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The p-median Problem and the Uncapacitated Facility Location Problem

by

Yu Wang, B.Sc., M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
Ottawa-Carleton Institute of Mathematics and Statistics
Carleton University
Ottawa, Ontario
August, 1996

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acceptance of the thesis.

The p-median Problem and
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submitted by
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in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

Dr. V. Dlab, Chairman
Department of Mathematics and Statistics

Dr. I. Pressman, Thesis Supervisor

Dr. R. Caron, External Examiner

Carleton University
August, 1996
Abstract

We study the $p$-median problem by expressing it as a 0-1 integer programming problem and investigate the optimal solution to the relaxed 0-1 integer programming problem (the relaxed problem). By using the Duality Theorem, we establish the connection between the $p$ selected facilities and the optimal solution to the relaxed problem. The value (distance-sum) of any $p$ selected facilities is equal to the optimal value to the relaxed problem plus the corresponding optimal dual values and dual slack values. If the optimal solution to the relaxed problem is integral, the optimal value of the relaxed problem is the value of the $p$-median.

Given an integral feasible solution, which can be easily obtained for the $p$-median problem, we construct a simplified integer programming problem for the $p$-median problem (the simplified problem). This procedure simultaneously reduces the number of variables and constraints. The optimal value of the relaxed simplified problem is less than or equal to the optimal value of the relaxed problem. After solving a sequence of these relaxed simplified problems, which are dynamically constructed, we have the same optimal solution as the relaxed problem.

To find the optimal integral solution to the relaxed problem, in Chapter 4, we
present two new branch and bound algorithms. One branches on variables according to the optimal dual slack values of the relaxed problem. The other method branches on variables according to the data generated by the problem and the structure of the optimal solution.

In Chapter 5, the results of the \( p \)-median problem are generalized to the uncapacitated facility location problem (UFL problem). We discuss the relationship between the \( p \)-median problem and the UFL problem. Under some conditions, the UFL problem is the same as the \( p \)-median problem. With the same ideas as in the \( p \)-median problem, we develop a series of approximations to the optimal solution of the relaxed UFL problem and provide a concrete methodology for doing this. We then extend the second branch and bound algorithm for the \( p \)-median problem to the UFL problem.
Acknowledgements

I would first like to thank my supervisor, Dr. I. Pressman. He introduced me to the field of mathematical programming and brought the topic of this thesis to me. I especially wish to thank Dr. Pressman for his carefully reviewing this thesis. I am deeply grateful for his guidance, kindness and much more.

I would like to thank Carleton University. The financial aid from the Faculty of Graduate Studies made it possible for me to finish my studies at Carleton. I enjoyed the friendly environment of the Department of Mathematics and Statistics.

I appreciate the referees for their suggestions and comments.

During the preparation of this thesis, my beloved father passed away. He would have been proud to see me complete this thesis, while my mother continuously encouraged and supported me. I dedicate this thesis to them.
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Chapter 1

Introduction

1.1 The p-median Problem

A network is an undirected graph $G(V, E)$ with node set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E$. Each node $v$ is associated with a nonnegative weight $w(v)$, and each edge $e$ is associated with a positive length $l(e)$. For each pair of nodes $v_i, v_j$, we let $d(v_i, v_j)$, the distance between $v_i$ and $v_j$, be the length of a shortest length path connecting $v_i$ and $v_j$. One of the classical problems in Location Theory is the $p$-median problem, which is defined as follows.

Given a network $G(V, E)$, let $X_p = \{x_1, x_2, \ldots, x_p\}$ be a set of $p$ nodes in $V$. Define the distance $d(v, X_p)$ between a node $v$ of $G$ and a subset $X_p$ of $V$ by

$$d(v, X_p) = \min_{1 \leq i \leq p} \{d(v, x_i)\}.$$  

For each set $X_p = \{x_1, x_2, \ldots, x_p\}$ of $p$ nodes in $V$, we define:

$$H(X_p) = \sum_{v \in V} w(v) \cdot d(v, X_p)$$  

(1.1.1)
and call \( H(X_p) \) the distance-sum of the set \( X_p \). If there exists \( X_p^* \) in \( V \) such that
\[
H(X_p^*) = \min_{X_p \subseteq V} \{H(X_p)\}
\]
(1.1.2)
then \( X_p^* \) is called a \( p \)-median of \( G \).

Let \( X_p \) be a set of \( p \) distinct points of \( G \). A point means a node of \( G \) or a point along any edge of \( G \). Hakimi(1965) showed that there exists a set of \( p \) nodes with the same optimal value. Therefore, in this thesis, a \( p \)-median means a set of \( p \) nodes \( X_p^* \) whose distance-sum satisfies (1.1.2).

We will consider the \( p \)-median problem on a connected network. However, we shall occasionally consider the \( p \)-median problem on a disconnected network. To make this meaningful, we assume that the number of components of the network \( G \) is less than or equal to \( p \), where each component contains at least one facility.

The \( p \)-median problem is a prototype for many realistic locational decision problems. One of them is to establish \( p \) facilities in \( p \) of the potential locations on the network and supply each client from the closest established facility so that the demands of all clients are met and the total costs incurred thereby are minimized. It has been traditional to use the phrase \emph{facility (node) \( j \) is open} to mean \( v_j \in X_p^* \) and words \emph{client \( i \) (\( v_i \))} to mean node \( v_i \).

The thesis is organized as follows:

In Chapter 1, we introduce the \( p \)-median problem, the notation and background that will be used for the rest of the thesis, and give a brief survey of the literature
of solving the p-median problem.

In Chapter 2, we consider the combinatorial properties of the p-median. show the connection between the p-median on the network and the 1-median of its region.

In Chapter 3, we investigate the integer programming problem for the p-median problem, and obtain some basic properties about the optimal solution to the relaxed integer programming problem. We establish the connection between the distance-sum of any selected p facilities and the optimal solution to the relaxed integer programming problem. Finally, we describe a procedure to construct a simplified integer programming problem for the p-median problem, which reduces the number of variables and constraints simultaneously.

In Chapter 4, we present two branching strategies. Both of them use the branch and bound algorithm to obtain the integral optimal solution of the p-median problem.

In Chapter 5, we generalize the results of the p-median problem to the uncapacitated facility location problem (UFL problem).

In Chapter 6, we indicate some applications of the p-median problem and the UFL problem, and some future work that we intend to do.

In the Appendix, we present a linear time algorithm to find the 1-median of the n-dimensional grid.
algorithm. Nothing is said when \( i \notin F \). \( \mathcal{NP} \) contains hardest problems, i.e. there is a subset of \( \mathcal{NP} \), called \( \mathcal{NPC} \), such that if there exists \( X \in \mathcal{NPC} \cap \mathcal{P} \), then every problem in \( \mathcal{NP} \) is in \( \mathcal{P} \), that is \( \mathcal{NP} = \mathcal{P} \). Problems in \( \mathcal{NPC} \) are called \( \mathcal{NP} \)-complete.

A problem \( X_1 \) is said to be **polynomially reducible** to another problem \( X_2 \), if there is an algorithm for \( X_1 \) that uses an algorithm for \( X_2 \) as subroutine and runs in polynomial time provided that each call of the subroutine takes unit time. A problem is called **\( \mathcal{NP} \)-hard**, if there is an \( \mathcal{NP} \)-complete problem that can be polynomially reduced to it. Thus, if a problem is \( \mathcal{NP} \)-hard, it is at least as difficult as any \( \mathcal{NP} \)-complete problem. It also follows that a polynomial time algorithm for an \( \mathcal{NP} \)-hard problem implies that \( \mathcal{P} = \mathcal{NP} \). Whether \( \mathcal{P} = \mathcal{NP} \) or not is a major unsolved problem in mathematics and computer science.

A **heuristic algorithm** is a process which seeks good (i.e. near-optimal) solutions at a reasonable computational time without guaranteeing either feasibility or optimality, and without even estimating how close a particular feasible solution is to optimality.

### 1.3 A Brief Survey

Hakimi(1964, 1965) first used the term **\( p \)-median** and introduced the \( p \)-median problem. The earliest reference for the \( p \)-median problem on a network is Hua et al(1962), who gave an algorithm in poetic form to find the 1-median on a tree net-
v and is denoted by deg_G(v). A node of degree 1 is called a leaf of G. A walk in G is a sequence \( v_0, e_1, v_1, \ldots, v_k \) such that \( v_0, \ldots, v_k \) are nodes and \( e_i = v_{i-1} v_i \) for \( i = 1, \ldots, k \). The walk is simple if \( v_0, \ldots, v_k \) are distinct. A simple walk is called a path. A walk is said to be closed if \( v_0 = v_k \). A closed walk is said to be a cycle if \( k \geq 3 \) and \( v_0, v_1, \ldots, v_{k-1} \) is a path.

**Proposition 1.2.1.** There is a unique partition of the nodes of a graph G into subsets \( V_1, \ldots, V_t \) with the property that nodes \( v_i \) and \( v_j \) are in the same subset if and only if G contains a path between \( v_i \) and \( v_j \).

Let \( V_k \) be a subset of the partition. The subgraph \( <V_k> \) is called a component of G. G is said to be connected if it has one component. A forest F of G is a subgraph that does not contain a cycle. A tree T of G is a connected forest. A rooted tree is a tree T with a root node r of T.

**Linear programming duality theorem**

A linear programming problem is to minimize or maximize a linear objective function subject to a finite number of linear constraints. A typical form is

\[
\min \quad c^T x \\
\text{s.t.} \\
Ax \geq b \\
x \geq 0
\]  

(1.2.1)
where $A$ is an $m \times n$ real matrix, $b \in \mathbb{R}^m, c \in \mathbb{R}^n$.

The minimization problem can be converted to a maximization problem since

$$\max \ c^T x = -\min \ -c^T x$$

subject to any constraints.

The **dual program** of (1.2.1) is

$$\max \ b^T y$$

subject to

$$A^T y \leq c \quad (1.2.2)$$

$$y \geq 0$$

The fundamental results of linear programming are the following theorems.

**Theorem 1.2.2. (Duality Theorem)** If one of the programs (1.2.1) and (1.2.2) has a finite optimal solution, then so does the other and the two optimal values are equal.

**Theorem 1.2.3.** Let $x$ and $y$ be feasible solutions of (1.2.1) and (1.2.2) respectively. Then, $x$ and $y$ optimize (1.2.1) and (1.2.2) if and only if the following conditions are satisfied:

(i) $$(A^T y - c)^T x = 0; \quad (1.2.3)$$

(ii) $$y^T(Ax - b) = 0.$$  \quad (1.2.4)$$

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Conditions (1.2.3) and (1.2.4) are called the complementary slackness conditions.

Since there are many different forms of constraints of linear programming problems, the duality theorem and the complementary slackness conditions appear in different forms. In this thesis, we shall use variants of the theorems without stating them.

Computational complexity

The input size of an integer $n$ is $\lfloor \log_2 (|n| + 1) \rfloor$, denoted by $s(n)$. For every rational number $r$, there are two unique integers $a, b$, with $\gcd(a, b) = 1$, and $b > 0$ such that $r = a/b$. The input size of $r$, $s(r)$, is $s(a) + s(b)$. If $a$ is a vector or a matrix, the size of $a$ is the sum of the input sizes of the entries. For a sequence $a_1, \ldots, a_k$ of vectors and matrices, its input size is $s(a_1, \ldots, a_k) = s(a_1) + \cdots + s(a_k)$. We define $|V| + |E|$ to be the input size of a graph $G = (V, E)$.

Given functions $f(n)$ and $g(n)$ from $\mathbb{Z}_+$ to $\mathbb{Z}_+$, $f(n)$ is said to be $O(g(n))$ if there is a constant $c > 0$ and $n' \in \mathbb{Z}_+$ such that $f(n) \leq cg(n)$ for all $n \geq n'$. Given a class of problems and an algorithm, the time complexity function $f: \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ of the algorithm is the maximum time $f(n)$ needed to solve any problem instance of input size $n \in \mathbb{Z}_+$. The algorithm is a polynomial-time algorithm if there exists a polynomial $g(x)$ such that $f(n) = O(g(n))$. Let $\mathcal{P}$ be the class of problems that can be solved in polynomial time.
A feasibility problem $X$ is a pair $(I, F)$ with $F \subseteq I$, where the elements of $I$ are finite binary strings. $I$ is called the set of instances of $X$, $F$ is called the set of feasible instances. Given an instance $i \in I$, we want to determine whether or not $i \in F$. The answer is either yes or no. The information that can be used to check feasibility in polynomial time is called a certificate of feasibility, which is denoted by $Q_i$, for each instance $i \in I$.

A nondeterministic algorithm consists of two stages: the guessing stage and the checking stage. The input to the algorithm is an $i \in I$. In the guessing stage, we guess a binary string $Q$. The checking stage is an algorithm that works with the pair $(i, Q)$ and may provide the output that $i \in F$. Two properties are required.

1. If $i \in F$, there is a certificate $Q_i$ such that when the pair $(i, Q_i)$ is given to the checking stage, the algorithm gives the answer that $i \in F$.

2. If $i \not\in F$, there is no output.

We say that the nondeterministic algorithm is polynomial if, for each $i \in F$, there is a polynomial time proof of feasibility, i.e. when $i \in F$, the running time in the checking stage is a polynomial function of the size of $i$ for some $Q_i$ for which it replies that $i \in F$.

We define $\mathcal{NP}$ to be the class of feasibility problems such that for each instance $i \in F$, the answer $i \in F$ is obtained in polynomial time by some nondeterministic
algorithm. Nothing is said when $i \not\in F$. $\mathcal{NP}$ contains hardest problems, i.e. there is a subset of $\mathcal{NP}$, called $\mathcal{NPC}$, such that if there exists $X \in \mathcal{NPC} \cap \mathcal{P}$, then every problem in $\mathcal{NP}$ is in $\mathcal{P}$, that is $\mathcal{NP} = \mathcal{P}$. Problems in $\mathcal{NPC}$ are called $\mathcal{NP}$-complete.

A problem $X_1$ is said to be polynomially reducible to another problem $X_2$, if there is an algorithm for $X_1$ that uses an algorithm for $X_2$ as subroutine and runs in polynomial time provided that each call of the subroutine takes unit time. A problem is called $\mathcal{NP}$-hard, if there is an $\mathcal{NP}$-complete problem that can be polynomially reduced to it. Thus, if a problem is $\mathcal{NP}$-hard, it is at least as difficult as any $\mathcal{NP}$-complete problem. It also follows that a polynomial time algorithm for an $\mathcal{NP}$-hard problem implies that $\mathcal{P} = \mathcal{NP}$. Whether $\mathcal{P} = \mathcal{NP}$ or not is a major unsolved problem in mathematics and computer science.

A heuristic algorithm is a process which seeks good (i.e. near-optimal) solutions at a reasonable computational time without guaranteeing either feasibility or optimality, and without even estimating how close a particular feasible solution is to optimality.

1.3 A Brief Survey

Hakimi(1964, 1965) first used the term $p$-median and introduced the $p$-median problem. The earliest reference for the $p$-median problem on a network is Hua et al(1962), who gave an algorithm in poetic form to find the 1-median on a tree net-
work. Since then, there have been numerous papers on the efficient computation of the $p$-median of a network. Many of these use an integer programming formulation for the $p$-median problem. We provide a compact version of this in Chapter 3.

Cornuejols, Fisher and Nemhauser(1977a) introduced a two-phase approach for generating and verifying near-optimal solutions for $p$-median problems. A greedy-interchange heuristic generated good upper bounds. A Lagrangian relaxation, obtained by dualizing (3.1.3), provided sharp lower bounds. Narula, Ogbu and Samuelsson(1977) used the same Lagrangian relaxation approach to get lower bounds for the $p$-median problem. Beasley(1993) presented a Lagrangian heuristic algorithm for location problems and showed computationally that it gave good quality solutions for $p$-median problems. His heuristic was based on Lagrangian relaxation and subgradient optimization.

Galvão(1980) presented a dual-based approach to solve the $p$-median problem. He modified Erlenkotter's(1978) method, DUALOC, for solving the uncapacitated facility location (UFL) problems. Mirchandani, Oudjit and Wong(1985) gave a "nested-dual" approach for the $p$-median problems. They used DUALOC as a subroutine. It is much faster than Galvão's(1980) algorithm, and can also be used to solve the $p$-median problem for several values of $p$. Captivo(1991) showed a fast primal and dual heuristic algorithm for the $p$-median problem.

Garfinkel, Neebe and Rao(1974) used Dantzig-Wolfe decomposition to solve the $p$-median problem. Magnanti and Wong(1981) developed techniques to ac-
celerate the convergence of Benders' decomposition. Thus, to find the \( p \)-median, they were able to reduce the number of integer programming problems to be solved. Nemhauser and Wolsey(1981) considered the Benders' cuts in the more general submodular set function. Conn and Cornuejols(1990) proposed a projection method for the \( p \)-median problem. Leung and Magnanti(1986) developed a strong cutting plane method to solve the capacitated plant location problems. Cheung(1980) described a minimum-cut approach.

The computational experience for general networks has shown that several algorithms mentioned above produce very good upper or lower bounds. The first formal probabilistic analysis of planar \( p \)-median algorithms was performed by Fisher and Hochbaum(1980). This was improved by Papadimitriou(1981). This type of analysis requires assumptions on the underlying probability distribution of problem instances. Different models give different results. For Euclidean model with uniform demand, Ahn, Cooper, Cornuejols and Frieze(1988) showed that a very good solution and a proof that the solution is within 0.2% of the optimal value can be found quickly almost surely.

Kariv and Hakimi(1979) proved that the \( p \)-median problem is \( \mathcal{NP} \)-hard even in the case where the network has a simple structure, (e.g. planar graph of maximum node degree 3). However, if the underlying graph of the network is a tree, Hua et al(1962), Goldman(1971) gave an \( O(n) \) algorithm for finding a 1-median. Matula and Kolde(1976), Kariv and Hakimi(1979), Hsu(1982) and Tamir(1996)
respectively described $O(p^2 n^3)$, $O(p^2 n^2)$, $O(p n^3)$ and $O(p n^2)$ algorithms for the $p$-median problem on the tree.
Chapter 2

On the 1-median Problem

2.1 Introduction

We have shown that the $p$-median problem has been extensively studied in the literature. In this Chapter, we will consider the combinatorial properties of $p$-medians which will be used in later chapters. The easiest case of the $p$-median problem is when $p = 1$. In Section 2.2, we show some connections between the $p$-median of the network and the 1-median of its region. Mirchandani and Oudjit (1980) considered the connection between 2-median and 1-median on the tree network. We introduce a definition of stable type, and then prove that $X^*_p$ is a $p$-median if and only if the distance-sum $H(X^*_p)$ is minimum among all the $(1, \cdots, 1)$ stable types.

In Section 2.3, we show some "negative" results about the $p$-median problem: the $\mathcal{NP}$-hardness of the $p$-median problem and some limitations of heuristic algo-
2.2 Some Properties of p-medians

For a network $G = (V, E)$, let $V = V_1 \cup V_2 \cup \cdots \cup V_k$ be a decomposition of $V$ into $k$ pairwise disjoint subsets, with induced subgraphs $\langle V_1 \rangle$, $\langle V_2 \rangle$, $\cdots$, $\langle V_k \rangle$ respectively. Let $X^*_i,p_i$ be a $p_i$-median of $\langle V_i \rangle$, for $1 \leq i \leq k$.

**Definition 2.2.1.** $(X^*_1,p_1, X^*_2,p_2, \cdots, X^*_k,p_k)$ is a $(p_1, p_2, \cdots, p_k)$ stable type of $G$, if $d(v, X^*_i,p_i) \leq d(v, X^*_j,p_j)$ for every $v \in V_i$ and $1 \leq i, j \leq k, j \neq i$.

If $X^*_p$ is a $p$-median of $G = (V, E)$, we say that a node $v \in V$ is served by a node $x \in X^*_p$, if $d(v, x) = d(v, X^*_p)$, i.e. $x$ is a node of $X^*_p$ which is closest to $v$. A subset $W \subset V$ is served by a node $x \in X^*_p$, if every node $v \in W$ is served by $x$. Suppose $X^*_q \subset X^*_p$, a subset $W \subset V$ is served by $X^*_q$, if for any $v \in W$, there exists $x \in X^*_q$ such that $v$ is served by $x$. We say that a subset $W$ affixes to $x \in X^*_p$, if $W$ is a maximal subset of $V$ that is served by $x$. $W$ affixes to $X^*_q \subset X^*_p$, if $W$ is a maximal subset of $V$ that is served by $X^*_q$. A node is called a tie node if it can be served by two or more equally distant open facilities. In the tie case, $W$ may contain the tie nodes or not. This does not affect the result.

Let $S \subseteq V, U \subseteq V$ be any subsets. Define

$$H(S, U) = \sum_{v \in S} w(v) \cdot d(v, U) \quad (2.2.1)$$

**Theorem 2.2.1.** Let $X^*_p$ be a $p$-median of $G$, $x \in X^*_p$, and let $W$ affix to $x$. 

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Then \( \{x\} \) is a 1-median of \(<W>\).

**Proof.** If \( \{x\} \) is not a 1-median of \(<W>\), then there is \( y \in W \) different from \( x \), such that

\[
H(W, \{y\}) < H(W, \{x\}).
\]

Let \( Y^* = (X_p \setminus \{x\}) \cup \{y\} \), then

\[
H(Y^*) = H(W, Y^*) + H(V \setminus W, Y^*) < H(W, \{x\}) + H(V \setminus W, X_p \setminus \{x\}) = H(X_p^*).
\]

This contradicts the assumption that \( X_p^* \) is a \( p \)-median of \( G \). \( \blacksquare \)

Generally, we have the following result.

**Theorem 2.2.2.** Let \( X_p^* \) be a \( p \)-median of \( G \). \( X_q^* \subset X_p^* \). If \( W \) affixes to \( X_q^* \), then \( X_q^* \) is a \( q \)-median of \(<W>\).

The proof is similar to that of Theorem 2.2.1.

With the previous two theorems, we see that if \( X \) is a median of \( G \), then any partition of \( X \) will be a stable type of \( G \), i.e.

**Theorem 2.2.3.** Let \( X_p^* \) be a \( p \)-median of \( G \). \( X_{1,p_1}, X_{2,p_2}, \ldots, X_{k,p_k} \) be pairwise disjoint subsets of \( X_p^* \) such that \( X_{1,p_1} \cup X_{2,p_2} \cup \cdots \cup X_{k,p_k} = X_p^* \), then \( (X_{1,p_1}, X_{2,p_2}, \ldots, X_{k,p_k}) \) is a \((p_1,p_2,\ldots,p_k)\) stable type of \( G \).

**Proof.** Let \( W_1, W_2, \ldots, W_k \) be subsets of \( V \) that affix to \( X_{1,p_1}, X_{2,p_2}, \ldots, X_{k,p_k} \) respectively, then \( W_1 \cup W_2 \cup \cdots \cup W_k = V \). By Theorem 2.2.2, \( X_{1,p_1}, X_{2,p_2}, \ldots, X_{k,p_k} \)
are \( p_1, p_2, \ldots, p_k \) medians of \( W_1, W_2, \ldots, W_k \) respectively. For any \( v \in W_i \),

\[
d(v, X^*_i, p_j) \leq d(v, X^*_j, p_i). \quad 1 \leq i, j \leq k, i \neq j. \quad (2.2.2)
\]

since \( W_i \) affixes to \( X^*_i, p_i \). It may happen that \( W_i \cap W_j = \emptyset \) for some \( i \) and \( j \). Let these nodes affix to either one, but not both, subsets. Then, \( W_i \cap W_j = \emptyset \) and (2.2.2) still holds. \( \blacksquare \)

From the above, we know that every partition of a \( p \)-median gives a stable type. If we can find all the stable types of the given network, then, the \( p \)-median must be one of these. Actually, we have the following result. Let \( \mathcal{X} = \{ (X^*_1, p_1, X^*_2, p_2, \ldots, X^*_k, p_k) \} \) be the set of all \( (p_1, p_2, \ldots, p_k) \) stable types of \( G \).

**Theorem 2.2.4.** \( X^*_p \) is a \( p \)-median of \( G \) if and only if

\[
H(X^*_p) = \min_{(X^*_1, p_1, X^*_2, p_2, \ldots, X^*_k, p_k) \in \mathcal{X}} H(X^*_1, p_1 \cup X^*_2, p_2 \cup \cdots \cup X^*_k, p_k)
\]

where \( p = p_1 + p_2 + \cdots + p_k \).

**Proof.** By definition, we know that

\[
H(X^*_p) \leq \min_{(X^*_1, p_1, X^*_2, p_2, \ldots, X^*_k, p_k) \in \mathcal{X}} H(X^*_1, p_1 \cup X^*_2, p_2 \cup \cdots \cup X^*_k, p_k).
\]

We prove the opposite inequality below.

Let \( X^*_p = X^*_1, p_1 \cup X^*_2, p_2 \cup \cdots \cup X^*_k, p_k \) be a partition of \( X^*_p \). By Theorem 2.2.3, we know that \( (X^*_1, p_1, X^*_2, p_2, \ldots, X^*_k, p_k) \) is a \( (p_1, p_2, \ldots, p_k) \) stable type. Hence

\[
H(X^*_p) = H(X^*_1, p_1 \cup X^*_2, p_2 \cup \cdots \cup X^*_k, p_k)
\]

\[
\geq \min_{(X^*_1, p_1, X^*_2, p_2, \ldots, X^*_k, p_k) \in \mathcal{X}} H(X^*_1, p_1 \cup X^*_2, p_2 \cup \cdots \cup X^*_k, p_k).
\]

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\[ H(X_p^*) = \min_{(X_{1,p_1}, X_{2,p_2}, \ldots, X_{k,p_k}) \in \mathcal{X}} H(X_{1,p_1} \cup X_{2,p_2} \cup \cdots \cup X_{k,p_k}). \]

**Corollary 2.2.5.** If \( X = \{\{v_1\}, \{v_2\}, \ldots, \{v_p\}\} \) is the set of all \((1, 1, \ldots, 1)\) stable types of \( G \), then \( X_p^* \) is a \( p \)-median of \( G \) if and only if
\[
H(X_p^*) = \min_{(\{v_1\}, \{v_2\}, \ldots, \{v_p\}) \in \mathcal{X}} H(\{v_1\} \cup \{v_2\} \cup \cdots \cup \{v_p\}).
\]

Theorem 2.2.4 and Corollary 2.2.5 tell us that when the size of the problem is huge, we can use the following approximation idea to solve the \( p \)-median problem. We divide the problem into several small size problems (say \( p \)) and solve each of them (say as \( 1 \)-median), then combine the solutions of the small size problems together.

### 2.3 The Complexity of the \( p \)-median Problem

It is easy to find a \( 1 \)-median of a network, however, the \( p \)-median problem is \( \mathcal{NP} \)-hard. Kariv and Hakimi(1979) proved the following result:

**Proposition 2.3.1.** The problem of finding a \( p \)-median is \( \mathcal{NP} \)-hard even when the network is planar of maximum node degree 3 with the length of all the edges and the weight of all the nodes are 1.

Although the \( p \)-median problem is \( \mathcal{NP} \)-hard, there are several approximate algorithms: greedy heuristic, interchange heuristic, greedy-interchange heuristic, dynamic programming heuristic(Cornejols, Fisher and Nemhauser,1977a), branch and bound, nested-dual approach(Mirchandani, Oudjit and Wong,1985), which de-
pends on the efficiency of solving the uncapacitated facility location subproblems, and some others.

One of the approximate algorithms that should be mentioned is the $q$-enumeration greedy-interchange heuristic, which has the best performance guarantee according to Nemhauser and Wolsey (1978). In general, however, we cannot expect this algorithm to yield an optimal solution.

The greedy heuristic first chooses a node that solves the 1 median problem and then proceeds recursively. Suppose $k \leq p$ nodes have been selected. Then, choose the next node that yields the maximum decrease in the distance sum, until $k = p$. The interchange heuristic attempts to decrease the distance sum of an initial feasible solution by interchanging an open node with one that is not open. This process continues until a solution is found such that the distance sum cannot be decreased further by such interchanges.

The $q$-enumeration greedy-interchange heuristic algorithm: $(q \leq p)$

Step 1. (q-enumeration): List all subsets of nodes of $V$ with cardinality $q$

Step 2. (greedy): Let $k = q$. Do the greedy heuristic until $k = p$.

Step 3. (interchange): Do the interchange to every solution obtained by the greedy heuristic.

Step 4. (output): Output the best solution among the interchange.
The $q$-enumeration greedy-interchange heuristic is quite effective, even when $q = 1$. Cornuejols, Fisher and Nemhauser (1977a) showed the comparisons and we tested some randomly generated problems and got the same conclusion as well. Cornuejols, Fisher and Nemhauser (1977a), Nemhauser and Wolsey (1978) have the following results for this algorithm:

**Proposition 2.3.2.** *The complexity of the $q$-enumeration greedy heuristic algorithm is $O(sn^{q+1})$ and is guaranteed to achieve at least $[1 - \left(\frac{p}{q}\right)\left(\frac{p}{q-1}\right)^{p-q}]$* 100% percent of the optimal value and the percentage is the best.

The $q$-enumeration greedy-interchange heuristic usually performs better than the $q$-enumeration greedy. However, in the worst case, we have:

**Proposition 2.3.3.** *In the worst case, the greedy-interchange heuristic is no better than the greedy heuristic.*

Whitaker (1983) presented a fast greedy-interchange heuristic algorithm and reported the comparison with the greedy-interchange heuristic algorithm on a number of large randomly generated networks and some difficult problem sets. The fast greedy algorithm was nearly six times faster than the greedy algorithm. The fast interchange algorithm itself, excluding the greedy initialization, was at least ten and in some instances more than thirty times faster than the interchange algorithm.

There is another approach to the $p$-median problem that we have not mentioned: the integer programming formulation, which is the topic of the next chapter.
Chapter 3

Integer Programming for the p-median Problem

3.1 The Integer Programming Formulation for the p-median Problem

Integer programming is a very useful tool for combinatorial optimization problems. Most combinatorial optimization problems can be expressed as integer programming problems. An integer programming formulation for the p-median problem is obtained by introducing the following variables. Suppose $G = (V, E)$ is a weighted network with $|V| = n$ and weight $w(v)$ for every node $v \in V$. The length of the shortest distance between node $v_i$ and node $v_j$ in the network is $d(v_i, v_j)$. Let $x_{jj} = 1$ if node $v_j$ is one of the $p$ selected facility location points and $x_{jj} = 0$ otherwise; $x_{ij} = 1$ if $v_i$ is served by a facility at $v_j$ and $x_{ij} = 0$ otherwise. The
The integer programming formulation of the $p$-median problem is:

\[ (p - MP) \quad Z = \min \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij} \quad (3.1.1) \]

\[ s.t. \]

\[ \sum_{j=1}^{n} x_{jj} = p \quad (3.1.2) \]

\[ \sum_{j=1}^{n} x_{ij} = 1, \text{ for } 1 \leq i \leq n \quad (3.1.3) \]

\[ x_{jj} - x_{ij} \geq 0, \text{ for } 1 \leq i, j \leq n \text{ and } i \neq j \quad (3.1.4) \]

\[ x_{ij} \in \{0, 1\}, \text{ for } 1 \leq i, j \leq n. \quad (3.1.5) \]

where $c_{ij} = w(v_i) \cdot d(v_i, v_j)$ for $1 \leq i, j \leq n$.

The constraints (3.1.3) guarantee that every client is served by exactly one facility, whereas (3.1.4) guarantees that it must be served by one of the $p$ selected facility locations. Relax the constraint (3.1.5) with

\[ x_{ij} \geq 0 \quad (3.1.6) \]

since, by (3.1.3), all $x_{ij} \leq 1$. Call (3.1.1), (3.1.2), (3.1.3), (3.1.4) and (3.1.6) the relaxed $p$-median problem (RPMP) and call (3.1.1), (3.1.2), (3.1.3) and (3.1.6) the basic relaxed $p$-median problem (BRPMP). We show in Section 3.4 that in order to solve the RPMP, we can start from the BRPMP and add some constraints of (3.1.4).

We shall use the following notations:

\[ x_{rec}^T = (x_{11}, x_{21}, \ldots, x_{n1}, \ldots, x_{1n}, x_{2n}, \ldots, x_{nn}), \]
\[ c_{vec}^T = (c_{11}, c_{21}, \ldots, c_{n1}, \ldots, c_{1n}, c_{2n}, \ldots, c_{nn}), \]

to represent the vector form of \( x_{ij} \) and \( c_{ij} \) respectively, while \( X_{mat} = (x_{ij})_{n \times n} \), 
\( C_{mat} = (c_{ij})_{n \times n} \) represent the matrix form of the variables \( x_{ij} \) and data \( c_{ij} \). It should be mentioned that the order of \( x_{vec}^T \) is slightly different from the normal order, which is appropriate for the matrix form expression of the \( p \)-median problem below. We shall sometimes use the notation \( X_{mat}(p) \) or \( x_{vec}(p) \) to represent the optimal solution for the given \( p \) with its optimal value \( Z(X_{mat}(p)) \) or \( Z(x_{vec}(p)) \) if we just consider the optimal solution. Sometimes, we shall use \( X_{mat}^* = (x_{ij}^*) \) or \( x_{vec}^* = (x_{ij}^*) \) to represent the optimal solution to distinguish it from other solutions. Let \( Z_{BRPM(p)}(X_{mat}) \) or \( Z_{BRPM(p)}(x_{vec}) \) represent the objective value of \( X_{mat} \) or \( x_{vec} \) to the BRPM.

Let

\[
A = \begin{pmatrix}
    e_1^T & e_2^T & \cdots & e_n^T \\
    I_n & I_n & \cdots & I_n \\
    A_1 & A_2 & \cdots & A_n
\end{pmatrix}_{(n^2+1) \times n^2}
\]

where \( e_i \) is an \( n \)-column vector with \( i \)-th component 1 and 0 elsewhere, \( I_n \) is an \( n \times n \) identity matrix, and

\[
A_i = \begin{pmatrix}
    -I_{i-1} & 1 & 0 \\
    0 & 1 & -I_{n-i}
\end{pmatrix}_{(n-1) \times n}
\]

where \( 1 \) is an appropriate column vector with each component 1, and \( 0 \) is an appropriate zero matrix or a zero column vector. When \( i = 1 \), \( I_0 \) is empty. The
problem (3.1.1) to (3.1.5) can be reformulated as follows:

\[
(p - MP) \quad Z = \min \ c^T_{vec} x_{vec} \tag{3.1.7}
\]

\[s.t.\]

\[
Ax_{vec} \geq b \tag{3.1.8}
\]

\[0 \leq x_{vec} \leq 1, \ x_{vec} \text{ integer} \tag{3.1.9}
\]

where \(b^T = (p, 1, \cdots, 1, 0, \cdots, 0)\). The constraint (3.1.9) can be relaxed to

\[x_{vec} \geq 0. \tag{3.1.10}
\]

Note that the first \(n + 1\) constraints in (3.1.8) are equality constraints, but we keep the simplified notation for convenience.

Let \(y_0, y_i\) and \(y_{ij}\), for \(1 \leq i, j \leq n\) and \(i \neq j\), correspond to the constraints (3.1.2), (3.1.3) and (3.1.4) respectively. The dual formulation of the relaxed \(p\)-median problem is:

\[(DPMP) \quad D_{LP} = \max \ p y_0 + \sum_{i=1}^{n} y_i \tag{3.1.11}\]

\[s.t.\]

\[
y_0 + y_j + \sum_{i \neq j}^{n} y_{ij} \leq c_{jj} = 0 \tag{3.1.12}
\]

\[
y_i - y_{ij} \leq c_{ij}, \text{ for } 1 \leq i, j \leq n, i \neq j \tag{3.1.13}
\]

\[
y_0, y_i \text{ unrestricted, } y_{ij} \geq 0, \text{ for } 1 \leq i, j \leq n. \tag{3.1.14}
\]

In integer programming, a starting basic feasible solution can be hard to find and the optimal integral solutions can be interior points of the feasible region of the relaxed problem. Two special properties of \(p\)-MP are:
(i) the optimal integral solution is an extreme point of the feasible region.

(ii) a starting basic feasible solution is easy to find.

We can fix any $p$ nodes as the selected facilities. These give a starting feasible solution to the $p$-MP. Very frequently, RPMP has an optimal integral solution. This observation is supported by results in the literature on random generated problems. Thus, an efficient method for solving RPMP can also be an efficient method for solving the $p$-median problem. There are various approaches for solving RPMP: Lagrangian duality; the subgradient approach; Dantzig-Wolfe’s decomposition; Benders’ decomposition; the projection method; the cutting plane method; the separation method and the branch-and-bound method. The polyhedron of the feasible solutions to (3.1.8) and (3.1.10) has fractional extreme points when $n > 3$. In Section 3.2, we characterize the fractional extreme points of this polyhedron. From Theorem 2.2.1, we know that if $X^*_p$ is the optimal set of $p$ facility points of $G$, then any node $v$ of $X^*_p$ is a 1-median of the subgraph that affixes to $v$. From Corollary 2.2.5, we know that any $p$-median has the minimal distance-sum among all the $(1,1,\cdots,1)$ stable types.

Let $X_{mat} = (x_{ij})$ be an optimal solution to the BRPMP for some $p$, which need not be integral. Let $S_{j_k} = \{v_i : x_{ij_k} \neq 0\}, k = 1, \cdots, q$, where $x_{ij_k}$ is in the optimal solution. In Section 3.2, we prove that $v_{j_k}$ is the 1-median of the subgraph $<S_{j_k}>$ if $v_{j_k} \in S_{j_k}$. If these $S_{j_k}$ are pairwise disjoint and \( \bigcup_{1 \leq k \leq q} S_{j_k} = V \), then $X_q = \{v_{j_1}, \cdots, v_{j_q}\}$ is a q-median of $G$. The results are still true if we add some of
the constraints of (3.1.4) to the BRPMP. With the Examples in that Section, we show that for a fractional $p$, the optimal solution $X_{mat}(p) = (x_{ij})$ may give two different optimal integral solutions at the same time: a $p_1$-median and a $p_2$-median.

In Section 3.3, we consider the fractional optimal solutions to the RPMP. If we have a fractional optimal solution $x_{vec}$ to the RPMP with optimal dual vector $y$ and optimal dual slack vector $z$, then, for any $p$ selected facilities $X_p$, the distance-sum $H(X_p)$ is a summation of $Z(x_{vec})$ and the corresponding parts of $y$ and $z$.

The disadvantage of the p-MP is that it needs $n^2$ variables and $n^2 + 1$ constraints. In Section 3.4, we deal with the reduction of variables and constraints. We find that many variables $x_{ij}$ can be fixed at zero before we solve the p-MP if we have a good integral feasible solution. We formulate a simplified relaxed $p$-median problem (SRPMP), which reduces both the number of variables and constraints at the same time. It also provides an easy method to test whether the integral feasible solution is an optimal integral solution to the RPMP.

### 3.2 Basic Properties of the BRPMP and the RPMP

In this Section, we show some basic properties of the optimal solutions to the BRPMP and RPMP. We are interested in the fractional optimal solutions, although these results hold for the integral optimal solutions as well. We will start from the
BRPMP and then move to the RPMP, from 1-median and 2-median instance to the general p-median situation. These results are true for any p, which need not be integral. We also give concrete examples to explain these results.

**Theorem 3.2.1.** Let $X_{max} = (x_{ij})$ be an optimal solution to the BRPMP for some $p$ and $S_j = \{v_i : x_{ij} \neq 0\}$. If $v_j \in S_j$, then $v_j$ is a 1-median of the subgraph $<S_j>$.

**Proof.** It is clear that $<S_j>$ is a connected subgraph. Suppose $v_k \neq v_j$ is a 1-median of the subgraph $<S_j>$, i.e.

$$H(S_j, v_k) < H(S_j, v_j)$$

where $H(S, U)$ is described by (2.2.1). Thus $v_k \in S_j$. Let

$$\alpha = \min_{v_i \in S_j} \{x_{ij} : x_{ij} \neq 0\} > 0.$$ 

Then we construct a new solution to the BRPMP as follows:

$$x'_{st} = \begin{cases} 
    x_{st} - \alpha, & \text{if } t = j \text{ and } v_s \in S_j, \\
    x_{st} + \alpha, & \text{if } t = k \text{ and } v_s \in S_j, \\
    x_{st}, & \text{otherwise}.
\end{cases}$$

Note that $x_{jj} \neq 0$ by the assumption $v_j \in S_j$. It is easy to check that $X'_{max} = (x'_{ij})$ is a feasible solution to the BRPMP. Since for each $s, 1 \leq s \leq n$, if $v_s \in S_j$, then

$$\sum_{t=1}^{n} x'_{st} = \sum_{\substack{1 \leq t \leq n \\text{ or } \text{t} \neq j \\text{ or } \text{t} \neq k}} x_{st} + (x_{sj} - \alpha) + (x_{sk} + \alpha)$$

$$= \sum_{t=1}^{n} x_{st} = 1.$$
If \( v_s \not\in S_j \), then \( x'_{st} = x_{st} \) for \( 1 \leq t \leq n \). So \( \sum_{t=1}^{n} x'_{st} = \sum_{t=1}^{n} x_{st} = 1 \). Thus, constraints (3.1.3) hold for \( X'_{\text{mat}} \). For constraint (3.1.2), we have

\[
\sum_{t=1}^{n} x'_{tt} = \sum_{1 \leq t \leq n} x_{tt} + (x_{jj} - \alpha) + (x_{kk} + \alpha) \\
= \sum_{t=1}^{n} x_{tt} = p.
\]

Thus, \( X'_{\text{mat}} \) satisfies the constraint (3.1.2) also. So, \( X'_{\text{mat}} \) is a feasible solution to the BRPMP. However,

\[
 Z_{\text{BRPMP}}(x'_{\text{vec}}) - Z_{\text{BRPMP}}(x_{\text{vec}}) = \alpha \cdot (H(S_j, v_k) - H(S_j, v_j)) < 0,
\]

which contradicts the assumption that \( X_{\text{mat}} \) is an optimal solution to BRPMP.

**Theorem 3.2.2.** Let \( X_{\text{mat}} = (x_{ij}) \) be an optimal solution to the BRPMP for some \( p \) and \( S_{j_k} = \{v_i : x_{ij_k} \neq 0\} \) for \( 1 \leq k \leq q \). If \( v_{j_k} \in S_{j_k} \) and the subsets \( S_{j_k} \) are pairwise disjoint for \( 1 \leq k \leq q \), then \( X_q = \{v_{j_1}, \ldots, v_{j_q}\} \) is a \( q \)-median of the subgraph \(< \bigcup_{1 \leq k \leq q} S_{j_k} > \).

**Proof.** Without loss of generality, suppose \( X_q = \{v_1, \ldots, v_q\} \), \( S_j = \{v_i : x_{ij} \neq 0\} \) and \( S_j \) are pairwise disjoint for \( j = 1, \ldots, q \). \( S = \bigcup_{1 \leq j \leq q} S_j \). Suppose \( Y_q = \{u_1, \ldots, u_q\} \neq X_q \) is a \( q \)-median of the subgraph \(<S>\), i.e. \( H(S, Y_q) < H(S, X_q) \).

Let \( T_j \) be affixed to \( u_j \) and \( T_j \) be pairwise disjoint in \( S \) for \( 1 \leq j \leq q \). Then \( S = \bigcup_{1 \leq j \leq q} T_j \). We can find these pairwise disjoint \( T_j \) easily since \( Y_q \) is given. Let

\[
\alpha = \min_{1 \leq j \leq q, v_i \in S_j} \{x_{ij} : x_{ij} \neq 0\} > 0.
\]

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We construct a new solution to the BRPMP as follows:

\[
x'_{ij} = \begin{cases} 
  x_{ij} - \alpha, & \text{if } v_j \in X_q \setminus Y_q \text{ and } v_i \in S_j, \text{ or } v_j \in X_q \cap Y_q \text{ and } v_i \in S_j \setminus T_j, \\
  x_{ij} + \alpha, & \text{if } v_j \in Y_q \setminus X_q \text{ and } v_i \in T_j, \text{ or } v_j \in X_q \cap Y_q \text{ and } v_i \in T_j \setminus S_j, \\
  x_{ij}, & \text{otherwise.}
\end{cases}
\]

In the following, we verify that \( X'_{mat} \) is a feasible solution to the BRPMP.

For each \( i \), \( 1 \leq i \leq n \). If \( v_i \notin <S> \), there is exactly one \( x'_{ij} = x_{ij} - \alpha \), one \( x'_{ij} = x_{ij} + \alpha \) and the remaining \( x'_{ij} = x_{ij} \). Therefore,

\[
\sum_{j=1}^{n} x'_{ij} = \sum_{1 \leq j \leq n, j \neq j_1} x_{ij} + (x_{ij_1} - \alpha) + (x_{ij_2} + \alpha) \\
= \sum_{j=1}^{n} x_{ij} = 1.
\]

If \( v_i \notin <S> \), then \( x'_{ij} = x_{ij} \) for \( 1 \leq j \leq n \). So, \( \sum_{i=1}^{n} x'_{st} = \sum_{t=1}^{n} x_{st} = 1 \). Thus, constraints (3.1.3) hold for \( X'_{mat} \). Since \( |X_q \setminus Y_q| = |Y_q \setminus X_q| \) and \( x'_{jj} = x_{jj} \) if \( v_j \in X_q \cap Y_q \), this means that the number of \( x'_{jj} = x_{jj} - \alpha \) is the same as the number of \( x'_k = x_k + \alpha \). So, \( \sum_{j=1}^{n} x'_{jj} = \sum_{j=1}^{n} x_{jj} = p \). Thus, \( X'_{mat} \) satisfies the constraint (3.1.2) also. Therefore, \( X'_{mat} = (x'_{ij}) \) is a feasible solution to the BRPMP. However,

\[
Z_{BRPMP}(x'_{vec}) - Z_{BRPMP}(x_{vec}) = \alpha \cdot (H(S,Y_q) - H(S,X_q)) < 0,
\]

which contradicts the assumption that \( X_{mat} \) is an optimal solution.

Remark: It should be mentioned that \( <S> = \bigcup_{1 \leq j \leq q} S_j \) need not be connected, but, there must be at least 1 median node in each component of the subgraph \( <S> \).
The optimal solution to the RPMP also satisfies Theorem 3.2.1 and Theorem 3.2.2. We can then omit the condition \( v_{j_k} \in S_{j_k} \) because of the constraints (3.1.4).

**Theorem 3.2.3.** Let \( X_{\text{mat}} = (x_{ij}) \) be an optimal solution to the RPMP for some \( p \) and \( S_{j_k} = \{ v_i : x_{ij} \neq 0 \} \) for \( 1 \leq k \leq q \), then \( v_{j_k} \) is a \( 1 \)-median of the subgraph \( <S_{j_k}> \). If \( S_{j_k} \) are pairwise disjoint for \( 1 \leq k \leq q \), then \( X_q = \{ v_{j_1}, \ldots, v_{j_q} \} \) is a \( q \)-median of the subgraph \( <\bigcup_{1 \leq k \leq q} S_{j_k}> \).

**Proof.** We use the same assumptions, notations, the constructions of \( \alpha \) and \( X'_{\text{mat}} = (x'_{ij}) \) as in the proof of Theorem 3.2.2. In the following, we shall check that \( X'_{\text{mat}} \) satisfies constraints (3.1.4).

If \( v_j \in X_q \setminus Y_q \), then \( x'_{jj} = x_{jj} - \alpha \) and every non-zero element of the \( j \)-th column of \( X_{\text{mat}} = (x_{ij}) \) is reduced by \( \alpha \), i.e. \( x'_{ij} = x_{ij} - \alpha \geq 0 \). Therefore, we have \( x'_{jj} - x'_{ij} \geq 0 \), for \( 1 \leq i \leq n \).

If \( v_j \in Y_q \setminus X_q \), then \( x'_{jj} = x_{jj} + \alpha \) and every increased element of the \( j \)-th column of \( X_{\text{mat}} \) is added by \( \alpha \). Therefore, \( x'_{jj} - x'_{ij} \geq 0 \), for \( 1 \leq i \leq n \).

If \( v_j \in X_q \cap Y_q \), then \( x'_{jj} = x_{jj} \). The non-zero elements \( x_{ij} \) in the \( X_{\text{mat}} \) will be decreased by \( \alpha \) if the corresponding \( v_i \in S_j \setminus T_j \). The elements \( x_{ij} \) in the \( X_{\text{mat}} \) will be increased by \( \alpha \) if the corresponding \( v_i \in T_j \setminus S_j \), but these \( x_{ij} = 0 \), so \( x'_{ij} = x_{ij} + \alpha = \alpha \leq x'_{jj} \). Therefore, \( x'_{jj} - x'_{ij} \geq 0 \), for \( 1 \leq i \leq n \).

For the remaining columns of \( X'_{\text{mat}} = (x'_{ij}) \), \( x'_{ij} = x_{ij} \). Therefore, \( x'_{jj} - x'_{ij} \geq 0 \), for \( 1 \leq i \leq n \).
From the above, we know that (3.1.4) is hold for \( X'_{mat} = (x'_{ij}) \). By Theorem 3.2.2, we know that \( X'_{mat} \) satisfies the constraints (3.1.2) and (3.1.3) also. Therefore, \( X'_{mat} \) is a feasible solution to the RPMP. However,

\[
Z(x'_{vec}) - Z(x_{vec}) = \alpha \cdot (H(S, Y_q) - H(S, X_q)) < 0,
\]

which contradicts the assumption that \( x_{vec} \) is an optimal solution.  

**Corollary 3.2.4.** Let \( X_{mat} = (x_{ij}) \) be an optimal solution to the RPMP with some \( p \) and \( S_{j_k} = \{v_i \mid x_{ij_k} \neq 0\} \), for \( 1 \leq k \leq q \). If the subsets \( S_{j_k} \) are pairwise disjoint and \( \bigcup_{1 \leq k \leq q} S_{j_k} = V \), then \( X_q = \{v_{j_1}, \ldots, v_{j_q}\} \) is a q-median of \( G \).

It is easy to see that Theorems 3.2.1 and 3.2.2 are still true if some of the constraints of (3.1.4) are added on to the BRPMP, i.e.

**Corollary 3.2.5.** The results of Theorems 3.2.1, 3.2.2 are still true if a subset of the constraints of (3.1.4) is added to the constraints of the BRPMP.

**Example 3.2.1.** The tree network \( G = (V, E) \) shown in Figure 1 consists of a set \( V = \{v_1, \ldots, v_6\} \) of 6 nodes with weights \( w(v_1), \ldots, w(v_6) \) (the bracketed numbers), associated with each node. The numbers along each edge are the lengths of the corresponding edge.

There is no theoretical restriction in the \( p \)-MP formulation which requires \( p \) to be an integer. We experimented with non-integral values of \( p \) and found that the results provided additional information at times. For example, the optimal solution
to the BRPMP with $p = 3.5$ is

$$X_{\text{mat}(3.5)} = \begin{pmatrix}
  x_{11} & x_{24} & x_{35} \\
  x_{42} & x_{44} & x_{55} \\
  x_{66}
\end{pmatrix}$$

where only the non-zero elements $x_{ij} \neq 0$ are shown. The missing elements indicate that $x_{ij} = 0$. By Theorem 3.2.1, $v_1, v_4, v_5$ and $v_6$ are 1-medians of $<S_1>$, $<S_4>$, $<S_5>$ and $<S_6>$ respectively, where $<S_1> = \{v_1\}$, $<S_4> = \{v_2, v_4\}$, $<S_5> = \{v_3, v_5\}$ and $<S_6> = \{v_6\}$. $v_2$ is not a 1-median of $<S_2> = \{v_4\}$, since $v_2 \not\in S_2$. By Theorem 3.2.2, $\{v_1, v_4, v_5, v_6\}$ is a 4-median of $<S_1 \cup S_4 \cup S_5 \cup S_6> = G$, since $S_1, S_4, S_5, S_6$ are pairwise disjoint. $\{v_4, v_5\}$ is a 2-median of the subgraph $<S_4 \cup S_5>$, even though it is disconnected.

The optimal solution to the RPMP with $p = 1.1$ is

$$X_{\text{mat}(1.1)} = \begin{pmatrix}
  x_{11} & x_{13} & x_{14} \\
  x_{21} & x_{24} & x_{33} \\
  x_{31} & x_{33} & x_{44} \\
  x_{51} & x_{53} & x_{63} \\
  x_{61} & x_{63}
\end{pmatrix}$$
where again only those elements shown have \( x_{ij} \neq 0 \) and the missing ones have \( x_{ij} = 0 \). From Theorem 3.2.3, we know that \( v_1 \) is the 1-median and \( \{v_2, v_4\} \) is the 2-median of \( G \). Thus, the optimal solution gives us two different medians at the same time.

The optimal solution to the RPMP has a strong connection with a \( p \)-median of the network, even when the optimal solution is fractional. We shall consider these cases in the next Section.

3.3 Fractional Optimal Solution to the RPMP

In this Section, we consider the fractional optimal solution to the RPMP. If the optimal solution to the RPMP is integral, then we have a solution to the \( p \)-median problem. In the following, we use the Duality Theorem to show that there is a close connection between the fractional optimal solution to the RPMP and the \( p \)-median of the network. Furthermore, the distance-sum of any \( p \) selected facilities can be expressed as a summation of the optimal value to the RPMP plus the penalty cost of using these selected facilities. This is described below. Thus, we extend the results from the integral optimal solution of the RPMP to the general fractional optimal solution.

For the remainder of this Section, we assume that \( X_{mat} = (x_{ij}) \) is the fractional optimal solution to the RPMP with optimal value \( Z(X_{mat}) \), except when we mention the optimal solution specifically. Let \( y, s, \tau \) be the corresponding optimal dual, slack
and dual slack vectors respectively, i.e.

\[ Ax_{vec} - s = b \]  \hspace{1cm} (3.3.1) \\
\[ A^T y + z = c \]  \hspace{1cm} (3.3.2)

\[ s \geq 0, \text{ and } z \geq 0. \] By the complementary slackness conditions, we have that 
\[ z^T x = 0, \text{ and } y^T s = 0. \] Let \( y_{ij} \) be the optimal dual values that correspond to the 
constraints \( x_{jj} - x_{ij} \geq 0 \) and let \( s_{ij} \) be the optimal slack values such that

\[ x_{ij} - x_{jj} + s_{ij} = 0. \]  \hspace{1cm} (3.3.3)

Let \( z_{ij} \) be the corresponding dual slack values, \( 1 \leq i, j \leq n. \) By the complementary slackness, we have

\[ x_{ij}z_{ij} = 0 \quad \text{and} \quad y_{ij}s_{ij} = 0, \]

for \( 1 \leq i, j \leq n, \) since \( y_0 \) and \( y_i, \) for \( 1 \leq i \leq n, \) correspond to the equality constraints in the primal problem.

For any feasible solution \( x'_{vec} \) to the RPMP with the slack vector \( s' = Ax'_{vec} - b, \)
we have

\[ Z(x'_{vec}) = c_{vec}^T x'_{vec} \]
\[ = (y^T A + z^T) x'_{vec} \]
\[ = y^T A x'_{vec} + z^T x'_{vec} \]
\[ = y^T (b + s') + z^T x'_{vec} \]
\[ = y^T b + y^T s' + z^T x'_{vec} \]
\[ = Z(x_{vec}) + y^T s' + z^T x'_{vec} \]  \hspace{1cm} (3.3.4)

The expression \( y^T s' + z^T x'_{vec} \) is called the penalty cost of \( x'_{vec}. \) This \( z^T x'_{vec} \) is
reminiscent of the duality gap in nonlinear programming. Thus, to find a $p$-median of $G$ is equivalent to finding an integral solution that minimizes the penalty cost. The difficulty is that sometimes the penalty cost is zero, sometimes it is not. In this Section, we show the connections between the optimal solution of the RPMP and the integral feasible solution of the $p$-MP. Let

$$S_j = \{v_i : x_{ij} \neq 0\}, \text{ for all } j.$$  

(3.3.5)

We start from the 2-median problem and then extend it to the general $p$-median problem.

**Lemma 3.3.1.** If $p = 2$, given $X_2 = \{v_{j_1}, v_{j_2}\}$ such that $S_{j_1} \cap S_{j_2} = \{v_h\}$ and $S_{j_1} \cup S_{j_2} = V$, then

(i) $H(\{v_{j_1}, v_{j_2}\}) = Z(x_{rec}) + \min\{y_{hj_1}, y_{hj_2}\}$;

(ii) $v_h$ is served by $v_{j_1}$ where $y_{hj_1} = \max\{y_{hj_1}, y_{hj_2}\}$;

(iii) For each $i$, $1 \leq i \leq n$, $i \neq h$, $v_i$ is served by $v_{j_1}$ where $x_{i j_1} \neq 0$.

**Proof.** Without loss of generality, suppose that $X_2 = \{v_1, v_2\}$, i.e. $x_{11} \neq 0, x_{22} \neq 0$ in the fractional optimal solution $X_{mat} = (x_{ij})$. Let $S_k = \{v_i : x_{ik} \neq 0\}, k = 1, 2; S_1 \cap S_2 = \{v_h\}, S_1 \cup S_2 = V$. To calculate $H(\{v_1, v_2\})$, each $v_i$ must be served either by $v_1$ or by $v_2$. By the assumption, we have that either $x_{i1} \neq 0, x_{i2} = 0$, or $x_{i1} = 0, x_{i2} \neq 0$ for each $i \neq h$. Note that $x_{h1} \neq 0$ and $x_{h2} \neq 0$. It is clear that $v_i$ is served by $v_1$ if $x_{i1} \neq 0$ or by $v_2$ if $x_{i2} \neq 0$. Otherwise, suppose that there is a node $v_t, t \neq h$, served by $v_1$ with $x_{t1} = 0, x_{t2} \neq 0$. Thus $c_{t1} < c_{t2}$. Let $\alpha = \min\{x_{11}, x_{22}, x_{12}\} > 0$. We construct a new feasible solution
\( X'_{\text{mat}} = (x'_{ij}) \) to the RPMP by setting \( x'_{i1} = \alpha, x'_{i2} = x_{i2} - \alpha; x'_{ij} = x_{ij} \) otherwise. Thus, \( Z(x'_\text{vec}) - Z(x_{\text{vec}}) = \alpha(c_{i1} - c_{i2}) < 0 \), which contradicts the assumption that \( x_{\text{vec}} \) is an optimal solution. Thus, (iii) holds.

If \( v_h \) is served by \( v_1 \), we construct an integral feasible solution \( X'_{\text{mat}} = (x'_{ij}) \) to the RPMP as follows:

\[
x'_{ij} = \begin{cases} 
1, & \text{if } j = 1 \text{ and } x_{i1} \neq 0, \\
1, & \text{if } j = 2 \text{ and } x_{i2} \neq 0, i \neq h, \\
0, & \text{otherwise}.
\end{cases}
\]

Since \( S_1 \cup S_2 = V \) and \( S_1 \cap S_2 = \{v_h\} \), then \( X'_{\text{mat}} \) is a feasible integral solution to the RPMP with \( p = 2 \). Actually, \( H(\{v_1, v_2\}) = Z(x'_\text{vec}) \) if \( v_h \) is served by \( v_1 \).

Let \( s' = Ax'_\text{vec} - b \), where \( s'_{ij} \) is the component of \( s' \) such that \( x'_{ij} - x'_{jj} + s'_{ij} = 0 \). It is clear from the construction that \( x'_{ij} = 0 \) if \( x_{ij} = 0 \). Thus,

\[
z^Tx'_\text{vec} = 0.
\]

Also \( s'_{ij} = 0 \) if \( s_{ij} = 0 \), for \( i \neq h \), where \( s_{ij} \) satisfies (3.3.3). The one exception is \( s'_{h2} = 1 \) while \( s_{h2} \) could be zero depending on whether or not \( x_{h2} - x_{22} = 0 \). However,

\[
y^{T}s' = y_{h2}
\]

is always true because of the complementary slackness conditions. Therefore, when \( v_h \) is served by \( v_1 \), we have

\[
Z(x'_\text{vec}) = Z(x_{\text{vec}}) + y^{T}s' + z^{T}x'_\text{vec} = Z(x_{\text{vec}}) + y_{h2}.
\]

With the same argument, if \( v_h \) is served by \( v_2 \), we could construct another
integral feasible solution $X_{\text{mat}}'' = (x''_{ij})$ such that $Z(x_{\text{vec}}'') = Z(x_{\text{vec}}) + y_{h1}$. Thus,

$$H(\{v_1, v_2\}) = \min \{Z(x_{\text{vec}}'), Z(x''_{\text{vec}})\} = Z(x_{\text{vec}}) + \min \{y_{h1}, y_{h2}\},$$

and $v_h$ is served by the $v_{j_h}$ with the larger value of $y_{h_j}$.

Remark: If $v_h = v_{j_1}$ or $v_h = v_{j_2}$, there is no $y_{jj}$. We set $y_{jj} = M$, for $1 \leq j \leq n$, where $M$ is a very large number. This is useful later.

If the intersection of $S_{j_1}$ and $S_{j_2}$ has more than one node, we have

Lemma 3.3.2. If $p = 2$, given $X_2 = \{v_{j_1}, v_{j_2}\}$ such that $S_{j_1} \cap S_{j_2} = \{v_i : i \in I\}$ and $S_{j_1} \cup S_{j_2} = V$, then

(i) $H(\{v_{j_1}, v_{j_2}\}) = Z(x_{\text{vec}}) + \sum_{i \in I} \min \{y_{ij_1}, y_{ij_2}\}$;

(ii) For each $i, i \in I$, $v_i$ is served by $v_{j_k}$ where $y_{ij_k} = \max \{y_{ij_1}, y_{ij_2}\}$;

(iii) For each $i, i \notin I$, $v_i$ is served by $v_{j_k}$ where $x_{ij_k} \neq 0$.

Proof. By Lemma 3.3.1, we see that for each $i \in I$, $v_i$ is served by $v_{j_1}$ if $y_{ij_1} \geq y_{ij_2}$, or by $v_{j_2}$ if $y_{ij_2} > y_{ij_1}$. For each $i \notin I$, $v_i$ is served by $v_{j_1}$ if $x_{ij_1} \neq 0$ or by $v_{j_2}$ if $x_{ij_2} \neq 0$. There is only (i) left. We claim that $H(\{v_{j_1}, v_{j_2}\}) = Z(x_{\text{vec}}')$, where $x_{\text{vec}}'$ is constructed as follows:

$$x'_{ij} = \begin{cases} 1, & \text{for } i \in I, \text{ if } j = j_1 \text{ and } y_{ij_1} \geq y_{ij_2}, \text{ or if } j = j_2 \text{ and } y_{ij_1} < y_{ij_2}; \\ 1, & \text{for } i \notin I, \text{ if } j = j_1 \text{ and } x_{ij_1} \neq 0, \text{ or if } j = j_2 \text{ and } x_{ij_2} \neq 0; \\ 0, & \text{otherwise}. \end{cases}$$

$X_{\text{mat}}' = (x'_{ij})$ is an integral feasible solution to the RPMP, because $S_{j_1} \cap S_{j_2} = \ldots$
\{v_i : i \in I\} and S_{j_1} \cup S_{j_2} = V. We also have

\[ z^T x'_{vec} = 0, \]

since \( x'_{ij} = 0 \) if \( x_{ij} = 0 \). Let \( s' = Ax'_{vec} - b \) and \( s'_{ij} \) be the component of \( s' \) such that \( x'_{ij} - x'_{jj} + s'_{ij} = 0 \). Thus, \( s'_{ij} = 0 \) if \( s_{ij} = 0 \), where \( s_{ij} \) satisfies (3.3.3), with the following exceptions: \( s'_{ij_1} = 1 \) and \( s'_{ij_2} = 0 \) if \( y_{ij_1} < y_{ij_2} \), or \( s'_{ij_2} = 1 \) and \( s'_{ij_1} = 0 \) if \( y_{ij_1} \geq y_{ij_2} \), for \( i \in I \). Thus,

\[ y^T s' = \sum_{i \in I} \min\{y_{ij_1}, y_{ij_2}\}. \]

Hence,

\[ H(\{v_{j_1}, v_{j_2}\}) = Z(x'_{vec}) = Z(x_{vec}) + y^T s' + z^T x'_{vec} \]

\[ = Z(x_{vec}) + \sum_{i \in I} \min\{y_{ij_1}, y_{ij_2}\}. \]

For the \( p \)-median problem, we first consider the special case where there is exactly one node \( v_h \) in the intersection of some of these \( S_{j_k} \), for \( 1 \leq k \leq p \). Let \( I \) be the set of all subsets of the index set \( \{j_1, \cdots, j_p\} \) with cardinality at least 2.

**Lemma 3.3.3.** Given \( X_p = \{v_{j_1}, \cdots, v_{j_p}\} \) such that \( S_{j_1} \cup \cdots \cup S_{j_p} = V \).

If \( \bigcup_{(\alpha_1, \cdots, \alpha_r) \in I} (S_{\alpha_1} \cap \cdots \cap S_{\alpha_r}) = \{v_h\} \), then

(i) \( H(\{v_{j_1}, \cdots, v_{j_p}\}) = Z(x_{vec}) + \sum_{k=1}^{p} y_{hj_k} - \max_{1 \leq k \leq p} \{y_{hj_k}\}; \)

(ii) \( v_h \) is served by \( v_{j_k} \) where \( y_{hj_k} = \max_{1 \leq k \leq p} \{y_{hj_k}\}; \)

(iii) For each \( i, i \neq h, v_i \) is served by \( v_{j_k} \) where \( x_{ij_k} \neq 0 \).

**Proof.** Without loss of generality, let \( r \) be the largest number such that \( S_{\alpha_1} \cap S_{\alpha_2} \cdots \cap S_{\alpha_r} = \{v_h\} \) with \( \{\alpha_1, \alpha_2, \cdots, \alpha_r\} \subset \{j_1, \cdots, j_p\} \) and \( \alpha_1, \alpha_2, \cdots, \alpha_r \).
are all distinct, \( r \geq 2 \). By the hypotheses, for any \( i \neq h \), there is only one \( j_k \) such that \( x_{ij_k} \neq 0 \) for \( 1 \leq k \leq p \). By the same argument as used in the proof of Lemma 3.3.1, \( v_i \) is served by \( v_{j_k} \) when we calculate \( H(\{v_{j_1}, \ldots, v_{j_p}\}) \). Suppose \( v_h \) is served by \( v_{\alpha_t} \), \( 1 \leq t \leq r \). Construct a set of integral feasible solution \( X_{mat}^{(t)} = (x_{ij}^{(t)}) \) for each \( t \), \( 1 \leq t \leq r \), as follows:

\[
x_{ij}^{(t)} = \begin{cases} 
1, & \text{if } i \neq h \text{ and } j \in \{j_1, \ldots, j_p\}, x_{ij} \neq 0, \\
1, & \text{if } i = h \text{ and } j = \alpha_t, \\
0, & \text{otherwise}.
\end{cases}
\]

Thus, \( X_{mat}^{(t)} = (x_{ij}^{(t)}) \) is an integral feasible solution to the RPMP, because \( S_{j_1} \cup \cdots \cup S_{j_p} = V \) and \( S_{\alpha_1} \cap S_{\alpha_2} \cdots \cap S_{\alpha_r} = \{v_h\} \). We also have

\[
z^T x_{vec}^{(t)} = 0,
\]

since \( x_{ij}^{(t)} = 0 \) if \( x_{ij} = 0 \). Let \( s^{(t)} = Ax_{vec}^{(t)} - b \), and let \( s_{ij}^{(t)} \) be the component of \( s^{(t)} \) such that \( x_{ij}^{(t)} - x_{jj}^{(t)} + s_{ij}^{(t)} = 0 \). Then \( s_{h\alpha_t}^{(t)} = 1 \), for \( 1 \leq q \leq r, q \neq t \); for the other \( s_{ij}^{(t)} \), \( s_{ij}^{(t)} = 0 \) if \( s_{ij} = 0 \), where \( s_{ij} \) satisfies (3.3.3). For those \( s_{ij} \neq 0 \), the corresponding \( y_{ij} = 0 \) by the complementary slackness. Thus,

\[
y^T s^{(t)} = \sum_{1 \leq q \leq r, q \neq i} y_{h\alpha_q}
\]

and

\[
H(\{v_{j_1}, \ldots, v_{j_p}\}) = \min_{1 \leq t \leq r} \{Z(x_{vec}^{(t)})\} = Z(x_{vec}) + \min_{1 \leq t \leq r} \{y^T s^{(t)} + z^T x_{vec}^{(t)}\}
\]

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\begin{align*}
= Z(x_{vec}) + \min \{ \sum_{1 \leq i \leq r} y_{k\alpha_i} \} \\
= Z(x_{vec}) + \sum_{t=1}^{r} y_{k\alpha_t} - \max_{1 \leq i \leq r} \{ y_{k\alpha_i} \}. \quad (3.3.6)
\end{align*}

\( v_h \) is served by \( v_{\alpha_i} \) with \( y_{k\alpha_i} = \max_{1 \leq i \leq r} \{ y_{k\alpha_i} \} \). For those \( j \in \{ j_1, \ldots, j_p \} - \{ \alpha_1, \ldots, \alpha_r \} \), we have \( s_{hj} \neq 0 \), since, by the assumption that \( r \) is the largest number of these \( S_{j_h} \) containing \( v_h \). Therefore \( y_{k\alpha_i} = 0 \), by the complementary slackness, so

\[ \sum_{t=1}^{r} y_{k\alpha_t} - \max_{1 \leq i \leq r} \{ y_{k\alpha_i} \} = \sum_{k=1}^{p} y_{k\alpha_i} - \max_{1 \leq k \leq p} \{ y_{k\alpha_i} \}. \]

Therefore,

\[ H(\{v_{j_1}, \ldots, v_{j_p}\}) = Z(x_{vec}) + \sum_{k=1}^{p} y_{k\alpha_k} - \max_{1 \leq k \leq p} \{ y_{k\alpha_k} \} \]

and \( v_h \) is served by \( v_{j_h} \) with \( y_{k\alpha_k} = \max_{1 \leq k \leq p} \{ y_{k\alpha_k} \} \). \( \Box \)

Remark: Sometimes, \( \max_{1 \leq i \leq r} \{ y_{k\alpha_i} \} = y_{hh} \) in (3.3.6). That is the reason why we set \( y_{jj} = M \) a very large number, for \( 1 \leq j \leq n \).

From Theorem 3.2.3, if these \( S_{j_h} \) are pairwise disjoint, then \( X_p \) is a \( p \)-median of the network, i.e. the penalty cost is zero. By Lemma 3.3.3, if there is only one node \( v_h \) in the intersection of some of these \( S_{j_h} \), the contribution to the penalty cost is \( \sum_{k=1}^{p} y_{k\alpha_k} - \max_{1 \leq k \leq p} \{ y_{k\alpha_k} \} \), when we calculate \( H(\{v_{j_1}, \ldots, v_{j_p}\}) \). Next, we examine the case with more than one node in the intersection of some of these \( S_{j_h} \).

**Lemma 3.3.4.** Given \( X_p = \{v_{j_1}, v_{j_2}, \ldots, v_{j_p}\} \) such that \( S_{j_1} \cup \cdots \cup S_{j_p} = V \).

If \( \bigcup_{(\alpha_1, \ldots, \alpha_r) \in I} (S_{\alpha_1} \cap \cdots \cap S_{\alpha_r}) = \{v_i : i \in I\} \), then
(i) \( H(\{v_{j_1}, \ldots, v_{j_p}\}) = Z(x_{vec}) + \sum_{i \in I}(\sum_{k=1}^{p} y_{i,j_k} - \max_{1 \leq k \leq p} \{y_{i,j_k}\}) \);

(ii) For each \( i \in I, v_i \) is served by \( v_{j_h} \) where \( y_{i,j_h} = \max_{1 \leq k \leq p} \{y_{i,j_k}\} \);

(iii) For each \( i \not\in I, v_i \) is served by \( v_{j_h} \) where \( x_{i,j_h} \neq 0 \).

**Proof.** By Lemma 3.3.3, for each \( i \in I, v_i \) is served by \( v_{j_h} \) with \( y_{i,j_h} = \max_{1 \leq k \leq p} \{y_{i,j_k}\} \). The contribution to the penalty cost is \( \sum_{k=1}^{p} y_{i,j_k} - \max_{1 \leq k \leq p} \{y_{i,j_k}\} \). Thus, the total penalty cost is \( \sum_{i \in I}(\sum_{k=1}^{p} y_{i,j_k} - \max_{1 \leq k \leq p} \{y_{i,j_k}\}) \). The proof is the same as that of Lemma 3.3.2. Therefore,

\[
H(\{v_{j_1}, \ldots, v_{j_p}\}) = Z(x_{vec}) + \sum_{i \in I}(\sum_{k=1}^{p} y_{i,j_k} - \max_{1 \leq k \leq p} \{y_{i,j_k}\}).
\]

For each \( i \not\in I, \) by Lemma 3.3.3, \( v_i \) is served by \( v_{j_h} \) with \( x_{i,j_h} \neq 0 \). \( \blacksquare \)

In the above, we only consider the case that the union of these \( S_{j_k}, 1 \leq k \leq p, \) is \( V \) and the intersection of some of these \( S_{j_k} \) is not empty. Suppose now that the \( S_{j_k}, 1 \leq k \leq p, \) are pairwise disjoint and \( \bigcup_{1 \leq k \leq p} S_{j_k} \subset V. \) For this situation, we have the analogous results. We start from the 2-median problem and extend it to the general \( p \)-median problem.

**Lemma 3.3.5.** If \( p = 2, \) given \( X_2 = \{v_{j_1}, v_{j_2}\} \) such that \( S_{j_1} \cap S_{j_2} = \phi \) and \( V - (S_{j_1} \cup S_{j_2}) = \{v_h\}, \) then

(i) \( H(\{v_{j_1}, v_{j_2}\}) = Z(x_{vec}) + \min\{z_{h,j_1}, z_{h,j_2}\} \),

(ii) \( v_h \) is served by \( v_{j_h} \) where \( z_{h,j_h} = \min\{z_{h,j_1}, z_{h,j_2}\} \),

(iii) For each \( i \neq h, v_i \) is served by \( v_{j_h} \) where \( x_{i,j_h} \neq 0 \).
Proof. Without loss of generality, suppose that \( X_2 = \{v_1, v_2\} \), i.e. \( x_{11} \neq 0, x_{22} \neq 0 \) in the fractional optimal solution \( X_{\text{mat}} = (x_{ij}), S_k = \{v_i : x_{ik} \neq 0\}, k = 1, 2; S_1 \cap S_2 = \emptyset \) and \( V - (S_1 \cup S_2) = \{v_h\} \). By the same argument as in the proof of Lemma 3.3.1, for any \( i \neq h, v_i \) is served by \( v_k \) if \( x_{ik} \neq 0, k = 1, 2, \) when we calculate \( H(\{v_1, v_2\}) \). Now suppose \( v_h \) is served by \( v_1 \). Construct an integral feasible solution \( X'_{\text{mat}} = (x'_{ij}) \) to the RPMP as follows:

\[
x'_{ij} = \begin{cases} 
1, & \text{if } j = 1 \text{ and } i = h \text{ or } x_{i1} \neq 0, \\
1, & \text{if } j = 2 \text{ and } x_{i2} \neq 0, \\
0, & \text{otherwise}.
\end{cases}
\]

Since \( S_1 \cap S_2 = \emptyset, V - S_1 \cup S_2 = \{v_h\}, X'_{\text{mat}} \) is a feasible integral solution to the RPMP with \( p = 2, H(\{v_1, v_2\}) = Z(x'_{vec}) \) when \( v_h \) is served by \( v_1 \).

Let \( s' = Ax'_{vec} - b \), where \( s'_{ij} \) is the component of \( s' \) such that \( x'_{ij} - x'_{jj} + s'_{ij} = 0 \). It is clear that \( s'_{ij} = 0 \) if \( s_{ij} = 0 \), where \( s_{ij} \) satisfies (3.3.3). Thus \( y^T s' = 0 \). Also \( x'_{ij} = 0 \) if \( x_{ij} = 0 \) with the one exception \( x'_{h1} = 1 \) and \( x_{h1} = 0 \). Thus \( z^T x'_{vec} = z_{h1} \).

Therefore, if \( v_h \) is served by \( v_1 \), we have

\[
Z(x'_{vec}) = Z(x_{vec}) + y^T s' + z^T x'_{vec} = Z(x_{vec}) + z_{h1}.
\]

With the same argument, if \( v_h \) is served by \( v_2 \), we construct another integral feasible solution \( X''_{\text{mat}} = (x''_{ij}) \) such that \( Z(x''_{vec}) = Z(x_{vec}) + z_{h2} \). Thus,

\[
H(\{v_1, v_2\}) = \min\{Z(x'_{vec}), Z(x''_{vec})\}
\]

\[
= Z(x_{vec}) + \min\{z_{h1}, z_{h2}\}
\]

and \( v_h \) is served by the node \( v_{jh} \) with the lower value of \( z_{hj} \). \( \blacksquare \)
Generally, if \( V - (S_{j_1} \cup S_{j_2}) \) has more than one node, we have

**Lemma 3.3.6.** If \( p = 2 \), given \( X_2 = \{v_{j_1}, v_{j_2}\} \) such that \( S_{j_k} \) are pairwise disjoint, and \( V - (S_{j_1} \cup S_{j_2}) = \{v_i : i \in L\} \), then

(i) \( H(\{v_{j_1}, v_{j_2}\}) = Z(x_{rec}) + \sum_{i \in L} \min \{z_{ij_1}, z_{ij_2}\} \).

(ii) For each \( i, i \in L \), \( v_i \) is served by \( v_{j_k} \) where \( z_{ij_k} = \min \{z_{ij_1}, z_{ij_2}\} \).

(iii) For each \( i, i \notin L \), \( v_i \) is served by \( v_{j_k} \) where \( x_{ij_k} \neq 0 \). ■

It is easy to follow the proof of Lemma 3.3.2 and use the result of Lemma 3.3.5 to prove this lemma. For the general \( p \)-median problem, we have

**Lemma 3.3.7.** Given \( X_p = \{v_{j_1}, v_{j_2}, \ldots, v_{j_p}\} \) such that \( S_{j_k} \) are pairwise disjoint and \( V - (S_{j_1} \cup \cdots \cup S_{j_p}) = \{v_i : i \in L\} \), then

(i) \( H(\{v_{j_1}, \cdots, v_{j_p}\}) = Z(x_{rec}) + \sum_{i \in L} \min_{1 \leq k \leq p} \{z_{ij_k}\} \).

(ii) For each \( i, i \in L \), \( v_i \) is served by \( v_{j_k} \) where \( z_{ij_k} = \min_{1 \leq k \leq p} \{z_{ij_k}\} \).

(iii) For each \( i, i \notin L \), \( v_i \) is served by \( v_{j_k} \) where \( x_{ij_k} \neq 0 \). ■

We can combine the above results together to obtain the following theorem for the general situation.

**Theorem 3.3.8.** Given \( X_p = \{v_{j_1}, v_{j_2}, \cdots, v_{j_p}\} \), if \( \bigcup_{\{a_1, \ldots, a_r\} \in I} (S_{a_1} \cap \cdots \cap S_{a_r}) = \{v_i : i \in I\} \), and \( V - (S_{j_1} \cup \cdots \cup S_{j_p}) = \{v_i : i \in L\} \), then

(i) \( H(\{v_{j_1}, \cdots, v_{j_p}\}) = Z(x_{rec}) + \sum_{i \in I} (\sum_{k=1}^{p} y_{ij_k} - \max_{1 \leq k \leq p} \{y_{ij_k}\}) + \sum_{i \in L} \min_{1 \leq k \leq p} \{z_{ij_k}\} \).

(ii) For each \( i, i \in I \), \( v_i \) is served by \( v_{j_k} \) where \( y_{ij_k} = \max_{1 \leq k \leq p} \{y_{ij_k}\} \).

(iii) For each \( i, i \in L \), \( v_i \) is served by \( v_{j_k} \) where \( z_{ij_k} = \min_{1 \leq k \leq p} \{z_{ij_k}\} \),
(iv) For each \( i, i \notin I \cup L \), \( v_i \) is served by \( v_{jh} \) where \( x_{ijh} \neq 0 \).

Remark: We have demonstrated a connection between the distance-sum of any \( p \) selected facilities \( X_p \) and the optimal solution to the RPMP. \( H(\{v_{j1}, \cdots, v_{jp}\}) \) is expressed as the summation of \( Z(X_{mat}) \) and the corresponding dual values and dual slack values. Let \( W = \sum_{i \in I} (\sum_{k=1}^{p} y_{ijh} - \max_{1 \leq k \leq p} \{y_{ihb}\}) \) be called the related overcovering penalty cost; and let \( U = \sum_{i \in L} \min_{1 \leq k \leq p} \{z_{ih}\} \) be called the related undercovering penalty cost. Then \( H(\{v_{j1}, \cdots, v_{jp}\}) = Z(x_{vec}) + W + U \), and the integral optimal solution occurs when \( W + U \) is minimized. Note that both of the \( W \) and \( U \) terms occur in various \( p \)-median problem instances.

The tie cases, i.e. a client could be served by two or more equally distance open facilities, happen only in cases (ii) and (iii) of Theorem 3.3.8. Because, in the case (iv), there is only one \( x_{ijh} \neq 0 \) for each \( i \notin I \cup L \).

If some \( x_{ij} = 0 \) in the optimal solution \( X_{mat} \), we have the following additional result.

Theorem 3.3.9. If \( x_{kh} = 0 \) in \( X_{mat} \), for any \( p \) node set \( X_p \) such that \( v_k \in X_p \), and \( K \) is an index set such that \( i \in K \) if and only if \( v_i \) is served by \( v_k \), then

\[
H(X_p) \geq Z(x_{vec}) + \sum_{i \in K} y_{ih} + \sum_{i \in K} z_{ih}.
\]  

(3.3.7)

Proof. Let \( x'_{vec} = (x'_{ij}) \) be the integral solution to the RPMP corresponding to \( X_p \), i.e. \( H(X_p) = Z(x'_{vec}) \). Then, \( x'_{ih} = 1 \), if \( i \in K \) and \( x'_{ih} = 0 \), if \( i \notin K \), i.e.
Thus, \( s'_{ik} = 1 \) if \( i \notin K \). Thus, \( z^T x_{vec} ^{'} \geq \sum_{i \in K} z_{ih} \) and \( y^T s' \geq \sum_{i \in K} y_{ih} \), so

\[
H(X_p) = Z(x_{vec} ^{'} ) \\
= Z(x_{vec} ) + y^T s' + z^T x_{vec} ^{'} \\
\geq Z(x_{vec} ) + \sum_{i \in K} y_{ih} + \sum_{i \in K} z_{ih} \text{.} \]

In the following, we develop a criterion to discard a node \( v_h \) from further consideration as potential open facility in the \( p \)-median problem. Hence, we reduce the size of the problem.

**Theorem 3.3.10.** Let \( x^{(0)}_{vec} = (x_{ij}^{(0)}) \) be a feasible integral solution and \( x^{*}_{vec} = (x_{ij}^{*}) \) a fractional optimal solution to the RPMP with objective values \( Z(x^{(0)}_{vec}) \) and \( Z(x^{*}_{vec}) \) respectively. If \( x^{*}_{kh} = 0 \) in the \( x^{*}_{vec} \) and there is a \( k \in \{1, \cdots , n\} \) such that \( z^{*}_{kh} > Z(x^{(0)}_{vec}) - Z(x^{*}_{vec}) \) and \( y^{*}_{kh} > Z(x^{(0)}_{vec}) - Z(x^{*}_{vec}) \) then \( v_h \) will not be an open facility in any \( p \)-medians.

**Proof.** Suppose we have a set of \( p \) facilities \( X_p \) with \( v_h \in X_p \). Let \( S = \{v_i : i \in K\} \) be the subset of \( V \) that affixes to \( v_h \). From Theorem 3.3.9, if \( k \in K \), we have

\[
H(X_p) \geq Z(x^{*}_{vec}) + \sum_{i \notin K} y^{*}_{ih} + \sum_{i \in K} z^{*}_{ih} \\
\geq Z(x^{*}_{vec}) + z^{*}_{kh} \\
> Z(x^{*}_{vec}) + Z(x^{(0)}_{vec}) - Z(x^{*}_{vec}) \\
= Z(x^{(0)}_{vec}) \text{.}
\]

If \( k \notin K \), we have
\[ H(X_p) \geq Z(x_{vec}^*) + \sum_{i \in K} y_{ih}^* + \sum_{i \in K} z_{ih}^* \]
\[ \geq Z(x_{vec}^*) + y_{kk}^* \]
\[ > Z(x_{vec}^*) + Z(x_{vec}^{(0)}) - Z(x_{vec}^*) \]
\[ = Z(x_{vec}^{(0)}). \]

Thus, if \( v_h \) is one of the \( p \) selected facilities, then \( H(X_p) > Z(x_{vec}^{(0)}) \), no matter what other \( p - 1 \) selected facilities are used. Therefore, we can fix \( x_{ih} = 0 \) for \( i = 1, \cdots, n \). That is, node \( v_h \) will not be an open facility in any \( p \)-medians.

Since we have set \( y_{jj} = M \) for \( j = 1, \cdots, n \), if \( z_{kk}^* > Z(x_{vec}^{(0)}) - Z(x_{vec}^*) \), then node \( v_h \) will not be an open facility in any \( p \)-medians because it is too "pricey".

**Example 3.2.1 (continued).** We give here an optimal solution \( X_{mat} \) of the relaxed 3-median problem for the tree network with optimal value 90.

\[ X_{mat}(3) = \begin{pmatrix}
0.5 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0.5 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0.5 \\
\end{pmatrix} \quad (3.3.8) \]

The optimal dual slack values in matrix form \( z_{mat} = (z_{ij}) \) and part of the optimal dual values in matrix form \( y_{mat} = (y_{ij}) \) with main diagonal \( M \) are given below, since we are interested in \( y_{ij}, i \neq j \).

\[ y_{mat} = \begin{pmatrix}
M & 2 & 2 & 0 & 0 & 0 \\
0 & M & 0 & 10 & 0 & 0 \\
0 & 0 & M & 0 & 4 & 0 \\
0 & 40 & 0 & M & 0 & 0 \\
0 & 0 & 44 & 0 & M & 0 \\
0 & 0 & 10 & 0 & 0 & M \\
\end{pmatrix}, \quad z_{mat} = \begin{pmatrix}
0 & 0 & 0 & 32 & 32 & 64 \\
0 & 8 & 20 & 0 & 30 & 40 \\
6 & 26 & 0 & 36 & 0 & 6 \\
0 & 0 & 40 & 0 & 60 & 80 \\
0 & 42 & 0 & 66 & 0 & 0 \\
50 & 110 & 0 & 140 & 20 & 0 \\
\end{pmatrix} \]
The dual values $y_i, 0 \leq i \leq 6$, that correspond to constraints $x_{11} + \cdots + x_{i6} = 3$ and $x_{i1} + \cdots + x_{i6} = 1$, for $1 \leq i \leq 6$, are $(y_0, y_1, \ldots, y_6) = (-70, 70, 20, 14, 60, 66, 70)$.

From (3.3.8) and Theorem 3.2.3, we know that each of the nodes $v_1, v_3, v_4, v_5, v_6$ is respectively a 1-median of the subgraph $<S_1>$, $<S_3>$, $<S_4>$, $<S_5>$, $<S_6>$, where $<S_1> = \{v_1\}$, $<S_3> = \{v_1, v_3, v_5, v_6\}$, $<S_4> = \{v_2, v_4\}$, $<S_5> = \{v_3, v_5\}$ and $<S_6> = \{v_6\}$. Since $\{S_1, S_4, S_5, S_6\}$ and $\{S_3, S_4\}$ are pairwise disjoint respectively and $S_1 \cup S_4 \cup S_5 \cup S_6 = V$, $S_3 \cup S_4 = V$, then by Corollary 3.2.4, $\{v_1, v_4, v_5, v_6\}$ is a 4-median and $\{v_3, v_4\}$ is a 2-median of the tree.

Since $S_3 \cap S_5 = \{v_3, v_5\}$, by Lemma 3.3.4,

$$H(\{v_3, v_4, v_5\}) = Z(x_{vcc}) + \min\{y_{33}, y_{35}\} + \min\{y_{53}, y_{55}\}$$

$$= 90 + 4 + 44 = 138.$$ 

$v_3, v_5$ are served by themselves respectively, $v_1, v_6$ are served by $v_3, v_2$ is served by $v_4$. On the other hand,

$$H(\{v_3, v_4, v_5\}) = w(v_1)d(v_1, v_3) + w(v_2)d(v_2, v_4) + w(v_6)d(v_6, v_3)$$

$$= 34 \cdot 2 + 10 \cdot 1 + 30 \cdot 2 = 138.$$ 

If we calculate $H(\{v_1, v_5, v_6\})$, using Lemma 3.3.7,

$$H(\{v_1, v_5, v_6\}) = Z(x_{vcc}) + \min\{z_{21}, z_{25}, z_{26}\} + \min\{z_{41}, z_{45}, z_{46}\}$$

$$= 90 + 0 + 0 = 90.$$ 

$v_2$ and $v_4$ are served by $v_1$, $v_3$ is served by $v_5$. On the other hand

$$H(\{v_1, v_5, v_6\}) = w(v_2)d(v_2, v_1) + w(v_3)d(v_3, v_5) + w(v_4)d(v_4, v_1)$$

$$= 10 \cdot 2 + 10 \cdot 1 + 20 \cdot 3 = 90.$$ 

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Since this is the optimal value to the RPMP, \( \{v_1, v_5, v_6\} \) is a 3-median of the tree.

If we calculate \( H(\{v_3, v_5, v_6\}) \), by Theorem 3.3.8,

\[
H(\{v_1, v_5, v_6\}) = Z(x_{vec}) + \min\{y_{33}, y_{35}\} + \min\{y_{53}, y_{55}\} + \min\{y_{63}, y_{66}\} \\
+ \min\{z_{23}, z_{25}, z_{26}\} + \min\{z_{43}, z_{45}, z_{46}\} \\
= 90 + 4 + 44 + 10 + 20 + 40 = 208,
\]

\( v_1, v_2, v_4 \) are served by \( v_3 \). On the other hand,

\[
H(\{v_3, v_5, v_6\}) = w(v_1)d(v_1, v_3) + w(v_2)d(v_2, v_3) + w(v_4)d(v_4, v_3) \\
= 34 \cdot 2 + 4 + 20 \cdot 5 = 208.
\]

Suppose \( v_2 \) is contained in some subset \( X_3 \), say \( X_3 = \{v_2, v_5, v_6\} \), thus, \( K = \{1, 2, 4\} \) in Theorem 3.3.9 and

\[
H(\{v_2, v_5, v_6\}) \geq Z(x_{vec}) + (z_{12} + z_{22} + z_{42}) + (y_{32} + y_{52} + y_{62}) \\
= 90 + (0 + 8 + 0) + (0 + 0 + 0) = 98.
\]

On the other hand,

\[
H(\{v_2, v_5, v_6\}) = w(v_1)d(v_1, v_2) + w(v_3)d(v_3, v_5) + w(v_4)d(v_4, v_2) \\
= 34 \cdot 2 + 10 \cdot 1 + 20 \cdot 1 = 98.
\]

Let \( x_{vec}^{(0)} \) be the integral feasible solution to the RPMP that corresponds to \( \{v_1, v_3, v_4\} \). Thus, \( Z(x_{vec}^{(0)}) = 92 \). Since

\[
z_{22} = 8 \\
> Z(x_{vec}^{(0)}) - Z(x_{vec}^*) \\
= 92 - 90 = 2,
\]

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we can fix $x_{i2} = 0$, for $1 \leq i \leq 6$, by Theorem 3.3.10. Hence, $v_2$ will never be open in any 3-medians.

For the tree network, Kolen (1983) has the following result:

**Proposition 3.3.11.** The distance-sum of the p-median is the same as the optimal value to the RPMP of the tree network.

Sometimes, it may happen that $x_{jj} = 1$, for $j = j_1, j_2, \cdots, j_p$ while some of $x_{ij}$ are fractional, i.e. $0 < x_{ij} < 1$. In such a situation, we have the following result:

**Theorem 3.3.12.** If the fractional optimal solution $X_{mat} = (x_{ij})$ to the RPMP is of the form: $x_{j_1j_1} = \cdots = x_{j_hp} = 1$ and $0 < x_{ijh} < 1$ for some $i$, $1 \leq h \leq p$, then \{v_{j_1}, \cdots, v_{j_p}\} is a p-median.

**Proof.** Without loss of generality, suppose $x_{11} = \cdots = x_{pp} = 1$ in $X_{mat}$ and $0 < x_{k1} < 1$. Then, some other $x_{kj}$ are also fractional with $j = 2, \cdots, p$, since $x_{k1} + \cdots + x_{kp} = 1$ and $x_{k,p+1} = \cdots = x_{kn} = 0$. If $x_{k2} \neq 0$, say, then $c_{k1} = c_{k2}$. This is true because if $c_{k1} < c_{k2}$, then we could set $x'_{k1} = x_{k1} + x_{k2}, x'_{k2} = 0$ and $x'_{ij} = x_{ij}$ otherwise. This would give a feasible solution to the RPMP with strictly lower objective value. Similarly, we cannot have $c_{k1} > c_{k2}$. Since this is the case, then we set $x_{k1} = 1$ and $x_{kj} = 0$ for $j \neq 1$. Thus, we adjust the solution so that $X_{mat}$ becomes an integral feasible solution to the RPMP with the same objective value. Therefore, \{v_1, \cdots, v_p\} is a p-median.

From now on, a fractional optimal solution to the RPMP means a solution
where some of the $x_{jj}$, $1 \leq j \leq n$, are fractional.

### 3.4 The Reduction of Variables and Constraints

The disadvantage of the p-MP is that it needs $n^2$ variables and $n^2 + 1$ constraints. When $n$ is large (e.g. $n = 100$), the RPMP is much harder to solve, even though it gives a very good bound (Beasley, 1985; Ahn, Cooper, Cornuejols and Frieze, 1988). Now, let us examine the structure of the optimal solution of the BRPMP.

**Proposition 3.4.1.** There exists an optimal solution $X_{mat} = (x_{ij})$ to the BRPMP such that for each $j$, $1 \leq j \leq n$, either $x_{jj} = 1$, or $x_{jj} = \gamma_j < 1$ and $x_{ji} = 1 - \gamma_j$, where $v_{ij}$ is one of the closest distance nodes to $v_j$. $\blacksquare$

This result follows from the observations in the proof of Theorem 3.3.12. We now add the constraints $x_{jj} - x_{ij} \geq 0, 1 \leq i \leq n$, to the BRPMP to formulate a new problem, where $v_j$ is one of the closest distance nodes to $v_i$. By Corollary 3.2.5, Theorems 3.2.1 and 3.2.2 both hold for the new optimal solution. This is better than the "weak" integer programming formulation for the p-median problem, which adds the constraints $\sum_{j=1}^{n} x_{ij} - nx_{jj} \leq 0$, for $1 \leq i \leq n$ (Nemhauser and Wolsey, 1988; Christofides and Beasley, 1982). Theorems 3.2.1 and 3.2.2 are no longer true for the optimal solution to the weak integer programming problem.

In the following, we construct a simplified relaxed p-median problem(SRPMP)
from the RPMP. We order the given data $c_{ij}$ as follows: for each $i = 1, \ldots, n$, let $L_i = \{i_1, i_2, \ldots, i_n\}$ be a sequence of indices such that if $i_j < i_k$, then $c_{i_j i} \leq c_{i_k i}$. If $c_{i_j i} = c_{i_k i}$, then $c_{i_j i}$ precedes $c_{i_k i}$ if $i_j < i_k$. It is clear that for every $i$, $c_{i_i i} = 0$, which corresponds to the variable $x_{ii}$. Suppose we have an integral feasible solution $X_{\text{mat}}^{(0)} = (x_{ij}^{(0)})$ to the RPMP. For each $i$, there is a unique $j$ such that $x_{ij}^{(0)} = 1$, and this is denoted by $j^{(0)}(i)$. If $j^{(0)}(i) = i_1$, set $j^{(0)}(i) = i_2$. We construct a new linear programming problem from the RPMP as follows.

For each $i = 1, \ldots, n$, define $r(i)$ to be the index such that $c_{i r(i)}^{(0)} = \cdots = c_{i r(i) + 1}$, $< c_{i r(i) + 2}$. If $r(i) = 2$, add constraints $x_{jj} - x_{ij} \geq 0$ with $j = i_2$, and $x_{i_1 i} + x_{i_2 i} + x_{i_2 i} = 1$. If $r(i) > 2$, add constraints $x_{jj} - x_{ij} \geq 0$ for $j = i_2, i_3, \ldots, i_{r(i) - 1}$, and $x_{i_1 i} + x_{i_2 i} + \cdots + x_{i_{r(i)} i} = 1$.

Finally, we have the constraint $x_{11} + x_{22} + \cdots + x_{nn} = p$.

The objective function is a function of only those $x_{ij}$ that appear in the above constraints. We call the above linear programming problem a simplified relaxed p-median problem (SRPMP). It should be mentioned that $r(i)$ is determined by the initial feasible integral solution $X_{\text{mat}}^{(0)}$. It is worth noting that we do not add constraints $x_{jj} - x_{ij} \geq 0$ for $j = i_{r(i)}$, $1 \leq i \leq n$, to the SRPMP. We will explain this below.

Let $X_{\text{SRPMP}} = (x_{ij})$ be an optimal solution to the SRPMP. $Z(X_{\text{SRPMP}})$ denotes the optimal value to the SRPMP and the objective value to the RPMP by setting the missing $x_{ij} = 0$ in $X_{\text{SRPMP}}$. These two values are equal even though
\( X_{SRPM} \) may not be feasible to the RPMP, since the objective functions are the same. Let \( X^*_{mat} \) be an optimal solution to the RPMP. Then,

**Theorem 3.4.2.** \( Z(X_{SRPM}) \leq Z(X^*_{mat}) \).

**Proof.** If \( X^*_{mat} \) is feasible to the SRPMP, then \( Z(X_{SRPM}) \leq Z(X^*_{mat}) \), since \( X_{SRPM} \) is optimal to the SRPMP. If \( X^*_{mat} \) is infeasible to the SRPMP, this means that there are some \( x^*_{kh} > 0 \) in the \( X^*_{mat} \) that do not appear in the SRPMP formulation.

Suppose, for the moment, that there is only one such \( x^*_{kh} > 0 \) that does not appear in the SRPMP formulation, thus \( h \neq k \). Let \( h = k_i \). There is an \( \alpha \), \( 2 < \alpha < t \), such that the corresponding constraint in the SRPMP is \( x_{kk_1} + x_{kk_2} + \cdots + x_{kk_\alpha} = 1 \). We construct a feasible solution \( X' = (x'_{ij}) \) to the SRPMP as follows:

\[
x'_{kk_\alpha} = x^*_{kk_\alpha} + x^*_{kk_i}, x'_{kk_i} = 0; x'_{ij} = x^*_{ij} \text{ otherwise.}
\]

Thus, all the \( x'_{ij} \) that do not appear in the SRPMP formulation have value zero. If \( i = k \), \( X' \) satisfies \( x'_{kk_1} + x'_{kk_2} + \cdots + x'_{kk_\alpha} = 1 \) and \( x'_{jj} - x'_{kj} \geq 0 \) for \( j = k_2, \ldots, k_{\alpha - 1} \). For the rest of \( i \), all the \( x'_{ij} \) that appear in the SRPMP satisfy the constraints \( x'_{ii_1} + x'_{ii_2} + \cdots + x'_{ii_{r(i)}} = 1 \) and \( x'_{jj} - x'_{ij} \geq 0 \) for \( j = i_2, i_3, \ldots, i_{r(i)-1} \). Thus, \( X' \) is a feasible solution to the SRPMP and \( Z(X') - Z(X^*_{mat}) = x^*_{kk}(c_{kk_\alpha} - c_{kk_i}) < 0 \), since \( c_{kk_\alpha} < c_{kk_i} \) in the construction of the SRPMP. Since \( Z(X_{SRPM}) \leq Z(X') \), it follows that \( Z(X_{SRPM}) < Z(X^*_{mat}) \).

If more than one variable \( x^*_{kh} \neq 0 \) in the \( X^*_{mat} \) do not appear in the SRPMP
formulation, we construct a feasible solution $X''$ to the SRPMP step by step using the same method to construct $X'$ above until all these $x_{kh}^*$ become zero in $X''$.

If the optimal solution to the SRPMP violates constraints $x_{jj} - x_{ij} \geq 0$, for some $j = r(i)$, then if $r(i) = n$, we add constraint $x_{jj} - x_{ij} \geq 0$ with $j = i_n$, if $r(i) < n$, we set $j^{(0)}(i) = r(i) + 1$, for those violated constraints. Following the procedure above, we construct a new SRPMP and solve the new SRPMP again until all the constraints $x_{jj} - x_{ij} \geq 0$ for current $j = r(i), 1 \leq i \leq n$, are satisfied. The number of such iterations is at most $n^2 - 2n$, because there are only $n^2 + 1$ constraints in the RPMP and there are at least $2n + 1$ constraints in the first formulation of the SRPMP. In each iteration of constructing new SRPMP, at least one new constraint is added, and

**Corollary 3.4.3.** The optimal value of the new SRPMP is greater than or equal to that of the old SRPMP.

**Proof.** Let $X^{(1)}$ and $X^{(2)}$ be the optimal solutions to the old and new SRPMP respectively. If $X^{(2)}$ is feasible to the old SRPMP, then $Z(X^{(1)}) \leq Z(X^{(2)})$. If $X^{(2)}$ is infeasible to the old SRPMP, from the proof of Theorem 3.4.2, there is a feasible solution $X'$ to the old SRPMP, such that $Z(X^{(1)}) \leq Z(X') < Z(X^{(2)})$.

It is easy to see that the integral feasible solution $X_{\text{mat}}^{(0)}$ to the RPMP is also a feasible solution to the SRPMP. The number of variables and constraints of the SRPMP depends on the $X_{\text{mat}}^{(0)}$. The closer the objective value of $X_{\text{mat}}^{(0)}$ is to the optimal value of the RPMP, the fewer the number of variables and constraints in
the SRPMP. On the other hand, if the integral feasible solution $X_{\text{mef}}^{(0)}$ is an optimal solution to the SRPMP, $X_{\text{mef}}^{(0)}$ must be also an optimal solution to the RPMP. Thus, we have a simple method to testify whether the integral feasible solution $X_{\text{mef}}^{(0)}$ is an optimal integral solution to the RPMP or not. It is easy to obtain an integral feasible solution $X_{\text{mef}}^{(0)}$.

Now, let us consider the following situation.

If we add constraints $x_{jj} - x_{ij} \geq 0$, for $j = r(i)$, $1 \leq i \leq n$, to the construction of SRPMP, then the optimal solution of SRPMP is feasible to the RPMP. Thus, the optimal value of SRPMP is greater than or equal to the optimal value of RPMP. It is an open problem to decide under which conditions these two optimal solutions will be the same.

\[\text{Figure 2}\]
Example 3.4.1. The network $G = (V, E)$ shown in Figure 2 consists of a set $V = \{v_1, x_2, \ldots, v_6\}$ of 6 nodes with weights $w(v_1), w(v_2), \ldots, w(v_6)$ (the bracketed numbers). The numbers along each edge are the lengths of the corresponding edge.

The distance between any two nodes is in the distance matrix $D = (d_{ij})$, where $d_{ij} = d(v_i, v_j)$ and $C = (c_{ij})$ where $c_{ij} = w(v_i)d(v_i, v_j) = w(v_i)d_{ij}$.

$$D = \begin{pmatrix} 0 & 3 & 5 & 9 & 7 & 4 \\ 3 & 0 & 4 & 6 & 8 & 9 \\ 5 & 4 & 0 & 3 & 5 & 8 \\ 9 & 6 & 3 & 0 & 6 & 7 \\ 7 & 8 & 5 & 6 & 0 & 5 \\ 4 & 9 & 8 & 7 & 5 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 21 & 35 & 63 & 49 & 28 \\ 27 & 0 & 36 & 54 & 72 & 81 \\ 25 & 20 & 0 & 15 & 25 & 40 \\ 99 & 66 & 33 & 0 & 66 & 77 \\ 63 & 72 & 45 & 54 & 0 & 45 \\ 40 & 90 & 80 & 70 & 50 & 0 \end{pmatrix}.$$ 

The 2-median RPMP formulation of the network is as follows:

$$Z = \min \sum_{j=1}^{6} \sum_{i=1}^{6} c_{ij}x_{ij}$$

s.t.

$$\sum_{j=1}^{6} x_{jj} = 2,$$

$$\sum_{j=1}^{6} x_{ij} = 1, \quad 1 \leq i \leq 6$$

$$x_{jj} - x_{ij} \geq 0, \quad 1 \leq i, j \leq 6 \text{ and } i \neq j$$

$$x_{ij} \geq 0, \quad 1 \leq i, j \leq 6.$$ 

Let $\{v_4, v_5\}$ be an integral feasible solution to the RPMP, i.e. $X_{max}^{(0)} = (x_{ij}^{(0)})$ with $x_{16}^{(0)} = x_{24}^{(0)} = x_{34}^{(0)} = x_{44}^{(0)} = x_{56}^{(0)} = x_{66}^{(0)} = 1; x_{ij}^{(0)} = 0$ otherwise, for $1 \leq i, j \leq 6$.

To construct the corresponding SRPMP, if $i = 2$, then $x_{22}^{(0)} = 0, j^{(0)}(2) = 4 = 2_4$ and $r(2) = 4$. Thus, we add constraints $x_{11} - x_{21} \geq 0, x_{33} - x_{23} \geq 0$ and $x_{21} + x_{22} + x_{23} + x_{24} = 1$. If $i = 4$, then $x_{44}^{(0)} = 1, j^{(0)}(4) = 3 = 4_2, r(4) = 2$. Thus,
we add constraints $x_{33} - x_{43} \geq 0$ and $x_{42} + x_{43} + x_{44} = 1$. So are the other values of $i$. The SRPMP formulation is as follows:

$$\min 21x_{12} + 28x_{16} + 27x_{21} + 36x_{23} + 54x_{24} + 20x_{32} + 15x_{34}$$

$$+ 66x_{42} + 33x_{43} + 45x_{53} + 45x_{56} + 40x_{61} + 50x_{66}$$

$$s.t.$$

$$x_{11} + x_{22} + x_{33} + x_{44} + x_{55} + x_{66} = 2$$
$$x_{11} + x_{12} + x_{16} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{32} + x_{33} + x_{34} = 1$$
$$x_{42} + x_{43} + x_{44} = 1$$
$$x_{53} + x_{55} + x_{56} = 1$$
$$x_{61} + x_{65} + x_{66} = 1$$
$$x_{22} - x_{12} \geq 0$$
$$x_{11} - x_{21} \geq 0$$
$$x_{33} - x_{23} \geq 0$$
$$x_{44} - x_{34} \geq 0$$
$$x_{33} - x_{43} \geq 0$$
$$x_{33} - x_{53} \geq 0$$
$$x_{11} - x_{61} \geq 0$$
$$x_{ij} \geq 0.$$

The optimal solution to the SRPMP is $x_{11}^{(1)} = x_{21}^{(1)} = x_{34}^{(1)} = x_{44}^{(1)} = x_{56}^{(1)} = x_{61}^{(1)} = 1; x_{ij}^{(1)} = 0$ otherwise, with $1 \leq i, j \leq 6$. There is a violation to the constraint $x_{66}^{(1)} - x_{56}^{(1)} \geq 0$, so we add a new variable $x_{54}$, new constraint $x_{66} - x_{56} \geq 0$ and change the objective function and constraint $x_{53} + x_{55} + x_{56} = 1$ accordingly to solve the new SRPMP. The new SRPMP has the same fractional optimal solution as the RPMP, i.e.

$$X_{mat(2)}^* = \begin{pmatrix}
0.5 & 0 & 0 & 0 & 0 & 0.5 \\
0.5 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0.5 & 0 \\
0.5 & 0 & 0 & 0 & 0.5 & 0
\end{pmatrix}.$$
It is interesting that we do not have the constraint $x_{66} - x_{16} \geq 0$ in the final SRPMP, but it is satisfied. Thus, we have reduced the number of variables from 36 to 20 and the number of constraints from 37 to 15.

If we have an optimal solution $x_{vec}^* = (x_{ij}^*)$ to the RPMP with the dual and dual slack value $y^*, z^*$ respectively, we can fix some variables at zero by using the dual slack values. Let $X_{mat}^{(0)}$ be the best integer solution found so far with the value $Z(X_{mat}^{(0)})$. $c_{vec} = A^Ty^* + z^*$. Any primal solution $x_{vec}$ to the RPMP gives the value:

\[ c_{vec}^T x_{vec} = (A^Ty^* + z^*)^T x_{vec} \]
\[ = (y^*)^TA x_{vec} + (z^*)^T x_{vec} \]
\[ \geq b^Ty^* + (z^*)^T x_{vec} \]
\[ = Z(x_{vec}^*) + (z^*)^T x_{vec} \]
\[ (Z(x_{vec}^*) = b^Ty^*) \]

Thus the $x_{ij}$ can be fixed at zero if $Z(x_{vec}^*) + z_{ij}^* \geq Z(X_{mat}^{(0)})$, since any feasible solution to the RPMP with $x_{ij} = 1$ will have at least $Z(x_{vec}^*) + z_{ij}^* \geq Z(X_{mat}^{(0)})$.

**Example 3.4.1 (continued).** The objective value of $X_{mat}^{(0)}$ is $Z(X_{mat}^{(0)}) = 142$, the optimal value of $X_{mat}^*$ is $Z(X_{mat}^*) = 134.5$. The optimal dual slack values in matrix form $z_{mat}^* = (z_{ij}^*)$ and part of the optimal dual values in matrix form $y_{mat}^* = (y_{ij}^*)$, with main diagonal $M$, are given below. We are only interested in the $y_{ij}, i \neq j$.

\[
y_{mat}^* = \begin{pmatrix}
M & 14 & 0 & 0 & 0 & 7 \\
19.5 & M & 10.5 & 0 & 0 & 0 \\
0 & 4 & M & 9 & 0 & 0 \\
0 & 0 & 22.5 & M & 0 & 0 \\
0 & 0 & 7.5 & 0 & M & 7.5 \\
10 & 0 & 0 & 0 & 0 & M \\
\end{pmatrix},
\]
\[
\begin{pmatrix}
0 & 0 & 0 & 28 & 14 & 0 \\
0 & 0 & 0 & 7.5 & 25.5 & 34.5 \\
1 & 0 & 0 & 0 & 1 & 16 \\
43.5 & 10.5 & 0 & 0 & 10.5 & 21.5 \\
10.5 & 19.5 & 0 & 1.5 & 12 & 0 \\
0 & 40 & 30 & 20 & 0 & 0 \\
\end{pmatrix}.
\]

The remaining dual values \( y_i^* \), \( 0 \leq i \leq 6 \) that correspond to constraints \( x_{i1} + \cdots + x_{i6} = 2 \) and \( x_{i1} + \cdots + x_{i6} = 1, 1 \leq i \leq 6 \), are \((y_0^*, y_1^*, \ldots, y_6^*) = (-64.5, 35, 46.5, 24, 55.5, 52.5, 50)\).

Since \( z_{42}^* = 10.5 > Z(X_{mat}^{(0)}) - Z(X_{mat}^*) \), we can fix \( x_{42} = 0 \). There are 14 other variables \( x_{ij} \), the corresponding \( z_{ij}^* \geq Z(X_{mat}^{(0)}) - Z(X_{mat}^*) \). Among them, 13 variables \( x_{ij} \) (except \( x_{24} \)) are not in the final SRPMP formulation. From this Example, we see that the SRPMP almost excludes those variables that will be fixed at zeros in finding a \( p \)-median.
Chapter 4

Algorithms for the p-median Problem

4.1 Introduction

From the previous Chapter, when the optimal solution to the SRPMP is integral and satisfies all the constraints $x_{ij} - x_{ij} \geq 0$ for $j = r(i), 1 \leq i \leq n$, then this solution is a p-median. Now, given that the optimal solution to the SRPMP is fractional and feasible to the RPMP, the problem of how to find an integral optimal solution remains. One method of proceeding is the branch and bound algorithm. Harvinen, Rajala and Sinervo (1972) used the branch and bound algorithm to solve the p-median problem. In this Section, we briefly discuss the branch and bound algorithm for integer programming problems. In Section 4.2, we give two different orderings in branch variable selection. This is very important to the running time of the algorithm. In Section 4.3, we present a complete algorithm for the p-median problem using one of the branching strategies in Section 4.2.
The branch and bound algorithm is a basic algorithm used by commercial codes for solving mixed-integer programming problems. Here, we give a brief description of the algorithm for our purposes. The general results are found in many standard mathematical programming or linear and integer programming books (e.g. Nemhauser and Wolsey, 1988; Schrijver, 1986).

The branch and bound algorithm provides a methodology to search for an integral optimal solution by doing only a partial enumeration. It solves an integer programming minimization problem by solving a sequence of linear programming problems. In the course of applying the branch and bound algorithm, the overall set of integral feasible solutions is partitioned into simpler subsets, called the subproblems. The subproblems are regarded as forming a tree, rooted at the original linear programming problem. When using the branch and bound algorithm, one of three things happens at each node of the tree.

(i) The subproblem has optimal value greater than the value of the best known integral feasible solution, so the node is fathomed by bounds.

(ii) The optimal solution is an integral solution with value less than the best known integral solution. Thus, we update the best known integral feasible solution and the node is fathomed.

(iii) The optimal solution to the subproblem is fractional and has optimal value less than the best known integral feasible solution. Thus, we have to branch the subproblem into two subproblems and repeat the same process again.

Two elementary strategies are used in the execution of the branch and bound
algorithm: the bounding strategy and the branching strategy.

The bounding strategy is used in selecting promising subproblem in the search for the integral optimal solution. It discards subproblems that cannot possibly contain the integral optimal solution. It should be both relatively easy to implement and also computationally efficient.

The branching strategy partitions the set of feasible solutions of the original problem into two subproblems. Each subset in the partition is the set of feasible solutions of a problem obtained by imposing additional constraints on the original problem. These problems are called candidate problems. The candidate problems must be generated in such a way that the bounding strategy can be applied to each of them. It should be possible to apply the branching strategy to any candidate problem and generate two candidate subproblems such that the following property holds: the potential increase of the optimal value of the candidate subproblems is as small as possible.

As we move down the tree, more and more integral variables are set at either 0 or 1. We can prune some candidate problems if their lower bound is greater than or equal to the objective value of the best known integral solution. The branch and bound algorithm is terminated if there are no candidate problems left and the best known integral feasible solution is the optimal solution.

For the $p$-median problem, we can obtain an integral feasible solution easily. If we use the greedy algorithm and other heuristic algorithms, we can get a better
integral feasible solution. Thus, the branching strategy is the main issue for the p-median problem. This is the topic of the next Section.

4.2 Branching Strategies for the p-median Problem

In this Section, we give two branching strategies for the RPMP, which will be later extended to the SRPMP. We are interested in the dual values $y_{ij}, i \neq j$ that correspond to the constraints $x_{jj} - x_{ij} \geq 0$. Let $y_{mat} = (y_{ij})$ represent the matrix form of dual values with the main diagonal $M$; $y$ represents the vector form of all dual values. The dual slack values are represented by $z_{mat} = (z_{ij})$.

Suppose $X_{mat}^{(0)} = (x_{ij}^{(0)})$ is the best known integral solution to the RPMP found so far. $X_{mat}^{*} = (x_{ij}^{*})$ is a fractional optimal solution to the RPMP with the corresponding optimal dual values $y_{mat}^{*}$ and dual slack values $z_{mat}^{*}$. Let $\delta^{(0)} = Z(X_{mat}^{(0)}) - Z(X_{mat}^{*})$ be the penalty cost of $X_{mat}^{(0)}$. We can fix the variables $x_{ij} = 0$, $1 \leq i \leq n$, for which $x_{jj}^{*} = 0$ in $X_{mat}^{*}$ and satisfy the condition of Theorem 3.3.10. Also fix the variable $x_{ij} = 0$ for which the corresponding $z_{ij}^{*} > \delta^{(0)}$, for $1 \leq i, j \leq n$.

First branching strategy

Suppose that there is a better integral solution $X_{mat}^{'} = (x_{ij}^{'})$ than $X_{mat}^{(0)}$. Then the penalty cost of $X_{mat}^{'}$ is less than the penalty cost of $X_{mat}^{(0)}$, i.e. $\delta^{(0)}$. Hence, the following variables might be 1 in $X_{mat}^{'}$: 

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(i) In the undercovering situation, some of the \( x'_{ij} \) with \( z^*_{ij} < \delta(0) \) and where \( x^*_{ij} = 0 \) in \( X_{mat}^* \).

(ii) In the over covering situation, some of the \( x'_{ik} \) with \( y^*_{ij} < \delta(0) \) for those \( x^*_{ij} > 0 \), where \( k \) is an index such that \( y^*_{ik} = \max_j \{ y^*_{ij} : x^*_{ij} > 0 \} \).

To simplify the branching strategy, we only consider the case (i) for which \( x^*_{ij} = 0 \) and \( z^*_{ij} < \delta(0) \). We do not consider the dual case (ii) here. Suppose that there is a better integral feasible solution \( X'_{mat} = (x'_{ij}) \) than \( X_{mat}^{(0)} \). If \( x'_{ij} = 1 \) in \( X_{mat}' \), then for \( x^*_{ik} \neq 0 \), we have \( x^*_{kk} = 0 \), or for \( x^*_{ik} = 0 \) and \( z^*_{ik} \leq z^*_{ij} \), we have \( x^*_{kk} = 0 \) as well. To verify this, if \( x^*_{kk} = 1 \) in \( X_{mat}' \), then, by Theorem 3.3.8, \( v_i \) is served by \( v_k \) in both cases, which contradicts that \( x^*_{ij} = 1 \), i.e. \( v_i \) is served by \( v_j \). It is equivalent to say that if we set the variable \( x_{ij} = 1 \), then \( x^*_{kk} = 0 \) simultaneously for those \( x^*_{ik} \neq 0 \), or \( x^*_{ik} = 0 \) and \( z^*_{ik} \leq z^*_{ij} \).

However, since \( x_{jj} - x_{ij} \geq 0 \), \( x_{ij} = 1 \) implies that \( x_{jj} = 1 \). Hence, the optimal value of the RPMP plus the constraint \( x_{jj} = 1 \) is less than or equal to the optimal value of the RPMP plus the constraint \( x_{ij} = 1 \). Therefore, we set \( x_{jj} = 1 \) instead of setting \( x_{ij} = 1 \). Sometimes, the optimal value of the RPMP plus the constraint \( x_{tt} = 1 \) is greater than or equal to \( Z(X_{mat}^{(0)}) \). Thus, we can fix the variables \( x_{it} = 0 \), \( 1 \leq i \leq n \), for further calculation, and discard the remaining unchosen values \( x^*_{ht} \) in column \( t \) of \( X_{mat}^* \) with \( z^*_{ht} < \delta(0) \). That is, we can reduce the computation time. Above all, we will consider the branching variable \( x_{jj} \), if \( 0 < x^*_{jj} < 1 \) in \( X_{mat}^* \) and some \( x^*_{ij} = 0 \) with the corresponding \( z^*_{ij} < \delta(0) \).
By Theorem 3.3.12, if all $x_{jj}$ are integral in the optimal solution $X_{mat} = (x_{ij})$ to the RPMP, then there is an integral optimal solution $\bar{X}_{mat} = (\bar{x}_{ij})$ such that $\bar{x}_{jj} = x_{jj}$ for all $j$. Replace $X_{mat}$ by $\bar{X}_{mat}$. To simplify the notation, we will say that $X_{mat}$ is integral. This will not cause any confusion, since the $p$ open facilities and the optimal values are the same. Thus, we only consider the variables $x_{jj}$ for $1 \leq j \leq n$. In order to minimize the penalty cost, we choose the fractional variable $x_{kk}^*$ first, if $x_{kk}^* = \min\{x_{ij}^* : x_{ij}^* = 0, 0 < x_{jj}^* < 1\}$. Set $x_{kk} = 1$ and $x_{kk} = 0$ respectively. Solve the RPMP again and get the new optimal solutions $X_{mat}^{(k,1)} = (x_{ij}^{(k,1)})$ and $X_{mat}^{(k,0)} = (x_{ij}^{(k,0)})$ respectively. There are three cases:

(i) $Z(X_{mat}^{(k,q)}) \geq Z(X_{mat}^{(0)})$, $q = 0$ or $1$. Follow the branching strategy above, choose the fractional variable $x_{ll}$ if one of the $z_{ll}^* = \min\{z_{ij}^* : x_{ij}^* = 0, 0 < x_{jj}^* < 1\}$ in the remaining columns of $X_{mat}$. Repeat the entire process again.

(ii) $Z(X_{mat}^{(k,q)}) < Z(X_{mat}^{(0)})$ and $x_{jj}^{(k,q)}$ are integral for $1 \leq j \leq n$. By Theorem 3.3.12, $X_{mat}^{(k,q)}$ is integral. Replace $X_{mat}^{(0)}$ by $X_{mat}^{(k,q)}$. Choose the next candidate problem. If there is no candidate problem, terminate the algorithm.

(iii) $Z(X_{mat}^{(k,q)}) < Z(X_{mat}^{(0)})$ and $X_{mat}^{(k,q)}$ is still fractional. Follow the branching strategy above, choose another fractional variable $x_{ll}$, if one of the $z_{ll}^* = \min\{z_{ij}^* : x_{ij}^* = 0, 0 < x_{jj}^* < 1\}$ in the remaining columns of $X_{mat}$. Repeat the same process again.

This is the standard procedure for the branch and bound algorithm.
Example 3.4.1 (continued). We have an integral feasible solution \( X_{\text{mat}}^{(0)} = (x_{ij}^{(0)}) \) and the optimal fractional solution \( X_{\text{mat}}^* = (x_{ij}^*) \) to the RPMP as follows.

\[
X_{\text{mat}}^{(0)} = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad X_{\text{mat}}^* = \begin{pmatrix}
    0.5 & 0 & 0 & 0 & 0 & 0.5 \\
    0.5 & 0 & 0.5 & 0 & 0 \\
    0 & 0 & 0.5 & 0.5 & 0 \\
    0 & 0 & 0.5 & 0 & 0.5 \\
    0.5 & 0 & 0 & 0 & 0.5 \\
\end{pmatrix}.
\]

\( Z(X_{\text{mat}}^{(0)}) = 142 \), \( Z(X_{\text{mat}}^*) = 134.5 \), \( \delta^{(0)} = 7.5 \). The optimal dual slack values are

\[
z_{\text{mat}}^* = \begin{pmatrix}
    0 & 0 & 0 & 28 & 14 & 0 \\
    0 & 0 & 0 & 7.5 & 25.5 & 34.5 \\
    1 & 0 & 0 & 0 & 1 & 16 \\
    43.5 & 10.5 & 0 & 0 & 10.5 & 21.5 \\
    10.5 & 19.5 & 0 & 1.5 & 12 & 0 \\
    0 & 40 & 30 & 20 & 0 & 0 \\
\end{pmatrix}.
\]

By Theorem 3.3.12, fix \( x_{i3} = 0 \) for \( 1 \leq i \leq 6 \), since \( z_{55}^* = 12 > \delta^{(0)} = 7.5 \). Since \( z_{i3}^* = \min_{i,j} \{ z_{ij}^* : x_{ij}^* = 0, 0 < x_{ij}^* < 1 \} = 0 \), set variable \( x_{33} = 1 \). Solving the RPMP again, we get an optimal solution

\[
X_{\text{mat}}^{(3,1)}(2) = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

\( Z(X_{\text{mat}}^{(3,1)}(2)) = 142 = Z(X_{\text{mat}}^{(0)}) \), which means that there is no improvement. Thus, we can fix variables \( x_{i3} = 0 \) for \( 1 \leq i \leq 6 \). In the remaining columns of \( X_{\text{mat}}^* \) where \( x_{jj}^* \) are fractional, \( z_{31}^* = 1 \) is the minimum. Set variable \( x_{11} = 1 \). Solving the RPMP again, we get an optimal solution

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\[X^{(1,1)}_{\text{mat}}(2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}
\] (4.2.1)

which is integral and \(Z(X^{(1,1)}_{\text{mat}}(2)) = 136 < Z(X^{(0)}_{\text{mat}})\). Replace \(X^{(0)}_{\text{mat}}\) by \(X^{(1,1)}_{\text{mat}}(2)\).

If we set \(x_{11} = 0\), and solve the RPMP again, we get a higher optimal value than \(Z(X^{(1,1)}_{\text{mat}}(2))\). Thus, \(X^{(1,1)}_{\text{mat}}(2)\) is an integral optimal solution, i.e. \(\{v_1, v_4\}\) is a 2-median of the network.

**Second branching strategy**

By Theorem 3.2.3, if \(x_{jj}^* \neq 0\) in \(X_{\text{mat}}^*\), then node \(v_j\) is a 1-median of the subgraph \(<S_j>\), where \(S_j = \{v_i : x_{ij}^* \neq 0\}\). By Corollary 2.2.5, \(X_p^*\) is a \(p\)-median of \(G\) if and only if \(H(X_p^*)\) has the minimal value among all the \((1, \cdots, 1)\) stable types. Thus, we can choose the branching variable \(x_{kk}\) if \(H(S_k, v_k) = \min_j \{H(S_j, v_j) : 0 < x_{jj}^* < 1\}\). Set \(x_{kk} = 1\) and 0 respectively, solve the RPMP, and get a new optimal solution \(X_{\text{mat}}^{**} = (x_{jj}^{**})\). There are three cases:

(i) \(Z(X_{\text{mat}}^{**}) \geq Z(X_{\text{mat}}^{(0)})\). Follow the branching strategy above, choose the fractional variable \(x_{\ell\ell}\) if \(H(S_\ell, v_\ell) = \min_j \{H(S_j, v_j) : 0 < x_{jj}^* < 1\}\) in the remaining columns of \(X_{\text{mat}}^*\). Repeat the entire process again.

(ii) \(Z(X_{\text{mat}}^{**}) < Z(X_{\text{mat}}^{(0)})\) and \(x_{jj}^{**}\) are integral for \(1 \leq j \leq n\). By Theorem 3.3.12, \(X_{\text{mat}}^{**}\) is integral. Replace \(X_{\text{mat}}^{(0)}\) by \(X_{\text{mat}}^{**}\). Choose the next candidate problem. If there is no candidate problem, terminate the algorithm.

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(iii) \( Z(X_{mat}^{**}) < Z(X_{mat}^{(0)}) \) and \( X_{mat}^{**} \) is still fractional. Follow the branching strategy above, choose the branching variable \( x_{hh} \) if \( H(S_h^{**}, v_h) = \min_j \{ H(S_j^{**}, v_j) : 0 < x_{jj}^{**} < 1 \} \) in the current optimal solution \( X_{mat}^{**} \), where \( S_j^{**} = \{ v_i : x^{**} \neq 0 \} \). Repeat the same process again.

It may happen that some variables \( x_{jj}^{*} = 0 \) in \( X_{mat}^{*} \) and \( x_{jj}^{**} \neq 0 \) in \( X_{mat}^{**} \), or \( S_h \) and \( S_h^{**} \) are different with respect to \( X_{mat}^{*} \) and \( X_{mat}^{**} \). We can follow the proof of Theorem 3.2.3 to prove that \( v_h \) is a 1-median of \( S_h^{**} \) under the condition that \( x_{hh} = 1 \) or \( 0 \). This means that the second branching strategy selects a 1-median node of a subgraph with the least distance-sum at each stage.

**Example 3.4.1 (continued).** From the optimal solution \( X_{mat}^{*} \) to the RPMP, we have \( H(S_1, v_1) = 67, H(S_3, v_3) = 114, H(S_4, v_4) = 15 \) and \( H(S_6, v_6) = 73 \). Hence, set \( x_{44} = 1 \). Solving the RPMP, we get the same optimal solution as (4.2.1). Set \( x_{44} = 0 \), and solve the RPMP again, we get an optimal solution with greater objective value than that of (4.2.1). Hence, \( \{ v_1, v_4 \} \) is the 2-median.

### 4.3 A Complete Algorithm for the p-median Problem

To extend the previous two branching strategies to the SRPMP of Section 3.4, replace solving the RPMP with solving the SRPMP. However, at each iteration of solving the SRPMP, we must verify that the optimal solution to the SRPMP is feasible to the RPMP.
With the above preparation, we have the following complete algorithm for the $p$-median problem using the first branching strategy. Let $P$ be the set of candidate problems, and let $P(X_{mat}^{(k,q)})$ be the subproblem with the optimal solution $X_{mat}^{(k,q)}$, $q = 0$ or 1. Let $F$ be the set of constraints to the SRPMP. The input data is $C = (c_{ij}), n$ and $p$.

Step 1. (Initialization): Use the greedy algorithm to find an integral feasible solution $X_{mat}^{(0)} = (x_{ij}^{(0)})$. For each $i$, let $L_i = \{i_1, i_2, \ldots, i_n\}$ be a sequence of indices such that if $i_j < i_k$, then $c_{ii_j} \leq c_{ii_k}$. If $c_{ii_j} = c_{ii_k}$, then $c_{ii_j}$ precedes $c_{ii_k}$ if $i_j < i_k$ in $C$. For each $i$, find the unique $j^{(0)}(i)$ such that $x_{ij}^{(0)} = 1$ with $j = j^{(0)}(i)$, if $j^{(0)}(i) = i_1$, set $j^{(0)}(i) = i_2$. $P = \emptyset$.

Step 2. (Relaxation): Call the SRPMP procedure (step 9) with $X_{mat}^{(0)}$, get an optimal solution $X_{mat}^* = (x_{ij}^*)$ to the RPMP. The current SRPMP is called the initial SRPMP, which will be solved repeatedly later. If $X_{mat}^*$ is integral go to step 5; else $P = P \cup \{P(X_{mat}^*)\}$, go to step 3.

Step 3. (Fixing variables): Let $\delta^{(0)} = Z(X_{mat}^{(0)}) - Z(X_{mat}^*)$. Fix $x_{ij} = 0$, $1 \leq i \leq n$, for those $x_{ij}^* = 0$ in $X_{mat}^*$ that satisfy the condition of Theorem 3.3.10. Fix $x_{ij} = 0$ if $z_{ij}^* \geq \delta^{(0)}$.

Step 4. (Complementary Slackness): By the complementary slackness condition $y^T(Ax_{vec}^* - b) = 0$, solve the system of linear equations for the dual vector $y$, obtain the optimal dual values $y^*$ and dual slack values $z_{mat}^*$. Go to step 8.
Step 5. (Termination test): If $P = \phi$, $X_{mat}^{(0)}$ is the optimal solution. STOP.

Step 6. (Problem selection): Select and delete a problem $P(X_{mat}^{(h,q)})$ from $P$. Call the SRPMP procedure (step 10) starting from the initial SRPMP with the additional constraint $x_{hh} = q$, get an optimal solution $X_{mat}^{(h,q)}$.

Step 7. (Pruning): (i) If $Z(X_{mat}^{(h,q)}) \geq Z(X_{mat}^{(0)})$, go to step 5.

(ii) If $Z(X_{mat}^{(h,q)}) < Z(X_{mat}^{(0)})$ and $x_{jj}^{(h,q)}$ are integral for $1 \leq j \leq n$, let $X_{mat}^{(u)} = X_{mat}^{(h,q)}$. Delete from $P$ all problems with $Z(X_{mat}^{(h,q)}) \geq Z(X_{mat}^{(0)})$. Go to step 5.

(iii) If $Z(X_{mat}^{(h,q)}) < Z(X_{mat}^{(0)})$ and $X_{mat}^{(h,q)}$ is fractional, go to step 8.

Step 8. (Branching): Choose the branching variable $x_{kk}$ if $z_{kk}^* = \min \{ z_{ij}^* : x_{ij}^* = 0, 0 < x_{ij}^* < 1 \}$ in the remaining columns of $X_{mat}^*$. Add problems $P(X_{mat}^{(k,0)})$ and $P(X_{mat}^{(k,1)})$ to $P$. Go to step 5.

The SRPMP procedure is as follows:

Step 9. (Formulating the SRPMP): Let $F = \{ x_{11} + x_{22} + \cdots + x_{nn} = p \}$. For each $i = 1, \ldots, n$. Suppose $c_{j_1(i)} = \cdots = c_{n_r(i)} < c_{n_r(i)+1}$, then

if $r(i) = 2$, then $F = F \cup \{ x_{jj} - x_{ij} \geq 0 \}$ with $j = i_2 \} \cup \{ x_{n_1} + x_{n_2} + x_{n_3} = 1 \}$;
else $F = F \cup \{ x_{jj} - x_{ij} \geq 0 \}$ for $j = i_2, \ldots, i_{r(i)-1} \} \cup \{ x_{n_1} + x_{n_2} + \cdots + x_{n_r(i)} = 1 \}$.

The objective function is the function of those $x_{ij}$ that appear in the above constraints.

Step 10. (Solving the SRPMP): Solve the SRPMP, get an optimal solution
\( X_{SRPMP} = (x_{ij}) \). If \( x_{jj} - x_{ij} < 0 \) for some \( j = r(i) \), then if \( r(i) < n \) set \( j^{(0)}(i) = r(i) + 1 \), go to step 9; if \( r(i) = n \), add constraint \( x_{jj} - x_{ij} \geq 0 \) with \( i = i_n \), go to step 9. If \( X_{SRPMP} \) is feasible to the RPMP, set missing variables \( x_{ij} = 0 \), return to the place that calls the SRPMP procedure with the optimal solution \( X_{SRPMP} \).

**Example 3.4.1 (continued).** Start from the \( X_{mat}^{(0)} = (x_{ij}^{(0)}) \) with \( x_{16}^{(0)} = x_{24}^{(0)} = x_{34}^{(0)} = x_{44}^{(0)} = x_{56}^{(0)} = x_{66}^{(0)} = 1 \); \( x_{ij}^{(0)} = 0 \) otherwise, for \( 1 \leq i, j \leq 6 \), we have the initial SRPMP as follows.

\[
\begin{align*}
\min \quad & 21x_{12} + 28x_{16} + 27x_{21} + 36x_{23} + 54x_{24} + 20x_{32} + 15x_{34} \\
& + 66x_{42} + 33x_{43} + 45x_{53} + 54x_{54} + 45x_{56} + 40x_{61} + 50x_{65} \\
\text{s.t.} \quad & \\
& x_{11} + x_{22} + x_{33} + x_{44} + x_{55} + x_{66} = 2 \\
& x_{11} + x_{12} + x_{16} = 1 \\
& x_{21} + x_{22} + x_{23} + x_{24} = 1 \\
& x_{32} + x_{33} + x_{34} = 1 \\
& x_{42} + x_{43} + x_{44} = 1 \\
& x_{53} + x_{54} + x_{55} + x_{56} = 1 \\
& x_{61} + x_{65} + x_{66} = 1 \\
& x_{22} - x_{12} \geq 0 \\
& x_{11} - x_{21} \geq 0 \\
& x_{33} - x_{23} \geq 0 \\
& x_{44} - x_{34} \geq 0 \\
& x_{33} - x_{43} \geq 0 \\
& x_{33} - x_{53} \geq 0 \\
& x_{66} - x_{56} \geq 0 \\
& x_{11} - x_{61} \geq 0 \\
& x_{ij} \geq 0.
\end{align*}
\]

The branch and bound algorithm part of the initial SRPMP is the same as that of the RPMP in Chapter 3, because the optimal solutions are feasible to the RPMP by setting \( x_{33} = 1 \) and 0.
The algorithm can be improved in several aspects. There are many heuristic algorithms to find an integral feasible solution $X_{\text{mat}}^{(0)}$, which are mentioned previously. There are different algorithms to solve the SRPMP because of its special structure. The normal methods are the simplex method if the size of the problem is small, and the interior method if the size of the problem is large. The matrix $A$ in (3.1.8) is a sparse matrix when $n > 10$. Sometimes, when we solve a SRPMP, we get an integral feasible solution which is better than $X_{\text{mat}}^{(0)}$, even though it is infeasible to the RPMP. Thus, we can save time for the remainder of the algorithm. For instance, in Example 3.4.1, when we solve the SRPMP (3.4.1), we get an integral optimal solution $X_{\text{mat}}^{(1)} = (x_{ij}^{(1)})$ with $x_{11}^{(1)} = x_{21}^{(1)} = x_{34}^{(1)} = x_{44}^{(1)} = x_{56}^{(1)} = x_{61}^{(1)} = 1; x_{ij}^{(1)} = 0$ otherwise, for $1 \leq i, j \leq 6$. Thus, we have a better integral feasible solution $\{v_1, v_4\}$ than $\{v_4, v_6\}$ with the objective value 136, although $X_{\text{mat}}^{(1)}$ is infeasible to the RPMP. Actually, $\{v_1, v_4\}$ is the optimal integral solution.
Chapter 5

The Uncapacitated Facility Location Problem

5.1 Introduction

Let $J = \{1, \cdots, n\}$ be a finite set of $n$ possible places to establish new facilities. The facility location problem deals with the supply from a subset of these potential facility locations to a set $I = \{1, \cdots, m\}$ of $m$ clients with prescribed demands. There are fixed costs for locating the facilities and transportation costs for shipping the supply between the facilities and the clients. We seek a minimum cost location and transportation plan to satisfy all clients' demands. This problem has been extensively studied in the literature and is commonly referred to as the plant location problem or facility location problem. If each potential facility has a capacity, i.e. there is a maximum demand that it can supply, the problem is called the capacitated facility location problem. If each potential facility is assumed to have unlimited capacity, the problem is called the uncapacitated facility location problem (UFL
Let \( f_j \) be the given fixed cost of opening facility \( j \), and let \( c_{ij} \) be the cost of serving client \( i \) from facility \( j \). Typically, \( c_{ij} \) is a cost of production plus transportation. We assume that \( f_j \geq 0 \) for all \( j \in J \) since if \( f_j < 0 \) every optimal solution contains facility \( j \).

For any given set \( X \) of open facilities, it is optimal to serve client \( i \) from a facility \( j \) such that \( c_{ij} \) is minimal over all \( j \in X \). Thus, given \( X \), the cost is \( Z(X) = \sum_{i \in I} \min_{j \in X} c_{ij} + \sum_{j \in X} f_j \). The UFL problem is to find a set \( X \) that gives the minimal cost \( Z \), i.e. \( Z = \min_X Z(X) \). This is a combinatorial optimization problem.

The integer programming formulation of the UFL problem is useful in the design of both heuristic and exact algorithms to solve it. An integer programming formulation is obtained by introducing the following variables. Let \( x_j = 1 \) if facility \( j \) is open and \( x_j = 0 \) otherwise; let \( x_{ij} = 1 \) if client \( i \) is served by facility \( j \) and \( x_{ij} = 0 \) otherwise. The integer programming problem is

\[
Z = \min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} f_j x_j
\]

\[\text{s.t.}\]

\[
\sum_{j \in J} x_{ij} = 1, \text{ for } i \in I \tag{5.1.2}
\]

\[
x_j - x_{ij} \geq 0, \text{ for } i \in I, j \in J \tag{5.1.3}
\]

\[
x_j, x_{ij} \in \{0, 1\}, \text{ for } i \in I, j \in J. \tag{5.1.4}
\]

The constraints (5.1.2) guarantee that every client is served, whereas (5.1.3)
guarantees that the clients are served only from open facilities.

Note that if we assume costs as negative profits, i.e. \( c_{ij} = -p_{ij} \), where \( p_{ij} \) is the profit of serving client \( i \) from facility \( j \), then the other integer programming formulation for the UFL problem is presented as

\[
\max \sum_{i \in I} \sum_{j \in J} p_{ij} x_{ij} - \sum_{j \in J} f_j x_j \tag{5.1.1'}
\]

subject to (5.1.2)-(5.1.4). These two formulations are mathematically equivalent since

\[
\max \sum_{i \in I} \sum_{j \in J} p_{ij} x_{ij} - \sum_{j \in J} f_j x_j = - \min \left( \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} f_j x_j \right).
\]

If \( c_{ij} \) is replaced by \( c'_{ij} = c_{ij} + c_i \) for all \( j \in J \), then, by (5.1.2), the cost of each feasible solution is changed by \( c_i \), whereas the set of optimal solutions remains the same. Hence, we can add or subtract a constant to any row of the cost matrix \( C = (c_{ij}) \) and not affect the optimal solution.

We can relax the constraints (5.1.4) to

\[
x_j \geq 0, x_{ij} \geq 0, \text{ for } i \in I, j \in J \tag{5.1.5}
\]

The problem (5.1.1), (5.1.2), (5.1.3) and (5.1.5) is called the relaxed UFL problem (RUFL problem), and the problem (5.1.1), (5.1.2) and (5.1.5) is called the basic relaxed UFL problem (BRUFL problem).

There are many published papers in the literature on the UFL problem. Krarup
and Pruzan (1983). Tansel, Francis and Lowe (1983) gave an up-to-date survey. Cor- 
nuejols, Nemhauser and Wolsey (in Mirchandani and Francis, 1990) discussed some 
 improved and new methods. Most algorithms mentioned in solving the \( p \)-median 
 problem can be used to solve the UFL problem. We will show the relationship be- 
 tween the \( p \)-median problem and the UFL problem in the next Section. Among these 
 algorithms, Erlenkotter's (1978) DUALOC appears to outperform all the others. His 
 algorithm is so effective that in 45 out of 48 problems that he tested, optimality was 
 reached at the first node of the branch and bound algorithm. Körkel (1989) mod-
 ified a primal-dual version of Erlenkotter's algorithm and obtained a significantly 
 improved procedure for the larger problems.

### 5.2 The \( p \)-median Problem and the UFL Problem

For a given UFL problem, we can construct a network \( G = (V, E) \) as follows:
\( V = \{v_i; i \in I\} \cup \{v_j; j \in J\} \), an edge \( e = v_i v_j \in E \) if \( e_{ij} \) exists and \( e_{ij} < \infty \).
The fixed cost \( f_i = \infty \) for \( i \in I \). For convenience, we will discuss both the \( p \) 
median problem and the UFL problem on the network \( G = (V, E) \) with \( |V| = n \),
\( C = (c_{ij})_{n \times n} \) and \( f = (f_1, \ldots, f_n) \). \( c_{ij} \) is the cost if client \( i \) is served by facili- 
ty \( j \). If client \( i \) is never served by facility \( j \), then set \( c_{ij} = \infty \). \( f_j \) is the fixed cost if facility 
\( j \) is open. If a facility is never open at \( j \), then set \( f_j = \infty \). Thus, in the integer 
programming formulation (5.1.1)-(5.1.4) of the UFL problem, \( I \subset J \), where \( I \) and 
\( J \) are the index sets of \( V \).
Two features distinguish the UFL problem from the p-median problem:

(i) A nonnegative fixed cost is associated with each potential facility location.

(ii) The number of facilities to be located is not prespecified.

Cornuejols, Fisher and Nemhauser (1977b) have the following result. If $X_{UFL} = (x_j, x_{ij})$ is a fractional optimal solution to the RUFL problem, let $J_1 = \{ j : 0 < x_j < 1 \}$ and $I_1 = \{ i : x_{ij} = 0 \text{ or } x_j \text{ for } 1 \leq j \leq n, \text{ and } x_{ij} \text{ is fractional for at least one } j \}$. Let $a_{ij} = 1$ if $x_{ij} > 0$ and $a_{ij} = 0$ otherwise, and denote by $A$ the $|I_1| \times |J_1|$ matrix whose entries are $a_{ij}$ for $i \in I_1, j \in J_1$.

**Proposition 5.2.1.** (Cornuejols, Fisher and Nemhauser, 1977b) $X_{UFL} = (x_j, x_{ij})$ is a fractional optimal solution to the RUFL problem if and only if

(i) $x_j = \max_i x_{ij}$ for all $j \in J_1$.

(ii) For each $i$, there is at most one $j$ with $0 < x_{ij} < x_j$.

(iii) The rank of $A$ equals $|J_1|$.

In the case where $f_j = 0$, $c_{ij} = w(.j) d(v_i, v_j)$ for all $i, j$, and $\sum_{j=1}^n x_j = p$, then the UFL problem becomes the p-median problem. If $x_j \neq 0$ in the optimal solution to the UFL problem, then client $j$ is served by itself, since $c_{jj} = w(v_j) d(v_j, v_j) = 0$, i.e. $x_j = x_{jj}$ always for all $j$. Thus, $x_j - x_{ij} \geq 0$ becomes $x_{jj} - x_{ij} \geq 0$ and $\sum_{j=1}^n x_j = p$ becomes $\sum_{j=1}^n x_{jj} = p$. This is precisely the integer programming formulation for the p-median problem. However, this is not our primary interest here.

It is clear that $x_j = x_{jj}$ in the optimal solution to the UFL problem, provided
that \( c_{jj} = \min_{1 \leq i \leq n} c_{ij} \), for all \( j \). A problem is called an extended \( p \)-median problem if it is a \( p \)-median problem and there is a fixed cost \( f_j \) if a facility is open at \( j \).

The integer programming formulation for the extended \( p \)-median problem is the same as (3.1.1)-(3.1.5). However, \( c_{jj} \) will be equal to \( f_j \), instead of \( c_{jj} = 0 \) in the \( p \)-median problem. Virtually all the results of Chapter 3 are still correct for the extended \( p \)-median problem since we do not assume \( c_{jj} = 0 \) in the proof of these results. The exception occurs in Section 3.4 and Chapter 4 in constructing SRPMP from an integral feasible solution \( X_{\text{mat}}^{(0)} \). At that time \( c_{jj} = 0 \), so \( j_1 = j \) for all \( L_j = \{ j_1, \ldots, j_n \} \). We now can fix \( j_1 = j \) even if \( c_{jj} \neq 0 \). The rest of the sequence of \( L_j \) is the same as before. Then all the results of Section 3.4 and algorithms in Chapter 4 remain true for the extended \( p \)-median problem.

In the general case of the UFL problem, \( c_{ij} \) could be any nonnegative number for all \( i, j \). We do not claim that if \( x_j \neq 0 \) in the optimal solution to the RUFL problem, then \( v_j \) is a 1-median of \( \langle S_j \rangle \), where \( S_j = \{ v_i : x_{ij} \neq 0 \} \). Sometimes, \( x_{jj} = 0 \) even if \( x_j \neq 0 \), which means \( v_j \not\in S_j \). Sometimes, \( \langle S_j \rangle \) is just the union of nodes \( v_i \) with no edges between them. However, some results for the \( p \)-median problem in Chapter 3 can be adapted for the UFL problem.

Given a network \( G = (V, E) \), \( X_p^* \) is called a \( p \)-UFL of \( G \), if

\[
Z(X_p^*) = \min_{X_p \subseteq V} \left( \sum_{i=1}^{n} \min_{j \in X_p} c_{ij} + \sum_{j \in X_p} f_j \right).
\]

The problem is called a \( p \)-UFL problem if it is a UFL problem and the number of open facilities is exactly \( p \). Nemhauser and Wolsey's (1988) definition (402) is
We start from the BRUFL problem and then move to the RUFL problem.

**Theorem 5.2.2.** If \( X_{UFL} = (x_j, x_{ij}) \) is an optimal solution to the BRUFL problem, \( S_j = \{ v_i : x_{ij} \neq 0 \} \) and \( x_j \neq 0 \), then \( v_j \) is a 1-UFL of the subset \( S_j \).

**Proof.** Suppose \( v_k \neq v_j \) is a 1-UFL of the subset \( S_j \), i.e.

\[
Z(S_j, v_k) < Z(S_j, v_j)
\]

where \( Z(S, U) = \sum_{v_i \in S} \min_{v_j \in U} c_{ij} + \sum_{v_j \in U} f_j \). By the assumption, \( x_j \neq 0 \). Let

\[
\alpha = \min \{ \min_{v_i \in S_j} \{ x_{ij} : x_{ij} \neq 0 \}, x_j \} > 0.
\]

Construct a new solution to the BRUFL problem as follows:

\[
x_t' = \begin{cases} 
  x_t - \alpha, & \text{if } t = j, \\
  x_t + \alpha, & \text{if } t = k, \\
  x_t, & \text{otherwise}.
\end{cases}
\]

\[
x_{st}' = \begin{cases} 
  x_{st} - \alpha, & \text{if } t = j \text{ and } v_s \in S_j, \\
  x_{st} + \alpha, & \text{if } t = k \text{ and } v_s \in S_j, \\
  x_{st}, & \text{otherwise}.
\end{cases}
\]

It is easy to verify that \( X'_{UFL} = (x_t', x_{st}') \) is a feasible solution to the BRUFL problem. By the assumption \( x_j \neq 0 \), hence \( x'_j = x_j - \alpha \geq 0 \). For each \( s, 1 \leq s \leq n \),
if $v_s \in S_j$, then

$$\sum_{t=1}^{n} x'_{st} = \sum_{1 \leq t \leq n, t \neq j} x_{st} + (x_{sj} - \alpha) + (x_{sk} + \alpha) = \sum_{t=1}^{n} x_{st} = 1$$

If $v_s \not\in S_j$, then $x'_{st} = x_{st}$ for $1 \leq t \leq n$. So $\sum_{t=1}^{n} x'_{st} = \sum_{t=1}^{n} x_{st} = 1$. Thus, the constraints of (5.1.2) hold for $X'_{UFL}$. However,

$$Z_{BRUFL}(X'_{UFL}) - Z_{BRUFL}(X_{UFL}) = \alpha \cdot (Z(S_j, v_k) - Z(S_j, v_j)) < 0,$$

which contradicts the assumption that $X_{UFL}$ is an optimal solution to the BRUFL problem. 

Let $Z_{BRUFL}(X_{UFL})$ be the objective value of $X_{UFL}$ with respect to the BRUFL problem, and let $I_j = \{ i : x_{ij} \neq 0 \}$ for all $j$, then we have

**Corollary 5.2.3.** If $X_{UFL} = (x_j, x_{ij})$ is an optimal solution to the BRUFL problem, $x_{jk} \neq 0$ and $I_j$ are pairwise disjoint for $1 \leq k \leq q$, then the set of $q$ nodes $X_q = \{ v_j, \cdots, v_{j_q} \}$ is a $q$-UFL of the subset $S = \{ v_i : i \in \bigcup_{1 \leq k \leq q} I_j \}$. 

Following the proof of Theorem 3.2.2, it is easy to prove this Corollary with the same construction of $X'_{UFL} = (x'_{i}, x'_{st})$ as in the proof of Theorem 5.2.2. The same argument applies to the corollaries below.

For the RUFL problem, we do not need the conditions $x_{jk} \neq 0$ as in the BRUFL problem.
Corollary 5.2.4. If \( X_{UFL} = (x_j, x_{ij}) \) is an optimal solution to the RUFL problem, \( I_{jk} \) are pairwise disjoint for \( 1 \leq k \leq q \), then \( X_q = \{v_{j_1}, \ldots, v_{j_q}\} \) is a \( q \)-UFL of the subset \( S = \{v_i : i \in \bigcup_{1 \leq k \leq q} I_{jk}\} \).

Corollary 5.2.5. If some of the constraints of (5.1.3) are added to the BRUFL problem, then the results of Theorem 5.2.2 and Corollary 5.2.3 are still true.

By Proposition 5.2.1, if \( X_{UFL} = (x_j, x_{ij}) \) is an optimal solution to the RUFL problem, then for each \( i \), there is at most one \( j \), such that \( 0 < x_{ij} < x_j \). However, if \( x_j \) is either 0 or 1 in the \( X_{UFL} \) for all \( j \), then we have

Theorem 5.2.6. If the fractional optimal solution \( X_{UFL} = (x_j, x_{ij}) \) to the RUFL problem is of the form: \( x_j \) is either 0 or 1 for all \( j \) and some of the \( x_{ij} \) are in \((0, 1)\), then there is an integral feasible solution to the RUFL problem with the same optimal value.

Proof. Without loss of generality, suppose \( x_1 = \cdots = x_q = 1, x_{q+1} = \cdots = x_n = 0 \) in the \( X_{UFL} \) and \( 0 < x_{k1} < 1 \) for some \( k \). Then, some other \( x_{kj} \) are also fractional with \( j = 2, \ldots, q \) since \( x_{k1} + \cdots + x_{kq} = 1 \) and \( x_{k(q+1)} = \cdots = x_{kn} = 0 \). If \( x_{k2} \neq 0 \), say, then \( c_{k1} = c_{k2} \), because if \( c_{k1} < c_{k2} \), let \( x_{k1}' = x_{k1} + x_{k2}, x_{k2}' = 0 \); the remaining \( x_{ij}' = x_{ij} \) and \( x_j' = x_j \) for all \( j \). We then get a feasible solution to the RUFL problem with strictly lower objective value. Similarly, we cannot have \( c_{k1} > c_{k2} \). Thus, if some other \( x_{kj} \) are in \((0, 1)\), then \( c_{k1} = c_{kj} \). Let \( X_{UFL} = (x_j, x_{ij}) \) with \( x_j = x_j \) for all \( j \) and for each \( i \), if \( c_{ij^*} = \min \{c_{ij} \} \), then set one \( x_{ij^*} = 1 \). Set the remaining \( x_{ij} = 0 \). Thus, \( X_{UFL} \) is an integral feasible solution to the RUFL
problem with the same optimal value.

By Theorem 5.2.6, we know that if the $x_j$ are either 0 or 1 for all $j$ in the optimal solution $X_{UFL} = (x_j, x_{ij})$ to the RUFL problem, then we have an integral optimal solution in hand. This is useful in the next Section.

5.3 An Algorithm for the UFL Problem

Given an integral feasible solution $X_{UFL}^{(0)}$ to the UFL problem, we construct a simplified relaxed UFL problem (SRUFL problem) from the RUFL problem. This is similar to our results for the $p$-median problem.

We order $c_{ij}$ as follows: for each $i = 1, \ldots, n$, let $L_i = \{i_1, i_2, \ldots, i_n\}$ be a sequence of indices such that if $i_j < i_k$, then $c_{i_j} \leq c_{i_k}$. If $c_{i_j} = c_{i_k}$, then $c_{i_j}$ precedes $c_{i_k}$ if $i_j < i_k$ in the existing order. By Proposition 5.2.1, if $c_{n_1} < c_{n_2}$ for some $i$, then $x_{n_1} = x_{i_1}$, since if $x_{n_1} \neq 0$, then $x_{n_1} = x_{i_1}$ in the optimal solution. However, if $c_{n_1} = c_{n_2}$, we could not say that $x_{n_1} = x_{i_1}$ and $x_{n_2} = x_{i_2}$ in the optimal solution. Suppose $X_{UFL}^{(0)} = (x_j^{(0)}, x_{ij}^{(0)})$ is an integral feasible solution to the RUFL problem. Then, for each $i$, there is a unique $j$ such that $x_{ij}^{(0)} = 1$. This $j$ is denoted by $j^{(0)}(i)$. If $j^{(0)}(i) = i_1$, set $j^{(0)}(i) = r_2$. We construct a linear programming problem from the RUFL problem as follows.

For each $i = 1, \ldots, n$, if $c_{n_1} < c_{n_2}$, then set $x_{n_1} = x_{i_1}$, which reduces the number of variables. If $c_{i_1} < c_{n_1} = \cdots = c_{n_{r_1}}, < c_{n_{r_1+1}}$, add constraints $x_j - x_{ij} > 0$.
for \( j = i_2, \ldots, i_{r(i)} \), and \( x_{i_1} + x_{i_2} + \cdots + x_{r(i)+1} = 1 \). If \( c_{ii_1} = c_{ii_2} \), then if \( c_{i(\sigma(i))} = \cdots = c_{i_{r(i)}} < c_{i_{r(i)+1}} \), add constraints \( x_j - x_{ij} \geq 0 \) for \( j = i_1, i_2, \ldots, i_{r(i)} \), and \( x_{i_1} + x_{i_2} + \cdots + x_{i_{r(i)+1}} = 1 \). If \( r(i) = n \) for some \( i \), then we add constraint \( x_{i_1} + \cdots + x_{i_{r(i)}} = 1 \) instead of \( x_{i_1} + x_{i_2} + \cdots + x_{i_{r(i)+1}} = 1 \). Finally, we add the constraint \( x_1 + \cdots + x_n \geq 1 \).

The objective function is a function of those \( x_{ij} \) that appear in the above constraints and all the \( x_j \). For those \( x_{i_1} \) that are replaced by \( x_{i_1} \), we add the coefficient \( c_{i_1} \) to the coefficient of \( x_{i_1} \). The above linear programming problem is called the simplified relaxed UFL problem (SRUFL problem). Like SRPMP, we do not add constraints \( x_j - x_{ij} \geq 0 \) for \( j = i_{r(i)+1}, 1 \leq i \leq n \), to the SRUFL problem.

Let \( X_{SRUFL} = (x_j, x_{ij}) \) be an optimal solution to the SRUFL. \( Z(X_{SRUFL}) \) denotes by the optimal value of the SRUFL problem. It also gives the objective value of \( X_{SRUFL} \) for the RUFL problem by setting the missing variables \( x_{ij} = 0 \). These two values are equal even though \( X_{SRUFL} \) may not be feasible to the RUFL problem. Let \( X_{UFL}^* = (x_j^*, x_{ij}^*) \) be an optimal solution to the RUFL problem. Then,

**Theorem 5.3.1.** \( Z(X_{SRUFL}) \leq Z(X_{UFL}^*) \).

The proof is the same as that of Theorem 3.4.2, where RPMP and SRPMP in the proof of Theorem 3.4.2 are replaced by RUFL problem and SRUFL problem respectively.

If the optimal solution to the SRUFL problem violates constraints \( x_j - x_{ij} \geq 0 \),
for some \( j = r(i) + 1 \), set \( j^{(i)}(i) = r(i) + 1 \), for those violated constraints. Following the procedure above, we construct a new SRUFL problem and solve the new SRUFL problem again until all the constraints \( x_j - x_{ij} \geq 0 \) for the current \( j = r(i) + 1 \), \( 1 \leq i \leq n \), are satisfied. The number of such iterations is at most \( n^2 - 2n \), because there are only \( n^2 \) constraints in the RUFL problem and at least \( 2n \) constraints in the first formulation of the SRUFL problem. In each iteration of constructing a new SRUFL problem, at least one new constraint is added.

**Corollary 5.3.2.** The optimal value of the new SRUFL problem is greater than or equal to that of the old SRUFL problem.

The proof is the same as that of Corollary 3.4.3, where SRUFL problem replaces SRPMP in the proof of Corollary 3.4.3.

**Example 5.3.1.** The example is from Erlenkotter(1978), which was presented by Khumawala(1973). It has five potential facility locations and eight clients, i.e. \( J = \{1, \ldots, 5\}, I = \{1, \ldots, 8\} \). The cost matrix \( C = (c_{ij}) \) is

\[
C = \begin{pmatrix}
120 & 210 & 180 & 210 & 170 \\
180 & +\infty & 190 & 190 & 150 \\
100 & 150 & 110 & 150 & 110 \\
+\infty & 240 & 195 & 180 & 150 \\
60 & 55 & 50 & 65 & 70 \\
+\infty & 210 & +\infty & 120 & 195 \\
180 & 110 & +\infty & 160 & 200 \\
+\infty & 165 & 195 & 120 & +\infty \\
\end{pmatrix}.
\]

The fixed cost vector is \( (f_1, f_2, f_3, f_4, f_5) = (200, 200, 200, 400, 300) \). Suppose that facilities located at \( j_2 \) and \( j_5 \) give an initial integral feasible solution. That is,
\[ X_{\text{uFL}}^{(0)} = (x_j^{(0)}, x_{ij}^{(0)}) \] with \( x_2^{(0)} = x_5^{(0)} = 1 \) and the other \( x_j^{(0)} = 0; x_{15}^{(0)} = x_{25}^{(0)} = x_{35}^{(0)} = x_{45}^{(0)} = x_{52}^{(0)} = x_{65}^{(0)} = x_{72}^{(0)} = x_{82}^{(0)} = 1 \), and the remaining \( x_{ij}^{(0)} = 0 \). From the \( X_{\text{uFL}}^{(0)} \), we have the following SRUFL problem.

\[
\begin{align*}
\text{min} & \quad 420x_1 + 310x_2 + 250x_3 + 640x_4 + 600x_5 + 180x_{13} + 170x_{15} \\
& + 180x_{21} + 190x_{23} + 150x_{32} + 110x_{33} + 110x_{35} + 195x_{43} \\
& + 180x_{44} + 60x_{51} + 55x_{52} + 210x_{62} + 195x_{65} + 180x_{71} \\
& + 160x_{74} + 165x_{82} + 195x_{83} \\
\text{s.t.} & \\
& \begin{align*}
x_1 + x_2 + x_3 + x_4 + x_5 & \geq 1 \\
x_1 + x_{15} + x_{13} & = 1 \\
x_5 + x_{21} + x_{23} & = 1 \\
x_1 + x_{33} + x_{35} + x_{32} & = 1 \\
x_5 + x_{44} + x_{43} & = 1 \\
x_3 + x_{52} + x_{51} & = 1 \\
x_4 + x_{65} + x_{62} & = 1 \\
x_2 + x_{74} + x_{71} & = 1 \\
x_4 + x_{82} + x_{83} & = 1 \\
x_5 - x_{15} & \geq 0 \\
x_1 - x_{21} & \geq 0 \\
x_3 - x_{33} & \geq 0 \\
x_5 - x_{35} & \geq 0 \\
x_4 - x_{44} & \geq 0 \\
x_2 - x_{52} & \geq 0 \\
x_5 - x_{65} & \geq 0 \\
x_4 - x_{74} & \geq 0 \\
x_2 - x_{82} & \geq 0 \\
x_{ij} & \geq 0.
\end{align*}
\]

In the problem (5.3.1), we have \( x_{11} = x_{31} = x_1, x_{72} = x_2, x_{53} = x_3, x_{64} = x_{84} = x_4 \) and \( x_{25} = x_{45} = x_5 \). The optimal solution of the problem (5.3.1) is

\[ X_{\text{uFL}}^{(1)} = (x_j^{(1)}, x_{ij}^{(1)}) \] with \( x_1^{(1)} = 1 \) and the rest of the \( x_j^{(1)} = 0; x_{21}^{(1)} = x_{43}^{(1)} = x_{51}^{(1)} = x_{62}^{(1)} = x_{71}^{(1)} = x_{83}^{(1)} = 1 \), and the rest of the \( x_{ij}^{(1)} = 0 \). There are three violations:
\[ x_3 - x_{43} \geq 0, \ x_2 - x_{62} \geq 0 \text{ and } x_3 - x_{43} \geq 0. \text{ Add these three constraints and new variable } x_{42} \text{ to the problem (5.3.1).} \]

The new optimal solution to the current SRUFL problem is \( X_{UFL}^{(2)} = (x_{ij}^{(2)}) \) with \( x_2^{(2)} = 1 \) and the rest of the \( x_j^{(2)} = 0; \ x_{13}^{(2)} = x_{23}^{(2)} = x_{32}^{(2)} = x_{42}^{(2)} = x_{52}^{(2)} = x_{62}^{(2)} = 1, \) and the rest of the \( x_{ij}^{(2)} = 0. \) There are two violations \( x_3 - x_{13} > 0 \) and \( x_3 - x_{23} \geq 0. \) Add these two constraints and the two new variables \( x_{12} \) and \( x_{24} \) to the current SRUFL problem.

The new optimal solution to the current SRUFL problem is \( X_{UFL}^{(1)} = (x_{ij}^{(1)}) \) with \( x_2^{(3)} = 1 \) and the rest of the \( x_j^{(3)} = 0; \ x_{12}^{(3)} = x_{24}^{(3)} = x_{32}^{(3)} = x_{42}^{(3)} = x_{52}^{(3)} = x_{62}^{(3)} = 1, \) and the rest of the \( x_{ij}^{(3)} = 0. \) There is one violation \( x_4 - x_{24} \geq 0. \) Add this constraint to the current SRUFL problem.

The new optimal solution to the current SRUFL problem is \( X_{UFL}^{(4)} = (x_{ij}^{(4)}) \) with \( x_2^{(4)} = x_3^{(4)} = x_5^{(4)} = \frac{1}{2} \) and the rest of the \( x_j^{(4)} = 0; \ x_{13}^{(4)} = x_{15}^{(4)} = x_{21}^{(4)} = x_{33}^{(4)} = x_{35}^{(4)} = x_{52}^{(4)} = x_{62}^{(4)} = x_{71}^{(4)} = x_{82}^{(4)} = x_{83}^{(4)} = \frac{1}{2}, \) and the rest of the \( x_{ij}^{(4)} = 0. \) There is one violation \( x_1 - x_{71} \geq 0. \) Add this constraint and new variable \( x_{75} \) to the current SRUFL problem.

Finally, the new optimal solution to the current SRUFL problem is \( X_{UFL}^{(5)} \)

\( (x_{ij}^{(5)}) \) with \( x_1^{(5)} = x_3^{(5)} = x_5^{(5)} = \frac{1}{3}, \ x_2^{(5)} = \frac{2}{3} \) and \( x_4^{(5)} = 0, \ x_{13}^{(5)} = x_{15}^{(5)} = x_{21}^{(5)} = x_{33}^{(5)} = x_{35}^{(5)} = x_{42}^{(5)} = x_{43}^{(5)} = x_{65}^{(5)} = x_{71}^{(5)} = x_{82}^{(5)} = x_{83}^{(5)} = \frac{1}{3}, \ x_{62}^{(5)} = x_{82}^{(5)} = \frac{2}{3} \) and the rest of the \( x_{ij}^{(5)} = 0. \) This is also the optimal solution to the RUFIL problem.

In the final SRUFL problem, there are 26 variables and 25 constraints. There
are 38 variables and 41 constraints in the RUFL problem.

If the optimal solution to the RUFL is fractional, we use the branch and bound algorithm with the second branching strategy that is used for the $p$-median problem. That is, we choose the branching variable $x_k$ with the smallest value $Z(S_k, v_k)$ over all the fractional variables $x_j$ in the current optimal solution. Thus, we have the following algorithm for the UFL problem. Let $\mathcal{P}$ be the set of candidate problems, and let $P, X_{UFL}^{(h)}$ be the subproblem with optimal solution $X_{UFL}^{(h)}$. Let $F$ be the set of constraints to the SRUFL problem. The input data is the cost matrix $C = (c_{ij})$, the fixed cost vector $f = (f_j)$, the number of clients $m$ and the number of potential facilities $n$.

Step 1. (Initialization): Use the DUALOC algorithm to find an integral feasible solution $X_{UFL}^{(0)} = (x_j^{(0)}, x_{ij}^{(0)})$. Let $L = \{i_1, i_2, \ldots, i_n\}$ be a sequence of indices such that if $i_j < i_k$, then $c_{i_j} \leq c_{i_k}$. If $c_{i_j} = c_{i_k}$, then $c_{i_j}$ precedes $c_{i_k}$ if $i_j < i_k$ in $C$. Find $j^{(0)}(i)$ such that $x_{ij}^{(0)} = 1$ with $j = j^{(0)}(i)$, for $1 \leq i \leq n$. If $j^{(0)}(i) = i_1$, set $j^{(0)}(i) = i_2$. $F = \phi$.

Step 2. (Relaxation): Call the SRUFL procedure (step 7) with $X_{UFL}^{(0)}$, and find an optimal solution $X_{UFL}^{*} = (x_j^{*}, x_{ij}^{*})$ to the RUFL problem. The current SRUFL problem is called the initial SRUFL problem, and it will be solved repeatedly later. If $X_{UFL}^{*}$ is integral go to step 5; else set $\mathcal{P} = \mathcal{P} \cup \{P(X_{UFL}^{*})\}$, go to step 3.

Step 3. (Termination test): If $\mathcal{P} = \phi$, $X_{UFL}^{(0)}$ is the optimal solution. STOP.
Step 4. (Problem selection): Select and delete a problem $P(X_{UFL}^{(h)})$ from $P$. Call the SRUFL procedure (step 8) starting from the initial SRUFL problem, get an optimal solution $X_{UFL}^{(h)}$.

Step 5. (Pruning): (i) If $Z(X_{UFL}^{(h)}) \geq Z(X_{UFL}^{(0)})$, go to step 3.

(ii) If $Z(X_{UFL}^{(h)}) < Z(X_{UFL}^{(0)})$ and $x_j^{(h)}$ are integral for $1 \leq j \leq n$, let $X_{UFL}^{(0)} = X_{UFL}^{(h)}$. Delete from $P$ all problems with $Z(X_{UFL}^{(h)}) \geq Z(X_{UFL}^{(0)})$. Go to step 3.

(iii) If $Z(X_{UFL}^{(h)}) < Z(X_{UFL}^{(0)})$ and $X_{UFL}^{(h)}$ is fractional, go to step 6.

Step 6. (Branching): Choose the branching variable $x_k$ if the corresponding

$$Z(S_k, v_k) = \min_j \{Z(S_j, v_j) : 0 < x_j^{(h)} < 1\} \text{ in } X_{UFL}^{(h)}.$$ Add problems $P(X_{UFL}^{(h)})$ and $P(X_{UFL}^{(k)})$ by setting $x_k = 0$ and $x_k = 1$ respectively to the $P$. Go to step 3.

The SRUFL procedure is as follows:

Step 7. (Formulating the SRUFL): Let $F = \{x_1 + x_2 + \cdots + x_n > 1\}$. For each

$$i = 1, \ldots, n,$$ if $c_{i1} < c_{i2}$, then set $x_{i1} = x_{i2}$, if $c_{i1} = \cdots = c_{i_{n+1}}$, then $F = F \cup \{x_j - x_{ij} \geq 0, \text{ for } j = i_2, \ldots, i_{n+1}\} \cup \{x_{i1} + x_{i2} + \cdots + x_{i_{n+1}} = 1\}$. If $c_{i1} = c_{i2}$, then if $c_{i1} = \cdots = c_{i_{n+1}} < c_{i_{n+1+1}}$, then $F = F \cup \{x_j - x_{ij} \geq 0, \text{ for } j = i_1, \ldots, i_{n+1}\} \cup \{x_{i1} + x_{i2} + \cdots + x_{i_{n+1+1}} = 1\}$. If $c_{i1} = c_{i2}$, then if $c_{i1} = \cdots = c_{i_{n+1}} < c_{i_{n+1+1}}$, then $F = F \cup \{x_j - x_{ij} \geq 0, \text{ for } j = i_1, \ldots, i_{n+1}\} \cup \{x_{i1} + x_{i2} + \cdots + x_{i_{n+1+1}} = 1\}$. If $c_{i1} = c_{i2}$, then if $c_{i1} = \cdots = c_{i_{n+1}} < c_{i_{n+1+1}}$, then $F = F \cup \{x_j - x_{ij} \geq 0, \text{ for } j = i_1, \ldots, i_{n+1}\} \cup \{x_{i1} + x_{i2} + \cdots + x_{i_{n+1+1}} = 1\}$.

The objective function is a linear combination of all $x_j$ and those $x_{ij}$ that appear in the above constraints. If some $x_{i_1}$ are replaced by $x_{i_1}$, replace these variables in the objective function also.

Step 8. (Solving the SRUFL problem): Solve the SRUFL problem, obtain an
optimal solution \( X_{SRUFL} = (x_j, x_{ij}) \).

If \( x_j - x_{ij} < 0 \) for some \( j = r(i) + 1 \), then

if \( r(i) + 1 = n \), add constraint \( x_{jj} - x_{ij} \geq 0 \) with \( j = i_n \).

else set \( j^{(i)}(i) = r(i) + 1 \);

go to step 7.

Else \( X_{SRUFL} \) is feasible to the RUFL problem, then set missing variables \( x_j = 0 \) and \( x_{ij} = 0 \), return to the calling place with the optimal solution \( X_{SRUFL} \).

**Example 5.3.1 (continued).** For the optimal solution \( X^{(5)}_{iFL} = (x_j^{(5)}, x_{ij}^{(5)}) \), we have \( Z(S_1, v_1) = 780 \), \( Z(S_2, v_2) = 980 \), \( Z(S_3, v_3) = 1120 \) and \( Z(S_5, v_5) = 1075 \). Thus, we choose the branching variable \( x_1 \). The new optimal solution to the current SRUFL problem plus the constraint \( x_1 = 1 \) is \( X^{(6)}_{iFL} = (x_j^{(6)}, x_{ij}^{(6)}) \) with \( x_1^{(6)} = x_2^{(6)} = 1 \), the rest of the \( x_j = 0 \); \( x_{11}^{(6)} = x_{21}^{(6)} = x_{31}^{(6)} = x_{42}^{(6)} = x_{52}^{(6)} = x_{62}^{(6)} = x_{72}^{(6)} = x_{82}^{(6)} = 1 \), the rest of the \( x_{ij} = 0 \). The objective value is 1580. This is an integral solution. If we set \( x_1 = 0 \), we get a greater optimal value than that of \( X^{(6)}_{iFL} \). Hence, \( X^{(6)}_{iFL} \) is the optimal integral solution, which is the same as Erlenkotter's(1978) result.
Chapter 6

Final Remarks

The $p$-median problem and the UFL problem have many applications. The $p$-median problem was proposed by Hakimi (1964, 1965) for the problem of locating switching centers in a communication network. It can be applied to locating computer facilities, programs or data files in a computer network to minimize annual storage and transmission costs. Mirzain (1985) applied the UFL problem to the star-star concentrator location problem, an important computer communication network design problem.

The UFL problem can also be used to study the bank account location problem (Cornuejols, Fisher and Nemhauser, 1977a) and portfolio management problem (Beck and Mulvey, 1982). The $p$-median problem and the UFL problem are useful models for location decision problems in emergency service systems such as medical services, accident rescue, fire and police stations. They are used in warehouse location problems as well. Anderson and Chelst (1986) reported an application of the $p$-median problem to the retail network in the automotive industry. In their decision
PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010b ANSI/ISO #2 EQUIVALENT

1.0 2.8 2.5
1.1 3.2 2.2
1.25 4.0 2.0

PRECISION RESOLUTION TARGETS
support system, it is easy to find the effect of moving a dealership to a new location or establishing \( p \) new locations and so on. They also reported the actual use of their system to reconfigure networks of bank branches in existing networks and to design new networks in new markets. In statistics, the \( p \)-median problem is called cluster analysis. The \( p \)-median problem and the UFL problem are subproblems for other problems such as covering problem (see Mirchandani and Francis, 1990).

In Chapter 3, we have established the connection between the \( p \) selected facilities and the optimal solution of the relaxed 0-1 integer programming problem. The distance-sum of the \( p \) selected facilities is the summation of the optimal value of the relaxed integer programming problem plus the corresponding optimal dual values and optimal dual slack values. We also construct a simplified integer programming problem for the \( p \)-median problem and the UFL problem. The optimal value of the relaxed simplified integer programming problem is the lower bound of these two problems. After solving a sequence of these simplified integer programming problems, we finally get the same optimal solution of the relaxed integer programming problems of these two problems. We plan to use these ideas for other combinatorial optimization problems such as quadratic assignment problem, set covering problem and so on. We also intend to compare the two branching strategies for the \( p \)-median problem with the other algorithms of the \( p \)-median problem. We are interested in comparing our algorithm for the UFL problem with other known algorithms as well.

We note that Erlenkotter's(1978) algorithm is used in the SONET Toolkit (Cosares, Deutsch, Saniee and Wasem,1995). SONET, which stands for Synchro-
nous Optical NETwork, is a new transmission standard that is rapidly being introduced by telecommunication carriers. The cost of the “information superhighway” for the United States of America will be approximately $5 trillion (Government Computer News, 1993), for which SONET will provide the backbone network. We hope that our results may be of some use to these designers.
Appendix

The 1-median Problem on the Grid

The easiest case for the \( p \)-median problem is the 1-median problem. Goldman (1971) has a linear time algorithm for 1-median on the tree. In this appendix, we use the Goldman’s algorithm to solve the 1-median problem on the \( n \)-dimensional grid. We proceed first with a 3-dimensional grid.

Let \( a_{i_1i_2i_3} \) be a node of an \( m_1 \times m_2 \times m_3 \) 3-dimensional grid, with weight \( w(a_{i_1i_2i_3}) \). Let \( a_j^{(1)} (1 \leq j \leq m_1) \) be the node of the line \( L_1 \) with the weight

\[
w^{(1)}(a_j^{(1)}) = \sum_{i_2=1}^{m_2} \sum_{i_3=1}^{m_3} w(a_{j_i_2i_3}),
\]

the sum of weight of nodes on the \( j \)-th plane along the first subscript. The line \( L_1 \) is a 1 dimensional projection of the graph onto the first coordinate. Similarly, the lines \( L_2, L_3 \) are defined respectively, i.e. let \( a_j^{(2)} (1 \leq j \leq m_2) \) be the node of the line \( L_2 \) with the weight

\[
w^{(2)}(a_j^{(2)}) = \sum_{i_1=1}^{m_1} \sum_{i_3=1}^{m_3} w(a_{i_1j_i_3}),
\]
and let $a_{j}^{(3)}(1 \leq j \leq m_{3})$ be the node of the line $L_{3}$ with the weight

$$w^{(3)}(a_{j}^{(3)}) = \sum_{i_{1}=1}^{m_{1}} \sum_{i_{2}=1}^{m_{2}} w(a_{i_{1},i_{2},j}).$$

With the above notations, we have the following Theorem.

**Theorem 1.** Let $G$ be an $m_{1} \times m_{2} \times m_{3}$ 3-dimensional grid, then $a_{i_{1}}^{(1)}, a_{i_{2}}^{(2)}, a_{i_{3}}^{(3)}$ is the 1-median of the line $L_{1}, L_{2}$ and $L_{3}$ respectively if and only if $a_{i_{1},i_{2},i_{3}}$ is the 1-median of the grid $G$.

**Proof.** Since

$$H(a_{i_{1},i_{2},i_{3}}) = \sum_{j=1}^{3} H(a_{j}^{(j)})$$

where

$$H(a_{j}^{(j)}) = \sum_{l=1}^{m_{j}} w^{(j)}(a_{l}^{(j)}) d^{(j)}(l, i_{j})$$

here $d^{(j)}(l, i_{j})$ is the distance between the node $l$ and node $i_{j}$ in the line $L_{j}$, for any node $a_{i_{1},i_{2},i_{3}}$ in $G$.

If the left side of (1) is the minimum, then each term of the right side must be the minimum, as $a_{i_{1}}^{(1)}, a_{i_{2}}^{(2)}, a_{i_{3}}^{(3)}$ are choosen independently, and vice versa.  

If there are several 1-median on the lines $L_{1}, L_{2}$ and $L_{3}$, we have the following:

**Corollary 2.** If $G$ is an $m_{1} \times m_{2} \times m_{3}$ 3-dimensional grid, and $a_{i_{1}}^{(1)}, \ldots, a_{i_{r}}^{(1)}; a_{j_{1}}^{(2)}, \ldots, a_{j_{s}}^{(2)}; a_{k_{1}}^{(3)}, \ldots, a_{k_{t}}^{(3)}$ are 1-mediens of lines $L_{1}, L_{2}, L_{3}$ respectively, then $a_{i_{1},i_{2},i_{3}}$ with $i \in \{ i_{1}, \ldots, i_{r} \}, j \in \{ j_{1}, \ldots, j_{s} \}, k \in \{ k_{1}, \ldots, k_{t} \}$ is a 1-median of $G$.

For the $n$-dimensional grid, we have the following two corollaries.
Corollary 3. Let $G$ be an $m_1 \times m_2 \times \cdots \times m_n$ $n$-dimensional grid, then $a_{i_1}^{(1)}, a_{i_2}^{(2)}, \ldots, a_{i_n}^{(n)}$ is a 1-median of the line $L_1, L_2, \ldots, L_n$ respectively if and only if $a_{i_1, i_n}$ is a 1-median of the grid $G$.

Corollary 4. If $G$ is an $m_1 \times m_2 \times \cdots \times m_n$ $n$-dimensional grid, and $a_{i_1}^{(1)}, \ldots, a_{i_r}^{(1)}; a_{j_1}^{(2)}, \ldots, a_{j_s}^{(2)}; a_{k_1}^{(n)}, \ldots, a_{k_t}^{(n)}$ are 1-medians of the line $L_1, L_2, \ldots, L_n$ respectively, then $a_{i_1, \ldots, i_r}$ with $i \in \{i_1, \ldots, i_r\}$, $j \in \{j_1, \ldots, j_s\}$, $k \in \{k_1, \ldots, k_t\}$ is a 1-median of $G$.

From Theorem 1, we see that to find a 1-median of the grid is equivalent to find 1-medians on the lines. To present the algorithm for the 1-median problem on the grid, we need Goldman's algorithm to find the 1-median on the tree.

Let $T = (V(T), E(T))$ be a tree, the weight $w(v) \geq 0$ for any $v \in V(T)$, then we have:

Algorithm 1. (Goldman, 1971)

Step 1: If $V(T)$ consists of a single node, then that node is the 1-median.

Step 2: Search for a leaf $v_i$, if $w(v_i) \geq \frac{1}{2} \sum_{v \in V'(T)} w(v)$, where $V'(T) = V(T) - \{v_i\}$, go to step 4, otherwise go to step 3.

Step 3: Let $v_j$ be the adjacent node of $v_i$, update $T$ by $V(T) \leftarrow V(T) - \{v_i\}$.

Step 4: STOP. $v_i$ is an 1-median.
Using Goldman's Algorithm as a subroutine, we have the following algorithm to find the 1-median of the grid. Let $G$ be an $m_1 \times m_2 \times \cdots \times m_n$ n-dimensional grid with the set of nodes $V(G) = \{a_{i_1,i_2,\ldots,i_n} : 1 \leq i_j \leq m_j, j = 1,2,\ldots,n\}$ and with the weights $w(a_{i_1,i_2,\ldots,i_n})$.

**Algorithm 2.**

Step 1: Form the lines $L_1, L_2, \ldots, L_n$ with the set of nodes $V(L_i) = \{a^{(i)}_{j} : 1 \leq j \leq m_i\} i = 1,2,\ldots,n$, and

$$w^{(1)}(a^{(1)}_{j}) = \sum_{i_2=1}^{m_2} \sum_{i_3=1}^{m_3} \cdots \sum_{i_n=1}^{m_n} w(a_{i_2,i_3,\ldots,i_n}),$$

$$w^{(2)}(a^{(2)}_{j}) = \sum_{i_1=1}^{m_1} \sum_{i_3=1}^{m_3} \cdots \sum_{i_n=1}^{m_n} w(a_{i_1,i_3,\ldots,i_n}),$$

$$\vdots$$

$$w^{(n)}(a^{(n)}_{j}) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} w(a_{i_1,i_2,\ldots,i_{n-1}}).$$

Step 2: Use Algorithm 1 to find the 1-medians $a^{(1)}_{t_1}, a^{(2)}_{t_2}, \ldots, a^{(n)}_{t_n}$ respectively of lines $L_1, L_2, \ldots, L_n$. Then $a_{t_1,t_2,\ldots,t_n}$ is a 1-median of $G$.

The complexity of Algorithm 2 is $O(\prod_{i=1}^{n} m_i)$ since there are $\prod_{i=1}^{n} m_i$ nodes and the complexity of Goldman's Algorithm is $O(|V(T)|)$. 

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Bibliography


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