)(1-x-z)x(x+z)]^{\frac{1}{2}} z} \\
& \quad \times \frac{\Delta^1(\Lambda, p_B, p_b - q, q)}{D_1(p_B, p_b - q, q)} C_0(p_B + q, p_b - q; \beta, \alpha) \left. \right] \quad (3.12)
\end{aligned}$$

with

$$\begin{aligned}
p_B \cdot (p_B - q)_{m_B} &= \frac{x}{2(x-z)} (m_B^2 + (\vec{p}_{B\perp} - \vec{q}_\perp)^2) + \frac{(x \cdot z)}{2x} (m_B^2 + \vec{p}_{B\perp}^2) \\
&\quad - \vec{p}_{B\perp} \cdot (\vec{p}_{B\perp} - \vec{q}_\perp) \\
p_b \cdot (p_b - q)_{m_b} &= \frac{(1-x)}{2(1-x-z)} (m_b^2 + (\vec{p}_{b\perp} - \vec{q}_\perp)^2) + \frac{(1-x-z)}{2(1-x)} (m_b^2 + \vec{p}_{b\perp}^2) \\
&\quad - \vec{p}_{b\perp} \cdot (\vec{p}_{b\perp} - \vec{q}_\perp) \quad (3.13)
\end{aligned}$$

where spinor completeness relations have been used to perform contractions

wherever possible.

It can be seen that the first two terms on the right hand side of the above equation are of the form of the mass corrections which we found in the one fermion case. They clearly correspond to the emission and reabsorption of a meson by the same fermion. The remaining terms correspond to the two distinct x^+ orderings of meson exchange between the two fermions. The mass correction terms can be grouped with the mass terms on the left hand side of the equation to which they correspond, and then we ostensibly have an equation whose only bare parameters are m_{B_0} and m_{b_0} . At this point, following Perry and Harindranath [9], we recall the lowest sector counterterm derived in the one fermion case (Equation 2.17). In order to use this counterterm in the two fermion case we need to take account of the fact that in the one fermion case the momentum fraction ' x ' did not appear, since it was obviously unity. Thus we need to modify the expression to take account of this fact, and having done this we obtain

$$\begin{aligned}
 m_{B_0}^2 &= m_B^2 - \frac{\lambda^2 \beta(x)}{2(2\pi)^3} \\
 &- \frac{\lambda^2}{2(2\pi)^3} \int d^2 q \int_0^1 dz \frac{\Delta(\Lambda, p_B - q, p_b, q)}{\tilde{D}_1(p_B, q)} \frac{1}{z(1-z)} (p_B \cdot (p_B - q)'_{m_B} - m_H^2)
 \end{aligned}
 \tag{3.14}$$

and

$$\begin{aligned}
m_{b_0}^2 &= m_b^2 - \frac{\lambda^2 \beta (1-x)}{2(2\pi)^3} \\
&- \frac{\lambda^2}{2(2\pi)^3} \int d^2 q \int_0^1 dz \frac{\Delta(\Lambda, p_B, p_b - q, q)}{\tilde{D}_1(p_b, q)} \frac{1}{z(1-z)} (p_b \cdot (p_b - q)'_{m_b} - m_b^2)
\end{aligned} \tag{3.15}$$

where we denote

$$\tilde{D}_1(p_B, q) = -\frac{zm_B^2}{(1-z)} - \frac{m_a^2}{z} - \frac{zp_{B\perp}^2}{(1-z)} - \frac{q_\perp^2}{z(1-z)} + \frac{2\vec{p}_{B\perp} \cdot \vec{q}_\perp}{(1-z)} \tag{3.16}$$

with an equivalent expression for $\tilde{D}_1(p_b, q)$. The term $p_b \cdot (p_b - q)'_{m_b}$ has been modified by the various changes of variables (in particular the shift of the upper limit of the 'z' integration from x to 1) and is now given by

$$\frac{1}{2(1-z)} (m_B^2 + (\vec{p}_{B\perp} - \vec{q}_\perp)^2) + \frac{(1-z)}{2} (m_B^2 + p_{B\perp}^2) - \vec{p}_{B\perp} \cdot (\vec{p}_{B\perp} - \vec{q}_\perp) \tag{3.17}$$

A similar expression exists for $p_b \cdot (p_b - q)'_{m_b}$. After substituting for the bare masses we perform several shifts of integration variables. First, in order to facilitate comparison of the mass correction terms and the counterterms, we alter the 'z' integration variables such that all mass correction term momentum fraction integrations are over the range $0 \rightarrow 1$. Second we shift the integration variables in the exchange terms such that we can combine them. Finally we notice that the momentum dependence of the spinors and coefficient function in the interaction terms can be simplified by a shift of $\vec{q}_\perp$ given

that we make the additional assumption that

$$\vec{p}_{b\perp} = -\vec{p}_{B\perp} \quad (3.18)$$

which selects the centre of mass frame with respect to the transverse momenta, and should not lead to a loss of generality. These operations produce the equation

$$\begin{aligned} & \left[M^2 - \frac{p_{B\perp}^2}{x(1-x)} \right. \\ & - \frac{m_B^2}{x} \left(1 + \frac{\lambda^2}{(2\pi)^3} \int d^2 q \int_0^1 dz \frac{(p_B \cdot (p_B - q)'_{m_B} - m_B^2)}{m_B^2 z(1-z)} \left[\frac{1}{D_B} - \frac{1}{D'_B} \right] \right) \\ & - \frac{m_b^2}{(1-x)} \left(1 + \frac{\lambda^2}{(2\pi)^3} \int d^2 q \int_0^1 dz \frac{(p_b \cdot (p_b - q)'_{m_b} - m_b^2)}{m_b^2 z(1-z)} \left[\frac{1}{D_b} - \frac{1}{D'_b} \right] \right) \Bigg] \\ & \quad \times C_0(x, p_{B\perp}; r, s) \\ & = \frac{2\lambda^2}{(2\pi)^3} m_B m_b \sum_{\alpha, \beta} \int d^2 q \int_0^1 \frac{dz}{(x-z)} \frac{1}{[x(1-x)z(1-z)]^{\frac{1}{2}}} \\ & \quad \times \bar{u}_r(m_B, x, p_{B\perp}) \Gamma u_\alpha(m_B, z, q_\perp) \bar{u}_s(m_b, 1-x, -p_{B\perp}) \Gamma u_\beta(m_b, 1-z, -q_\perp) \\ & \quad \times \left[\frac{\theta(z-x)}{\tilde{D}_a} - \frac{\theta(x-z)}{\tilde{D}_b} \right] C_0(z, q_\perp; \alpha, \beta) \end{aligned} \quad (3.19)$$

where we have also dropped the cutoff functions in order to simplify the notation. We express the arguments of the coefficient functions in terms of momentum fractions and transverse components, and include the appropriate mass as an argument of the spinors for clarity. Several denominator functions

have been defined in the writing of this equation. Explicitly these are

$$\begin{aligned}
 D_B &= xM^2 - \frac{m_B^2}{(1-z)} - \frac{xm_b^2}{(1-x)} - \frac{m_a^2}{z} - \frac{(1-xz)p_{B\perp}^2}{(1-x)(1-z)} - \frac{q_\perp^2}{z(1-z)} \\
 &\quad + \frac{2\vec{p}_{B\perp} \cdot \vec{q}_\perp}{(1-z)} \\
 D'_B &= -\frac{zm_B^2}{(1-z)} - \frac{m_a^2}{z} - \frac{zp_{B\perp}^2}{(1-z)} - \frac{q_\perp^2}{z(1-z)} + \frac{2\vec{p}_{B\perp} \cdot \vec{q}_\perp}{(1-z)} \\
 \tilde{D}_a &= M^2 - \frac{m_B^2}{x} - \frac{m_b^2}{(1-z)} - \frac{m_a^2}{(z-x)} - \frac{zp_{B\perp}^2}{x(z-x)} - \frac{(1-x)q_\perp^2}{(z-x)(1-z)} \\
 &\quad + \frac{2\vec{p}_{B\perp} \cdot \vec{q}_\perp}{(z-x)}
 \end{aligned} \tag{3.20}$$

where D_b and D'_b are related to D_B and D'_B respectively, by the transformation $x \rightarrow 1-x$, $m_B \leftrightarrow m_b$ and $\vec{p}_{B\perp} \rightarrow -\vec{p}_{B\perp}$, while $\tilde{D}_a$ and $\tilde{D}_b$ are related by $x \rightarrow 1-x$, $m_B \leftrightarrow m_b$ and $z \rightarrow 1-z$. These transformations have no fundamental significance (rather they are just a matter of definition), but they do have the consequence that the right hand side is continuous (though not differentiable) at $z = x$.

Chapter 4

Coupling Constant Renormalisation

Up to this point we have closely followed Perry and Harindranath [9], and have found that most aspects of the procedure they outline can be repeated in 3+1 dimensions provided only that the transverse momentum terms are included. At this point, however, we encounter a significant difference. The mass correction terms, on the left hand side of Equation 3.19, which in the 1+1 case were rendered finite by the use of the mass counterterms from the one fermion case, are still divergent in 3+1 dimensions. Inspection of these mass correction terms reveals that it is the integration over the transverse momentum which diverges logarithmically. Since it is in the inclusion of the transverse momenta that the 3+1 treatment differs from the 1+1 case, it seems reasonable that this new divergence should be related to these momenta.

In order to examine the mass correction terms it is convenient to define a parameter

$$\delta = M^2 - \frac{m_B^2}{x} - \frac{m_b^2}{(1-x)} - \frac{p_{B\perp}^2}{x(1-x)} \quad (4.1)$$

which is analogous to the binding energy of the two fermion bound state, in particular vanishing when the fermions are on shell. This quantity arises very naturally in this analysis, and is, for example, the coefficient of C_0 on the left hand side of Equation 3.19 if the mass correction terms are ignored. It is also of use when considering the mass correction terms themselves, since it can easily be seen that

$$D_B = D'_B + x\delta$$

$$D_b = D'_b + (1-x)\delta$$

We can immediately see, therefore, that the mass correction terms are zero at, or above, the two particle threshold. This indicates that the renormalisation procedure has at least been successful in fixing the fermion masses in the only regime where they can be unambiguously checked, namely when they are propagating as free particles.

From Equation 3.19 we have as the explicit expression for the m_B correc-

tion term

$$-\frac{1}{x} \frac{\lambda^2}{(2\pi)^3} \int d^2 q_\perp \int_0^1 dz \frac{(p_B \cdot (p_B - q)'_{m_B} - m_B^2)}{z(1-z)} \left[\frac{1}{D'_B + x\delta} - \frac{1}{D'_B} \right] \quad (4.2)$$

This expression can be greatly simplified by use of the transformation $\vec{q}_\perp \rightarrow \vec{q}_\perp - z\vec{p}_{B\perp}$. Performing the transformation we find that the integrand becomes independent of ϕ_q , enabling us to trivially perform the angular integration, leaving

$$-\frac{1}{x} \frac{\lambda^2}{(2\pi)^2} \int dq_\perp q_\perp \int_0^1 dz \frac{1}{2z(1-z)^2} (q_\perp^2 + z^2 m_B^2) \left[\frac{1}{D'_B + x\delta} - \frac{1}{D'_B} \right] \quad (4.3)$$

where D'_B has been transformed

$$D'_B \rightarrow -\frac{1}{z(1-z)} (z^2 m_B^2 + (1-z)m_a^2 + q_\perp^2) \quad (4.4)$$

In order to try and isolate the source of the divergence, and in light of the observation that the mass correction terms are well defined (zero) at zeroth order in δ , we expand in powers of δ . Thus we find, for the m_B case

$$\begin{aligned} \left[\frac{1}{D'_B + x\delta} - \frac{1}{D'_B} \right] &= -\frac{x\delta}{D_B'^2} \left[1 - \frac{x\delta}{D'_B} + \left(\frac{x\delta}{D'_B} \right)^2 - \dots \right] \\ &= -\frac{x\delta}{D_B'^2} + \frac{x^2 \delta^2}{D_B'^2 (D'_B + x\delta)} \end{aligned} \quad (4.5)$$

The advantage of isolating the first order term in δ is that it is the only term which diverges, and thus in isolating it we have isolated the divergence. The same expansion for the m_b term yields

$$\left[\frac{1}{D_b} - \frac{1}{D'_b} \right] = -\frac{(1-x)\delta}{D_b'^2} + \frac{(1-x)^2 \delta^2}{D_b'^2 (D'_b + (1-x)\delta)} \quad (4.6)$$

which behaves similarly.

Denoting the contribution of the first term to the mass correction by $\lambda^2 \delta I_B$ we find that

$$I_B = \frac{1}{2(2\pi)^2} \int dq_\perp q_\perp dz z \frac{(q_\perp^2 + z^2 m_B^2)}{(z^2 m_B^2 + (1-z)m_a^2 + q_\perp^2)^2} > 0 \quad (4.7)$$

If we regulate this expression by imposing an upper limit Λ on the $q_\perp$ integration, this transverse integration can be readily performed

$$I_B = \frac{1}{4(2\pi)^2} \int_0^1 dz z \left[\log \left(1 + \frac{\Lambda^2}{(z^2 m_B^2 + (1-z)m_a^2)} \right) - \frac{(1-z)m_a^2 \Lambda^2}{(z^2 m_B^2 + (1-z)m_a^2 + \Lambda^2)} \right] \quad (4.8)$$

Note that the second term in is finite as $\Lambda \rightarrow \infty$ whereas the first gives a logarithmic divergence. We denote them separately through $I_B = I_B^{\log} + I_B^f$

Denoting the full (finite) mass correction term by Δ we have

$$\Delta(\Lambda) = \left(\frac{-1}{(2\pi)^2 x} \int dq_\perp q_\perp \int_0^1 dz \frac{1}{2z(1-z)^2} (q_\perp^2 + z^2 m_B^2) \frac{x^2 \delta^2}{D_B'^2 (D_B' + x\delta)} + \delta I_B^f \right) + (x \rightarrow 1-x, m_B \rightarrow m_b) \quad (4.9)$$

We can then rewrite the coefficient of C_0 on the left hand side of Equation 3.19

as

$$\delta + \lambda^2 \delta (I_B^{\log} + I_b^{\log}) + \lambda^2 \Delta(\Lambda) \quad (4.10)$$

Now if we write the first two terms of this equation as

$$\delta(1 + \lambda^2(I_B^{\log} + I_b^{\log}))$$

we can divide the whole of Equation 3.19 by the divergent term, and recover a well defined equation, provided we identify

$$\tilde{\lambda}^2 = \frac{\lambda^2}{(1 + \lambda^2(I_B^{\log} + I_b^{\log}))} \quad (4.11)$$

as the renormalised coupling constant.

We thus find, in 3+1 dimensions with a $\Gamma = i\gamma_5$ coupling, the bound state equation

$$\begin{aligned} & \left[M^2 - \frac{m_B^2}{x} - \frac{m_b^2}{(1-x)} - \frac{p_{B\perp}^2}{x(1-x)} + \tilde{\lambda}^2 \Delta \right] C_0(x, p_{B\perp}; r, s) \\ &= \frac{2\tilde{\lambda}^2}{(2\pi)^3} m_B m_b \sum_{\alpha, \beta} \int d^2 q \int_0^1 \frac{dz}{(x-z)} \frac{1}{[x(1-x)z(1-z)]^{\frac{1}{2}}} \\ & \quad \times \bar{u}_r(m_B, x, p_{B\perp}) \Gamma u_\alpha(m_B, z, q_\perp) \bar{u}_s(m_b, 1-x, -p_{B\perp}) \Gamma u_\beta(m_b, 1-z, -q_\perp) \\ & \quad \times \left[\frac{\theta(z-x)}{\tilde{D}_a} - \frac{\theta(x-z)}{\tilde{D}_b} \right] C_0(z, q_\perp; \alpha, \beta) \end{aligned} \quad (4.12)$$

in the lowest (non-trivial) Tamm-Dancoff approximation.

4.1 Explicit Mass Correction Calculation

In solving the bound state equation (Equation 4.12) we will need an explicit form for the mass correction term (Equation 4.9). From Equation 4.8 we already have

$$I_B^f = -\frac{1}{4(2\pi)^2} \int_0^1 dz \frac{z(1-z)m_a^2\Lambda^2}{(z^2m_B^2 + (1-z)m_a^2 + \Lambda^2)} \quad (4.13)$$

This can be rewritten as

$$I_B^f = -\frac{1}{4(2\pi)^2} \frac{m_a^2}{m_B^2} \Lambda^2 \int_0^1 dz \frac{z(1-z)}{(z - \alpha(\Lambda))(z - \alpha^*(\Lambda))} \quad (4.14)$$

where

$$\alpha(\Lambda) = \frac{m_a^2}{2m_B^2} + \sqrt{\left(\frac{m_a^2}{2m_B^2}\right)^2 - \left(\frac{m_a^2 + \Lambda^2}{m_B^2}\right)} \quad (4.15)$$

(Note that α is invariably complex).

This can now be integrated to give

$$I_B^f = -\frac{1}{4(2\pi)^2} \frac{m_a^2}{m_B^2} \Lambda^2 (\beta(\alpha) + \beta(\alpha^*) - 1) \quad (4.16)$$

where

$$\beta(\alpha) = \frac{(\alpha^2 - \alpha)}{(\alpha^* - \alpha)} \log \left(\frac{\alpha - 1}{\alpha} \right) \quad (4.17)$$

We are now left to integrate the rest of the expression for Δ . Using the change of variables $u = q_\perp^2$ we can perform the transverse integration, leaving

us with terms involving quadratics and logarithms of quadratics in 'z'. This integration, and the subsequent 'z' integration are relatively straightforward but messy, and there seems to be little point in reproducing it here.

The result of these manipulations is the expression for the mass correction term

$$\begin{aligned} \Delta = & \left(\frac{\delta}{4(2\pi)^2} \left[\frac{\Lambda^2 m_a^2}{m_B^4} I_1 - I_2 \right] + \delta I_B^f \right) \\ & + (x \rightarrow 1-x; m_B \rightarrow m_b) \end{aligned} \quad (4.18)$$

where

$$\begin{aligned} I_1 = & \frac{\alpha_3(1-\alpha_3)}{(\alpha_3-\alpha_4)(\alpha_3-\beta_3)(\alpha_3-\beta_4)} \log \left(\frac{\alpha_3-1}{\alpha_3} \right) \\ & + \frac{\alpha_4(1-\alpha_4)}{(\alpha_4-\alpha_3)(\alpha_4-\beta_3)(\alpha_4-\beta_4)} \log \left(\frac{\alpha_4-1}{\alpha_4} \right) \\ & + \frac{\beta_3(1-\beta_3)}{(\beta_3-\alpha_3)(\beta_3-\alpha_4)(\beta_3-\beta_4)} \log \left(\frac{\beta_3-1}{\beta_3} \right) \\ & + \frac{\beta_4(1-\beta_4)}{(\beta_4-\alpha_3)(\beta_4-\alpha_4)(\beta_4-\beta_3)} \log \left(\frac{\beta_4-1}{\beta_4} \right) \end{aligned} \quad (4.19)$$

and

$$I_2 = f(\alpha_1) + f(\beta_1) - f(\alpha_2) - f(\beta_2) + f(\alpha_3) + f(\beta_3) - f(\alpha_4) - f(\beta_4) \quad (4.20)$$

with

$$\begin{aligned} f(\alpha) = & \frac{1-\alpha^2}{2} \log(1-\alpha) + \frac{\alpha^2}{2} \log(-\alpha) - \frac{\alpha}{2} \\ & - \frac{m_a^2}{x\delta} \left((1-\alpha) \log(1-\alpha) + \alpha \log(\alpha) \right) \end{aligned} \quad (4.21)$$

In writing these expressions we have made use of the notation

$$\frac{\alpha_i}{\beta_i} = \frac{b_i}{2} \pm \sqrt{\left(\frac{b_i}{2}\right)^2 - c_i} \quad (4.22)$$

with

$$\begin{aligned} b_1 &= b_2 = \frac{m_a^2 + x\delta}{m_B^2 + x\delta} \\ b_3 &= b_4 = c_4 = \frac{m_a^2}{m_B^2} \\ c_1 &= \frac{m_a^2}{m_B^2 + x\delta} \\ c_2 &= \frac{m_a^2 + \Lambda^2}{m_B^2 + x\delta} \\ c_3 &= \frac{m_a^2 + \Lambda^2}{m_B^2} \end{aligned} \quad (4.23)$$

These terms are essentially the roots of the quadratics appearing in the original expression.

4.2 Triviality

Recall the equation which defines the renormalised coupling constant in terms of the remaining logarithmically divergent part of the mass correction

$$\tilde{\lambda}^2 = \frac{\lambda^2}{(1 + \lambda^2(I_B^{\log} + I_b^{\log}))} \quad (4.24)$$

Provided that we regulate the $I_B^{\log}$ and $I_b^{\log}$ terms, with, for example, a cutoff on the transverse momentum integration, we can rewrite this equation to give us the bare coupling in terms of the renormalised coupling

$$\lambda^2 = \frac{\tilde{\lambda}^2}{(1 - \tilde{\lambda}^2(I_B^{\log} + I_b^{\log}))} \quad (4.25)$$

Again it should be noted that this is only possible when the divergent terms are regulated, since then the renormalisation is a finite one. The form of this equation is interesting since it implies that for any finite value of $(I_B^{\log} + I_b^{\log})$ there will be a maximum value of $\tilde{\lambda}$ above which λ^2 will be negative, and hence λ imaginary.

An imaginary bare coupling is not acceptable because it is the bare coupling which appears in the Lagrangian, and hence the Hamiltonian. With an imaginary coupling constant the Hamiltonian would not be Hermitian. The only consistent way that we could allow the cutoff to go to infinity would be to insist that the coupling went to zero. The coupling constant does go to zero at high momentum scales in theories possessing asymptotic freedom,

and so this is not a concern in such theories. In the case we are considering however, we do have to be concerned about this possibility of an imaginary bare coupling constant. This means that for a given cutoff, for example, we have a constraint on the largest acceptable renormalised coupling constant. If the coupling constant goes to zero the theory becomes one of free particles, and so is trivial in this sense. Hence the constraint is called the ‘triviality constraint’. This behaviour was first observed in the Lee model (see for example Ref [18]).

The existence of this constraint means that we will have to find an explicit expression for $(I_B^{\log}$ and $I_b^{\log})$. We have already obtained (Equation 4.8)

$$I_B^{\log} = \frac{1}{4(2\pi)^2} \int_0^1 dz z \log \left(1 + \frac{\Lambda^2}{(z^2 m_B^2 + (1-z)m_a^2)} \right) \quad (4.26)$$

The ‘log’ term can easily be split into the difference of the logarithms of two quadratics in ‘z’. These quadratics can be factorised and the integrand further split into terms proportional to the logarithm of terms linear in ‘z’. These can be straightforwardly integrated to give

$$I_B^{\log} = \frac{1}{4(2\pi)^2} [\beta(\alpha(\Lambda)) + \beta(\alpha^*(\Lambda)) + \beta(\alpha(0)) + \beta(\alpha^*(0))] \quad (4.27)$$

where

$$\beta(\alpha) = \frac{1}{2}(1 - \alpha^2) \log(1 - \alpha) + \frac{\alpha^2}{2} \log(-\alpha) \quad (4.28)$$

and

$$\alpha(\Lambda) = \frac{m_a^2}{2m_B^2} + \sqrt{\left(\frac{m_a^2}{2m_B^2}\right)^2 - \left(\frac{m_a^2 + \Lambda^2}{m_B^2}\right)} \quad (4.29)$$

There exists a similar expression for I_b^{log} , differing only by $m_H \rightarrow m_b$. The α term is invariably complex, but this presents no difficulty in practice.

This triviality condition turns out to be quite a stringent one, and as we will see in Chapter 7 actually permits only one bound state.

Chapter 5

Asymptotic Behaviour

At this point we have rendered the mass correction terms finite, using a coupling constant renormalisation, and as a consequence there are no ‘explicit’ divergences remaining in the bound state equation. At first sight it appears that we should be able to proceed to solve the equation for the bound state wavefunctions, but when this is attempted it is found that we still encounter divergence problems.

The reason that the equation is more badly behaved than it appears, lies in the transverse momentum dependence of free particle spinors when expressed in Light Front variables. That this dependence is different from the usual equal time case can be demonstrated from a consideration of the Lorentz transformation properties of the spinors.

We know that

$$\bar{u}(p)\gamma^\mu u(p) = u^\dagger(p)\gamma^0\gamma^\mu u(p) \sim p^\mu \quad (5.1)$$

Writing

$$u^\dagger(p)u(p) = u^\dagger(p)(\Lambda^+ + \Lambda^-)u(p) = u^{(+)\dagger}(p)u^{(+)}(p) + u^{(-)\dagger}(p)u^{(-)}(p) \quad (5.2)$$

with $\Lambda^\pm = \frac{1}{\sqrt{2}}\gamma^0\gamma^\pm$, and using the fact that

$$u^{(\pm)\dagger}(p)u^{(\pm)}(p) = \frac{1}{\sqrt{2}}u^\dagger(p)\gamma^0\gamma^\pm u(p) \sim p^\pm \quad (5.3)$$

we can see that since $p^- = (m^2 + p_\perp^2)/2p^+$, we must have

$$u(p) \sim p_\perp \quad (\sim \sqrt{p_\perp} \text{ in } E.T.) \quad (5.4)$$

for large $p_\perp$. Explicit forms for the spinors are given in Appendix A.

With this transverse momentum dependence in the spinors, a naive power counting analysis reveals that there may be divergences remaining. In order to establish this with more certainty, we will examine the asymptotic form of the solutions of Equation 4.12 in the same spirit as, for example, Dalitz, Sundaresan and Bethe [19].

As a first step towards obtaining an explicit representation of Equation 4.12, we expand the amplitude $C_0(x, \vec{p}_{B\perp}; \alpha, \beta) \equiv C_0(x, p_{B\perp}, \phi_B; \alpha, \beta)$ in terms of its azimuthal angular variable, ϕ_B , in which we expect it to be periodic. Prior to doing this however we will redefine

$$C_0(x, p_{B\perp}, \phi_B; \alpha, \beta) \rightarrow e^{i(\alpha+\beta)\phi_B} C_0(x, p_{B\perp}, \phi_B; \alpha, \beta) \quad (5.5)$$

with $\alpha, \beta = \pm 1/2$. The reason for this transformation will become clear presently. Now we expand C_0 (we will drop the suffix '0' in what follows),

which will still be periodic in ϕ_B , as

$$C(x, p_{B\perp}, \phi_B; \alpha, \beta) = \sum_{m=-\infty}^{\infty} C_m(x, p_{B\perp}; \alpha, \beta) e^{im\phi_B} \quad (5.6)$$

Next we substitute for the spinors, using the explicit forms given in Appendix A, and group the resulting terms into a matrix form, using a vector representation for the amplitude

$$\vec{C}_m = \begin{pmatrix} C_m(+\frac{1}{2}, +\frac{1}{2}) \\ C_m(+\frac{1}{2}, -\frac{1}{2}) \\ C_m(-\frac{1}{2}, +\frac{1}{2}) \\ C_m(-\frac{1}{2}, -\frac{1}{2}) \end{pmatrix} \quad (5.7)$$

We find, for the spinor terms (omitting normalisation factors)

$$-(\underline{\Gamma}^{(+)} e^{i(\phi_B - \phi_q)} + \underline{\Gamma}^{(0)} + \underline{\Gamma}^{(-)} e^{-i(\phi_B - \phi_q)}) \quad (5.8)$$

with

$$\begin{aligned} \underline{\Gamma}^{(+)} &= \begin{pmatrix} A\bar{A} & A\bar{B} & B\bar{A} & B\bar{B} \\ A\bar{C} & 0 & B\bar{C} & 0 \\ C\bar{A} & C\bar{B} & 0 & 0 \\ C\bar{C} & 0 & 0 & 0 \end{pmatrix} \\ \underline{\Gamma}^{(-)} &= \begin{pmatrix} 0 & 0 & 0 & C\bar{C} \\ 0 & 0 & C\bar{B} & -C\bar{A} \\ 0 & B\bar{C} & 0 & -A\bar{C} \\ B\bar{B} & -B\bar{A} & -A\bar{B} & A\bar{A} \end{pmatrix} \\ \underline{\Gamma}^{(0)} &= \begin{pmatrix} 0 & A\bar{C} & C\bar{A} & B\bar{C} + C\bar{B} \\ A\bar{B} & -A\bar{A} & B\bar{B} + C\bar{C} & -B\bar{A} \\ B\bar{A} & B\bar{B} + C\bar{C} & -A\bar{A} & -A\bar{B} \\ B\bar{C} + C\bar{B} & -C\bar{A} & -A\bar{C} & 0 \end{pmatrix} \end{aligned} \quad (5.9)$$

where

$$A = m_B(x - z)$$

$$\begin{aligned}
B &= -xq_{\perp} \\
C &= zp_{B\perp} \\
\bar{A} &= -m_b(x-z) \\
\bar{B} &= (1-x)q_{\perp} \\
\bar{C} &= -(1-z)p_{B\perp}
\end{aligned} \tag{5.10}$$

Since the only other other angular dependence in the kernel comes from $\vec{p}_{B\perp} \cdot \vec{q}_{\perp} = p_{B\perp} q_{\perp} \cos(\phi_B - \phi_q)$ terms in the denominators, it can be seen that the transformation used on the amplitude (Eqn 5.5) has had the effect of making the kernel depend on the azimuthal angles through $\phi_B - \phi_q$ only. Given this fact, and the the periodicity of the integrand, we can easily see that the equation is diagonal in 'm', with the right hand side taking the form

$$\begin{aligned}
& -\frac{2\tilde{\lambda}^2}{(2\pi)^2} m_B m_b \frac{1}{2\pi} \int_0^{2\pi} d\phi_B \int dq_{\perp} q_{\perp} \int_0^1 \frac{dz}{(x-z)} \frac{1}{[x(1-x)z(1-z)]^{\frac{1}{2}}} \\
& \times \frac{1}{\Pi N} (\underline{\Gamma}^{(+)} e^{-i(m-1)\phi_B} + \underline{\Gamma}^{(0)} e^{-im\phi_B} + \underline{\Gamma}^{(-)} e^{-i(m+1)\phi_B}) \\
& \times \left[\frac{\theta(z-x)}{\bar{D}_a} - \frac{\theta(x-z)}{\bar{D}_b} \right] \bar{C}_m(z, q_{\perp})
\end{aligned} \tag{5.11}$$

after shifting ϕ_B so as to remove the ϕ_q dependence, and performing the, now trivial, ϕ_q integration.

The fact that the equation has become diagonal in 'm' can be explained by noticing that because of the transformation of the amplitude (Eqn 5.5), 'm' is now the eigenvalue of the third component of the total angular momentum,

J_3 , rather than that of the orbital angular momentum, L_3 , (which it would have been if we had expanded the original amplitude). Thus we are seeing a consequence of the azimuthal symmetry which is still explicit in light front variables.

In order to facilitate the asymptotic analysis some further transformations will be performed. First we absorb the $(x - z)$ factor in the denominator into $\tilde{D}_a$ and $\tilde{D}_b$. We then isolate the angular dependence in the denominator by defining A_a and A_b through

$$\begin{aligned}\tilde{D}_a &= -2p_{B\perp}q_{\perp} (A_a - \cos(\phi_B)) \\ \tilde{D}_b &= 2p_{B\perp}q_{\perp} (A_b - \cos(\phi_B))\end{aligned}\quad (5.12)$$

Explicitly

$$\begin{aligned}A_a &= \frac{1}{2p_{B\perp}q_{\perp}} \left(\frac{(z-x)}{x} m_B^2 + \frac{(z-x)}{(1-z)} m_b^2 + m_a^2 + \frac{z}{x} p_{B\perp}^2 + \frac{(1-x)}{(1-z)} q_{\perp}^2 \right. \\ &\quad \left. - (z-x)M^2 \right) \\ A_b &= \frac{1}{2p_{B\perp}q_{\perp}} \left(\frac{(x-z)}{z} m_B^2 + \frac{(x-z)}{(1-x)} m_b^2 + m_a^2 + \frac{x}{z} q_{\perp}^2 + \frac{(1-z)}{(1-x)} p_{B\perp}^2 \right. \\ &\quad \left. - (x-z)M^2 \right)\end{aligned}\quad (5.13)$$

where $A_a, A_b > 1$ for a bound state.

The angular integration can now be performed (by contour integration for example) to give, for the right hand side of the equation

$$-\frac{\bar{\lambda}^2}{2(2\pi)^2} \int dq_{\perp} q_{\perp} \int_0^1 \frac{dz}{(x-z)x(1-x)z(1-z)}$$

$$\begin{aligned}
& \times \left[\frac{\theta(z-x)}{\sqrt{A_a^2-1}} \left(\underline{\Gamma}^{(+)} F_a^{|m-1|} + \underline{\Gamma}^{(0)} F_a^{|m|} + \underline{\Gamma}^{(-)} F_a^{|m+1|} \right) + \right. \\
& \left. \frac{\theta(x-z)}{\sqrt{A_b^2-1}} \left(\underline{\Gamma}^{(+)} F_b^{|m-1|} + \underline{\Gamma}^{(0)} F_b^{|m|} + \underline{\Gamma}^{(-)} F_b^{|m+1|} \right) \right] \frac{1}{2p_{B\perp} q_{\perp}} \vec{C}_m(z, q_{\perp})
\end{aligned} \tag{5.14}$$

where $F_a = 1/(A_a + \sqrt{A_a^2 - 1})$ (with a similar expression for F_b), and where we have used the explicit expression for the spinor normalisation factors

$$\Pi N = 4m_B m_b [x(1-x)z(1-z)]^{\frac{1}{2}} \tag{5.15}$$

We can remove the $x(1-x)z(1-z)$ factors from the kernel, through the definition

$$\vec{\psi}_m = \frac{1}{x(1-x)} \vec{C}_m \tag{5.16}$$

In order to establish the asymptotic behaviour we are going to examine the contributions to a particular amplitude, ψ_0 , from regions where either $p_{B\perp} \gg q_{\perp}$ or $q_{\perp} \gg p_{B\perp}$. This being the case we will consider terms of the form $p_{B\perp} q_{\perp}$ small compared to $p_{B\perp}^2$ or $q_{\perp}^2$. From inspection of Eqn 5.9, it is clear that if either $p_{B\perp}$ or $q_{\perp}$ is large, the Γ 's are dominated by the $B\bar{B}$ and $C\bar{C}$ terms. Further, for $m = 0$ the F_a and F_b terms asymptotically suppress the contribution of $\Gamma^{(+)}$ and $\Gamma^{(-)}$ by a factor of $\text{Max}(p_{B\perp}, q_{\perp})$, leaving the kernel dominated by $\Gamma^{(0)}$.

Taking this limit we obtain the equation

$$\begin{aligned}
& (x(1-x))^2 (\delta + \tilde{\lambda}^2 \Delta) \vec{\psi}_m(x, p_{B\perp}) \\
&= \frac{\tilde{\lambda}^2}{2(2\pi)^2} \int d\mathbf{q}_\perp q_\perp \int_0^1 dz [\theta(z-x)x(1-z) + \theta(x-z)z(1-x)] \\
&\quad \times \underline{\Gamma}^{(0)'} \vec{\psi}_m(z, q_\perp)
\end{aligned} \tag{5.17}$$

where

$$\underline{\Gamma}^{(0)'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.18}$$

Only two of the components are non-zero in this limit, and by redefining

$$\begin{aligned}
\psi_0^S &= \frac{1}{\sqrt{2}} \left(\psi_0(+\frac{1}{2}, -\frac{1}{2}) + \psi_0(-\frac{1}{2}, +\frac{1}{2}) \right) \\
\psi_0^A &= \frac{1}{\sqrt{2}} \left(\psi_0(+\frac{1}{2}, -\frac{1}{2}) - \psi_0(-\frac{1}{2}, +\frac{1}{2}) \right)
\end{aligned} \tag{5.19}$$

we can diagonalise $\underline{\Gamma}^{(0)'}$ which decouples the equations. The $\underline{\Gamma}^{(0)'}$ now contributes a 1 for the ψ^S equation and a -1 for the ψ^A equation.

The left hand side of Equation 5.17 asymptotically goes as $p_{B\perp}^2$. Thus considering either of the amplitudes, generically denoted by ψ , we find

$$\begin{aligned}
\psi(x, p_{B\perp}) &\sim \frac{\tilde{\lambda}^2}{p_{B\perp}^2} \int_0^\infty d\mathbf{q}_\perp q_\perp \int_0^1 dz [\theta(z-x)x(1-z) + \theta(x-z)z(1-x)] \\
&\quad \times \psi(z, q_\perp)
\end{aligned} \tag{5.20}$$

At this point it is clear that $\psi \sim p_{B\perp}^{-2}$, since the integral term is independent of $p_{B\perp}$. This momentum dependence is not enough to keep the integral

finite. Had we attempted to regulate the upper limit with a cutoff, q_{max} , we would have found that solutions are possible for any finite q_{max} , but not for $q_{max} = \infty$. Thus no solution is possible for the $m = 0$ case.

For different values of 'm' the situation is different. If we consider $m = 1$, we find that $\Gamma^{(0)}$ and $\Gamma^{(-)}$ are suppressed, and the problematic terms are the $B\bar{B}$ and $C\bar{C}$ in $\Gamma^{(+)}$ (the reverse is true for $m = -1$). For $|m| \geq 2$ all the Γ 's are suppressed, and there seems to be no obstacle to solving for the amplitudes. This association of the divergence problems of the equation with the behaviour of the (unsuppressed) $B\bar{B}$ and $C\bar{C}$ terms is made more interesting when we note the spin components that they correspond to.

For $m = 0$ (or $J_3 = 0$) these terms couple the spin states $(+\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, +\frac{1}{2})$, i.e. $S_3 = 0$, (since $\Gamma^{(0)}$ dominates). For $m = \pm 1$ (or $J_3 = \pm 1$) they couple the states $(\pm\frac{1}{2}, \pm\frac{1}{2})$, i.e. $S_3 = \pm 1$ (since $\Gamma^{(\pm)}$ dominates) respectively. Thus in all cases the problematic terms seem to be related to $L_3 = 0$.

This analysis leads us to state that the bound state equation is well defined for states with $L_3 \neq 0$ and which are not coupled to states with $L_3 = 0$. A simple classical picture of states with $L_3 = 0$ would indicate that they have a zero impact parameter projection in the transverse plane, and consequently can become arbitrarily close. Thus it is for these states that we would expect any problem with high transverse momenta to manifest itself most clearly.

Chapter 6

Normalisation and Consistency Considerations

Recalling the starting point for the lowest order Tamm-Dancoff approximation, we expanded the bound state in the truncated Fock basis 3.2. Diagonalising the Hamiltonian within this truncated basis ultimately resulted in the bound state equation 4.12 for the amplitude C_0 . This equation is essentially an eigenvalue equation for C_0 . The eigenvalues (which are related to the coupling constant) can be unambiguously determined from this equation, but the eigenvectors (which represent the amplitude itself) are only determined up to a constant. Thus we are still left with the problem of the normalisation of the bound state amplitudes.

Ultimately this normalisation is fixed by the normalisation of the bound state itself

$$\langle \Psi(P') | \Psi(P) \rangle = \delta(P^{+'} - P^+) \delta^2(\vec{P}'_{\perp} - \vec{P}_{\perp}) \quad (6.1)$$

or, using $p \equiv (p^+, \vec{p}_\perp) = (xP^+, \vec{p}_\perp)$ as a notational convenience

$$\langle \Psi(P') | \Psi(P) \rangle = \delta(P' - P) \quad (6.2)$$

From this condition and the fact that the expansion of $|\Psi\rangle$ includes both C_0 and C_1 , we see that the normalisation of the two amplitudes cannot be separated.

Inserting the expansion for $|\Psi\rangle$ gives, symbolically (ignoring spin for the moment)

$$\begin{aligned} \langle \Psi(P') | \Psi(P) \rangle &= \delta(P' - P) \\ &\times \left[\int dp \theta(P^+ - p^+) |C_0(p, P - p)|^2 + \right. \\ &\quad \left. \int dp dq \theta(P^+ - p^+ - q^+) |C_1(p, P - p - q, q)|^2 \right] \end{aligned} \quad (6.3)$$

where the positivity of the individual p^+, q^+ is assumed. For future reference we denote

$$\begin{aligned} N_0^2 &= \int dp \theta(P^+ - p^+) |C_0(p, P - p)|^2 \\ N_1^2 &= \int dp dq \theta(P^+ - p^+ - q^+) |C_1(p, P - p - q, q)|^2 \end{aligned} \quad (6.4)$$

and the normalisation condition becomes

$$N_0^2 + N_1^2 = 1 \quad (6.5)$$

Thus consideration of the normalisation of the bound state necessarily involves the normalisation of both C_0 and C_1 and leads us to consider the

consistency of the Tamm-Dancoff approximation. Expanding the bound state in terms of a Fock basis is in itself just an alternative representation. However the validity of the Tamm-Dancoff approach rests on the assumption not only that the series converges, but that it converges quite rapidly, with the lowest Fock space sectors being the leading terms. If this is not the case, and in fact the higher sectors make a significant contribution, then the Tamm-Dancoff method cannot be expected to be a useful approximation method.

We are working in the lowest non-trivial Tamm-Dancoff approximation, and thus are not able to investigate the general problem of the rate of convergence. However we can obtain some information from the relative sizes of N_0^2 and N_1^2 , since these are the probabilities of finding the two fermion and two fermion/one meson states in the bound state. We require at least that $N_1 < N_0$. Qualitatively we expect that the smaller the ratio N_1/N_0 , the better the approximation.

The expression for N_1^2 naturally involves C_1 which is at present unknown. We can, however, relate it to C_0 through Equation 3.11 which resulted from the coupling of the high sector to the low sector.

Inserting this expression for C_1 , and introducing the additional notation $\tilde{p} = ((1-x)P^+, -\vec{p}_\perp)$ we obtain

$$N_1^2 = P^+ \frac{2\lambda^2}{(2\pi)^3} \sum_{r,s,\alpha,\beta} \int d^2 p_\perp d^2 q_\perp \int_0^1 dx dz \frac{1}{D'^2}$$

$$\begin{aligned}
& \times \left[\frac{\bar{u}_s(\tilde{q}, m_b) \Gamma u_\alpha(\tilde{p}, m_b)}{[(1-z)(1-x)(z-x)]^{\frac{1}{2}}} m_b C_0(p; r, \alpha) + \right. \\
& \quad \left. \frac{\bar{u}_r(p, m_B) \Gamma u_\alpha(q, m_B)}{[xz(z-x)]^{\frac{1}{2}}} m_B C_0(q; \alpha, s) \right] \\
& \times [H.C.(\alpha \rightarrow \beta)] \tag{6.6}
\end{aligned}$$

where

$$D' = M^2 - \frac{m_B^2}{x} - \frac{m_b^2}{(1-z)} - \frac{m_a^2}{(z-x)} - \frac{zp_\perp^2}{x(z-x)} - \frac{(1-x)q_\perp^2}{(1-z)(z-x)} + \frac{2\tilde{p}_\perp \cdot \tilde{q}_\perp}{(z-x)}$$

Expanding this equation produces four terms, on two of which we can use spinor completeness relations to obtain (relabelling spin indices)

$$\begin{aligned}
& P^+ \frac{\lambda^2}{(2\pi)^3} \sum_{r,s} \int d^2 p_\perp d^2 q_\perp \int_0^1 dx dz \frac{1}{D'^2} \\
& \left[\frac{(\tilde{p} \cdot \tilde{q} - m_b^2)}{(1-z)(1-x)(z-x)} |C_0(p; r, s)|^2 + \frac{(p \cdot q - m_B^2)}{xz(z-x)} |C_0(q; r, s)|^2 \right] \tag{6.7}
\end{aligned}$$

In writing the 4-vector products $\tilde{p} \cdot \tilde{q}$ and $p \cdot q$ we have implicitly assumed the extension of the vectors to four vectors, using the mass shell conditions

$$p^\mu p_\mu = p \cdot p = m_B^2 \quad \text{and} \quad \tilde{p}^\mu \tilde{p}_\mu = \tilde{p} \cdot \tilde{p} = m_b^2 \tag{6.8}$$

The remaining (cross) terms are complex conjugates of each other, and so we need only consider one of them explicitly.

We have, within one such cross term the spinor product

$$\bar{u}_s(\tilde{q}, m_b) \Gamma u_\alpha(\tilde{p}, m_b) \bar{u}_\beta(q, m_B) \Gamma u_r(p, m_B) \tag{6.9}$$

In order to represent these cross terms we perform the same phase transformation as in Equation 5.5, and then introduce a vector representation (Equation 5.7) for the spin components of the amplitudes. Then using the representation of the spinors given by Equation A.7 we find that the spinor product can be written as

$$\frac{1}{4m_B m_b} \frac{1}{[x(1-x)z(1-z)]^{\frac{1}{2}}} (\underline{\Gamma}^{(+)} e^{i(\phi_p - \phi_q)} + \underline{\Gamma}^{(0)} + \underline{\Gamma}^{(-)} e^{-i(\phi_p - \phi_q)}) \quad (6.10)$$

with

$$\begin{aligned} \underline{\Gamma}^{(+)} &= \begin{pmatrix} 0 & 0 & 0 & -C\bar{C} \\ 0 & 0 & -C\bar{B} & C\bar{A} \\ 0 & -B\bar{C} & 0 & -A\bar{C} \\ -B\bar{B} & B\bar{A} & -A\bar{B} & A\bar{A} \end{pmatrix} \\ \underline{\Gamma}^{(-)} &= \begin{pmatrix} A\bar{A} & A\bar{B} & -B\bar{A} & -B\bar{B} \\ A\bar{C} & 0 & -B\bar{C} & 0 \\ -C\bar{A} & -C\bar{B} & 0 & 0 \\ -C\bar{C} & 0 & 0 & 0 \end{pmatrix} \\ \underline{\Gamma}^{(0)} &= \begin{pmatrix} 0 & A\bar{C} & -C\bar{A} & -(B\bar{C} + C\bar{B}) \\ A\bar{B} & -A\bar{A} & -(B\bar{B} + C\bar{C}) & B\bar{A} \\ -B\bar{A} & -(B\bar{B} + C\bar{C}) & -A\bar{A} & -A\bar{B} \\ -(B\bar{C} + C\bar{B}) & C\bar{A} & -A\bar{C} & 0 \end{pmatrix} \end{aligned} \quad (6.11)$$

and

$$A = m_B(z - x)$$

$$B = zp_{\perp}$$

$$C = -xq_{\perp}$$

$$\begin{aligned}
\bar{A} &= m_B(x - z) \\
\bar{B} &= (1 - z)p_\perp \\
\bar{C} &= -(1 - x)q_\perp
\end{aligned} \tag{6.12}$$

This representation is similar to that given in Equation 5.9, differing only due to the different organisation of spin components.

Since we are now dealing exclusively with C_0 we will drop the suffix '0' and expand the amplitude in terms of its azimuthal angular dependence (Equation 5.6).

As a result of the phase transformation Equation 5.5, the azimuthal angular dependence of the spinor product and denominator terms is entirely of the form $\phi_p - \phi_q$, and by shifting $\phi_p \rightarrow \phi_p - \phi_q$ we can trivially perform the ϕ_q integration. The result of these manipulations is the expression

$$\begin{aligned}
N_1^2 = P^+ \sum_m \int dp_\perp dq_\perp dx dz d\phi_p p_\perp q_\perp & \frac{\lambda^2}{(2\pi)^2} \frac{1}{D'^2(\cos(\phi_p))} \\
& \left[\frac{(\tilde{p} \cdot \tilde{q} - m_b^2)}{(1 - z)(1 - x)(z - x)} \tilde{C}_m^\dagger(p) \tilde{C}_m(p) + \frac{(p \cdot q - m_B^2)}{xz(z - x)} \tilde{C}_m^\dagger(q) \tilde{C}_m(q) \right. \\
& \left. - \frac{1}{x(1 - x)z(1 - z)(z - x)} \right. \\
& \left. \tilde{C}_m^\dagger(q) \left(\underline{\Gamma}^{(+)} e^{i(m+1)\phi_p} + \underline{\Gamma}^{(0)} e^{im\phi_p} + \underline{\Gamma}^{(-)} e^{i(m-1)\phi_p} \right) \tilde{C}_m(p) + C.C. \right]
\end{aligned} \tag{6.13}$$

where we have showed the ϕ_p dependence of the denominator.

Performing a similar, although much more straightforward, analysis for

N_0 gives

$$N_0^2 = P^+ 2\pi \sum_m \int dp_\perp dx p_\perp \vec{C}_m^\dagger(p) \vec{C}_m(p) \quad (6.14)$$

The fact that both N_0 and N_1 reduce to a sum over 'm' (which is the eigenvalue of J_3 , suggests that we could define

$$N_i^2 = \sum_m N_i^{(m)2} \quad (6.15)$$

and check the validity of the approximation by comparing $N_0^{(m)}$ and $N_1^{(m)}$ for given 'm'.

The remaining azimuthal angular integration in Equation 6.13 can be performed relatively straightforwardly (using contour integration for example), leaving a four dimensional integration. This is the form which has been used to calculate the normalisation factors for the amplitudes to be shown in Chapter 7.

Chapter 7

Numerical Results

The bound state equation (Equation 4.12) is an integral equation which, for the lack of an analytic solution, needs to be solved by numerical methods. In examining the asymptotic behaviour the azimuthal integration was performed, at the expense of expanding the amplitudes into a series in J_3 . The equation thus obtained (Equation 5.14) is only a two dimensional integral equation, and becomes the starting point for numerical work. Details of the numerical methods employed are contained in Appendix C and here we will merely outline them, and present the results obtained.

The basic method employed to solve the integral equation is to perform the longitudinal, ' x ', and transverse, $q_\perp$, integrations by Gaussian quadrature. This procedure transforms the integral equation into a matrix equation which can be diagonalised. Due to the nature of the equation it is easier to fix the bound state mass, ' M ', and solve for the coupling as an eigenvalue.

As a preliminary, Figure 7.1 shows the variation of the maximum coupling $\alpha = \frac{\lambda^2}{4\pi}$ allowed by the triviality constraint as the cutoff is varied. From Equation 4.27 we can see that this constraint depends only on the fermion and meson masses and the cutoff. Thus in particular the same maximum α applies for all the different J_3 components of a given amplitude.

In Figure 7.2 we show the ratio of the coupling for a typical bound state to the maximum coupling allowed by triviality. There are several features to note from this figure. The first is the observation that for a cutoff greater than ≈ 100 no bound states are possible. This is not surprising given the asymptotic behaviour of the kernel discussed earlier, and merely serves to remind us of the necessity of a cutoff. A second, more surprising, feature is that for small cutoffs there is a problem also. This seems to contradict ones intuition that in this regime one should be approaching nonrelativistic behaviour. The most likely explanation for this problem is that, for calculational expedience, we have employed a cutoff on the transverse momenta alone. Thus reducing the cutoff does not place any restriction on the longitudinal momenta. Thus we would not expect to reach a nonrelativistic regime in this manner. This seems to support quite strongly the use of a regulator, such as that introduced by Equation 2.4, which acts on all momenta. With such a regulator it seems reasonable to expect that we would not have this low cutoff behaviour. Assuming then that this is an artifact of the cutoff

Figure 7.1

Plot of the maximum coupling allowed by triviality (α_T) vs. the cutoff.
(data is for $m_B = m_b = 1$, $m_a = 0.1$)

procedure, we take the Figure 7.2 as indicating that the optimum cutoff is at a value of ≈ 40 .

We now move to the amplitudes themselves. The fact that the bound state equation still possesses explicit azimuthal angular symmetry manifests itself in the fact that we can solve for equal J_3 multiplets. Figures 7.3 and 7.4 show the four spin components of the of a two fermion amplitude corresponding to $J_3 = 0$ and $M = 1.998m_B$ (we use $m_B = m_b = 1$ throughout the numerical work, and denote $p_{B\perp}$ by Btr). We show also the value of the normalisation constant N_1^2 which can be calculated for this state using the techniques described in Chapter 6. The values of these normalisation constants indicate that the contribution of the two fermion/one meson state is only about 20% of that of the two fermion state. This increases confidence in the entire procedure. A point to note is that this multiplet is dominated by the $(+\frac{1}{2}, -\frac{1}{2})$ (or $(+, -)$) and $(-\frac{1}{2}, +\frac{1}{2})$ (or $(-, +)$) spin index components. These components also appear to be identical (that they are actually identical can be indicated by an analysis of the integral equation itself). To the extent that we can use the interpretation of the spin indices as spin components we can see that if we had chosen to work with the linear combinations

$$\begin{aligned} C^S &= \frac{1}{\sqrt{2}} (C(+, -) + C(-, +)) \\ C^A &= \frac{1}{\sqrt{2}} (C(+, -) - C(-, +)) \end{aligned} \tag{7.1}$$

Figure 7.2

Plot of the ratio of the bound state coupling (α) to the maximum allowed by the triviality constraint (α_T). The triviality constraint is only satisfied for the range of cutoffs where this ratio is less than one. (Data is for $M = 1.998$, $m_B = m_b = 1$, $m_a = 0.1$, $J_3 = 0$)

$\{C0++\}$ $\{C0+-\}$

Figure 7.3

Two fermion bound state amplitudes for $J_3 = 0$, $M = 1.998$, $\alpha = 1.4773$.
 $(+,+)$ and $(+,-)$ spin components. ($m_B = m_b = 1$, $m_a = 0.1$, cutoff = 40, $N_1^2 = 0.818$)

$\{C0-+\}$

 $\{C0--\}$

Figure 7.4

Two fermion bound state amplitudes for $J_3 = 0$, $M = 1.998$, $\alpha = 1.4773$.
 (-,+) and (-,-) spin components. ($m_B = m_b = 1$, $m_a = 0.1$, cutoff = 40, $N_1^2 = 0.818$)

we would have found that C^S dominated over C^A . This indicates that the bound state shown is related to the spin zero component of the $J = 1, J_3 = 0$ triplet, rather than the $J = 0, J_3 = 0$ singlet.

The state shown in Figures 7.3 and 7.4 is in fact the only bound state allowed by the triviality constraint for the given parameter values. This seems to be a general feature. There is at most one $J_3 = 0$ bound state for any value of the bound state mass, M . We will return to this point in the discussion. Figures 7.5 and 7.6 show the $J_3 = 1$ sector of the state with the same mass. It is shown for completeness since the coupling constant required to produce it violates the triviality constraint. Unfortunately this has been found to be a general feature. All $J_3 = 1$ components that have been examined have been found to require couplings which are too large. It is not clear why this should be the case, although from the asymptotic analysis it is clear that there are terms in the kernel for $J_3 = 1$ which are comparable to those in the $J_3 = 0$ kernel. This is a matter which requires further investigation, but may only be resolved when a more consistent method is developed for dealing with the divergences in the kernel. This point is discussed further in the Discussion.

Presented also are the bound state amplitudes for the $J_3 = 0$ and $J_3 = 1$ with the bound state mass $M = 1.940$ (Figures 7.7, 7.8, 7.9 and 7.10). This value of the bound state mass is chosen since for the $J_3 = 0$ sector it is the most deeply bound state allowed by the triviality constraint. These

$\{C0++\}$

$\{C0+-\}$

Figure 7.5

Two fermion bound state amplitudes for $J_3 = 1$, $M = 1.998$, $\alpha = 1.7754$.
 $(+,+)$ and $(+,-)$ spin components. ($m_B = m_b = 1$, $m_a = 0.1$, cutoff = 40, $N_1^2 = 0.795$)

Figure 7.6

Two fermion bound state amplitudes for $J_3 = 1$, $M = 1.998$, $\alpha = 1.7754$.
 $(-,+)$ and $(-,-)$ spin components. ($m_B = m_b = 1$, $m_a = 0.1$, cutoff = 40, $N_1^2 = 0.795$)

amplitudes are superficially very similar to those for $M = 1.998$. The only significant difference is that they are more extended in both the longitudinal and transverse momenta. Thus there is an increased probability of finding a constituent with a high momentum. Intuitively this seems reasonable since from the uncertainty principle we would expect that a more tightly bound a state would have components with higher momenta.

In Figures 7.11, 7.12 and 7.13 we show the effect of the variation of the coupling, α , on the bound state mass, for a boson mass of m_H , $0.1m_H$ and $0.01m_H$ respectively. In all cases there was only one bound state which satisfied the triviality constraint. The range of coupling shown is from that which produces a bound state mass of $1.998m_H$ (which is chosen simply as a mass close to threshold), up to the maximum value allowed by triviality. Much as one might expect, the bound state mass decreases (state becomes more tightly bound) as the coupling is increased. It is clear from these figures that there is not much sensitivity to the boson mass. As a final check on the numerical procedures we have employed the Lanczos algorithm (see Appendix C) to check the sensitivity of these results to the order of the Gaussian quadrature. The same order of quadrature is employed for both integrations, and is thus just generically referred to as the number of gaussian points. Figure 7.14 shows this dependence for the state shown in Figure 7.3. We find that the overall change over the range shown is less than 1%.

$\{C0++\}$ $\{C0+-\}$

Figure 7.7

Two fermion bound state amplitudes for $J_3 = 0$, $M = 1.940$, $\alpha = 1.5033$.
 (+,+) and (+,-) spin components. ($m_B = m_b = 1$, $m_a = 0.1$, cutoff = 40, $N_1^2 = 0.817$)

$\{C0-+\}$ $\{C0--\}$

Figure 7.8

Two fermion bound state amplitudes for $J_3 = 0$, $M = 1.940$, $\alpha = 1.5033$.
 $(-,+)$ and $(-,-)$ spin components. ($m_B = m_b = 1$, $m_a = 0.1$, cutoff = 40, $N_1^2 = 0.817$)

$\{C0++\}$ $\{C0+-\}$

Figure 7.9

Two fermion bound state amplitudes for $J_3 = 1$, $M = 1.940$, $\alpha = 1.7904$.
 $(+,+)$ and $(+,-)$ spin components. ($m_B = m_b = 1$, $m_a = 0.1$, cutoff = 40, $N_1^2 = 0.802$)

$\{C0-+\}$ $\{C0--\}$

Figure 7.10

Two fermion bound state amplitudes for $J_3 = 1$, $M = 1.940$, $\alpha = 1.7904$.
 $(-,+)$ and $(-,-)$ spin components. ($m_B = m_b = 1$, $m_a = 0.1$, cutoff = 40, $N_1^2 = 0.802$)

Figure 7.11

Plot of the bound state mass (M) vs. the coupling (α) for $m_a = 1$. The graph is terminated at the point where the coupling exceeds the maximum coupling allowed by the triviality constraint. Note that the bound state threshold is $M = m_B + m_b = 2$.

(Data is for $m_B = m_b = 1, J_3 = 0$, cutoff = 40)

Figure 7.12

Plot of the bound state mass (M) vs. the coupling (α) for $m_a = 0.1$. The graph is terminated at the point where the coupling exceeds the maximum coupling allowed by the triviality constraint. Note that the bound state threshold is $M = m_B + m_b = 2$.

(Data is for $m_B = m_b = 1, J_3 = 0$, cutoff = 40)

Figure 7.13

Plot of the bound state mass (M) vs. the coupling (α) for $m_a = 0.01$. The graph is terminated at the point where the coupling exceeds the maximum coupling allowed by the triviality constraint. Note that the bound state threshold is $M = m_B + m_b = 2$.

(Data is for $m_B = m_b = 1, J_3 = 0$, cutoff = 40)

Figure 7.14

Plot of the coupling (α) vs. the number of gaussian points used for both the 'x' and transverse integration (note that dimension of matrix for 'n' gaussian points is $4n^2$). Variation is less than 1% for the range shown.
 (Data is for $M = 1.998$, $m_B = m_b = 1$, $m_a = 0.1$, $J_3 = 0$, cutoff = 40)

Chapter 8

General Considerations

We have seen that, when taken at face value, the bound state equation we have derived has no solutions for amplitudes with, or coupled to amplitudes with, $L_3 = 0$. This is somewhat unexpected given that the equation has the form one might expect for a bound state equation in the 'equal time' formalism.

The factor that seems to be responsible for this deviation from the behaviour we might expect, and more importantly for the bad behaviour of the kernel, is the transverse momentum dependence of the light front 'energy' for on-shell momenta

$$p^- = \frac{m^2 + p_\perp^2}{2p^+} \quad (8.1)$$

As discussed earlier, it is this behaviour which is responsible for the asymptotically linear dependence of the free particle spinors on the transverse momentum, which in turn is the cause of the high momentum dependence of

the diagonal terms in Eqn 5.9. It would seem to be of interest therefore, to try and gain some insight into the origin of this behaviour.

The difference between ‘equal time’ and ‘light front’ coordinates lies in the definition of the ‘time’ coordinate (and one of the space coordinates). Hornbostel [20] uses a coordinate system which interpolates between these two sets, and which will prove useful here. We define

$$\begin{pmatrix} x^+ \\ x^- \end{pmatrix} = \begin{pmatrix} \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \quad (8.2)$$

where we consider $\theta \in [\pi/2, \pi]$, the extremes corresponding to ‘light front’ and ‘equal time’ coordinates respectively. The transverse coordinates, $\vec{x}_\perp$, are unchanged.

Within this framework the mass-shell condition reads

$$\cos(\theta)(p_-^2 - p_+^2) + 2\sin(\theta)p_+p_- - p_\perp^2 = m^2 \quad (8.3)$$

For the ‘equal time’ case ($\theta = \pi$) we have

$$p_+^2 - p_-^2 = m^2 + p_\perp^2 \quad (8.4)$$

or

$$p_+ = \sqrt{m^2 + p_\perp^2 + p_-^2} \quad (8.5)$$

which, given that $p_+ = p^0$ gives us the usual asymptotic linear dependence on the transverse momenta.

For the general case we solve the mass-shell equation for $1/p_+$ (to avoid problems when $\cos(\theta) \rightarrow 0$). Denoting $\cos(\theta) = C$ and $\sin(\theta) = S$ we find

$$\frac{1}{p_+} = \frac{Sp_- \pm \sqrt{(Sp_-)^2 - [m^2 + p_\perp^2 + Cp_-^2]C}}{[m^2 + p_\perp^2 - Cp_-^2]} \quad (8.6)$$

For $\theta \in (\pi/2, \pi)$ we find that asymptotically $p_+ \sim p_\perp^2 / \sqrt{p_\perp^2} = p_\perp$. It is only when $\theta = \pi/2$ that the $\sim \sqrt{p_\perp^2}$ in the numerator of Eqn 8.6 is lost, and we recover $p_+ (\equiv p^-)$ as shown in Eqn 8.1 (the second root goes to zero, implying that p_+ goes to infinity).

Thus we have established that the high transverse momentum dependence only occurs exactly on the light front.

One of the primary motivations for using light front quantisation is the triviality of the vacuum. This triviality is due to the fact that p^+ and p^- are either both positive or both negative. If we define p^0 to be positive, it then follows that p^+ and p^- are both positive also. At a general interpolating angle this is not the case. For $\theta \in (\pi/2, \pi]$ we know that $\sin(\theta/2) > \cos(\theta/2)$. It follows that p^+ and p^- can be of the same, or different, sign depending on the value of p^3 , and therefore that it is possible to have disconnected vacuum diagrams which do not violate p^+ conservation. It is only when $\theta = \pi/2$ that we have the required behaviour, and hence the triviality of the vacuum.

It is therefore the case that we have either a non-trivial vacuum and

linear dependence of the 'energy' on the transverse momenta, or a trivial vacuum and a quadratic dependence of 'energy' on the transverse momenta. It thus appears as though it is the same feature of the light front coordinates which renders the vacuum trivial, that causes divergences in the bound state equation we have derived. This can be seen as a concrete manifestation of the fact that the subject of divergences, and hence renormalisation, is a very different one in the light front, as compared to the more usual equal time, formalism.

Chapter 9

Discussion and Conclusion

9.1 Discussion

The choice a Yukawa model with a pseudoscalar meson rather than a scalar meson was not an arbitrary one. The choice was motivated by the fact that the deuteron can be considered to be a two fermion (proton and neutron) state, bound by pion exchange (at least in low energy regimes). Given the fact that a pion is a pseudoscalar meson the intention was that we would have some physical data against which to compare our results. The comparison is not exact, since isospin has not been taken into account, but qualitative agreement might still be expected.

The ground state of the deuteron is known to be [21] predominantly a $J = 1$ spin triplet 3S_1 (with a small 3D_1 component). It is weakly bound, its binding energy being about 0.1% of the mass of its constituents. There are no known more deeply bound states.

In the numerical work we took $m_H = m_b = 1$, and thus to approximate the pion we should use a boson mass $m_a = 0.1$. The mass of the deuteron can then be approximated by $M = 1.998$. The amplitude calculated with these parameters is shown in Figures 7.3 and 7.4. Following the discussion in Chapter 7 we find that it is dominated by spin components that can be interpreted as the zero component of a spin triplet state. We also found only one state that was bound. This is an important feature, since, as mentioned above, only one bound state exists in nature.

It should be stressed that these identifications cannot be performed unambiguously since the actual angular momenta are not known. Thus the $J_3 = 0$ state could represent, for example, an admixture of states with various J . To establish the angular momenta would require the simultaneous solution of the Einstein equation 2.1, and the equation $J^2|\Psi\rangle = j(j+1)|\Psi\rangle$. This is a non-trivial task since $J^2 = J_3^2 + J_\perp^2$, and the $J_\perp$ are dynamic operators. This simultaneous solution has not been attempted.

A further obstacle in identifying the calculated bound state with a triplet spin state lies in the fact the $J_3 = 1$ state (shown in Figures 7.5 and 7.6) does not satisfy the triviality constraint. The cause of this failure is undoubtedly related to the divergences in the kernel. Treating these divergences by truncating only the transverse momenta was a calculational expedient, and it is clear that such a truncation does not respect rotational invariance, for

example.

If this was the only part of the treatment which violated Lorentz invariance, the obvious path would be to switch to a more covariant cutoff. In practice however not only have we discarded instantaneous terms in deriving the bound state equation, but the very act of truncating the Fock space violates Lorentz invariance. Thus what is needed is a more sophisticated approach for dealing with the kernel divergences. The nature of this approach is far from clear.

We can at least state that our results are not contradicted by the experimental data in that we see evidence of a unique, $J = 1$ spin triplet state.

Further, from Figure 7.12 we see that it is not possible to increase the binding energy of the state indefinitely. The triviality constraint precludes the existence of a solution with a mass below ≈ 1.94 . Thus we have a relatively small range of allowed bound state masses, and the fact that the mass we require lies within this range is gratifying. Subsequent to the completion of the majority of this work, we became aware of some work performed by Glazek et al [22] which is similar to the that presented above in many respects. These authors examine the Yukawa model in 3+1 dimensions with a scalar meson. The first point of note is that the triviality constraint is exactly the same in the scalar and pseudoscalar cases. The procedure used for renormalising the coupling constant is also identical to that used here,

although there are differences in the finite part of the mass correction term due to the difference in the nature of the meson.

The results obtained are broadly similar to those obtained here. Some differences do exist however. They find two bound states in the $J_3 = 0$ sector rather than one, and they find one bound state in the $J_3 = 1$ sector, where we found none. Another difference found is that as they reduced the meson mass they found more bound states. An argument proposed for this is that as meson mass decreases, the Yukawa potential becomes Coulomb like, hence more bound states would be expected (an infinite number in the limit). Our results indicate that the meson mass has some (small) effect on the mass of the bound state, but no effect on the number of bound states, which stays at one. In fact the data for a graph of bound state mass against ' α ' was calculated for a meson mass of $0.001m_H$, but was not included since it was indistinguishable from Figure 7.13. For a pseudoscalar meson the classical Yukawa potential is not applicable [12] in any regime, and thus there is no reason to expect the same behaviour.

We also note that Brodsky and Pauli [23, 24] have examined a bound state Tamm-Dancoff equation on the light front for Q.E.D. and have found similar divergences to those discussed here. This seems to confirm that the problems we have encountered are not restricted to the Yukawa type theory we have considered. These authors also proceed by imposing a cutoff on the transverse

momenta and comment on the fact that their numerical solutions seem quite stable against changes in this cutoff, provided that the coupling constant is kept small. This behaviour is not too surprising, since the divergences are quite weak (i.e. logarithmic).

The preprint by Glazek et al [22] also contains a possible approach for treating the divergences in the kernel in a more fundamental manner. They note that from power counting arguments there are many more terms with the correct canonical dimensions to appear in the Hamiltonian, than actually appear in the canonical Hamiltonian. They consider the possibility that the effective Hamiltonian in a Tamm-Dancoff truncated theory may contain terms that do not appear in the canonical Hamiltonian. They hope to be able to use these non-canonical counterterms to restore the Lorentz invariance of the theory. This work is at an early stage, but the idea seems interesting.

On a heuristic level there seems to be a relationship between this idea and the recognition of the fact that the free particle Fock states do not seem to be an ideal basis in which to expand a bound state amplitude. It is this expansion which introduces the free particle spinors which seem to be responsible for the bad high momentum behaviour (see Chapter 8). This suggests the possibility of expanding the amplitude in a more suitable basis. Such an approach would be equivalent to modifying the functional dependence of p^- on the transverse momentum, and would necessarily introduce

new terms into the effective Hamiltonian.

9.2 Conclusion

We have examined in the lowest non-trivial light front Tamm-Dancoff approximation the case of fermions and pseudoscalar mesons coupled via a Yukawa interaction in 3+1 dimensions.

The primary motivation for this work was to establish whether or not the sector dependent renormalisation scheme proposed [9] for the 1+1 dimensional case could also be used in 3+1 dimensions. We find that most of the properties of this scheme are retained in 3+1 dimensions. There are spurious divergences produced by the use of light front variables. These are eliminated in 3+1 dimensions much as they were in 1+1 dimensions. This is undoubtedly due to the fact that in 3+1 dimensions the 'extra' transverse dimensions are unaffected by the change to light front variables, and do not change in number or form these particular divergences. The mass renormalisations are also still successful, as is evidenced by the fact that in the only regime where the individual particle masses have any unambiguous meaning, namely when they are on shell, the particles are seen to have their physical masses.

Notwithstanding these positive features, we find that the transverse degrees of freedom demand that the coupling constant be renormalised (in order

to render to mass correction terms finite), and also stop the kernel from decreasing fast enough to formally admit any solutions in cases associated with $L_3 = 0$.

Proceeding with a cutoff we find that there is evidence of a unique, deuteron like, bound state in the $J_3 = 0$ sector (but no allowed bound state in higher J_3 sectors). Within the stated context of this work, we will conclude with the observation that the sector dependent renormalisation scheme seems not to generalise to 3+1 dimensions in a straightforward way, as had been initially hoped. There are recent proposals for how the scheme might be generalised, but much work remains to be done if these are to be brought to a definitive form.

Appendix A

Light Front Variables

In order to facilitate comparison, our definitions follow those of Perry and Harindranath [9] wherever possible. Assuming the same ‘normal’ space-time metric as Bjorken and Drell [25] we define the light front time and longitudinal space variables as

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3) \quad (\text{A.1})$$

with the transverse space components $\vec{x}_{\perp} \equiv (x^1, x^2)$ unchanged. With these definitions it follows that the scalar product becomes

$$x.y = x^+y^- + x^-y^+ - \vec{x}_{\perp}.\vec{y}_{\perp} \quad (\text{A.2})$$

Note that this implies that the ‘light front’ metric is

$$g_{\mu\nu} = g^{\mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (\text{A.3})$$

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with $\mu = \{+, 1, 2, -\}$, which means that we have $x^\pm = x_\mp$. As an immediate consequence of the mixing of '+' and '-' indices which occurs in the scalar product we are lead to define p^- as the light front energy (i.e. p^- is conjugate to x^+ , the light front time).

We similarly define

$$\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^3) \quad (\text{A.4})$$

and thus find that, for example,

$$\gamma \cdot p = \gamma^+ p^- + \gamma^- p^+ - \vec{\gamma}_\perp \cdot \vec{p}_\perp \quad (\text{A.5})$$

Where we need an explicit representation for the gamma matrices, we have used the Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.6})$$

We will also need, on occasion, an explicit representation for the spinors. We have chosen this to be

$$u_\uparrow(p) = \frac{1}{N} \begin{pmatrix} m \\ 0 \\ \sqrt{2}p^+ \\ p_\perp e^{i\phi_p} \end{pmatrix}, \quad u_\downarrow(p) = \frac{1}{N} \begin{pmatrix} -p_\perp e^{-i\phi_p} \\ \sqrt{2}p^+ \\ 0 \\ m \end{pmatrix} \quad (\text{A.7})$$

where

$$N = \sqrt{2\sqrt{2}mp^+} \quad (\text{A.8})$$

These spinors are normalised to [26]

$$u^\dagger(p)u(p) = \frac{p^0}{m} = \frac{p^+ + p^-}{\sqrt{2}m} \quad (\text{A.9})$$

We will subsequently (Equation B.5) introduce the projection operators $\Lambda^{(\pm)}$. For the sake of completeness we will mention here that the spinors defined by $u^{(\pm)}(p) = \Lambda^{(\pm)}u(p)$ are, as a consequence of the above, normalised as

$$u^{(\pm)\dagger}(p)u^{(\pm)}(p) = \frac{p^\pm}{\sqrt{2}m} \quad (\text{A.10})$$

These spinors are obtained by explicitly solving the momentum space Dirac equation in Light Front coordinates. This procedure does not uniquely define the spinors, and we have exercised this freedom to choose a representation which does not explicitly contain p^- .

A further difference between the Light Front and Instant forms can be found in the distribution of Kinematic and Dynamic Poincare generators [16, 27]. Dynamic operators evolve the system in time, and since the identification of the ‘time’ coordinate is different in Light Front and Instant coordinates, it follows that the Dynamic operators may also be different. The most straightforward way to see this is to examine the Poincare generators at a coordinate point in the initial surface. If the generators cause a displacement off the surface (i.e. contain an ∂_+) then they are producing ‘time’ evolution and are

dynamic. The ten Poincare generators are P_μ and $M_{\mu,\nu}$. Acting on a scalar field (considering a field with intrinsic degrees of freedom would only mean the inclusion of a term acting on these degrees of freedom, and would not alter the space time properties) these have a representation

$$\begin{aligned} P_\mu &= i\partial_\mu \\ M_{\mu,\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) \end{aligned} \quad (\text{A.11})$$

At a general point on the light front surface $x^\mu = (0, \vec{x}_\perp, x^-)$ we see immediately that P_i, P_- ($i = 1, 2$) are unaffected, generating displacements within the surface, and are thus kinematic. The remaining momenta $P_+ = P^- = i\frac{\partial}{\partial x^+}$ generates displacement in Light Front time, and is hence dynamic.

Of the remaining generators, the only possible candidates for dynamic operators are M_{+i} and M_{+-} , since these are the only generators which contain ∂_+ . However for M_{+-} we have

$$M_{+-} = i(x_+P_- - x_-P_+) = ix_+P_- = ix^-P^+ \quad (\text{A.12})$$

since $x_- = x^+ = 0$. Thus the kinematic operators are

$$P_i, P_-, M_{-i}, M_{ij}, M_{+-} \quad (\text{A.13})$$

and the dynamic operators are

$$P_+ = P^-(\equiv H_{LF}), M_{+i} \quad (\text{A.14})$$

Note that there are seven kinematic operators, and three dynamic ones. In Instant coordinates there are only six kinematic operators (and of course four dynamic ones).

The physical interpretation of these operators is complicated by the unusual coordinate system. If we write

$$\begin{aligned} J_a &= \frac{1}{2}\epsilon_{abc}M_{bc} \\ K_a &= M_{0a} \end{aligned} \tag{A.15}$$

where $a = 1, 2, 3$, as the usual definition of the rotation and boost operators, we find that the kinematic operators can be written as

$$P_i, P^+, J_3, \frac{1}{\sqrt{2}}(K^1 + J^2), \frac{1}{\sqrt{2}}(K^2 - J^1) \tag{A.16}$$

showing, for example that ‘longitudinal’ boosts and the third component of angular momentum are kinematic. Conspicuously absent from this list are the transverse components of the angular momentum, which appear mixed with boost operators. In the Instant case, of course, all three components of the angular momentum are kinematic; here only one of them is. Thus, unlike the Instant case, angular momentum cannot be separated from the dynamics.

As a final point we can note that using the commutation relations for the

Poincare generators

$$\begin{aligned}
 [M_{\mu\nu}, M_{\rho\sigma}] &= i (g_{\mu\rho} M_{\nu\sigma} - g_{\nu\rho} M_{\mu\sigma} + g_{\mu\sigma} M_{\rho\nu} - g_{\nu\sigma} M_{\rho\mu}) \\
 [M_{\mu\nu}, P_\rho] &= i (g_{\mu\rho} P_\nu - g_{\nu\rho} P_\mu) \\
 [P_\mu, P_\nu] &= 0
 \end{aligned}
 \tag{A.17}$$

we can easily see that the subset of kinematic operators A.13 is closed in the sense that no commutator of one member of the set with another member produces an element outside of the set. Thus the kinematic generators generate a subgroup of the Poincare group.

Appendix B

Light Front Hamiltonian

Our starting point for the derivation of the Hamiltonian is the Lagrangian density describing fermions and bosons interacting via a Yukawa interaction term

$$\begin{aligned}\mathcal{L} = & \frac{i}{2}(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi) - m_F\bar{\psi}\psi \\ & + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_B^2\phi^2 - \lambda\bar{\psi}\psi\Gamma\phi\end{aligned}\tag{B.1}$$

where we are using $\Gamma = i\gamma_5$, although for the most part we do not need to use this explicitly in deriving the Hamiltonian. In what follows we will often implicitly use a lagrangian with fermion kinetic term

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi\tag{B.2}$$

which differs from the corresponding term in Eqn B.1 by a surface term.

The field equation for ψ can be found from the Euler Lagrange equation

for a variation in $\bar{\psi}$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} \right) \quad (\text{B.3})$$

which yields

$$i\gamma^\mu \partial_\mu \psi - (m_F + \lambda \Gamma \phi) \psi = 0 \quad (\text{B.4})$$

Using the projection operators

$$\begin{aligned} \Lambda^{(\pm)} &= \frac{1}{2} \gamma^\mp \gamma^\pm = \frac{1}{\sqrt{2}} \gamma^0 \gamma^\pm \\ \Lambda^{(\pm)} \Lambda^{(\pm)} &= \Lambda^{(\pm)} \\ \Lambda^{(\pm)} \Lambda^{(\mp)} &= 0 \end{aligned} \quad (\text{B.5})$$

and defining $\psi^{(\pm)} = \Lambda^{(\pm)} \psi$, the field equation can be separated into the two coupled equations

$$\begin{aligned} i\sqrt{2} \partial_+ \psi^{(+)} + i\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp \psi^{(-)} - \gamma^0 (m_F + \lambda \Gamma \phi) \psi^{(-)} &= 0 \\ i\sqrt{2} \partial_- \psi^{(-)} + i\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp \psi^{(+)} - \gamma^0 (m_F + \lambda \Gamma \phi) \psi^{(+)} &= 0 \end{aligned} \quad (\text{B.6})$$

The second of these equations involves no light front ‘time’ (x^+) derivative and can be interpreted as a constraint equation for $\psi^{(-)}$ as follows. Rewriting this equation as

$$\partial_- \psi^{(-)} = \partial^+ \psi^{(-)} = \frac{1}{i\sqrt{2}} \left(\gamma^0 (m_F + \lambda \Gamma \phi) - i\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp \right) \psi^{(+)} \quad (\text{B.7})$$

we can formally invert the differential operator to obtain

$$\psi^{(-)} = \frac{1}{\partial^+} \frac{1}{i\sqrt{2}} \left(\gamma^0(m_F + \lambda\Gamma\phi) - i\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp \right) \psi^{(+)} \quad (\text{B.8})$$

This formal inversion can be realised by the relation

$$\frac{1}{\partial^+} f(x^-) = \frac{1}{2} \int dy^- \epsilon(x^- - y^-) f(y^-) \quad (\text{B.9})$$

up to a constant with respect to x^- (which of course means in general an arbitrary function of $\vec{x} = (x^1, x^2)$). This constant is conventionally taken to be zero. Note that $\epsilon(x)$ is the antisymmetric step function $\epsilon(x) = \theta(x) - \theta(-x)$.

The fact that $\psi^{(-)}$ can be obtained once $\psi^{(+)}$ is known demonstrates that ψ has fewer degrees of freedom than is the case in the equal time formalism. Thus $\psi^{(+)}$ represents the dynamical variable, and we should take care to eliminate $\psi^{(-)}$ from the Hamiltonian. This point is further emphasised by the fact that the Schwinger Action principle gives anticommutation relations for $\psi^{(+)}$ only, and so in practice the only way to deal with $\psi^{(-)}$ is through the constraint equation. To obtain the light front Hamiltonian, $\mathcal{P}^-$, we start by defining the energy momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \phi_{r,\mu}} \frac{\partial \phi_r}{\partial x_\nu} - \mathcal{L} g^{\mu\nu} \quad \text{where } \phi_r = \phi, \psi, \psi^\dagger \quad (\text{B.10})$$

With the lagrangian Eqn B.1 we find (neglecting total derivatives for notational simplicity)

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi + i\bar{\psi} \gamma^\mu \partial^\nu \psi - \mathcal{L} g^{\mu\nu} \quad (\text{B.11})$$

Now use the definition for the four momentum operator

$$P^\mu = \int d\sigma_\nu T^{\nu\mu} \quad (\text{B.12})$$

where $d\sigma_\nu$ is an infinitesimal element of the initial (hyper) surface, which in this case has only one element $d\sigma_+ = d^2x dx^-$. So

$$P^\mu = \int d^2x dx^- T^{+\mu} \quad (\text{B.13})$$

and in particular

$$P^- = \int d^2x dx^- T^{+-} = \int \mathcal{P}^- \quad (\text{B.14})$$

Now from the expression for $T^{\mu\nu}$ we can evaluate $T^{+-} = \mathcal{P}^-$ (the Light Front Hamiltonian density) as

$$\mathcal{P}^- = \frac{1}{2}m_B^2\phi^2 + \bar{\psi}(m_F + \lambda\Gamma\phi)\psi - i\bar{\psi}\gamma^- \partial^+ \psi - i\bar{\psi}\vec{\gamma}_\perp \cdot \vec{\nabla}_\perp \psi + \frac{1}{2}\vec{\nabla}_\perp \phi \cdot \vec{\nabla}_\perp \phi \quad (\text{B.15})$$

We can obtain a more convenient form for the Hamiltonian by using the field equations (Eqn B.6)

$$P^- = \int d^2\vec{x}_\perp dx^- \left(\frac{1}{2}m_B^2\phi^2 + \frac{1}{2}\vec{\nabla}_\perp \phi \cdot \vec{\nabla}_\perp \phi + i\sqrt{2}\psi^{(+)\dagger} \partial^- \psi^{(+)} \right) \quad (\text{B.16})$$

We should note that this expression for P^- is really only a formal one, in that it avoids appearing to contain an interaction term by instead containing

a ‘time’ derivative, $\partial^- = \partial_+ = \frac{\partial}{\partial x^+}$.

In order to remove this ‘time’ derivative from the Hamiltonian we can use the field equations (Eqn B.6) once again. The first can be used to substitute for $\partial^- \psi^{(+)}$. This introduces $\psi^{(-)}$ once again, which can be removed by use of the constraint equation (Equation B.8). Using the integral prescription (Equation B.9) for the inverse of the derivative we obtain

$$\begin{aligned} \partial^- \psi^{(+)} &= -\frac{1}{4} \int dx'^- \epsilon(x^- - x'^-) \\ &\quad \times \left(\gamma^0 m_F + \lambda \gamma^0 \Gamma \phi - i \vec{\alpha}_\perp \cdot \vec{\nabla}_\perp \right) \left(\gamma^0 m_F + \lambda \gamma^0 \Gamma \phi' - i \vec{\alpha}_\perp \cdot \vec{\nabla}_\perp \right) \psi^{(+)' } \end{aligned} \quad (\text{B.17})$$

where we are using $f' \equiv f(x^+, \vec{x}_\perp, x'^-)$ to denote the dependence on x'^- rather than x^- . Notice that this expression involves terms both linear and quadratic in the coupling λ .

The Hamiltonian can be split into terms of order zero in the coupling constant and those of order one or greater, i.e. into a ‘free’ (or mass term) part, P_M^- , and an interaction part, P_I^-

$$\begin{aligned} P_M^- &= \int d^2 x dx^- \left(\frac{1}{2} m_B^2 + \frac{1}{2} (\vec{\nabla}_\perp \phi)^2 \right) \\ &\quad + \frac{1}{i 2 \sqrt{2}} \int d^2 x dx^- dx'^- \epsilon(x^- - x'^-) \psi^{(+)\dagger} \left(m_F^2 - (\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp)^2 \right) \psi^{(+)' } \end{aligned} \quad (\text{B.18})$$

and

$$\begin{aligned}
P_I^- &= \frac{1}{i2\sqrt{2}} \int d^2x dx^- dx'^- \epsilon(x^- - x'^-) \\
&\times \left[\lambda \psi^{(+)\dagger} \left((\gamma^0 m_F - i\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp) \gamma^0 \Gamma \phi' + \gamma^0 \Gamma \phi (\gamma^0 m_F - i\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp) \right) \psi^{(+)} \right. \\
&\left. + \lambda^2 \psi^{(+)\dagger} \phi \phi' \psi^{(+)} \right] \quad (B.19)
\end{aligned}$$

To proceed any further we need to know the commutation relations for the fields. Following Chang, Root and Yan [28] we use Schwingers action principle to derive the equal x^+ commutation relations (more details on this principle can be found in, for example, Ref [29]). We find

$$[\phi(x), \phi(y)]_{x^+=y^+} = -\frac{i}{4} \delta^2(\vec{x}_\perp - \vec{y}_\perp) \epsilon(x^- - y^-) \quad (B.20)$$

where $\epsilon(x) = \theta(x) - \theta(-x)$, and

$$\{\psi_\alpha^{(+)\dagger}(x), \psi_\beta^{(+)}(y)\}_{x^+=y^+} = \frac{1}{\sqrt{2}} \Lambda_{\beta\alpha}^{(+)} \delta^2(\vec{x}_\perp - \vec{y}_\perp) \delta(x^- - y^-) \quad (B.21)$$

In order to obtain a usable form for the Hamiltonian we need to expand the fields ψ and ϕ in terms of a free field basis.

For the fermion field we write

$$\begin{aligned}
\psi^{(+)}(x) &= \sum_s \int \frac{d^2\vec{p}_\perp dp^+}{(2\pi)^{\frac{3}{2}}} \left(\frac{m}{p^+} \right)^{\frac{1}{2}} \theta(p^+) \\
&\times \left[b(p, s) u^{(+)}(p, s) e^{-i(p^+ x^- - \vec{p}_\perp \cdot \vec{x}_\perp)} + d^\dagger(p, s) v^{(+)}(p, s) e^{+i(p^+ x^- - \vec{p}_\perp \cdot \vec{x}_\perp)} \right] \quad (B.22)
\end{aligned}$$

where

$$\{b(p, r), b^\dagger(q, s)\} = \{d(p, r), d^\dagger(q, s)\} = \delta(p^- - q^-) \delta^2(\vec{p}_\perp - \vec{q}_\perp) \delta_{r,s} \quad (\text{B.23})$$

with all other anticommutators equal to zero. Similarly for the boson field we write

$$\begin{aligned} \phi(x) = & \int \frac{d^2 \vec{p}_\perp dp^+}{(2\pi)^{\frac{3}{2}}} \left(\frac{1}{2p^+} \right)^{\frac{1}{2}} \theta(p^+) \\ & \times \left[a(p) e^{-i(p^+ x^- - \vec{p}_\perp \cdot \vec{x}_\perp)} + a^\dagger(p) e^{+i(p^+ x^- - \vec{p}_\perp \cdot \vec{x}_\perp)} \right] \quad (\text{B.24}) \end{aligned}$$

where

$$[a(p), a^\dagger(q)] = \delta(p^- - q^-) \delta^2(\vec{p}_\perp - \vec{q}_\perp) \quad (\text{B.25})$$

with all other commutators equal to zero. It should be noted that we perform our expansions at a fixed light front time, $x^+ = 0$. Since the Hamiltonian should be ‘time’ independent this is the obvious point at which to evaluate it. The substitution of these field expansions into the expression for the Hamiltonian is relatively straightforward and will not be detailed here. There are a number of points which should be mentioned concerning it however. The first is the usual point that by expanding in a plane wave basis can cause trouble in the sense that the exponentials do not vanish at infinity. This can cause problems when partially integrating some of the terms, and is always dealt with here by assuming the implicit existence of a function multiplying

the exponentials which is one everywhere except at very large x where it vanishes 'sufficiently' fast to render our manipulations meaningful.

The $x^-, x^{-'}$ part of a generic exponential integration appearing in the expansion of the Hamiltonian is (for the λ^2 quadrilinear terms for example)

$$\int dx^- dx^{-'} \epsilon(x^- - x^{-'}) e^{i(\alpha p^+ x^- + \beta q^+ x^{-'} + \gamma r^+ x^- + \delta s^+ x^{-'})} \quad (\text{B.26})$$

where p, q, r, s are field momenta, and $\alpha, \beta, \gamma, \delta = \pm 1$.

To perform the $x^{-'}$ integration we use a partial integration

$$\int_{-\infty}^{\infty} dy \epsilon(x - y) e^{ia y} = \frac{1}{ia} \left[e^{ia y} \epsilon(x - y) \right]_{-\infty}^{\infty} - \frac{1}{ia} \int_{-\infty}^{\infty} dy e^{ia y} \frac{\partial}{\partial y} \epsilon(x - y) \quad (\text{B.27})$$

then discarding surface terms as outlined above and identifying

$$\frac{\partial}{\partial y} \epsilon(x - y) = -2\delta(x - y) \quad (\text{B.28})$$

we have

$$\int_{-\infty}^{\infty} dy \epsilon(x - y) e^{ia y} = \frac{2}{ia} e^{ia y} \quad (\text{B.29})$$

We can note that this result is perhaps more obvious if the inverse derivative notation is retained, and that we have to take care to exclude the case $a = 0$. If 'a' is subsequently integrated over this exclusion will manifest itself as a principal value prescription.

Equation B.26 can now be written as

$$\begin{aligned}
 & -\frac{2i}{(\beta q^+ + \delta s^+)} \int dx^- e^{i(\alpha p^+ + \beta q^+ + \gamma r^+ + \delta s^+)x^-} \\
 = & -\frac{4\pi i}{(\beta q^+ + \delta s^+)} \delta(\alpha p^+ + \beta q^+ + \gamma r^+ + \delta s^+) \quad (B.30)
 \end{aligned}$$

A further unusual feature arises because of the appearance of the $O(\lambda^2)$ quadrilinear term. Recall that this arose through the use of the constraint equation for $\psi^{(-)}$. Because of this term we will obtain products of creation and annihilation operators like

$$b_s^\dagger(p) b_{s'}^+(q) a(r) a^\dagger(s) \quad (B.31)$$

Such terms must be normal ordered, and so we use the identity

$$a(r) a^\dagger(s) = [a(r), a^\dagger(s)] + a^\dagger(s) a(r) \quad (B.32)$$

which gives a normal ordered term, plus

$$b_s^\dagger(p) b_{s'}^+(q) \delta(r^- - s^-) \delta^2(\vec{r}_\perp - \vec{s}_\perp) \quad (B.33)$$

This is an operator rather than a ‘c’ number, and as such cannot be simply discarded. As can be seen from the example above, these terms have the operator structure of mass terms, and give rise to terms in the Hamiltonian called “self induced inertias”.

The final point which should be noted is that because of the fact that the plus component, p^+ , of real massive particles is always positive,

some of the terms in the Hamiltonian will never contribute, and hence will not be shown. These are the terms responsible for creating particles from the vacuum (and annihilating particles into the vacuum). They always appear with a factor of the form

$$\delta(p^+ + q^+ + r^+ + s^+) \quad (\text{B.34})$$

where all the arguments are of the same sign. Such a term can never have support if momentum is to be conserved. It is in this manner that the triviality of the vacuum manifests itself.

In exhibiting the Hamiltonian which results from these considerations, we follow Perry and Harindranath [9], and split the ‘interaction’ Hamiltonian P_I^- into three parts,

$$P_I^- = P_V^- + P_F^- + P_S^- \quad (\text{B.35})$$

The division is such that P_V^- changes the particle number by one, P_F^- changes the particle number by two and P_S^- leaves the particle number unchanged.

We find

$$\begin{aligned} P_V^- = & \frac{\lambda m_F}{\sqrt{2}(2\pi)^{\frac{3}{2}}} \sum_{s,t} \int dp dq dr \frac{1}{(p^+ q^+ r^+)^{\frac{1}{2}}} \theta(p^+) \theta(q^+) \theta(r^+) \\ & \left\{ b^\dagger(p, s) b(q, t) a(r) \bar{u}(p, s) \Gamma u(q, t) \delta(p - q - r) \right. \\ & \left. + b^\dagger(p, s) b(q, t) a^\dagger(r) \bar{u}(p, s) \Gamma u(q, t) \delta(p - q + r) \right\} \end{aligned}$$

$$\begin{aligned}
& + b^\dagger(p, s) d^\dagger(q, t) a(r) \bar{u}(p, s) \Gamma v(q, t) \delta(p + q - r) \\
& + d(p, t) b(q, s) a^\dagger(r) \bar{v}(p, s) \Gamma u(q, t) \delta(-p - q + r) \\
& - d^\dagger(q, t) d(p, s) a(r) \bar{v}(p, r) \Gamma v(q, s) \delta(-p + q - r) \\
& - d^\dagger(q, s) d(p, r) a^\dagger(r) \bar{v}(p, r) \Gamma v(q, s) \delta(-p + q + r) \} \quad (\text{B.36})
\end{aligned}$$

where for compactness we have used the notation $dp \equiv d\vec{p}_\perp dp^+$ and $\delta(p) \equiv \delta(\vec{p}_\perp) \delta(p^+)$.

Next we find

$$\begin{aligned}
P_F^- = & -\frac{\lambda^2 m_F}{4(2\pi)^3} \sum_{s,t} \int dp dq dr dw \frac{1}{(p^+ q^+ r^+ w^+)^{\frac{1}{2}}} \theta(p^+) \theta(q^+) \theta(r^+) \theta(w^+) \\
& \left\{ b^\dagger(p, s) b(q, t) a(r) a(w) \bar{u}(p, s) \gamma^+ u(q, t) \frac{\delta(p - q - r - w)}{(-q^+ - w^+)} \right. \\
& + b^\dagger(p, s) b(q, t) a^\dagger(r) a^\dagger(w) \bar{u}(p, s) \gamma^+ u(q, t) \frac{\delta(p - q + r + w)}{(-q^+ + w^+)} \\
& + b^\dagger(p, s) d^\dagger(q, t) a^\dagger(w) a(r) \bar{u}(p, s) \gamma^+ v(q, t) \frac{\delta(p + q - r + w)}{(q^+ + w^+)} \\
& + b^\dagger(p, s) d^\dagger(q, t) a^\dagger(r) a(w) \bar{u}(p, s) \gamma^+ v(q, t) \frac{\delta(p + q + r - w)}{(q^+ - w^+)} \\
& + d^\dagger(p, s) b(q, t) a^\dagger(w) a(r) \bar{v}(p, s) \gamma^+ u(q, t) \frac{\delta(-p - q - r + w)}{(-q^+ + w^+)} \\
& + d(p, s) b(q, t) a^\dagger(r) a(w) \bar{v}(p, s) \gamma^+ u(q, t) \frac{\delta(-p - q + r - w)}{(-q^+ - w^+)} \\
& - d^\dagger(q, t) d(p, s) a(r) a(w) \bar{v}(p, s) \gamma^+ v(q, t) \frac{\delta(-p + q - r - w)}{(q^+ - w^+)} \\
& \left. - d^\dagger(q, t) d(p, s) a^\dagger(r) a^\dagger(w) \bar{v}(p, s) \gamma^+ v(q, t) \frac{\delta(-p + q + r + w)}{(q^+ + w^+)} \right\} \quad (\text{B.37})
\end{aligned}$$

and for the final part of the interaction term,

$$\begin{aligned}
 P_S^- = & -\frac{\lambda^2 m_F}{4(2\pi)^3} \sum_{s,t} \int dp dq dr dw \frac{1}{(p^+ q^+ r^+ w^+)^{\frac{1}{2}}} \theta(p^+) \theta(q^+) \theta(r^+) \theta(w^+) \\
 & \left\{ b^\dagger(p, s) b(q, t) a^\dagger(w) a(r) \bar{u}(p, s) \gamma^+ u(q, t) \frac{\delta(p - q - r + w)}{(-q^+ + w^+)} \right. \\
 & + b^\dagger(p, s) b(q, t) a^\dagger(r) a(w) \bar{u}(p, s) \gamma^+ u(q, t) \frac{\delta(p - q + r - w)}{(-q^+ - w^+)} \\
 & + b^\dagger(p, s) d^\dagger(q, t) a(r) a(w) \bar{u}(p, s) \gamma^+ v(q, t) \frac{\delta(p + q - r - w)}{(q^+ - w^+)} \\
 & + d(p, s) b(q, t) a^\dagger(r) a^\dagger(w) \bar{v}(p, s) \gamma^+ u(q, t) \frac{\delta(-p - q + r + w)}{(-q^+ + w^+)} \\
 & - d^\dagger(q, t) d(p, s) a^\dagger(w) a(r) \bar{v}(p, s) \gamma^+ v(q, t) \frac{\delta(-p + q - r + w)}{(q^+ + w^+)} \\
 & \left. - d^\dagger(q, t) d(p, s) a^\dagger(r) a(w) \bar{v}(p, s) \gamma^+ v(q, t) \frac{\delta(-p + q + r - w)}{(q^+ - w^+)} \right\}
 \end{aligned}
 \tag{B.38}$$

The terms in P_F^- and P_S^- are referred to as instantaneous. The reason for this lies in the fact that the Lagrangian contains no meson-meson interaction, which implies that a term of the form, for example, $b^\dagger(p, s) b(q, t) a(r) a(w)$ must correspond to fermion exchange. Since all fields in the Hamiltonian are defined to be at equal light front time, this exchange is considered to be instantaneous.

For P_M^- , in which we include all terms diagonal in creation and annihila-

tion operators, we find

$$\begin{aligned}
P_M^- &= \int d^2 \vec{p}_\perp dp^+ \theta(p^+) \frac{1}{2p^+} a^\dagger(p) a(p) \left(m_B^2 + p_\perp^2 + \frac{\lambda^2}{2(2\pi)^3} \alpha(p^+) \right) \\
&+ \int d^2 \vec{p}_\perp dp^+ \theta(p^+) \frac{1}{2p^+} \sum_s b^\dagger(p, s) b(p, s) \left(m_F^2 + p_\perp^2 + \frac{\lambda^2}{2(2\pi)^3} \beta(p^+) \right) \\
&+ \int d^2 \vec{p}_\perp dp^+ \theta(p^+) \frac{1}{2p^+} \sum_s d^\dagger(p, s) d(p, s) \left(m_F^2 + p_\perp^2 + \frac{\lambda^2}{2(2\pi)^3} \gamma(p^+) \right)
\end{aligned}$$

where

$$\begin{aligned}
\alpha(p^+) &= \wp \int d^2 \vec{r}_\perp dr^+ \theta(r^+) \left(\frac{1}{(p^+ - r^+)} - \frac{1}{(p^+ + r^+)} \right) \\
\beta(p^+) &= \wp \int d^2 \vec{r}_\perp dr^+ \theta(r^+) \frac{p^+}{r^+(p^+ - r^+)} \\
\gamma(p^+) &= \int d^2 \vec{r}_\perp dr^+ \theta(r^+) \frac{p^+}{r^+(p^+ + r^+)} \tag{B.39}
\end{aligned}$$

where $\wp$ denotes the principal part. It should be noted that α, β and γ , the ‘self induced inertias’, are all divergent, and as such we must assume that the above expressions are regulated in some way, such that we can manipulate them.

Appendix C

Numerical Methods

The starting point for the numerical work is the two dimensional integral equation with right hand side given by Equation 5.14. Adopting a symbolic notation for the kernel of this equation we can write (we will denote $p_{H\perp}$ by $p_\perp$ etc., and drop the tilde on the renormalised coupling constant in what follows)

$$(\delta + \lambda^2 \Delta) \vec{C}_m(x, p_\perp) = \lambda^2 \int_0^\Lambda dq_\perp \int_0^1 dz \underline{K}^{(m)}(x, p_\perp; z, q_\perp) \vec{C}_m(z, q_\perp) \quad (\text{C.1})$$

where $\delta = \delta(x, p_\perp)$ and $\Delta = \Delta(x, p_\perp)$. A cutoff on the transverse momentum integration has been assumed, in order that the solution of this equation be consistent with the treatment of the self energy terms.

As a first step the ‘ $q_\perp$ ’ integration is mapped onto the range $0 \rightarrow 1$ by

the transformation

$$q_{\perp} = \frac{q}{(c_{\Lambda} - q)} \quad \text{where} \quad c_{\Lambda} = 1 + \frac{1}{\Lambda} \quad (\text{C.2})$$

This transformation introduces a Jacobian factor

$$J(q) = \frac{c_{\Lambda}}{(c_{\Lambda} - q)^2} \quad (\text{C.3})$$

A similar transformation is performed on $p_{\perp}$ to ensure that we have functions of the same variables on both sides of the equation. We will adopt the somewhat dubious convention of referring to the new functions of the transformed variables by their old names, for simplicity of presentation. Each of the integrals is then approximated by a gaussian quadrature. In the work performed the possibility of a different order of quadrature for each of the integrals was allowed for. In practice the same order was used for both, and thus we will assume this in the following. Since a matrix equation will result from this procedure, and since the 'z' and 'q' are just the analogue on the right hand side of the equation of 'x' and 'p', we will denote the indices of the 'z' and 'q' abscissas as j_x and j_p . We obtain the expression for the right hand side of the equation

$$\lambda^2 \sum_{j_x, j_p=0}^{N-1} J(q_{j_p}) w_{j_x} w_{j_p} \underline{K}^{(m)}(x, p; z_{j_x}, q_{j_p}) \vec{C}_m(z_{j_x}, q_{j_p}) \quad (\text{C.4})$$

In order to obtain a matrix equation the amplitude on the left hand side of the equation is evaluated at the same points. We use the indices i_x and i_p

for the 'x' and 'p' abscissas. This yields the equation

$$\begin{aligned} & \left(\delta(x_{i_x}, p_{i_p}) + \lambda^2 \Delta(x_{i_x}, p_{i_p}) \right) \vec{C}_m(x_{i_x}, p_{i_p}) \\ &= \lambda^2 \sum_{j_x, j_p=0}^{N-1} J(q_{j_p}) w_{j_x} w_{j_p} \underline{K}^{(m)}(x_{i_x}, p_{i_p}; z_{j_x}, q_{j_p}) \vec{C}_m(z_{j_x}, q_{j_p}) \end{aligned} \quad (\text{C.5})$$

We will now adopt the notation $f(x_{i_x}) = f_{j_x}$ and replace the vector notation $\vec{C}$ with an index notation C_{i_s} (where we are using a combined spin index ranging from $0 \rightarrow 3$ to cover the components of $\vec{C}$). This results in the equation

$$\left(\delta_{i_x, i_p} + \lambda^2 \Delta_{i_x, i_p} \right) C_{m, i_x, i_p, i_s} = \lambda^2 \sum_{j_x, j_p, j_s} J_{j_p} w_{j_x} w_{j_p} K_{i_x, i_p, i_s; j_x, j_p, j_s}^{(m)} C_{m, j_x, j_p, j_s} \quad (\text{C.6})$$

Since the equation is diagonal in the index 'm' we will not show it in subsequent equations, although of course the kernel depends on it's value. We will also include the Jacobian and gaussian weights into a redefined kernel

$$K'_{i_x, i_p, i_s; j_x, j_p, j_s} = J_{j_p} w_{j_x} w_{j_p} K_{i_x, i_p, i_s; j_x, j_p, j_s} \quad (\text{C.7})$$

Equation C.6 is not in the form we require, since the coupling constant λ appears on both sides of the equation. We can remedy this relatively straightforwardly by moving the term proportional to λ^2 to the right hand side of the equation. The only slight complication is that the indices on this term are 'i' indices, and not summed over. In order to include this new term into the matrix structure (ie. into the sum) we need to change the indices form 'i' to 'j' and introduce Kronecker delta's to ensure that the term is

functionally unaltered (we will denote these by $1_{i,j}$ to distinguish them from the $\delta = M^2 - M_0^2$ already in use). Performing this procedure we obtain

$$\delta_{i_x, i_p} C_{i_x, i_p, i_s} = \lambda^2 \sum_{j_x, j_p, j_s} \left(K'_{i_x, i_p, i_s; j_x, j_p, j_s} - 1_{i_x, j_x} 1_{i_p, j_p} 1_{i_s, j_s} \Delta_{i_x, i_p} \right) C_{j_x, j_p, j_s} \quad (\text{C.8})$$

The final rearrangement required is to multiply both sides of the equation by δ_{i_x, i_p}^{-1} and make one further redefinition of the kernel

$$K''_{i_x, i_p, i_s; j_x, j_p, j_s} = \delta_{i_x, i_p}^{-1} \left(K'_{i_x, i_p, i_s; j_x, j_p, j_s} - 1_{i_x, j_x} 1_{i_p, j_p} 1_{i_s, j_s} \Delta_{i_x, i_p} \right) \quad (\text{C.9})$$

to obtain

$$C_{i_x, i_p, i_s} = \lambda^2 \sum_{j_x, j_p, j_s} K''_{i_x, i_p, i_s; j_x, j_p, j_s} C_{j_x, j_p, j_s} \quad (\text{C.10})$$

This equation is now in the form of an eigenvalue equation for λ^{-2} . The only remaining problem is that we have a triple sum which makes the equation appear unfamiliar. This can be remedied by mapping the three indices onto a single index in much the same way as we have already done for the ‘spin’ index. Defining

$$i_T = N^2 i_s + N i_p + i_x \quad \text{with} \quad i_T : 0 \rightarrow 4N^2 - 1 \quad (\text{C.11})$$

we have an invertible mapping which enables us to write

$$C_{i_T} = \lambda^2 \sum_{j_T=0}^{4N^2-1} K''_{i_T, j_T} C_{j_T} \quad (\text{C.12})$$

This is the form of the equation which we use to obtain the eigenvalues λ^{-2} and the eigenvectors C_{i_T} . Each eigenvector represents all four spin states of

the amplitude evaluated at the gaussian abscissas for 'x' and 'p'. Once these are obtained it is a fairly trivial matter to disentangle the various indices, invert the mapping for $p_{\perp}$, and interpolate between the gaussian abscissas in order to find the amplitudes at equally spaced points. As a final point we will note that for $m = 0$ (i.e. for the $J_3 = 0$ case) the spin index part of Equation 5.14 assumes the generic form

$$\Gamma^{(0)} + (\Gamma^{(+)} + \Gamma^{(-)}) F \quad (\text{C.13})$$

Closer inspection reveals that the combined operations of transposition, $x \leftrightarrow z$ and $p_{\perp} \leftrightarrow q_{\perp}$ leave this term unchanged. This is of interest since it corresponds to the change $i_T \leftrightarrow j_T$ which is just transposition. Overall this symmetry is spoiled by the $q_{\perp}$ arising from $d^2 q_{\perp}$, the gaussian weights, the Jacobian factor and the δ^{-1} term. It is possible however by defining the amplitude

$$C'_{i_x, i_p, i_s} = \frac{\sqrt{\delta_{i_x, i_p} p_{B\perp, i_p} w_{i_x} w_{i_p}}}{(c_{\Lambda} - p_{i_x})} C_{i_x, i_p, i_s} \quad (\text{C.14})$$

to arrive at a symmetric representation for the kernel. Note that this is only possible in the $J_3 = 0$ case.

The advantage of this transformation is that techniques exist for finding the eigenvalues of very large symmetric matrices using only modest amounts of computer storage. One such technique is the Lanczos algorithm which we have used to check the eigenvalues up to $N = 50$. Given the mapping

employed, this corresponds to a matrix dimension of $4 \times 50^2 = 10000$. Finding the eigenvalues of a 10000×10000 matrix would be intractable using conventional eigenvalue routines. We conclude this appendix with a brief description of the Lanczos algorithm since it is relatively unfamiliar.

C.1 The Lanczos Algorithm

It should be stated at the start of this section that the Lanczos Algorithm is not an algorithm for finding eigenvalues, rather it is an algorithm for tridiagonalising a symmetric matrix. Once this tridiagonalisation has been performed one must use a conventional eigensystem routine to find the eigenvalues. This being said, finding the eigenvalues of a tridiagonal matrix does not require much storage, since all that is required is to eliminate the subdiagonal (or superdiagonal) terms. The advantage of the Lanczos algorithm is that if the matrix elements themselves can be readily calculated, the tridiagonalisation can be performed (in theory) with a storage requirement equivalent to five rows of the matrix. In practice to find all the eigenvalues requires more than this, but to find only a few requires less.

The principle behind any such tridiagonalisation routine is to perform a similarity transformation on the given matrix which brings it to tridiagonal form. Such a transformation will not affect the eigenvalues. A similarity transformation is equivalent to finding the representation of a matrix afforded

by a different set of basis vectors. The Lanczos algorithm generates a set of orthonormal basis vectors sequentially, at each step needing only the previous two vectors to be known.

The algorithm is defined by the relation [30]

$$\beta_{i+1}v_{i+1} = Av_i - \alpha_i v_i - \beta_i v_{i-1} \quad (\text{C.15})$$

where 'A' is the matrix, v_i is a basis vector and α_i, β_i are vectors of coefficients. α_i is defined by

$$\alpha_i = v_i^T Av_i \quad (\text{C.16})$$

The algorithm is started with an initial, normalised, vector, v_1 (generated randomly for example) and the definition $\beta_1 = 0$. This is sufficient to start the algorithm. In practice one calculates

$$z = Av_i - \alpha_i v_i - \beta_i v_{i-1} \quad (\text{C.17})$$

and then equates

$$\begin{aligned} \beta_{i+1} &= \|z\| \\ v_{i+1} &= \frac{z}{\|z\|} \end{aligned} \quad (\text{C.18})$$

One can readily show that the (normalised) vectors produced by this algorithm are mutually orthogonal. Rewriting Equation C.15 in the form

$$Av_i = \alpha_i v_i + \beta_i v_{i-1} + \beta_{i+1} v_{i+1} \quad (\text{C.19})$$

makes it clear that we are generating a tridiagonal representation of the matrix 'A'. Explicitly we have

$$A \sim \begin{pmatrix} \alpha_1 & \beta_2 & 0 & 0 & \cdots \\ \beta_2 & \alpha_2 & \beta_3 & 0 & \\ 0 & \beta_3 & \alpha_3 & \beta_4 & \\ 0 & 0 & \beta_4 & \alpha_5 & \\ \vdots & & & & \ddots \end{pmatrix} \quad (\text{C.20})$$

This procedure is equivalent to producing the set of vectors

$$u_i = \{A^i v_1\} \equiv \{v_1, Av_1, A^2 v_1, A^3 v_1, \dots\} \quad (\text{C.21})$$

and orthogonalising them. Now we expect $A^n v_1$ to tend to the eigenvector of 'A' with the largest eigenvalue, provided v_1 has a non-zero projection onto this vector (this can easily be seen by expanding v_1 in terms of the eigenvectors of 'A'). Thus as we iterate Equation C.15 we expect to quite rapidly (the actual speed of convergence depends upon the spacing of the eigenvalues) produce a v_i close to this eigenvector. From Equation C.19 and the definition of α_i , we can see that when this occurs we expect α_i to be a good approximation of the maximum eigenvalue, and the surrounding β to be small. This implies that α_i , and hence the maximum eigenvalue of 'A', is an eigenvalue of the tridiagonal matrix produced up to that point.

To restate this, even if the matrix dimension is 'n', and we have iterated 'm' times, with $m < n$, we will still have a tridiagonal matrix, albeit of smaller dimensions, and we can calculate its eigenvalues. Examining these

eigenvalues as the number of iterations is increased, we are likely to see the largest eigenvalue appear, and converge, long before the 'n' iterations. In practice with a 10000×10000 matrix, the largest eigenvalue was completely stable at ≈ 50 iterations.

Since in our work only the largest few eigenvalues are required, discussion of further properties of the Lanczos algorithm and its possible limitations will be omitted.

Bibliography

- [1] E. Leader and E. Predazzi, *Gauge Theories and the 'New Physics'*, Cambridge University Press (1982) and references therein
- [2] R.P. Feynman, *Photon Hadron Interactions*, Benjamin, New York (1972)
- [3] F.E. Close, *An Introduction to Quarks and Partons*, Academic Press (1979)
- [4] G. Altarelli and G. Parisi, Nucl. Phys. B126, 298 (1977)
- [5] M. Gell-Mann and Y. Ne'eman, *The Eightfold Way*, Benjamin, New York (1964)
- [6] T. Cheng and L. Li, *Gauge Theory of Elementary Particle Physics*, Oxford University Press (1984)
- [7] R.J.Perry, A.Harindranath and K.G.Wilson, Phys. Rev. Lett. 65,2959 (1990)
- [8] S. Weinberg, Phys. Rev. 150, 1313 (1966)
- [9] R.J. Perry and A. Harindranath, Phys. Rev. D43, 4051 (1991)
- [10] I.Tamm, J. Phys (USSR) 9, 449 (1945)
- [11] S.M.Dancoff, Phys Rev. 78, 382 (1950)
- [12] H.A.Bethe and F.de Hoffmann, *Mesons and Fields*, Row, Peterson and Company (1955)
- [13] F.J. Dyson, M. Ross, E.E. Salpeter, S.S. Schweber, M.K. Sundaresan, W.M.Visscher and H.A. Bethe. Phys. Rev. 95 1644 (1954)

- [14] F.J. Dyson, Phys. Rev. 91, 1543 (1953)
- [15] R.H. Dalitz and F.J. Dyson, Phys. Rev. 99, 301 (1955)
- [16] P.A.M. Dirac, Rev. Mod. Phys. 21, 392 (1949)
- [17] A. Harindranath, *Sources For Light Front Physics*, Ohio State University document. (This document contains an extensive list of references for Light Front Physics)
- [18] S.S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Harper and Row (1961)
- [19] R.H. Dalitz, M.K. Sundareshan and H.A. Bethe, Proc. Cambridge. Phil. Soc. 52, 251 (1956)
- [20] K. Hornbostel, Cornell University Preprint CLNS 91/1078 (1991)
- [21] R. G. Sachs, *Nuclear Theory*, Addison Wesley (1953)
- [22] S. Glazek, A. Harindranath, S. Pinsky, J. Shigemitsu and K. Wilson, Ohio State University preprint (preliminary) OHSTPY-HEP-T-92-004 (1992)
- [23] S.J. Brodsky and H.C. Pauli, SLAC-PUB-5558 (1991), and private communication.
- [24] M. Krautgärtner, H.C. Pauli and F. Wölz, Heidelberg Univ. Preprint MPH-V36-1991
- [25] Bjorken and Drell, *Relativistic Quantum Mechanics*, McGraw Hill (1964)
- [26] S. Chang and T. Yan, Phys. Rev. D7, 1147 (1973)
- [27] K. Hornbostel, *Constructing Hadrons on the Light Cone*, Cornell University Preprint CLNS 90/1038 (1990)
- [28] S. Chang, R.G. Root and T. Yan, Phys. Rev. D7, 1133 (1973)
- [29] Roman, *Introduction to Quantum Field Theory*, Wiley (1969)
- [30] J.G. Lewis, *Algorithms for Sparse Matrix problems*, Stanford University Technical Report STAN-CS-77-595 (1977)

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25 | 05 | 93 |

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