The p-median Problem and the Uncapacitated Facility Location Problem

by

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Abstract

We study the p-median problem by expressing it as a 0-1 integer programming problem and investigate the optimal solution to the relaxed 0-1 integer programming problem (the relaxed problem). By using the Duality Theorem, we establish the connection between the p selected facilities and the optimal solution to the relaxed problem. The value (distance-sum) of any p selected facilities is equal to the optimal value to the relaxed problem plus the corresponding optimal dual values and dual slack values. If the optimal solution to the relaxed problem is integral, the optimal value of the relaxed problem is the value of the p-median.

Given an integral feasible solution, which can be easily obtained for the p-median problem, we construct a simplified integer programming problem for the p-median problem (the simplified problem). This procedure simultaneously reduces the number of variables and constraints. The optimal value of the relaxed simplified problem is less than or equal to the optimal value of the relaxed problem. After solving a sequence of these relaxed simplified problems, which are dynamically constructed, we have the same optimal solution as the relaxed problem.

To find the optimal integral solution to the relaxed problem, in Chapter 4, we

present two new branch and bound algorithms. One branches on variables according to the optimal dual slack values of the relaxed problem. The other method branches on variables according to the data generated by the problem and the structure of the optimal solution.

In Chapter 5, the results of the p-median problem are generalized to the uncapacitated facility location problem (UFL problem). We discuss the relationship between the p-median problem and the UFL problem. Under some conditions, the UFL problem is the same as the p-median problem. With the same ideas as in the p-median problem, we develop a series of approximations to the optimal solution of the relaxed UFL problem and provide a concrete methodology for doing this. We then extend the second branch and bound algorithm for the p-median problem to the UFL problem.

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I would first like to thank my supervisor, Dr. I. Pressman. He introduced me to the field of mathematical programming and brought the topic of this thesis to me. I especially wish to thank Dr. Pressman for his carefully reviewing this thesis. I am deeply grateful for his guidance, kindness and much more.

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During the preparation of this thesis, my beloved father passed away. He would have been proud to see me complete this thesis, while my mother continuously encouraged and supported me. I dedicate this thesis to them.

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Chapter 1

Introduction

1.1 The p-median Problem

A *network* is an undirected graph $G(V, E)$ with node set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. Each node v is associated with a nonnegative weight $w(v)$, and each edge e is associated with a positive length $l(e)$. For each pair of nodes v_i, v_j, we let $d(v_i, v_j)$, the distance between v_i and v_j, be the length of a shortest length path connecting v_i and v_j. One of the classical problems in Location Theory is the p-median problem, which is defined as follows.

Given a network $G(V, E)$, let $X_p = \{x_1, x_2, \cdots, x_p\}$ be a set of p nodes in V. Define the distance $d(v, X_p)$ between a node v of G and a subset X_p of V by

$$d(v, X_p) = \min_{1 \le i \le p} \{d(v, x_i)\}.$$

For each set $X_p = \{x_1, x_2, \cdots, x_p\}$ of p nodes in V, we define:

$$H(X_p) = \sum_{v \in V} w(v) \cdot d(v, X_p) \tag{1.1.1}$$

and call $H(X_p)$ the *distance-sum* of the set X_p. If there exists X_p^* in V, such that

$$H(X_p^*) = \min_{X_p \subset V} \{H(X_p)\} \tag{1.1.2}$$

then X_p^* is called a *p-median* of G.

Let X_p be a set of p distinct points of G. A point means a node of G or a point along any edge of G. Hakimi(1965) showed that there exists a set of p nodes with the same optimal value. Therefore, in this thesis, a *p-median* means a set of p nodes X_p^* whose distance-sum satisfies (1.1.2).

We will consider the p-median problem on a connected network. However, we shall occasionally consider the p-median problem on a disconnected network. To make this meaningful, we assume that the number of components of the network G is less than or equal to p, where each component contains at least one facility.

The p-median problem is a prototype for many realistic locational decision problems. One of them is to establish p facilities in p of the potential locations on the network and supply each client from the closest established facility so that the demands of all clients are met and the total costs incurred thereby are minimized. It has been traditional to use the phrase *facility* (*node*) *j is open* to mean $v_j \in X_p^*$ and words *client i* (v_i) to mean node v_i.

The thesis is organized as follows:

In Chapter 1, we introduce the p-median problem, the notation and background that will be used for the rest of the thesis, and give a brief survey of the literature

of solving the p-median problem.

In Chapter 2, we consider the combinatorial properties of the p-medians, show the connection between the p-median on the network and the 1-median of its region.

In Chapter 3, we investigate the integer programming problem for the p-median problem, and obtain some basic properties about the optimal solution to the relaxed integer programming problem. We establish the connection between the distance-sum of any selected p facilities and the optimal solution to the relaxed integer programming problem. Finally, we describe a procedure to construct a simplified integer programming problem for the p-median problem, which reduces the number of variables and constraints simultaneously.

In Chapter 4, we present two branching strategies. Both of them use the branch and bound algorithm to obtain the integral optimal solution of the p-median problem.

In Chapter 5, we generalize the results of the p-median problem to the uncapacitated facility location problem(UFL problem).

In Chapter 6, we indicate some applications of the p-median problem and the UFL problem, and some future work that we intend to do.

In the Appendix, we present a linear time algorithm to find the 1-median of the n-dimensional grid.

algorithm. Nothing is said when $i \notin F$. $\mathcal{NP}$ contains hardest problems, i.e. there is a subset of $\mathcal{NP}$, called $\mathcal{NPC}$, such that if there exists $X \in \mathcal{NPC} \cap \mathcal{P}$, then every problem in $\mathcal{NP}$ is in $\mathcal{P}$, that is $\mathcal{NP} = \mathcal{P}$. Problems in $\mathcal{NPC}$ are called $\mathcal{NP}$-**complete**.

A problem X_1 is said to be **polynomially reducible** to another problem X_2, if there is an algorithm for X_1 that uses an algorithm for X_2 as subroutine and runs in polynomial time provided that each call of the subroutine takes unit time. A problem is called $\mathcal{NP}$-**hard**, if there is an $\mathcal{NP}$-complete problem that can be polynomially reduced to it. Thus, if a problem is $\mathcal{NP}$-hard, it is at least as difficult as any $\mathcal{NP}$-complete problem. It also follows that a polynomial time algorithm for an $\mathcal{NP}$-hard problem implies that $\mathcal{P} = \mathcal{NP}$. Whether $\mathcal{P} = \mathcal{NP}$ or not is a major unsolved problem in mathematics and computer science.

A **heuristic algorithm** is a process which seeks good (i.e. near-optimal) solutions at a reasonable computational time without guaranteeing either feasibility or optimality, and without even estimating how close a particular feasible solution is to optimality.

1.3 A Brief Survey

Hakimi(1964, 1965) first used the term *p-median* and introduced the p-median problem. The earliest reference for the p-median problem on a network is Hua et al(1962), who gave an algorithm in poetic form to find the 1-median on a tree net-

v and is denoted by $deg_G(v)$. A node of degree 1 is called a **leaf** of G. A **walk** in G is a sequence $v_0, e_1, v_1, \cdots, v_{k-1}, e_k, v_k$ such that $v_0, \cdots, v_k$ are nodes and $e_i = v_{i-1}v_i$ for $i = 1, \cdots, k$. The walk is **simple** if $v_0, \cdots, v_k$ are distinct. A simple walk is called a **path**. A walk is said to be **closed** if $v_0 = v_k$. A closed walk is said to be a **cycle** if $k \geq 3$ and $v_0, v_1, \cdots, v_{k-1}$ is a path.

Proposition 1.2.1. *There is a unique partition of the nodes of a graph G into subsets $V_1, \cdots, V_t$ with the property that nodes v_i and v_j are in the same subset if and only if G contains a path between v_i and v_j.*

Let V_k be a subset of the partition. The subgraph $<V_k>$ is called a **component** of G. G is said to be **connected** if it has one component. A **forest** F of G is a subgraph that does not contain a cycle. A **tree** T of G is a connected forest. A **rooted tree** is a tree T with a root node r of T.

Linear programming duality theorem

A linear programming problem is to minimize or maximize a linear objective function subject to a finite number of linear constraints. A typical form is

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & \\ & Ax \geq b \\ & x \geq 0 \end{aligned} \tag{1.2.1}$$

where A is an $m \times n$ real matrix, $b \in \mathbf{R}^m, c \in \mathbf{R}^n$.

The minimization problem can be converted to a maximization problem since

$$\max\ c^T x = -\min\ -c^T x$$

subject to any constraints.

The **dual program** of (1.2.1) is

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & \\ & A^T y \leq c \\ & y \geq \mathbf{0} \end{aligned} \tag{1.2.2}$$

The fundamental results of linear programming are the following theorems.

Theorem 1.2.2. (Duality Theorem) *If one of the programs (1.2.1) and (1.2.2) has a finite optimal solution, then so does the other and the two optimal values are equal.*

Theorem 1.2.3. *Let x and y be feasible solutions of (1.2.1) and (1.2.2) respectively. Then, x and y optimize (1.2.1) and (1.2.2) if and only if the following conditions are satisfied:*

(i) $(A^T y - c)^T x = 0;$ (1.2.3)

(ii) $y^T(Ax - b) = 0.$ (1.2.4)

Conditions (1.2.3) and (1.2.4) are called the **complementary slackness conditions**.

Since there are many different forms of constraints of linear programming problems, the duality theorem and the complementary slackness conditions appear in different forms. In this thesis, we shall use variants of the theorems without stating them.

Computational complexity

The **input size** of an integer n is $\lceil \log_2(|n|+1) \rceil$, denoted by $s(n)$. For every rational number r, there are two unique integers a, b, with $\gcd(a,b) = 1$, and $b > 0$ such that $r = a/b$. The input size of r, $s(r)$, is $s(a) + s(b)$. If a is a vector or a matrix, the size of a is the sum of the input sizes of the entries. For a sequence $a_1, \cdots, a_k$ of vectors and matrices, its input size is $s(a_1, \cdots, a_k) = s(a_1) + \cdots + s(a_k)$. We define $|V| + |E|$ to be the input size of a graph $G = (V, E)$.

Given functions $f(n)$ and $g(n)$ from $\mathbf{Z}_+$ to $\mathbf{Z}_+$, $f(n)$ is said to be $O(g(n))$ if there is a constant $c > 0$ and $n' \in \mathbf{Z}_+$ such that $f(n) \leq cg(n)$ for all $n \geq n'$. Given a class of problems and an algorithm, the time complexity function $f: \mathbf{Z}_+ \to \mathbf{Z}_+$ of the algorithm is the maximum time $f(n)$ needed to solve any problem instance of input size $n \in \mathbf{Z}_+$. The algorithm is a **polynomial-time** algorithm if there exists a polynomial $g(x)$ such that $f(n) = O(g(n))$. Let $\mathcal{P}$ be the *class of problems that can be solved in polynomial time.*

A **feasibility problem** X is a pair (I, F) with $F \subseteq I$, where the elements of I are finite binary strings. I is called the *set of instances* of X, F is called the *set of feasible instances*. Given an instance $i \in I$, we want to determine whether or not $i \in F$. The answer is either yes or no. The information that can be used to check feasibility in polynomial time is called a *certificate of feasibility*, which is denoted by Q_i, for each instance $i \in I$.

A **nondeterministic algorithm** consists of two stages: the guessing stage and the checking stage. The input to the algorithm is an $i \in I$. In the guessing stage, we guess a binary string Q. The checking stage is an algorithm that works with the pair (i, Q) and may provide the output that $i \in F$. Two properties are required.

1. If $i \in F$, there is a certificate Q_i such that when the pair (i, Q_i) is given to the checking stage, the algorithm gives the answer that $i \in F$.

2. If $i \notin F$, there is no output.

We say that the nondeterministic algorithm is polynomial if, for each $i \in F$, there is a polynomial time proof of feasibility, i.e. when $i \in F$, the running time in the checking stage is a polynomial function of the size of i for some Q_i for which it replies that $i \in F$.

We define $\mathcal{NP}$ to be the class of feasibility problems such that for each instance $i \in F$, the answer $i \in F$ is obtained in polynomial time by some nondeterministic

algorithm. Nothing is said when $i \notin F$. $\mathcal{NP}$ contains hardest problems, i.e. there is a subset of $\mathcal{NP}$, called $\mathcal{NPC}$, such that if there exists $X \in \mathcal{NPC} \cap \mathcal{P}$, then every problem in $\mathcal{NP}$ is in $\mathcal{P}$, that is $\mathcal{NP} = \mathcal{P}$. Problems in $\mathcal{NPC}$ are called $\mathcal{NP}$-**complete**.

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work. Since then, there have been numerous papers on the efficient computation of the p-median of a network. Many of these use an integer programming formulation for the p-median problem. We provide a compact version of this in Chapter 3.

Cornuejols, Fisher and Nemhauser(1977a) introduced a two-phase approach for generating and verifying near-optimal solutions for p-median problems. A greedy-interchange heuristic generated good upper bounds. A Lagrangian relaxation, obtained by dualizing (3.1.3), provided sharp lower bounds. Narula, Ogbu and Samuelsson(1977) used the same Lagrangian relaxation approach to get lower bounds for the p-median problem. Beasley(1993) presented a Lagrangian heuristic algorithm for location problems and showed computationally that it gave good quality solutions for p-median problems. His heuristic was based on Lagrangian relaxation and subgradient optimization.

Galvão(1980) presented a dual-based approach to solve the p-median problem. He modified Erlenkotter's(1978) method, DUALOC, for solving the uncapacitated facility location (UFL) problems. Mirchandani, Oudjit and Wong(1985) gave a "nested-dual" approach for the p-median problems. They used DUALOC as a subroutine. It is much faster than Galvão's(1980) algorithm, and can also be used to solve the p-median problem for several values of p. Captivo(1991) showed a fast primal and dual heuristic algorithm for the p-median problem.

Garfinkel, Neebe and Rao(1974) used Dantzig-Wolfe decomposition to solve the p-median problem. Magnanti and Wong(1981) developed techniques to ac-

celerate the convergence of Benders' decomposition. Thus, to find the p-median, they were able to reduce the number of integer programming problems to be solved. Nemhauser and Wolsey(1981) considered the Benders' cuts in the more general submodular set function. Conn and Cornuejols(1990) proposed a projection method for the p-median problem. Leung and Magnanti(1986) developed a strong cutting plane method to solve the capacitated plant location problems. Cheung(1980) described a minimum-cut approach.

The computational experience for general networks has shown that several algorithms mentioned above produce very good upper or lower bounds. The first formal probabilistic analysis of planar p-median algorithms was performed by Fisher and Hochbaum(1980). This was improved by Papadimitriou(1981). This type of analysis requires assumptions on the underlying probability distribution of problem instances. Different models give different results. For Euclidean model with uniform demand, Ahn, Cooper, Cornuejols and Frieze(1988) showed that a very good solution and a proof that the solution is within 0.2% of the optimal value can be found quickly almost surely.

Kariv and Hakimi(1979) proved that the p-median problem is $\mathcal{NP}$-hard even in the case where the network has a simple structure, (e.g. planar graph of maximum node degree 3). However, if the underlying graph of the network is a tree, Hua et al(1962), Goldman(1971) gave an $O(n)$ algorithm for finding a 1-median. Matula and Kolde(1976), Kariv and Hakimi(1979), Hsu(1982) and Tamir(1996)

respectively described $O(p^2n^3)$, $O(p^2n^2)$, $O(pn^3)$ and $O(pn^2)$ algorithms for the p-median problem on the tree.

Chapter 2

On the 1-median Problem

2.1 Introduction

We have shown that the p-median problem has been extensively studied in the literature. In this Chapter, we will consider the combinatorial properties of p-medians which will be used in later chapters. The easiest case of the p-median problem is when $p = 1$. In Section 2.2, we show some connections between the p-median of the network and the 1-median of its region. Mirchandani and Oudjit(1980) considered the connection between 2-median and 1-median on the tree network. We introduce a definition of stable type, and then prove that X_p^* is a p-median if and only if the distance-sum $H(X_p^*)$ is minimum among all the $(1, \cdots, 1)$ stable types.

In Section 2.3, we show some "negative" results about the p-median problem: the $\mathcal{NP}$-hardness of the p-median problem and some limitations of heuristic algo-

rithms for the p-median problem.

2.2 Some Properties of p-medians

For a network $G = (V, E)$, let $V = V_1 \cup V_2 \cup \cdots \cup V_k$ be a decomposition of V into k pairwise disjoint subsets, with induced subgraphs $<V_1>$, $<V_2>$, $\cdots$, $<V_k>$ respectively. Let X^*_{i,p_i} be a p_i-median of $<V_i>$, for $1 \leq i \leq k$.

Definition 2.2.1. $(X^*_{1,p_1}, X^*_{2,p_2}, \cdots, X^*_{k,p_k})$ is a $(p_1, p_2, \cdots, p_k)$ *stable type* of G, if $d(v, X^*_{i,p_i}) \leq d(v, X^*_{j,p_j})$ for every $v \in V_i$ and $1 \leq i, j \leq k, j \neq i$.

If X^*_p is a p-median of $G = (V, E)$, we say that a node $v \in V$ is *served* by a node $x \in X^*_p$, if $d(v, x) = d(v, X^*_p)$, i.e. x is a node of X^*_p which is closest to v. A subset $W \subset V$ is served by a node $x \in X^*_p$, if every node $v \in W$ is served by x. Suppose $X^*_q \subset X^*_p$, a subset $W \subset V$ is served by X^*_q, if for any $v \in W$, there exists $x \in X^*_q$ such that v is served by x. We say that a subset W *affixes* to $x \in X^*_p$, if W is a maximal subset of V that is served by x. W affixes to $X^*_q \subset X^*_p$, if W is a maximal subset of V that is served by X^*_q. A node is called a tie node if it can be served by two or more equally distant open facilities. In the tie case, W may contain the tie nodes or not. This does not affect the result.

Let $S \subseteq V$, $U \subseteq V$ be any subsets. Define

$$H(S, U) = \sum_{v \in S} w(v) \cdot d(v, U) \tag{2.2.1}$$

Theorem 2.2.1. *Let X^*_p be a p-median of G, $x \in X^*_p$, and let W affix to x.*

Then $\{x\}$ is a 1-median of $<W>$.

Proof. If $\{x\}$ is not a 1-median of $<W>$, then there is $y \in W$ different from x, such that

$$H(W, \{y\}) < H(W, \{x\}).$$

Let $Y^* = (X_p^* \backslash \{x\}) \cup \{y\}$, then

$$H(Y^*) = H(W, Y^*) + H(V \backslash W, Y^*) < H(W, \{x\}) + H(V \backslash W, X_p \backslash \{x\}) = H(X_p^*).$$

This contradicts the assumption that X_p^* is a p-median of G. ∎

Generally, we have the following result.

Theorem 2.2.2. *Let X_p^* be a p-median of G, $X_q^* \subset X_p^*$. If W affixes to X_q^*, then X_q^* is a q-median of $<W>$.*

The proof is similar to that of Theorem 2.2.1. ∎

With the previous two theorems, we see that if X is a median of G, then any partition of X will be a stable type of G, i.e.

Theorem 2.2.3. *Let X_p^* be a p-median of G. $X_{1,p_1}, X_{2,p_2}, \cdots, X_{k,p_k}$ be pairwise disjoint subsets of X_p^* such that $X_{1,p_1} \cup X_{2,p_2} \cup \cdots \cup X_{k,p_k} = X_p^*$, then $(X_{1,p_1}, X_{2,p_2}, \cdots, X_{k,p_k})$ is a $(p_1, p_2, \cdots, p_k)$ stable type of G.*

Proof. Let $W_1, W_2, \cdots, W_k$ be subsets of V that affix to $X_{1,p_1}, X_{2,p_2}, \cdots, X_{k,p_k}$ respectively, then $W_1 \cup W_2 \cup \cdots \cup W_k = V$. By Theorem 2.2.2, $X_{1,p_1}, X_{2,p_2}, \cdots, X_{k,p_k}$

are $p_1, p_2, \cdots, p_k$ medians of $W_1, W_2, \cdots, W_k$ respectively. For any $v \in W_i$,

$$d(v, X^*_{i,p_i}) \leq d(v, X^*_{j,p_j}), \quad 1 \leq i, j \leq k, i \neq j. \tag{2.2.2}$$

since W_i affixes to X_{i,p_i}. It may happen that $W_i \cap W_j \neq \phi$ for some i and j. Let these nodes affix to either one, but not both, subsets. Then, $W_i \cap W_j = \phi$, and (2.2.2) still holds. ∎

From the above, we know that every partition of a p-median gives a stable type. If we can find all the stable types of the given network, then, the p-median must be one of these. Actually, we have the following result. Let $\mathcal{X} = \{(X_{1,p_1}, X_{2,p_2}, \cdots, X_{k,p_k})\}$ be the set of all $(p_1, p_2, \cdots, p_k)$ stable types of G.

Theorem 2.2.4. *X^*_p is a p-median of G if and only if*

$$H(X^*_p) = \min_{(X_{1,p_1}, X_{2,p_2}, \cdots, X_{k,p_k}) \in \mathcal{X}} H(X_{1,p_1} \cup X_{2,p_2} \cup \cdots \cup X_{k,p_k})$$

where $p = p_1 + p_2 + \cdots + p_k$.

Proof. By definition, we know that

$$H(X^*_p) \leq \min_{(X_{1,p_1}, X_{2,p_2}, \cdots, X_{k,p_k}) \in \mathcal{X}} H(X_{1,p_1} \cup X_{2,p_2} \cup \cdots \cup X_{k,p_k}).$$

We prove the opposite inequality below.

Let $X^*_p = X^*_{1,p_1} \cup X^*_{1,p_2} \cup \cdots \cup X^*_{k,p_k}$ be a partition of X^*_p. By Theorem 2.2.3, we know that $(X^*_{1,p_1}, X^*_{2,p_2}, \cdots, X^*_{k,p_k})$ is a $(p_1, p_2, \cdots, p_k)$ stable type. Hence

$$\begin{aligned} H(X^*_p) &= H(X^*_{1,p_1} \cup X^*_{2,p_2} \cup \cdots \cup X^*_{k,p_k}) \\ &\geq \min_{(X_{1,p_1}, X_{2,p_2}, \cdots, X_{k,p_k}) \in \mathcal{X}} H(X_{1,p_1} \cup X_{2,p_2} \cup \cdots \cup X_{k,p_k}), \end{aligned}$$

so

$$H(X_p^*) = \min_{(X_{1,p_1}, X_{2,p_2}, \cdots, X_{k,p_k}) \in \mathcal{X}} H(X_{1,p_1} \cup X_{2,p_2} \cup \cdots \cup X_{k,p_k}). \quad \blacksquare$$

Corollary 2.2.5. *If $\mathcal{X} = \{(\{v_1\}, \{v_2\}, \cdots, \{v_p\})\}$ is the set of all $(1, 1, \cdots, 1)$ stable types of G, then X_p^* is a p-median of G if and only if*

$$H(X_p^*) = \min_{(\{v_1\}, \{v_2\}, \cdots, \{v_p\}) \in \mathcal{X}} H(\{v_1\} \cup \{v_2\} \cup \cdots \cup \{v_p\}). \quad \blacksquare$$

Theorem 2.2.4 and Corollary 2.2.5 tell us that when the size of the problem is huge, we can use the following approximation idea to solve the p-median problem. We divide the problem to several small size problems (say p) and solve each of them (say as 1-median), then combine the solutions of the small size problems together.

2.3 The Complexity of the p-median Problem

It is easy to find a 1-median of a network, however, the p-median problem is $\mathcal{NP}$-hard. Kariv and Hakimi(1979) proved the following result:

Proposition 2.3.1. *The problem of finding a p-median is $\mathcal{NP}$-hard even when the network is planar of maximum node degree 3 with the length of all the edges and the weight of all the nodes are 1.*

Although the p-median problem is $\mathcal{NP}$-hard, there are several approximate algorithms: greedy heuristic, interchange heuristic, greedy-interchange heuristic, dynamic programming heuristic(Cornuejols, Fisher and Nemhauser,1977a), branch and bound, nested-dual approach(Mirchandani, Oudjit and Wong,1985), which de-

pends on the efficiency of solving the uncapacitated facility location subproblems, and some others.

One of the approximate algorithms that should be mentioned is the q enumeration greedy-interchange heuristic, which has the best performance guarantee according to Nemhauser and Wolsey(1978). In general, however, we cannot expect this algorithm to yield an optimal solution.

The *greedy heuristic* first chooses a node that solves the 1 median problem and then proceeds recursively. Suppose $k \leq p$ nodes have been selected. Then, choose the next node that yields the maximum decrease in the distance sum, until $k = p$. The *interchange heuristic* attempts to decrease the distance-sum of an initial feasible solution by interchanging an open node with one that is not open. This process continues until a solution is found such that the distance sum cannot be decreased further by such interchanges.

The q-enumeration greedy-interchange heuristic algorithm: $(q < p)$

Step 1. (q-enumeration): List all subsets of nodes of V with cardinality q

Step 2. (greedy): Let $k = q$. Do the greedy heuristic until $k = p$.

Step 3. (interchange): Do the interchange to every solution obtained by the greedy heuristic.

Step 4. (output): Output the best solution among the interchange.

The q-enumeration greedy-interchange heuristic is quite effective, even when $q = 1$. Cornuejols, Fisher and Nemhauser(1977a) showed the comparisons and we tested some randomly generated problems and got the same conclusion as well. Cornuejols, Fisher and Nemhauser(1977a), Nemhauser and Wolsey(1978) have the following results for this algorithm:

Proposition 2.3.2. *The complexity of the q-enumeration greedy heuristic algorithm is $O(pn^{q+1})$ and is guaranteed to achieve at least $[1 - (\frac{p-q}{p})(\frac{p-q-1}{p-q})^{p-q}] \cdot 100\%$ percent of the optimal value and the percentage is the best.*

The q-enumeration greedy-interchange heuristic usually performs better than the q-enumeration greedy. However, in the worst case, we have:

Proposition 2.3.3. *In the worst case, the greedy-interchange heuristic is no better than the greedy heuristic.*

Whitaker(1983) presented a fast greedy-interchange heuristic algorithm and reported the comparison with the greedy-interchange heuristic algorithm on a number of large randomly generated networks and some difficult problem sets. The fast greedy algorithm was nearly six times faster than the greedy algorithm. The fast interchange algorithm itself, excluding the greedy initialization, was at least ten and in some instances more than thirty times faster than the interchange algorithm.

There is another approach to the p-median problem that we have not mentioned: the integer programming formulation, which is the topic of the next chapter.

Chapter 3

Integer Programming for the p-median Problem

3.1 The Integer Programming Formulation for the p-median Problem

Integer programming is a very useful tool for combinatorial optimization problems. Most combinatorial optimization problems can be expressed as integer programming problems. An integer programming formulation for the p-median problem is obtained by introducing the following variables. Suppose $G = (V, E)$ is a weighted network with $|V| = n$ and weight $w(v)$ for every node $v \in V$. The length of the shortest distance between node v_i and node v_j in the network is $d(v_i, v_j)$. Let $x_{jj} = 1$ if node v_j is one of the p selected facility location points and $x_{jj} = 0$ otherwise; $x_{ij} = 1$ if v_i is served by a facility at v_j and $x_{ij} = 0$ otherwise. The

integer programming formulation of the p-median problem is:

$$(p-MP) \qquad Z = \min \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij} \tag{3.1.1}$$

$s.t.$

$$\sum_{j=1}^{n} x_{jj} = p \tag{3.1.2}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \text{ for } 1 \leq i \leq n \tag{3.1.3}$$

$$x_{jj} - x_{ij} \geq 0, \text{ for } 1 \leq i, j \leq n \text{ and } i \neq j \tag{3.1.4}$$

$$x_{ij} \in \{0, 1\}, \text{ for } 1 \leq i, j \leq n. \tag{3.1.5}$$

where $c_{ij} = w(v_i) \cdot d(v_i, v_j)$ for $1 \leq i, j \leq n$.

The constraints (3.1.3) guarantee that every client is served by exactly one facility, whereas (3.1.4) guarantees that it must be served by one of the p selected facility locations. Relax the constraint (3.1.5) with

$$x_{ij} \geq 0 \tag{3.1.6}$$

since, by (3.1.3), all $x_{ij} \leq 1$. Call (3.1.1), (3.1.2), (3.1.3), (3.1.4) and (3.1.6) the *relaxed p-median problem* (RPMP) and call (3.1.1), (3.1.2), (3.1.3) and (3.1.6) the *basic relaxed p-median problem* (BRPMP). We show in Section 3.4 that in order to solve the RPMP, we can start from the BRPMP and add some constraints of (3.1.4).

We shall use the following notations:

$$x_{vec}^T = (x_{11}, x_{21}, \cdots, x_{n1}, \cdots\cdots, x_{1n}, x_{2n}, \cdots, x_{nn}),$$

$$c_{vec}^T = (c_{11}, c_{21}, \cdots, c_{n1}, \cdots\cdots, c_{1n}, c_{2n}, \cdots, c_{nn}),$$

to represent the vector form of x_{ij} and c_{ij} respectively, while $X_{mat} = (x_{ij})_{n\times n}$, $C_{mat} = (c_{ij})_{n\times n}$ represent the matrix form of the variables x_{ij} and data c_{ij}. It should be mentioned that the order of x_{vec}^T is slightly different from the normal order, which is appropriate for the matrix form expression of the p-median problem below. We shall sometimes use the notation $X_{mat}(p)$ or $x_{vec}(p)$ to represent the optimal solution for the given p with its optimal value $Z(X_{mat}(p))$ or $Z(x_{vec}(p))$ if we just consider the optimal solution. Sometimes, we shall use $X_{mat}^* = (x_{ij}^*)$ or $x_{vec}^* = (x_{ij}^*)$ to represent the optimal solution to distinguish it from other solutions. Let $Z_{BRPMP}(X_{mat})$ or $Z_{BRPMP}(x_{vec})$ represent the objective value of X_{mat} or x_{vec} to the BRPMP.

Let

$$A = \begin{pmatrix} e_1^T & e_2^T & \cdots & e_n^T \\ I_n & I_n & \cdots & I_n \\ A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{pmatrix}_{(n^2+1)\times n^2}$$

where e_i is an n-column vector with i-th component 1 and 0 elsewhere, I_n is an $n \times n$ identity matrix, and

$$A_i = \begin{pmatrix} -I_{i-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & -I_{n-i} \end{pmatrix}_{(n-1)\times n}$$

where $\mathbf{1}$ is an appropriate column vector with each component 1, and $\mathbf{0}$ is an appropriate zero matrix or a zero column vector. When $i = 1$, I_0 is empty. The

problem (3.1.1) to (3.1.5) can be reformulated as follows:

$$(p-MP) \qquad Z = \min c_{vec}^T x_{vec} \tag{3.1.7}$$

$$s.t.$$

$$Ax_{vec} \geq b \tag{3.1.8}$$

$$\mathbf{0} \leq x_{vec} \leq \mathbf{1}, \quad x_{vec} \text{ integer} \tag{3.1.9}$$

where $b^T = (p, 1, \cdots, 1, 0, \cdots, 0)$. The constraint (3.1.9) can be relaxed to

$$x_{vec} \geq \mathbf{0}. \tag{3.1.10}$$

Note that the first $n+1$ constraints in (3.1.8) are equality constraints, but we keep the simplified notation for convenience.

Let y_0, y_i and y_{ij}, for $1 \leq i, j \leq n$ and $i \neq j$, correspond to the constraints (3.1.2), (3.1.3) and (3.1.4) respectively. The dual formulation of the relaxed p-median problem is:

$$(DPMP) \qquad D_{LP} = \max \quad py_0 + \sum_{i=1}^{n} y_i \tag{3.1.11}$$

$$s.t.$$

$$y_0 + y_j + \sum_{\substack{i=1 \\ i \neq j}}^{n} y_{ij} \leq c_{jj} = 0 \tag{3.1.12}$$

$$y_i - y_{ij} \leq c_{ij}, \quad \text{for } 1 \leq i, j \leq n, i \neq j \tag{3.1.13}$$

$$y_0, y_i \quad \text{unrestricted}, \quad y_{ij} \geq 0, \quad \text{for } 1 \leq i, j \leq n. \tag{3.1.14}$$

In integer programming, a starting basic feasible solution can be hard to find and the optimal integral solutions can be interior points of the feasible region of the relaxed problem. Two special properties of p-MP are:

(i) the optimal integral solution is an extreme point of the feasible region.

(ii) a starting basic feasible solution is easy to find.

We can fix any p nodes as the selected facilities. These give a starting feasible solution to the p-MP. Very frequently, RPMP has an optimal integral solution. This observation is supported by results in the literature on random generated problems. Thus, an efficient method for solving RPMP can also be an efficient method for solving the p-median problem. There are various approaches for solving RPMP: Lagrangian duality; the subgradient approach; Dantzig-Wolfe's decomposition; Benders' decomposition; the projection method; the cutting plane method; the separation method and the branch-and-bound method. The polyhedron of the feasible solutions to (3.1.8) and (3.1.10) has fractional extreme points when $n > 3$. In Section 3.2, we characterize the fractional extreme points of this polyhedron. From Theorem 2.2.1, we know that if X_p^* is the optimal set of p facility points of G, then any node v of X_p^* is a 1-median of the subgraph that affixes to v. From Corollary 2.2.5, we know that any p-median has the minimal distance-sum among all the $(1, 1, \cdots, 1)$ stable types.

Let $X_{mat} = (x_{ij})$ be an optimal solution to the BRPMP for some p, which need not be integral. Let $S_{j_k} = \{v_i : x_{ij_k} \neq 0\}, k = 1, \cdots, q$, where x_{ij_k} is in the optimal solution. In Section 3.2, we prove that v_{j_k} is the 1-median of the subgraph $<S_{j_k}>$ if $v_{j_k} \in S_{j_k}$. If these S_{j_k} are pairwise disjoint and $\bigcup_{1 \leq k \leq q} S_{j_k} = V$, then $X_q = \{v_{j_1}, \cdots, v_{j_q}\}$ is a q-median of G. The results are still true if we add some of

the constraints of (3.1.4) to the BRPMP. With the Examples in that Section, we show that for a fractional p, the optimal solution $X_{mat}(p) = (x_{ij})$ may give two different optimal integral solutions at the same time: a p_1-median and a p_2-median.

In Section 3.3, we consider the fractional optimal solutions to the RPMP. If we have a fractional optimal solution x_{vec} to the RPMP with optimal dual vector y and optimal dual slack vector z, then, for any p selected facilities X_p, the distance-sum $H(X_p)$ is a summation of $Z(x_{vec})$ and the corresponding parts of y and z.

The disadvantage of the p-MP is that it needs n^2 variables and $n^2 + 1$ constraints. In Section 3.4, we deal with the reduction of variables and constraints. We find that many variables x_{ij} can be fixed at zero before we solve the p-MP if we have a good integral feasible solution. We formulate a simplified relaxed p-median problem (SRPMP), which reduces both the number of variables and constraints at the same time. It also provides an easy method to test whether the integral feasible solution is an optimal integral solution to the RPMP.

3.2 Basic Properties of the BRPMP and the RPMP

In this Section, we show some basic properties of the optimal solutions to the BRPMP and RPMP. We are interested in the fractional optimal solutions, although these results hold for the integral optimal solutions as well. We will start from the

BRPMP and then move to the RPMP, from 1-median and 2-median instance to the general p-median situation. These results are true for any p, which need not be integral. We also give concrete examples to explain these results.

Theorem 3.2.1. *Let $X_{mat} = (x_{ij})$ be an optimal solution to the BRPMP for some p and $S_j = \{v_i : x_{ij} \neq 0\}$. If $v_j \in S_j$, then v_j is a 1-median of the subgraph $<S_j>$.*

Proof. It is clear that $<S_j>$ is a connected subgraph. Suppose $v_k \neq v_j$ is a 1-median of the subgraph $<S_j>$, i.e.

$$H(S_j, v_k) < H(S_j, v_j)$$

where $H(S, U)$ is described by (2.2.1). Thus $v_k \in S_j$. Let

$$\alpha = \min_{v_i \in S_j} \{x_{ij} : x_{ij} \neq 0\} > 0.$$

Then we construct a new solution to the BRPMP as follows:

$$x'_{st} = \begin{cases} x_{st} - \alpha, & \text{if } t = j \text{ and } v_s \in S_j, \\ x_{st} + \alpha, & \text{if } t = k \text{ and } v_s \in S_j, \\ x_{st}, & \text{otherwise.} \end{cases}$$

Note that $x_{jj} \neq 0$ by the assumption $v_j \in S_j$. It is easy to check that $X'_{mat} = (x'_{ij})$ is a feasible solution to the BRPMP. Since for each $s, 1 \leq s \leq n$, if $v_s \in S_j$, then

$$\begin{aligned} \sum_{t=1}^{n} x'_{st} &= \sum_{\substack{1 \leq t \leq n \\ t \neq j \\ t \neq k}} x_{st} + (x_{sj} - \alpha) + (x_{sk} + \alpha) \\ &= \sum_{t=1}^{n} x_{st} = 1. \end{aligned}$$

If $v_s \notin S_j$, then $x'_{st} = x_{st}$ for $1 \le t \le n$. So $\sum_{t=1}^{n} x'_{st} = \sum_{t=1}^{n} x_{st} = 1$. Thus, constraints (3.1.3) hold for X'_{mat}. For constraint (3.1.2), we have

$$\sum_{t=1}^{n} x'_{tt} = \sum_{\substack{1 \le t \le n \\ t \ne j \\ t \ne k}} x_{tt} + (x_{jj} - \alpha) + (x_{kk} + \alpha)$$

$$= \sum_{t=1}^{n} x_{tt} = p.$$

Thus, X'_{mat} satisfies the constraint (3.1.2) also. So, X'_{mat} is a feasible solution to the BRPMP. However,

$$Z_{BRPMP}(x'_{vec}) - Z_{BRPMP}(x_{vec}) = \alpha \cdot (H(S_j, v_k) - H(S_j, v_j)) < 0,$$

which contradicts the assumption that X_{mat} is an optimal solution to BRPMP. ∎

Theorem 3.2.2. *Let $X_{mat} = (x_{ij})$ be an optimal solution to the BRPMP for some p and $S_{j_k} = \{v_i : x_{ij_k} \ne 0\}$ for $1 \le k \le q$. If $v_{j_k} \in S_{j_k}$ and the subsets S_{j_k} are pairwise disjoint for $1 \le k \le q$, then $X_q = \{v_{j_1}, \cdots, v_{j_q}\}$ is a q-median of the subgraph $< \bigcup_{1 \le k \le q} S_{j_k} >$.*

Proof. Without loss of generality, suppose $X_q = \{v_1, \cdots, v_q\}$, $S_j = \{v_i : x_{ij} \ne 0\}$ and S_j are pairwise disjoint for $j = 1, \cdots, q$. $S = \bigcup_{1 \le j \le q} S_j$. Suppose $Y_q = \{u_1, \cdots, u_q\} \ne X_q$ is a q-median of the subgraph $<S>$, i.e. $H(S, Y_q) < H(S, X_q)$. Let T_j be affixed to u_j and T_j be pairwise disjoint in S for $1 \le j \le q$. Then $S = \bigcup_{1 \le j \le q} T_j$. We can find these pairwise disjoint T_j easily since Y_q is given. Let

$$\alpha = \min_{\substack{1 \le j \le q \\ v_i \in S_j}} \{x_{ij} : x_{ij} \ne 0\} > 0.$$

We construct a new solution to the BRPMP as follows:

$$x'_{ij} = \begin{cases} x_{ij} - \alpha, & \text{if } v_j \in X_q \backslash Y_q \text{ and } v_i \in S_j, \text{ or } v_j \in X_q \cap Y_q \text{ and } v_i \in S_j \backslash T_j, \\ x_{ij} + \alpha, & \text{if } v_j \in Y_q \backslash X_q \text{ and } v_i \in T_j, \text{ or } v_j \in X_q \cap Y_q \text{ and } v_i \in T_j \backslash S_j, \\ x_{ij}, & \text{otherwise.} \end{cases}$$

In the following, we verify that X'_{mat} is a feasible solution to the BRPMP. For each i, $1 \leq i \leq n$. If $v_i \in <S>$, there is exactly one $x'_{ij_1} = x_{ij_1} - \alpha$, one $x'_{ij_2} = x_{ij_2} + \alpha$ and the remaining $x'_{ij} = x_{ij}$. Therefore,

$$\begin{aligned} \sum_{j=1}^{n} x'_{ij} &= \sum_{\substack{1 \leq j \leq n \\ j \neq j_1 \\ i \neq j_2}} x_{ij} + (x_{ij_1} - \alpha) + (x_{ij_2} + \alpha) \\ &= \sum_{j=1}^{n} x_{ij} = 1. \end{aligned}$$

If $v_i \notin <S>$, then $x'_{ij} = x_{ij}$ for $1 \leq j \leq n$. So, $\sum_{i=1}^{n} x'_{si} = \sum_{i=1}^{n} x_{si} = 1$. Thus, constraints (3.1.3) hold for X'_{mat}. Since $|X_q \backslash Y_q| = |Y_q \backslash X_q|$ and $x'_{jj} = x_{jj}$ if $v_j \in X_q \cap Y_q$, this means that the number of $x'_{jj} = x_{jj} - \alpha$ is the same as the number of $x'_{kk} = x_{kk} + \alpha$. So, $\sum_{j=1}^{n} x'_{jj} = \sum_{j=1}^{n} x_{jj} = p$. Thus, X'_{mat} satisfies the constraint (3.1.2) also. Therefore, $X'_{mat} = (x'_{ij})$ is a feasible solution to the BRPMP. However,

$$Z_{BRPMP}(x'_{vec}) - Z_{BRPMP}(x_{vec}) = \alpha \cdot (H(S, Y_q) - H(S, X_q)) < 0,$$

which contradicts the assumption that X_{mat} is an optimal solution. ∎

Remark: It should be mentioned that $<S> = <\bigcup_{1 \leq j \leq q} S_j>$ need not be connected, but, there must be at least 1 median node in each component of the subgraph $<S>$.

The optimal solution to the RPMP also satisfies Theorem 3.2.1 and Theorem 3.2.2. We can then omit the condition $v_{j_k} \in S_{j_k}$ because of the constraints (3.1.4).

Theorem 3.2.3. *Let $X_{mat} = (x_{ij})$ be an optimal solution to the RPMP for some p and $S_{j_k} = \{v_i : x_{ij_k} \neq 0\}$ for $1 \leq k \leq q$, then v_{j_k} is a 1-median of the subgraph $<S_{j_k}>$. If S_{j_k} are pairwise disjoint for $1 \leq k \leq q$, then $X_q = \{v_{j_1}, \cdots, v_{j_q}\}$ is a q-median of the subgraph $< \bigcup_{1 \leq k \leq q} S_{j_k} >$.*

Proof. We use the same assumptions, notations, the constructions of α and $X'_{mat} = (x'_{ij})$ as in the proof of Theorem 3.2.2. In the following, we shall check that X'_{mat} satisfies constraints (3.1.4).

If $v_j \in X_q \backslash Y_q$, then $x'_{jj} = x_{jj} - \alpha$ and every non-zero element of the j-th column of $X_{mat} = (x_{ij})$ is reduced by α, i.e. $x'_{ij} = x_{ij} - \alpha \geq 0$. Therefore, we have $x'_{jj} - x'_{ij} \geq 0$, for $1 \leq i \leq n$.

If $v_j \in Y_q \backslash X_q$, then $x'_{jj} = x_{jj} + \alpha$ and every increased element of the j-th column of X_{mat} is added by α. Therefore, $x'_{jj} - x'_{ij} \geq 0$, for $1 \leq i \leq n$.

If $v_j \in X_q \cap Y_q$, then $x'_{jj} = x_{jj}$. The non-zero elements x_{ij} in the X_{mat} will be decreased by α if the corresponding $v_i \in S_j \backslash T_j$. The elements x_{ij} in the X_{mat} will be increased by α if the corresponding $v_i \in T_j \backslash S_j$, but these $x_{ij} = 0$, so $x'_{ij} = x_{ij} + \alpha = \alpha \leq x'_{jj}$. Therefore, $x'_{jj} - x'_{ij} \geq 0$, for $1 \leq i \leq n$.

For the remaining columns of $X'_{mat} = (x'_{ij})$, $x'_{ij} = x_{ij}$. Therefore, $x'_{jj} - x'_{ij} \geq 0$, for $1 \leq i \leq n$.

From the above, we know that (3.1.4) is hold for $X'_{mat} = (x'_{ij})$. By Theorem 3.2.2, we know that X'_{mat} satisfies the constraints (3.1.2) and (3.1.3) also. Therefore, X'_{mat} is a feasible solution to the RPMP. However,

$$Z(x'_{vec}) - Z(x_{vec}) = \alpha \cdot (H(S, Y_q) - H(S, X_q)) < 0,$$

which contradicts the assumption that x_{vec} is an optimal solution. ∎

Corollary 3.2.4. *Let $X_{mat} = (x_{ij})$ be an optimal solution to the RPMP with some p and $S_{j_k} = \{v_i : x_{ij_k} \neq 0\}$, for $1 \leq k \leq q$. If the subsets S_{j_k} are pairwise disjoint and $\bigcup_{1 \leq k \leq q} S_{j_k} = V$, then $X_q = \{v_{j_1}, \cdots, v_{j_q}\}$ is a q-median of G.* ∎

It is easy to see that Theorems 3.2.1 and 3.2.2 are still true if some of the constraints of (3.1.4) are added on to the BRPMP, i.e.

Corollary 3.2.5. *The results of Theorems 3.2.1, 3.2.2 are still true if a subset of the constraints of (3.1.4) is added to the constraints of the BRPMP.* ∎

Example 3.2.1. The tree network $G = (V, E)$ shown in Figure 1 consists of a set $V = \{v_1, \cdots, v_6\}$ of 6 nodes with weights $w(v_1), \cdots, w(v_6)$ (the bracketed numbers), associated with each node. The numbers along each edge are the lengths of the corresponding edge.

There is no theoretical restriction in the p-MP formulation which requires p to be an integer. We experimented with non-integral values of p and found that the results provided addtional information at times. For example, the optimal solution

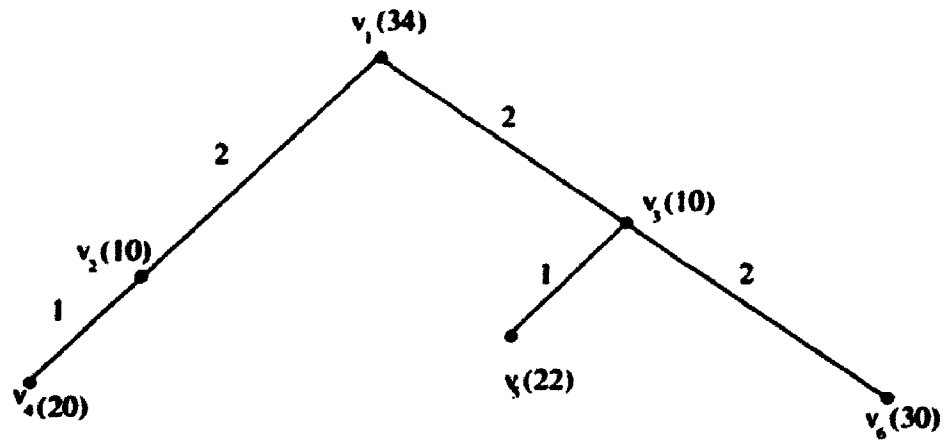

Figure 1

to the BRPMP with $p = 3.5$ is

$$X_{mat}(3.5) = \begin{pmatrix} x_{11} & & & & \\ & & x_{24} & & \\ & & & x_{35} & \\ & x_{42} & x_{44} & & \\ & & & x_{55} & \\ & & & & x_{66} \end{pmatrix}$$

where only the non-zero elements $x_{ij} \neq 0$ are shown. The missing elements indicate that $x_{ij} = 0$. By Theorem 3.2.1, v_1, v_4, v_5 and v_6 are 1-medians of $<S_1>$, $<S_4>$, $<S_5>$ and $<S_6>$ respectively, where $<S_1> = \{v_1\}$, $<S_4> = \{v_2, v_4\}$, $<S_5> = \{v_3, v_5\}$ and $<S_6> = \{v_6\}$. v_2 is not a 1-median of $<S_2> = \{v_4\}$, since $v_2 \notin S_2$. By Theorem 3.2.2, $\{v_1, v_4, v_5, v_6\}$ is a 4-median of $<S_1 \cup S_4 \cup S_5 \cup S_6> = G$, since S_1, S_4, S_5, S_6 are pairwise disjoint. $\{v_4, v_5\}$ is a 2-median of the subgraph $<S_4 \cup S_5>$, even though it is disconnected.

The optimal solution to the RPMP with $p = 1.1$ is

$$X_{mat}(1.1) = \begin{pmatrix} x_{11} & x_{13} & \\ x_{21} & & x_{24} \\ x_{31} & x_{33} & \\ x_{41} & & x_{44} \\ x_{51} & x_{53} & \\ x_{61} & x_{63} & \end{pmatrix}$$

where again only those elements snown have $x_{ij} \neq 0$ and the missing ones have $x_{ij} = 0$. From Theorem 3.2.3, we know that v_1 is the 1-median and $\{v_3, v_4\}$ is the 2-median of G. Thus, the optimal solution gives us two different medians at the same time.

The optimal solution to the RPMP has a strong connection with a p-median of the network, even when the optimal solution is fractional. We shall consider these cases in the next Section.

3.3 Fractional Optimal Solution to the RPMP

In this Section, we consider the fractional optimal solution to the RPMP. If the optimal solution to the RPMP is integral, then we have a solution to the p-median problem. In the following, we use the Duality Theorem to show that there is a close connection between the fractional optimal solution to the RPMP and the p-median of the network. Furthermore, the distance-sum of any p selected facilities can be expressed as a summation of the optimal value to the RPMP plus the penalty cost of using these selected facilities. This is described below. Thus, we extend the results from the integral optimal solution of the RPMP to the general fractional optimal solution.

For the remainder of this Section, we assume that $X_{mat} = (x_{ij})$ is the fractional *optimal* solution to the RPMP with optimal value $Z(X_{mat})$, except when we mention the optimal solution specifically. Let y, s, z be the corresponding optimal dual, slack

and dual slack vectors respectively, i.e.

$$Ax_{vec} - s = b \tag{3.3.1}$$

$$A^T y + z = c \tag{3.3.2}$$

$s \geq 0$, and $z \geq 0$. By the complementary slackness conditions, we have that $z^T x = 0$, and $y^T s = 0$. Let y_{ij} be the optimal dual values that correspond to the constraints $x_{jj} - x_{ij} \geq 0$ and let s_{ij} be the optimal slack values such that

$$x_{ij} - x_{jj} + s_{ij} = 0. \tag{3.3.3}$$

Let z_{ij} be the corresponding dual slack values, $1 \leq i, j \leq n$. By the complementary slackness, we have

$$x_{ij} z_{ij} = 0 \quad \text{and} \quad y_{ij} s_{ij} = 0,$$

for $1 \leq i, j \leq n$, since y_0 and y_i, for $1 \leq i \leq n$, correspond to the equality constraints in the primal problem.

For any feasible solution x'_{vec} to the RPMP with the slack vector $s' = Ax'_{vec} - b$, we have

$$\begin{aligned} Z(x'_{vec}) &= c^T_{vec} x'_{vec} \\ &= (y^T A + z^T) x'_{vec} \\ &= y^T A x'_{vec} + z^T x'_{vec} \\ &= y^T (b + s') + z^T x'_{vec} \\ &= y^T b + y^T s' + z^T x'_{vec} \\ &= Z(x_{vec}) + y^T s' + z^T x'_{vec} \end{aligned} \tag{3.3.4}$$

The expression $y^T s' + z^T x'_{vec}$ is called the *penalty cost* of x'_{vec}. This $z^T x'_{vec}$ is

reminiscent of the duality gap in nonlinear programming. Thus, to find a p-median of G is equivalent to finding an integral solution that minimizes the penalty cost. The difficulty is that sometimes the penalty cost is zero, sometimes it is not. In this Section, we show the connections between the optimal solution of the RPMP and the integral feasible solution of the p-MP. Let

$$S_j = \{v_i : x_{ij} \neq 0\}, \text{ for all } j. \tag{3.3.5}$$

We start from the 2-median problem and then extend it to the general p-median problem.

Lemma 3.3.1. *If $p = 2$, given $X_2 = \{v_{j_1}, v_{j_2}\}$ such that $S_{j_1} \cap S_{j_2} = \{v_h\}$ and $S_{j_1} \cup S_{j_2} = V$, then*

(i) $H(\{v_{j_1}, v_{j_2}\}) = Z(x_{vec}) + \min\{y_{hj_1}, y_{hj_2}\}$;

(ii) v_h is served by v_{j_k} where $y_{hj_k} = \max\{y_{hj_1}, y_{hj_2}\}$;

(iii) For each i, $1 \leq i \leq n, i \neq h$, v_i is served by v_{j_k} where $x_{ij_k} \neq 0$.

Proof. Without loss of generality, suppose that $X_2 = \{v_1, v_2\}$, i.e. $x_{11} \neq 0, x_{22} \neq 0$ in the fractional optimal solution $X_{mat} = (x_{ij})$. Let $S_k = \{v_i : x_{ik} \neq 0\}, k = 1, 2; S_1 \cap S_2 = \{v_h\}$, $S_1 \cup S_2 = V$. To calculate $H(\{v_1, v_2\})$, each v_i must be served either by v_1 or by v_2. By the assumption, we have that either $x_{i1} \neq 0$, $x_{i2} = 0$, or $x_{i1} = 0$, $x_{i2} \neq 0$ for each $i \neq h$. Note that $x_{h1} \neq 0$ and $x_{h2} \neq 0$. It is clear that v_i is served by v_1 if $x_{i1} \neq 0$ or by v_2 if $x_{i2} \neq 0$. Otherwise, suppose that there is a node $v_t, t \neq h$, served by v_1 with $x_{t1} = 0, x_{t2} \neq 0$. Thus $c_{t1} < c_{t2}$. Let $\alpha = \min\{x_{11}, x_{22}, x_{t2}\} > 0$. We construct a new feasible solution

$X'_{mat} = (x'_{ij})$ to the RPMP by setting $x'_{t1} = \alpha, x'_{t2} = x_{t2} - \alpha; x'_{ij} = x_{ij}$ otherwise. Thus, $Z(x'_{vec}) - Z(x_{vec}) = \alpha(c_{t1} - c_{t2}) < 0$, which contradicts the assumption that x_{vec} is an optimal solution. Thus, (iii) holds.

If v_h is served by v_1, we construct an integral feasible solution $X'_{mat} = (x'_{ij})$ to the RPMP as follows:

$$x'_{ij} = \begin{cases} 1, & \text{if } j = 1 \text{ and } x_{i1} \neq 0, \\ 1, & \text{if } j = 2 \text{ and } x_{i2} \neq 0, i \neq h, \\ 0, & \text{otherwise.} \end{cases}$$

Since $S_1 \cup S_2 = V$ and $S_1 \cap S_2 = \{v_h\}$, then X'_{mat} is a feasible integral solution to the RPMP with $p = 2$. Actually, $H(\{v_1, v_2\}) = Z(x'_{vec})$ if v_h is served by v_1.

Let $s' = Ax'_{vec} - b$, where s'_{ij} is the component of s' such that $x'_{ij} - x'_{jj} + s'_{ij} = 0$. It is clear from the construction that $x'_{ij} = 0$ if $x_{ij} = 0$. Thus,

$$z^T x'_{vec} = 0.$$

Also $s'_{ij} = 0$ if $s_{ij} = 0$, for $i \neq h$, where s_{ij} satisfies (3.3.3). The one exception is $s'_{h2} = 1$ while s_{h2} could be zero depending on whether or not $x_{h2} - x_{22} = 0$. However,

$$y^T s' = y_{h2}$$

is always true because of the complementary slackness conditions. Therefore, when v_h is served by v_1, we have

$$Z(x'_{vec}) = Z(x_{vec}) + y^T s' + z^T x'_{vec} = Z(x_{vec}) + y_{h2}.$$

With the same argument, if v_h is served by v_2, we could construct another

integral feasible solution $X''_{mat} = (x''_{ij})$ such that $Z(x''_{vec}) = Z(x_{vec}) + y_{h1}$. Thus,

$$H(\{v_1, v_2\}) = \min\{Z(x'_{vec}), Z(x''_{vec})\}$$

$$= Z(x_{vec}) + \min\{y_{h1}, y_{h2}\},$$

and v_h is served by the v_{j_k} with the larger value of y_{hj_k}. ∎

Remark: If $v_h = v_{j_1}$ or $v_h = v_{j_2}$, there is no y_{jj}. We set $y_{jj} = M$, for $1 \le j \le n$, where M is a very large number. This is useful later.

If the intersection of S_{j_1} and S_{j_2} has more than one node, we have

Lemma 3.3.2. *If $p = 2$, given $X_2 = \{v_{j_1}, v_{j_2}\}$ such that $S_{j_1} \cap S_{j_2} = \{v_i : i \in I\}$ and $S_{j_1} \cup S_{j_2} = V$, then*

(i) $H(\{v_{j_1}, v_{j_2}\}) = Z(x_{vec}) + \sum_{i \in I} \min\{y_{ij_1}, y_{ij_2}\}$;

(ii) For each i, $i \in I$, v_i is served by v_{j_k} where $y_{ij_k} = \max\{y_{ij_1}, y_{ij_2}\}$;

(iii) For each i, $i \notin I$, v_i is served by v_{j_k} where $x_{ij_k} \neq 0$.

Proof. By Lemma 3.3.1, we see that for each $i \in I$, v_i is served by v_{j_1} if $y_{ij_1} \ge y_{ij_2}$, or by v_{j_2} if $y_{ij_2} > y_{ij_1}$. For each $i \notin I$, v_i is served by v_{j_1} if $x_{ij_1} \neq 0$ or by v_{j_2} if $x_{ij_2} \neq 0$. There is only (i) left. We claim that $H(\{v_{j_1}, v_{j_2}\}) = Z(x'_{vec})$, where x'_{vec} is constructed as follows:

$$x'_{ij} = \begin{cases} 1, & \text{for } i \in I, \text{ if } j = j_1 \text{ and } y_{ij_1} \ge y_{ij_2}, \text{ or if } j = j_2 \text{ and } y_{ij_1} < y_{ij_2}; \\ 1, & \text{for } i \notin I, \text{ if } j = j_1 \text{ and } x_{ij_1} \neq 0, \text{ or if } j = j_2 \text{ and } x_{ij_2} \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

$X'_{mat} = (x'_{ij})$ is an integral feasible solution to the RPMP, because $S_{j_1} \cap S_{j_2} =$

$\{v_i : i \in I\}$ and $S_{j_1} \cup S_{j_2} = V$. We also have

$$z^T x'_{vec} = 0,$$

since $x'_{ij} = 0$ if $x_{ij} = 0$. Let $s' = Ax'_{vec} - b$ and s'_{ij} be the component of s' such that $x'_{ij} - x'_{jj} + s'_{ij} = 0$. Thus, $s'_{ij} = 0$ if $s_{ij} = 0$, where s_{ij} satisfies (3.3.3), with the following exceptions: $s'_{ij_1} = 1$ and $s'_{ij_2} = 0$ if $y_{ij_1} < y_{ij_2}$, or $s'_{ij_2} = 1$ and $s'_{ij_1} = 0$ if $y_{ij_1} \geq y_{ij_2}$, for $i \in I$. Thus,

$$y^T s' = \sum_{i \in I} \min\{y_{ij_1}, y_{ij_2}\}.$$

Hence,

$$\begin{aligned} H(\{v_{j_1}, v_{j_2}\}) = Z(x'_{vec}) &= Z(x_{vec}) + y^T s' + z^T x'_{vec} \\ &= Z(x_{vec}) + \sum_{i \in I} \min\{y_{ij_1}, y_{ij_2}\}. \quad \blacksquare \end{aligned}$$

For the p-median problem, we first consider the special case where there is exactly one node v_h in the intersection of some of these S_{j_k}, for $1 \leq k \leq p$. Let $\mathcal{I}$ be the set of all subsets of the index set $\{j_1, \cdots, j_p\}$ with cardinality at least 2.

Lemma 3.3.3. *Given $X_p = \{v_{j_1}, v_{j_2}, \cdots, v_{j_p}\}$ such that $S_{j_1} \cup \cdots \cup S_{j_p} = V$. If $\bigcup_{\{\alpha_1, \cdots, \alpha_r\} \in \mathcal{I}} (S_{\alpha_1} \cap \cdots \cap S_{\alpha_r}) = \{v_h\}$, then*

(i) $H(\{v_{j_1}, \cdots, v_{j_p}\}) = Z(x_{vec}) + \sum_{k=1}^{p} y_{hj_k} - \max_{1 \leq k \leq p} \{y_{hj_k}\}$;

(ii) v_h is served by v_{j_k} where $y_{hj_k} = \max_{1 \leq k \leq p} \{y_{hj_k}\}$;

(iii) For each i, $i \neq h$, v_i is served by v_{j_k} where $x_{ij_k} \neq 0$.

Proof. Without loss of generality, let r be the largest number such that $S_{\alpha_1} \cap S_{\alpha_2} \cdots \cap S_{\alpha_r} = \{v_h\}$ with $\{\alpha_1, \alpha_2, \cdots \alpha_r\} \subset \{j_1, \cdots, j_p\}$ and $\alpha_1, \alpha_2, \cdots, \alpha_r$

are all distinct, $r \geq 2$. By the hypotheses, for any $i \neq h$, there is only one j_k such that $x_{ij_k} \neq 0$ for $1 \leq k \leq p$. By the same argument as used in the proof of Lemma 3.3.1, v_i is served by v_{j_k} when we calculate $H(\{v_{j_1}, \cdots, v_{j_p}\})$. Suppose v_h is served by $v_{\alpha_t}, 1 \leq t \leq r$. Construct a set of integral feasible solution $X^{(t)}_{mat} = (x^{(t)}_{ij})$ for each t, $1 \leq t \leq r$, as follows:

$$x^{(t)}_{ij} = \begin{cases} 1, & \text{if } i \neq h \text{ and } j \in \{j_1, \cdots, j_p\}, x_{ij} \neq 0, \\ 1, & \text{if } i = h \text{ and } j = \alpha_t, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $X^{(t)}_{mat} = (x^{(t)}_{ij})$ is an integral feasible solution to the RPMP, because $S_{j_1} \cup \cdots \cup S_{j_p} = V$ and $S_{\alpha_1} \cap S_{\alpha_2} \cdots \cap S_{\alpha_r} = \{v_h\}$. We also have

$$z^T x^{(t)}_{vec} = 0,$$

since $x^{(t)}_{ij} = 0$ if $x_{ij} = 0$. Let $s^{(t)} = Ax^{(t)}_{vec} - b$, and let $s^{(t)}_{ij}$ be the component of $s^{(t)}$ such that $x^{(t)}_{ij} - x^{(t)}_{jj} + s^{(t)}_{ij} = 0$. Then $s^{(t)}_{h\alpha_q} = 1$, for $1 \leq q \leq r, q \neq t$; for the other $s^{(t)}_{ij}$, $s^{(t)}_{ij} = 0$ if $s_{ij} = 0$, where s_{ij} satisfies (3.3.3). For those $s_{ij} \neq 0$, the corresponding $y_{ij} = 0$ by the complementary slackness. Thus,

$$y^T s^{(t)} = \sum_{\substack{1 \leq q \leq r \\ q \neq t}} y_{h\alpha_q}$$

and

$$\begin{aligned} H(\{v_{j_1}, \cdots, v_{j_p}\}) &= \min_{1 \leq t \leq r} \{Z(x^{(t)}_{vec})\} \\ &= Z(x_{vec}) + \min_{1 \leq t \leq r} \{y^T s^{(t)} + z^T x^{(t)}_{vec}\} \end{aligned}$$

$$= Z(x_{vec}) + \min_{1\le t\le r}\{\sum_{\substack{1\le q\le r\\ q\ne t}} y_{h\alpha_q}\}$$

$$= Z(x_{vec}) + \sum_{t=1}^{r} y_{h\alpha_t} - \max_{1\le t\le r}\{y_{h\alpha_t}\}. \qquad (3.3.6)$$

v_h is served by v_{α_t} with $y_{h\alpha_t} = \max_{1\le t\le r}\{y_{h\alpha_t}\}$. For those $j \in \{j_1,\cdots,j_p\} - \{\alpha_1,\cdots,\alpha_r\}$, we have $s_{hj} \ne 0$, since, by the assumption that r is the largest number of these S_{j_k} containing v_h. Therefore $y_{hj} = 0$, by the complementary slackness, so

$$\sum_{t=1}^{r} y_{h\alpha_t} - \max_{1\le t\le r}\{y_{h\alpha_t}\} = \sum_{k=1}^{p} y_{hj_k} - \max_{1\le k\le p}\{y_{hj_k}\}.$$

Therefore,

$$H(\{v_{j_1},\cdots,v_{j_p}\}) = Z(x_{vec}) + \sum_{k=1}^{p} y_{hj_k} - \max_{1\le k\le p}\{y_{hj_k}\}$$

and v_h is served by v_{j_k} with $y_{hj_k} = \max_{1\le k\le p}\{y_{hj_k}\}$. ∎

Remark: Sometimes, $\max_{1\le t\le r}\{y_{h\alpha_t}\} = y_{hh}$ in (3.3.6). That is the reason why we set $y_{jj} = M$ a very large number, for $1 \le j \le n$.

From Theorem 3.2.3, if these S_{j_k} are pairwise disjoint, then X_p is a p-median of the network, i.e. the penalty cost is zero. By Lemma 3.3.3, if there is only one node v_h in the intersection of some of these S_{j_k}, the contribution to the penalty cost is $\sum_{k=1}^{p} y_{hj_k} - \max_{1\le k\le p}\{y_{hj_k}\}$, when we calculate $H(\{v_{j_1},\cdots,v_{j_p}\})$. Next, we examine the case with more than one node in the intersection of some of these S_{j_k}.

Lemma 3.3.4. *Given $X_p = \{v_{j_1}, v_{j_2},\cdots,v_{j_p}\}$ such that $S_{j_1}\cup\cdots\cup S_{j_p} = V$. If $\bigcup_{\{\alpha_1,\cdots,\alpha_r\}\in\mathcal{I}} (S_{\alpha_1}\cap\cdots\cap S_{\alpha_r}) = \{v_i : i \in I\}$, then*

(i) $H(\{v_{j_1}, \cdots, v_{j_p}\}) = Z(x_{vec}) + \sum_{i \in I}(\sum_{k=1}^{p} y_{ij_k} - \max_{1 \le k \le p}\{y_{ij_k}\})$;

(ii) For each i, $i \in I$, v_i *is served by* v_{j_k} *where* $y_{ij_k} = \max_{1 \le k \le p}\{y_{ij_k}\}$;

(iii) For each i, $i \notin I$, v_i *is served by* v_{j_k} *where* $x_{ij_k} \neq 0$.

Proof. By Lemma 3.3.3, for each $i \in I$, v_i is served by v_{j_k} with $y_{ij_k} = \max_{1 \le k \le p}\{y_{ij_k}\}$. The contribution to the penalty cost is $\sum_{k=1}^{p} y_{ij_k} - \max_{1 \le k \le p}\{y_{ij_k}\}$. Thus, the total penalty cost is $\sum_{i \in I}(\sum_{k=1}^{p} y_{ij_k} - \max_{1 \le k \le p}\{y_{ij_k}\})$. The proof is the same as that of Lemma 3.3.2. Therefore,

$$H(\{v_{j_1}, \cdots, v_{j_p}\}) = Z(x_{vec}) + \sum_{i \in I}(\sum_{k=1}^{p} y_{ij_k} - \max_{1 \le k \le p}\{y_{ij_k}\}).$$

For each $i \notin I$, by Lemma 3.3.3, v_i is served by v_{j_k} with $x_{ij_k} \neq 0$. ∎

In the above, we only consider the case that the union of these $S_{j_k}, 1 \le k \le p$, is V and the intersection of some of these S_{j_k} is not empty. Suppose now that the $S_{j_k}, 1 \le k \le p$, are pairwise disjoint and $\bigcup_{1 \le k \le p} S_{j_k} \subset V$. For this situation, we have the analogous results. We start from the 2-median problem and extend it to the general p-median problem.

Lemma 3.3.5. *If* $p = 2$, *given* $X_2 = \{v_{j_1}, v_{j_2}\}$ *such that* $S_{j_1} \cap S_{j_2} = \phi$ *and* $V - (S_{j_1} \cup S_{j_2}) = \{v_h\}$, *then*

(i) $H(\{v_{j_1}, v_{j_2}\}) = Z(x_{vec}) + \min\{z_{hj_1}, z_{hj_2}\}$,

(ii) v_h *is served by* v_{j_k} *where* $z_{hj_k} = \min\{z_{hj_1}, z_{hj_2}\}$,

(iii) For each i, $i \neq h$, v_i *is served by* v_{j_k} *where* $x_{ij_k} \neq 0$.

Proof. Without loss of generality, suppose that $X_2 = \{v_1, v_2\}$, i.e. $x_{11} \neq 0, x_{22} \neq 0$ in the fractional optimal solution $X_{mat} = (x_{ij}), S_k = \{v_i : x_{ik} \neq 0\}, k = 1,2; S_1 \cap S_2 = \phi$ and $V - (S_{j_1} \cup S_{j_2}) = \{v_h\}$. By the same argument as in the proof of Lemma 3.3.1, for any $i \neq h$, v_i is served by v_k if $x_{ik} \neq 0, k = 1,2$, when we calculate $H(\{v_1, v_2\})$. Now suppose v_h is served by v_1. Construct an integral feasible solution $X'_{mat} = (x'_{ij})$ to the RPMP as follows:

$$x'_{ij} = \begin{cases} 1, & \text{if } j = 1 \text{ and } i = h \text{ or } x_{i1} \neq 0, \\ 1, & \text{if } j = 2 \text{ and } x_{i2} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $S_1 \cap S_2 = \phi, V - S_1 \cup S_2 = \{v_h\}$, X'_{mat} is a feasible integral solution to the RPMP with $p = 2$. $H(\{v_1, v_2\}) = Z(x'_{vec})$ when v_h is served by v_1.

Let $s' = Ax'_{vec} - b$, where s'_{ij} is the component of s' such that $x'_{ij} - x'_{jj} + s'_{ij} = 0$. It is clear that $s'_{ij} = 0$ if $s_{ij} = 0$, where s_{ij} satisfies (3.3.3). Thus $y^T s' = 0$. Also $x'_{ij} = 0$ if $x_{ij} = 0$ with the one exception $x'_{h1} = 1$ and $x_{h1} = 0$. Thus $z^T x'_{vec} = z_{h1}$. Therefore, if v_h is served by v_1, we have

$$Z(x'_{vec}) = Z(x_{vec}) + y^T s' + z^T x'_{vec} = Z(x_{vec}) + z_{h1}.$$

With the same argument, if v_h is served by v_2, we construct another integral feasible solution $X''_{mat} = (x''_{ij})$ such that $Z(x''_{vec}) = Z(x_{vec}) + z_{h2}$. Thus,

$$\begin{aligned} H(\{v_1, v_2\}) &= \min\{Z(x'_{vec}), Z(x''_{vec})\} \\ &= Z(x_{vec}) + \min\{z_{h1}, z_{h2}\} \end{aligned}$$

and v_h is served by the node v_{j_k} with the lower value of z_{hj_k}. ∎

Generally, if $V - (S_{j_1} \cup S_{j_2})$ has more than one node, we have

Lemma 3.3.6. *If $p = 2$, given $X_2 = \{v_{j_1}, v_{j_2}\}$ such that S_{j_k} are pairwise disjoint, and $V - (S_{j_1} \cup S_{j_2}) = \{v_i : i \in L\}$, then*

(i) $H(\{v_{j_1}, v_{j_2}\}) = Z(x_{vec}) + \sum_{i \in L} \min\{z_{ij_1}, z_{ij_2}\}$,

(ii) For each i, $i \in L$, v_i is served by v_{j_k} where $z_{ij_k} = \min\{z_{ij_1}, z_{ij_2}\}$.

(iii) For each i, $i \notin L$, v_i is served by v_{j_k} where $x_{ij_k} \neq 0$. ∎

It is easy to follow the proof of Lemma 3.3.2 and use the result of Lemma 3.3.5 to prove this lemma. For the general p-median problem, we have

Lemma 3.3.7. *Given $X_p = \{v_{j_1}, v_{j_2}, \cdots, v_{j_p}\}$ such that S_{j_k} are pairwise disjoint and $V - (S_{j_1} \cup \cdots \cup S_{j_p}) = \{v_i : i \in L\}$, then*

(i) $H(\{v_{j_1}, \cdots, v_{j_p}\}) = Z(x_{vec}) + \sum_{i \in L} \min_{1 \le k \le p} \{z_{ij_k}\}$,

(ii) For each i, $i \in L$, v_i is served by v_{j_k} where $z_{ij_k} = \min_{1 \le k \le p} \{z_{ij_k}\}$,

(iii) For each i, $i \notin L$, v_i is served by v_{j_k} where $x_{ij_k} \neq 0$. ∎

We can combine the above results together to obtain the following theorem for the general situation.

Theorem 3.3.8. *Given $X_p = \{v_{j_1}, v_{j_2}, \cdots, v_{j_p}\}$, if $\bigcup_{\{\alpha_1, \cdots, \alpha_r\} \in \mathcal{I}} (S_{\alpha_1} \cap \cdots \cap S_{\alpha_r}) = \{v_i : i \in I\}$, and $V - (S_{j_1} \cup \cdots \cup S_{j_p}) = \{v_i : i \in L\}$, then*

(i) $H(\{v_{j_1}, \cdots, v_{j_p}\}) = Z(x_{vec}) + \sum_{i \in I} (\sum_{k=1}^{p} y_{ij_k} - \max_{1 \le k \le p} \{y_{ij_k}\}) + \sum_{i \in L} \min_{1 \le k \le p} \{z_{ij_k}\}$,

(ii) For each i, $i \in I$, v_i is served by v_{j_k} where $y_{ij_k} = \max_{1 \le k \le p} \{y_{ij_k}\}$,

(iii) For each i, $i \in L$, v_i is served by v_{j_k} where $z_{ij_k} = \min_{1 \le k \le p} \{z_{ij_k}\}$,

(iv) For each i, $i \notin I \cup L$, v_i is served by v_{j_k} where $x_{ij_k} \neq 0$. ∎

Remark: We have demonstrated a connection between the distance-sum of any p selected facilities X_p and the optimal solution to the RPMP. $H(\{v_{j_1}, \cdots, v_{j_p}\})$ is expressed as the summation of $Z(X_{mat})$ and the corresponding dual values and dual slack values. Let $W = \sum_{i \in I}(\sum_{k=1}^{p} y_{ij_k} - \max_{1 \leq k \leq p}\{y_{ij_k}\})$ be called the related ***overcovering penalty cost***; and let $U = \sum_{i \in L} \min_{1 \leq k \leq p}\{z_{ij_k}\}$ be called the related ***undercovering penalty cost***. Then $H(\{v_{j_1}, \cdots, v_{j_p}\}) = Z(x_{vec}) + W + U$, and the integral optimal solution occurs when $W + U$ is minimized. Note that both of the W and U terms occur in various p-median problem instances.

The tie cases, i.e. a client could be served by two or more equally distance open facilities, happen only in cases (ii) and (iii) of Theorem 3.3.8. Because, in the case (iv), there is only one $x_{ij_k} \neq 0$ for each $i \notin I \cup L$.

If some $x_{jj} = 0$ in the optimal solution X_{mat}, we have the following additional result.

Theorem 3.3.9. *If $x_{hh} = 0$ in X_{mat}, for any p node set X_p such that $v_h \in X_p$, and K is an index set such that $i \in K$ if and only if v_i is served by v_h, then*

$$H(X_p) \geq Z(x_{vec}) + \sum_{i \notin K} y_{ih} + \sum_{i \in K} z_{ih}. \tag{3.3.7}$$

Proof. Let $x'_{vec} = (x'_{ij})$ be the integral solution to the RPMP corresponding to X_p, i.e. $H(X_p) = Z(x'_{vec})$. Then, $x'_{ih} = 1$, if $i \in K$ and $x'_{ih} = 0$, if $i \notin K$, i.e.

$s'_{ih} = 1$, if $i \notin K$. Thus, $z^T x'_{vec} \geq \sum_{i \in K} z_{ih}$ and $y^T s' \geq \sum_{i \notin K} y_{ih}$, so

$$\begin{aligned} H(X_p) &= Z(x'_{vec}) \\ &= Z(x_{vec}) + y^T s' + z^T x'_{vec} \\ &\geq Z(x_{vec}) + \sum_{i \notin K} y_{ih} + \sum_{i \in K} z_{ih}. \qquad \blacksquare \end{aligned}$$

In the following, we develop a criterion to discard a node v_h from further consideration as potential open facility in the p-median problem. Hence, we reduce the size of the problem.

Theorem 3.3.10. *Let $x^{(0)}_{vec} = (x^{(0)}_{ij})$ be a feasible integral solution and $x^*_{vec} = (x^*_{ij})$ a fractional optimal solution to the RPMP with objective values $Z(x^{(0)}_{vec})$ and $Z(x^*_{vec})$ respectively. If $x^*_{hh} = 0$ in the x^*_{vec} and there is a $k \in \{1, \cdots, n\}$ such that $z^*_{kh} > Z(x^{(0)}_{vec}) - Z(x^*_{vec})$ and $y^*_{kh} > Z(x^{(0)}_{vec}) - Z(x^*_{vec})$, then v_h will not be an open facility in any p-medians.*

Proof. Suppose we have a set of p facilities X_p with $v_h \in X_p$. Let $S = \{v_i : i \in K\}$ be the subset of V that affixes to v_h. From Theorem 3.3.9, if $k \in K$, we have

$$\begin{aligned} H(X_p) &\geq Z(x^*_{vec}) + \sum_{i \notin K} y^*_{ih} + \sum_{i \in K} z^*_{ih} \\ &\geq Z(x^*_{vec}) + z^*_{kh} \\ &> Z(x^*_{vec}) + Z(x^{(0)}_{vec}) - Z(x^*_{vec}) \\ &= Z(x^{(0)}_{vec}). \end{aligned}$$

If $k \notin K$, we have

$$
\begin{aligned}
H(X_p) &\geq Z(x^*_{vec}) + \sum_{i \notin K} y^*_{ih} + \sum_{i \in K} z^*_{ih} \\
&\geq Z(x^*_{vec}) + y^*_{kh} \\
&> Z(x^*_{vec}) + Z(x^{(0)}_{vec}) - Z(x^*_{vec}) \\
&= Z(x^{(0)}_{vec}).
\end{aligned}
$$

Thus, if v_h is one of the p selected facilities, then $H(X_p) > Z(x^{(0)}_{vec})$, no matter what other $p-1$ selected facilities are used. Theretore, we can fix $x_{ih} = 0$ for $i = 1, \cdots, n$. That is, node v_h will not be an open facility in any p-medians. ∎

Since we have set $y_{jj} = M$ for $j = 1, \cdots, n$, if $z^*_{hh} > Z(x^{(0)}_{vec}) - Z(x^*_{vec})$, then node v_h will not be an open facility in any p-medians because it is too "pricey".

Example 3.2.1(continued). We give here an optimal solution X_{mat} of the realxed 3-median problem for the tree network with optimal value 90.

$$
X_{mat}(3) = \begin{pmatrix} 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \end{pmatrix} \tag{3.3.8}
$$

The optimal dual slack values in matrix form $z_{mat} = (z_{ij})$ and part of the optimal dual values in matrix form $y_{mat} = (y_{ij})$ with main diagonal M are given below, since we are interested in $y_{ij}, i \neq j$.

$$
y_{mat} = \begin{pmatrix} M & 2 & 2 & 0 & 0 & 0 \\ 0 & M & 0 & 10 & 0 & 0 \\ 0 & 0 & M & 0 & 4 & 0 \\ 0 & 40 & 0 & M & 0 & 0 \\ 0 & 0 & 44 & 0 & M & 0 \\ 0 & 0 & 10 & 0 & 0 & M \end{pmatrix}, \quad z_{mat} = \begin{pmatrix} 0 & 0 & 0 & 32 & 32 & 64 \\ 0 & 8 & 20 & 0 & 30 & 40 \\ 6 & 26 & 0 & 36 & 0 & 6 \\ 0 & 0 & 40 & 0 & 60 & 80 \\ 0 & 44 & 0 & 66 & 0 & 0 \\ 50 & 110 & 0 & 140 & 20 & 0 \end{pmatrix}
$$

The dual values $y_i, 0 \leq i \leq 6$, that correspond to constraints $x_{11} + \cdots + x_{18} = 3$ and $x_{i1} + \cdots + x_{i6} = 1$, for $1 \leq i \leq 6$, are $(y_0, y_1, \cdots, y_6) = (-70, 70, 20, 14, 60, 66, 70)$.

From (3.3.8) and Theorem 3.2.3, we know that each of the nodes v_1, v_3, v_4, v_5, v_6 is respectively a 1-median of the subgraph $<S_1>$, $<S_3>$, $<S_4>$, $<S_5>$, $<S_6>$, where $<S_1> = \{v_1\}$, $<S_3> = \{v_1, v_3, v_5, v_6\}$, $<S_4> = \{v_2, v_4\}$, $<S_5> = \{v_3, v_5\}$ and $<S_6> = \{v_6\}$. Since $\{S_1, S_4, S_5, S_6\}$ and $\{S_3, S_4\}$ are pairwise disjoint respectively and $S_1 \cup S_4 \cup S_5 \cup S_6 = V$, $S_3 \cup S_4 = V$, then by Corollary 3.2.4, $\{v_1, v_4, v_5, v_6\}$ is a 4-median and $\{v_3, v_4\}$ is a 2-median of the tree.

Since $S_3 \cap S_5 = \{v_3, v_5\}$, by Lemma 3.3.4,

$$H(\{v_3, v_4, v_5\}) = Z(x_{vec}) + \min\{y_{33}, y_{35}\} + \min\{y_{53}, y_{55}\}$$

$$= 90 + 4 + 44 = 138,$$

v_3, v_5 are served by themselves respectively, v_1, v_6 are served by v_3, v_2 is served by v_4. On the other hand,

$$H(\{v_3, v_4, v_5\}) = w(v_1)d(v_1, v_3) + w(v_2)d(v_2, v_4) + w(v_6)d(v_6, v_3)$$

$$= 34 \cdot 2 + 10 \cdot 1 + 30 \cdot 2 = 138.$$

If we calculate $H(\{v_1, v_5, v_6\})$, using Lemma 3.3.7,

$$H(\{v_1, v_5, v_6\}) = Z(x_{vec}) + \min\{z_{21}, z_{25}, z_{26}\} + \min\{z_{41}, z_{45}, z_{46}\}$$

$$= 90 + 0 + 0 = 90,$$

v_2 and v_4 are served by v_1, v_3 is served by v_5. On the other hand

$$H(\{v_1, v_5, v_6\}) = w(v_2)d(v_2, v_1) + w(v_3)d(v_3, v_5) + w(v_4)d(v_4, v_1)$$

$$= 10 \cdot 2 + 10 \cdot 1 + 20 \cdot 3 = 90.$$

Since this is the optimal value to the RPMP, $\{v_1, v_5, v_6\}$ is a 3-median of the tree.

If we calculate $H(\{v_3, v_5, v_6\})$, by Theorem 3.3.8,

$$\begin{aligned} H(\{v_3, v_5, v_6\}) =& Z(x_{vec}) + \min\{y_{33}, y_{35}\} + \min\{y_{53}, y_{55}\} + \min\{y_{63}, y_{66}\} \\ &+ \min\{z_{23}, z_{25}, z_{26}\} + \min\{z_{43}, z_{45}, z_{46}\} \\ =& 90 + 4 + 44 + 10 + 20 + 40 = 208, \end{aligned}$$

v_1, v_2, v_4 are served by v_3. On the other hand,

$$\begin{aligned} H(\{v_3, v_5, v_6\}) &= w(v_1)d(v_1, v_3) + w(v_2)d(v_2, v_3) + w(v_4)d(v_4, v_3) \\ &= 34 \cdot 2 + \quad 4 + 20 \cdot 5 = 208. \end{aligned}$$

Suppose v_2 is contained in some subset X_3, say $X_3 = \{v_2, v_5, v_6\}$, thus, $K = \{1, 2, 4\}$ in Theorem 3.3.9 and

$$\begin{aligned} H(\{v_2, v_5, v_6\}) &\geq Z(x_{vec}) + (z_{12} + z_{22} + z_{42}) + (y_{32} + y_{52} + y_{62}) \\ &= 90 + (0 + 8 + 0) + (0 + 0 + 0) = 98. \end{aligned}$$

On the other hand,

$$\begin{aligned} H(\{v_2, v_5, v_6\}) &= w(v_1)d(v_1, v_2) + w(v_3)d(v_3, v_5) + w(v_4)d(v_4, v_2) \\ &= 34 \cdot 2 + 10 \cdot 1 + 20 \cdot 1 = 98. \end{aligned}$$

Let $x_{vec}^{(0)}$ be the integral feasible solution to the RPMP that corresponds to $\{v_1, v_3, v_4\}$. Thus, $Z(x_{vec}^{(0)}) = 92$. Since

$$\begin{aligned} z_{22} &= 8 \\ &> Z(x_{vec}^{(0)}) - Z(x_{vec}^{*}) \\ &= 92 - 90 = 2, \end{aligned}$$

we can fix $x_{i2} = 0$, for $1 \leq i \leq 6$, by Theorem 3.3.10. Hence, v_2 will never be open in any 3-medians.

For the tree network, Kolen(1983) has the following result:

Proposition 3.3.11. *The distance-sum of the p-median is the same as the optimal value to the RPMP of the tree network.* ∎

Sometimes, it may happen that $x_{jj} = 1$, for $j = j_1, j_2, \cdots, j_p$ while some of x_{ij} are fractional, i.e. $0 < x_{ij} < 1$. In such a situation, we have the following result:

Theorem 3.3.12. *If the fractional optimal solution $X_{mat} = (x_{ij})$ to the RPMP is of the form: $x_{j_1 j_1} = \cdots = x_{j_p j_p} = 1$ and $0 < x_{ij_h} < 1$ for some i, $1 \leq h \leq p$, then $\{v_{j_1}, \cdots, v_{j_p}\}$ is a p-median.*

Proof. Without loss of generality, suppose $x_{11} = \cdots = x_{pp} = 1$ in X_{mat} and $0 < x_{k1} < 1$. Then, some other x_{kj} are also fractional with $j = 2, \cdots, p$, since $x_{k1} + \cdots + x_{kp} = 1$ and $x_{k,p+1} = \cdots = x_{kn} = 0$. If $x_{k2} \neq 0$, say, then $c_{k1} = c_{k2}$. This is true because if $c_{k1} < c_{k2}$, then we could set $x'_{k1} = x_{k1} + x_{k2}, x'_{k2} = 0$ and $x'_{ij} = x_{ij}$ otherwise. This would give a feasible solution to the RPMP with strictly lower objective value. Similarly, we cannot have $c_{k1} > c_{k2}$. Since this is the case, then we set $x_{k1} = 1$ and $x_{kj} = 0$ for $j \neq 1$. Thus, we adjust the solution so that X_{mat} becomes an integral feasible solution to the RPMP with the same objective value. Therefore, $\{v_1, \cdots, v_p\}$ is a p-median. ∎

From now on, a fractional optimal solution to the RPMP means a solution

where some of the x_{jj}, $1 \leq j \leq n$, are fractional.

3.4 The Reduction of Variables and Constraints

The disadvantage of the p-MP is that it needs n^2 variables and $n^2 + 1$ constraints. When n is large (e.g. $n = 100$), the RPMP is much harder to solve, even though it gives a very good bound (Beasley, 1985; Ahn, Cooper, Cornuejols and Frieze, 1988). Now, let us examine the structure of the optimal solution of the BRPMP.

Proposition 3.4.1. *There exists an optimal solution $X_{mat} = (x_{ij})$ to the BRPMP such that for each j, $1 \leq j \leq n$, either $x_{jj} = 1$, or $x_{jj} = \gamma_j < 1$ and $x_{ji_j} = 1 - \gamma_j$, where v_{i_j} is one of the closest distance nodes to v_j.* ∎

This result follows from the observations in the proof of Theorem 3.3.12. We now add the constraints $x_{jj} - x_{ij} \geq 0, 1 \leq i \leq n$, to the BRPMP to formulate a new problem, where v_j is one of the closest distance nodes to v_i. By Corollary 3.2.5, Theorems 3.2.1 and 3.2.2 both hold for the new optimal solution. This is better than the "weak" integer programming formulation for the p-median problem, which adds the constraints $\sum_{j=1}^{n} x_{ij} - nx_{jj} \leq 0$, for $1 \leq i \leq n$ (Nemhauser and Wolsey, 1988; Christofides and Beasley, 1982). Theorems 3.2.1 and 3.2.2 are no longer true for the optimal solution to the weak integer programming problem.

In the following, we construct a simplified relaxed p-median problem(SRPMP)

from the RPMP. We order the given data c_{ij} as follows: for each $i = 1, \cdots, n$, let $L_i = \{i_1, i_2, \cdots, i_n\}$ be a sequence of indices such that if $i_j < i_k$, then $c_{ii_j} \leq c_{ii_k}$. If $c_{ii_j} = c_{ii_k}$, then c_{ii_j} precedes c_{ii_k} if $i_j < i_k$. It is clear that for every i, $c_{ii_1} = 0$, which corresponds to the variable x_{ii}. Suppose we have an integral feasible solution $X^{(0)}_{mat} = (x^{(0)}_{ij})$ to the RPMP. For each i, there is a unique j such that $x^{(0)}_{ij} = 1$, and this is denoted by $j^{(0)}(i)$. If $j^{(0)}(i) = i_1$, set $j^{(0)}(i) = i_2$. We construct a new linear programming problem from the RPMP as follows.

For each $i = 1, \cdots, n$, define $r(i)$ to be the index such that $c_{ij^{(0)}(i)} = \cdots = c_{ii_{r(i)}} < c_{ii_{r(i)+1}}$. If $r(i) = 2$, add constraints $x_{jj} - x_{ij} \geq 0$ with $j = i_2$, and $x_{ii_1} + x_{ii_2} + x_{ii_3} = 1$. If $r(i) > 2$, add constraints $x_{jj} - x_{ij} \geq 0$ for $j = i_2, i_3, \cdots, i_{r(i)-1}$, and $x_{ii_1} + x_{ii_2} + \cdots + x_{ii_{r(i)}} = 1$.

Finally, we have the constraint $x_{11} + x_{22} + \cdots + x_{nn} = p$.

The objective function is a function of only those x_{ij} that appear in the above constraints. We call the above linear programming problem a *simplified relaxed p-median problem*(SRPMP). It should be mentioned that $r(i)$ is determined by the initial feasible integral solution $X^{(0)}_{mat}$. It is worth noting that we do not add constraints $x_{jj} - x_{ij} \geq 0$ for $j = i_{r(i)}$, $1 \leq i \leq n$, to the SRPMP. We will explain this below.

Let $X_{SRPMP} = (x_{ij})$ be an optimal solution to the SRPMP. $Z(X_{SRPMP})$ denotes the optimal value to the SRPMP and the objective value to the RPMP by setting the missing $x_{ij} = 0$ in X_{SRPMP}. These two values are equal even though

X_{SRPMP} may not be feasible to the RPMP, since the objective functions are the same. Let X^*_{mat} be an optimal solution to the RPMP. Then,

Theorem 3.4.2. $Z(X_{SRPMP}) \leq Z(X^*_{mat})$.

Proof. If X^*_{mat} is feasible to the SRPMP, then $Z(X_{SRPMP}) \leq Z(X^*_{mat})$, since X_{SRPMP} is optimal to the SRPMP. If X^*_{mat} is infeasible to the SRPMP, this means that there are some $x^*_{kh} > 0$ in the X^*_{mat} that do not appear in the SRPMP formulation.

Suppose, for the moment, that there is only one such $x^*_{kh} > 0$ that does not appear in the SRPMP formulation, thus $h \neq k$. Let $h = k_t$. There is an α, $2 < \alpha < t$, such that the corresponding constraint in the SRPMP is $x_{kk_1} + x_{kk_2} + \cdots + x_{kk_\alpha} = 1$. We construct a feasible solution $X' = (x'_{ij})$ to the SRPMP as follows:

$$x'_{kk_\alpha} = x^*_{kk_\alpha} + x^*_{kk_t}, x'_{kk_t} = 0; x'_{ij} = x^*_{ij} \text{ otherwise.}$$

Thus, all the x'_{ij} that do not appear in the SRPMP formulation have value zero. If $i = k$, X' satisfies $x'_{kk_1} + x'_{kk_2} + \cdots + x'_{kk_\alpha} = 1$ and $x'_{jj} - x'_{kj} \geq 0$ for $j = k_2, \cdots, k_{\alpha-1}$. For the rest of i, all the x'_{ij} that appear in the SRPMP satisfy the constraints $x'_{ii_1} + x'_{ii_2} + \cdots + x'_{ii_{r(i)}} = 1$ and $x'_{jj} - x'_{ij} \geq 0$ for $j = i_2, i_3, \cdots, i_{r(i)-1}$. Thus, X' is a feasible solution to the SRPMP and $Z(X') - Z(X^*_{mat}) = x^*_{kh}(c_{kk_\alpha} - c_{kk_t}) < 0$, since $c_{kk_\alpha} < c_{kk_t}$ in the construction of the SRPMP. Since $Z(X_{SRPMP}) \leq Z(X')$, it follows that $Z(X_{SRPMP}) < Z(X^*_{mat})$.

If more than one variable $x^*_{kh} \neq 0$ in the X^*_{mat} do not appear in the SRPMP

formulation, we construct a feasible solution X'' to the SRPMP step by step using the same method to construct X' above until all these x^*_{kh} become zero in X''. ∎

If the optimal solution to the SRPMP violates constraints $x_{jj} - x_{ij} \geq 0$, for some $j = r(i)$, then if $r(i) = n$, we add constraint $x_{jj} - x_{ij} \geq 0$ with $j = i_n$, if $r(i) < n$, we set $j^{(0)}(i) = r(i) + 1$, for those violated constraints. Following the procedure above, we construct a new SRPMP and solve the new SRPMP again until all the constraints $x_{jj} - x_{ij} \geq 0$ for current $j = r(i), 1 \leq i \leq n$, are satisfied. The number of such iterations is at most $n^2 - 2n$, because there are only $n^2 + 1$ constraints in the RPMP and there are at least $2n + 1$ constraints in the first formulation of the SRPMP. In each iteration of constructing new SRPMP, at least one new constraint is added, and

Corollary 3.4.3. *The optimal value of the new SRPMP is greater than or equal to that of the old SRPMP.*

Proof. Let $X^{(1)}$ and $X^{(2)}$ be the optimal solutions to the old and new SRPMP respectively. If $X^{(2)}$ is feasible to the old SRPMP, then $Z(X^{(1)}) \leq Z(X^{(2)})$. If $X^{(2)}$ is infeasible to the old SRPMP, from the proof of Theorem 3.4.2, there is a feasible solution X' to the old SRPMP, such that $Z(X^{(1)}) \leq Z(X') < Z(X^{(2)})$. ∎

It is easy to see that the integral feasible solution $X^{(0)}_{mat}$ to the RPMP is also a feasible solution to the SRPMP. The number of variables and constraints of the SRPMP depends on the $X^{(0)}_{mat}$. The closer the objective value of $X^{(0)}_{mat}$ is to the optimal value of the RPMP, the fewer the number of variables and constraints in

the SRPMP. On the other hand, if the integral feasible solution $X_{mat}^{(0)}$ is an optimal solution to the SRPMP, $X_{mat}^{(0)}$ must be also an optimal solution to the RPMP. Thus, we have a simple method to testify whether the integral feasible solution $X_{mat}^{(0)}$ is an ontimal integral solution to the RPMP or not. It is easy to obtain an integral feasible solution $X_{mat}^{(0)}$.

Now, let us consider the following situation.

If we add constraints $x_{jj} - x_{ij} \geq 0$, for $j = r(i), 1 \leq i \leq n$, to the construction of SRPMP, then the optimal solution of SRPMP is feasible to the RPMP. Thus, the optimal value of SRPMP is greater than or equal to the optimal value of RPMP. It is an open problem to decide under which conditions these two optimal solutions will be the same.

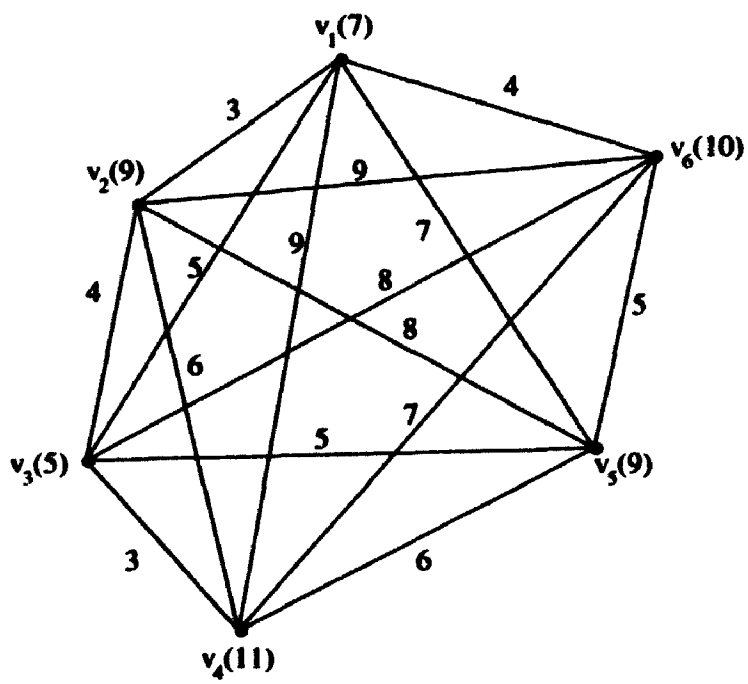

Figure 2

Example 3.4.1. The network $G = (V, E)$ shown in Figure 2 consists of a set $V = \{v_1, x_2, \cdots, v_6\}$ of 6 nodes with weights $w(v_1), w(v_2), \cdots, w(v_6)$ (the bracketed numbers). The numbers along each edge are the lengths of the corresponding edge. The distance between any two nodes is in the distance matrix $D = (d_{ij})$, where $d_{ij} = d(v_i, v_j)$ and $C = (c_{ij})$ where $c_{ij} = w(v_i)d(v_i, v_j) = w(v_i)d_{ij}$.

$$D = \begin{pmatrix} 0 & 3 & 5 & 9 & 7 & 4 \\ 3 & 0 & 4 & 6 & 8 & 9 \\ 5 & 4 & 0 & 3 & 5 & 8 \\ 9 & 6 & 3 & 0 & 6 & 7 \\ 7 & 8 & 5 & 6 & 0 & 5 \\ 4 & 9 & 8 & 7 & 5 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 21 & 35 & 63 & 49 & 28 \\ 27 & 0 & 36 & 54 & 72 & 81 \\ 25 & 20 & 0 & 15 & 25 & 40 \\ 99 & 66 & 33 & 0 & 66 & 77 \\ 63 & 72 & 45 & 54 & 0 & 45 \\ 40 & 90 & 80 & 70 & 50 & 0 \end{pmatrix}.$$

The 2-median RPMP formulation of the network is as follows:

$$Z = \min \sum_{j=1}^{6} \sum_{i=1}^{6} c_{ij} x_{ij}$$

$$s.t.$$

$$\sum_{j=1}^{6} x_{jj} = 2,$$

$$\sum_{j=1}^{6} x_{ij} = 1, \quad 1 \le i \le 6$$

$$x_{jj} - x_{ij} \ge 0, \quad 1 \le i, j \le 6 \text{ and } i \ne j$$

$$x_{ij} \ge 0, \quad 1 \le i, j \le 6.$$

Let $\{v_4, v_6\}$ be an integral feasible solution to the RPMP, i.e. $X^{(0)}_{mat} = (x^{(0)}_{ij})$ with $x^{(0)}_{16} = x^{(0)}_{24} = x^{(0)}_{34} = x^{(0)}_{44} = x^{(0)}_{56} = x^{(0)}_{66} = 1; x^{(0)}_{ij} = 0$ otherwise, for $1 \le i, j \le 6$. To construct the corresponding SRPMP, if $i = 2$, then $x^{(0)}_{22} = 0$, $j^{(0)}(2) = 4 = 2_4$ and $r(2) = 4$. Thus, we add constraints $x_{11} - x_{21} \ge 0$, $x_{33} - x_{23} \ge 0$ and $x_{21} + x_{22} + x_{23} + x_{24} = 1$. If $i = 4$, then $x^{(0)}_{44} = 1$, $j^{(0)}(4) = 3 = 4_2$, $r(4) = 2$. Thus,

we add constraints $x_{33} - x_{43} \geq 0$ and $x_{42} + x_{43} + x_{44} = 1$. So are the other values of i. The SRPMP formulation is as follows:

$$\begin{aligned} \min\ & 21x_{12} + 28x_{16} + 27x_{21} + 36x_{23} + 54x_{24} + 20x_{32} + 15x_{34} \\ & + 66x_{42} + 33x_{43} + 45x_{53} + 45x_{56} + 40x_{61} + 50x_{65} \end{aligned} \tag{3.4.1}$$

$s.t.$

$$\begin{aligned} & x_{11} + x_{22} + x_{33} + x_{44} + x_{55} + x_{66} = 2 \\ & x_{11} + x_{12} + x_{16} = 1 \\ & x_{21} + x_{22} + x_{23} + x_{24} = 1 \\ & x_{32} + x_{33} + x_{34} = 1 \\ & x_{42} + x_{43} + x_{44} = 1 \\ & x_{53} + x_{55} + x_{56} = 1 \\ & x_{61} + x_{65} + x_{66} = 1 \\ & x_{22} - x_{12} \geq 0 \\ & x_{11} - x_{21} \geq 0 \\ & x_{33} - x_{23} \geq 0 \\ & x_{44} - x_{34} \geq 0 \\ & x_{33} - x_{43} \geq 0 \\ & x_{33} - x_{53} \geq 0 \\ & x_{11} - x_{61} \geq 0 \\ & x_{ij} \geq 0. \end{aligned}$$

The optimal solution to the SRPMP is $x_{11}^{(1)} = x_{21}^{(1)} = x_{34}^{(1)} = x_{44}^{(1)} = x_{56}^{(1)} = x_{61}^{(1)} = 1; x_{ij}^{(1)} = 0$ otherwise, with $1 \leq i, j \leq 6$. There is a violation to the constraint $x_{66}^{(1)} - x_{56}^{(1)} \geq 0$, so we add a new variable x_{54}, new constraint $x_{66} - x_{56} \geq 0$ and change the objective function and constraint $x_{53} + x_{55} + x_{56} = 1$ accordingly to solve the new SRPMP. The new SRPMP has the same fractional optimal solution as the RPMP, i.e.

$$X^*_{mat}(2) = \begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0 & 0.5 \end{pmatrix}.$$

It is interesting that we do not have the constraint $x_{66} - x_{16} \geq 0$ in the final SRPMP, but it is satisfied. Thus, we have reduced the number of variables from 36 to 20 and the number of constraints from 37 to 15.

If we have an optimal solution $x^*_{vec} = (x^*_{ij})$ to the RPMP with the dual and dual slack value y^*, z^* respectively, we can fix some variables at zero by using the dual slack values. Let $X^{(0)}_{mat}$ be the best integer solution found so far with the value $Z(X^{(0)}_{mat})$. $c_{vec} = A^T y^* + z^*$. Any primal solution x_{vec} to the RPMP gives the value

$$\begin{aligned} c^T_{vec} x_{vec} =& (A^T y^* + z^*)^T x_{vec} \\ =& (y^*)^T A x_{vec} + (z^*)^T x_{vec} \\ \geq& b^T y^* + (z^*)^T x_{vec} \\ =& Z(x^*_{vec}) + (z^*)^T x_{vec} \qquad (Z(x^*_{vec}) = b^T y^*) \end{aligned}$$

Thus the x_{ij} can be fixed at zero if $Z(x^*_{vec}) + z^*_{ij} \geq Z(X^{(0)}_{mat})$, since any feasible solution to the RPMP with $x_{ij} = 1$ will have at least $Z(x^*_{vec}) + z^*_{ij} \geq Z(X^{(0)}_{mat})$.

Example 3.4.1(continued). The objective value of $X^{(0)}_{mat}$ is $Z(X^{(0)}_{mat}) = 142$, the optimal value of X^*_{mat} is $Z(X^*_{mat}) = 134.5$. The optimal dual slack values in matrix form $z^*_{mat} = (z^*_{ij})$ and part of the optimal dual values in matrix form $y^*_{mat} = (y^*_{ij})$, with main diagonal M, are given below. We are only interested in the $y_{ij}, i \neq j$.

$$y^*_{mat} = \begin{pmatrix} M & 14 & 0 & 0 & 0 & 7 \\ 19.5 & M & 10.5 & 0 & 0 & 0 \\ 0 & 4 & M & 9 & 0 & 0 \\ 0 & 0 & 22.5 & M & 0 & 0 \\ 0 & 0 & 7.5 & 0 & M & 7.5 \\ 10 & 0 & 0 & 0 & 0 & M \end{pmatrix},$$

$$z^*_{mat} = \begin{pmatrix} 0 & 0 & 0 & 28 & 14 & 0 \\ 0 & 0 & 0 & 7.5 & 25.5 & 34.5 \\ 1 & 0 & 0 & 0 & 1 & 16 \\ 43.5 & 10.5 & 0 & 0 & 10.5 & 21.5 \\ 10.5 & 19.5 & 0 & 1.5 & 12 & 0 \\ 0 & 40 & 30 & 20 & 0 & 0 \end{pmatrix}.$$

The remaining dual values y_i^*, $0 \leq i \leq 6$ that correspond to constraints $x_{11} + \cdots + x_{66} = 2$ and $x_{i1} + \cdots + x_{i6} = 1$, $1 \leq i \leq 6$, are $(y_0^*, y_1^*, \cdots, y_6^*) = (-64.5, 35, 46.5, 24, 55.5, 52.5, 50)$.

Since $z_{42}^* = 10.5 > Z(X_{mat}^{(0)}) - Z(X_{mat}^*)$, we can fix $x_{42} = 0$. There are 14 other variables x_{ij}, the corresponding $z_{ij}^* \geq Z(X_{mat}^{(0)}) - Z(X_{mat}^*)$. Among them, 13 variables x_{ij} (except x_{24}) are not in the final SRPMP formulation. From this Example, we see that the SRPMP almost excludes those variables that will be fixed at zeros in finding a p-median.

Chapter 4

Algorithms for the p-median Problem

4.1 Introduction

From the previous Chapter, when the optimal solution to the SRPMP is integral and satisfies all the constraints $x_{jj} - x_{ij} \geq 0$ for $j = r(i), 1 \leq i \leq n$, then this solution is a p-median. Now, given that the optimal solution to the SRPMP is fractional and feasible to the RPMP, the problem of how to find an integral optimal solution remains. One method of proceeding is the branch and bound algorithm. Harvinen, Rajala and Sinerva(1972) used the branch and bound algorithm to solve the p-median problem. In this Section, we briefly discuss the branch and bound algorithm for integer programming problems. In Section 4.2, we give two different orderings in branch variable selection. This is very important to the running time of the algorithm. In Section 4.3, we present a complete algorithm for the p-median problem using one of the branching strategies in Section 4.2.

The branch and bound algorithm is a basic algorithm used by commercial codes for solving mixed-integer programming problems. Here, we give a brief description of the algorithm for our purposes. The general results are found in many standard mathematical programming or linear and integer programming books (e.g. Nemhauser and Wolsey,1988; Schrijver,1986).

The branch and bound algorithm provides a methodology to search for an integral optimal solution by doing only a *partial enumeration*. It solves an integer programming minimization problem by solving a sequence of linear programming problems. In the course of applying the branch and bound algorithm, the overall set of integral feasible solutions is partitioned into simpler subsets, called the *subproblems*. The subproblems are regarded as forming a tree, rooted at the original linear programming problem. When using the branch and bound algorithm, one of three things happens at each node of the tree.

(i) The subproblem has optimal value greater than the value of the best known integral feasible solution, so the node is fathomed by bounds.

(ii) The optimal solution is an integral solution with value less than the best known integral solution. Thus, we update the best known integral feasible solution and the node is fathomed.

(iii) The optimal solution to the subproblem is fractional and has optimal value less than the best known integral feasible solution. Thus, we have to branch the subproblem into two subproblems and repeat the same process again.

Two elementary strategies are used in the execution of the branch and bound

algorithm: *the bounding strategy* and *the branching strategy.*

The bounding strategy is used in selecting promising subproblem in the search for the integral optimal solution. It discards subproblems that cannot possibly contain the integral optimal solution. It should be both relatively easy to implement and also computationally efficient.

The branching strategy partitions the set of feasible solutions of the original problem into two subproblems. Each subset in the partition is the set of feasible solutions of a problem obtained by imposing additional constraints on the original problem. These problems are called *candidate problems.* The candidate problems must be generated in such a way that the bounding strategy can be applied to each of them. It should be possible to apply the branching strategy to any candidate problem and generate two candidate subproblems such that the following property holds: the potential increase of the optimal value of the candidate subproblems is as small as possible.

As we move down the tree, more and more integral variables are set at either 0 or 1. We can prune some candidate problems if their lower bound is greater than or equal to the objective value of the best known integral solution. The branch and bound algorithm is terminated if there are no candidate problems left and the best known integral feasible solution is the optimal solution.

For the p-median problem, we can obtain an integral feasible solution easily. If we use the greedy algorithm and other heuristic algorithms, we can get a better

integral feasible solution. Thus, the branching strategy is the main issue for the p-median problem. This is the topic of the next Section.

4.2 Branching Strategies for the p-median Problem

In this Section, we give two branching strategies for the RPMP, which will be later extended to the SRPMP. We are interested in the dual values $y_{ij}, i \neq j$ that correspond to the constraints $x_{jj} - x_{ij} \geq 0$. Let $y_{mat} = (y_{ij})$ represent the matrix form of dual values with the main diagonal M; y represents the vector form of all dual values. The dual slack values are represented by $z_{mat} = (z_{ij})$.

Suppose $X^{(0)}_{mat} = (x^{(0)}_{ij})$ is the best known integral solution to the RPMP found so far. $X^*_{mat} = (x^*_{ij})$ is a fractional optimal solution to the RPMP with the corresponding optimal dual values y^*_{mat} and dual slack values z^*_{mat}. Let $\delta^{(0)} = Z(X^{(0)}_{mat}) - Z(X^*_{mat})$ be the penalty cost of $X^{(0)}_{mat}$. We can fix the variables $x_{ij} = 0$, $1 \leq i \leq n$, for which $x^*_{jj} = 0$ in X^*_{mat} and satisfy the condition of Theorem 3.3.10. Also fix the variable $x_{ij} = 0$ for which the corresponding $z^*_{ij} > \delta^{(0)}$, for $1 \leq i, j \leq n$.

First branching strategy

Suppose that there is a better integral solution $X'_{mat} = (x'_{ij})$ than $X^{(0)}_{mat}$. Then the penalty cost of X'_{mat} is less than the penalty cost of $X^{(0)}_{mat}$, i.e. $\delta^{(0)}$. Hence, the following variables might be 1 in X'_{mat}:

(i) In the undercovering situation, some of the x'_{ij} with $z^*_{ij} < \delta^{(0)}$ and where $x^*_{ij} = 0$ in X^*_{mat}.

(ii) In the over covering situation, some of the x'_{ik} with $y^*_{ij} < \delta^{(0)}$ for those $x^*_{ij} > 0$, where k is an index such that $y^*_{ik} = \max_j\{y^*_{ij} : x^*_{ij} > 0\}$.

To simplify the branching strategy, we only consider the case (i) for which $x^*_{ij} = 0$ and $z^*_{ij} < \delta^{(0)}$. We do not consider the dual case (ii) here. Suppose that there is a better integral feasible solution $X'_{mat} = (x'_{ij})$ than $X^{(0)}_{mat}$. If $x'_{ij} = 1$ in X'_{mat}, then for $x^*_{ik} \neq 0$, we have $x'_{kk} = 0$, or for $x^*_{ik} = 0$ and $z^*_{ik} \leq z^*_{ij}$, we have $x'_{kk} = 0$ as well. To verify this, if $x'_{kk} = 1$ in X'_{mat}, then, by Theorem 3.3.8, v_i is served by v_k in both cases, which contradicts that $x'_{ij} = 1$, i.e. v_i is served by v_j. It is equivalent to say that if we set the variable $x_{ij} = 1$, then $x_{kk} = 0$ simultaneouly for those $x^*_{ik} \neq 0$, or $x^*_{ik} = 0$ and $z^*_{ik} \leq z^*_{ij}$.

However, since $x_{jj} - x_{ij} \geq 0$, $x_{ij} = 1$ implies that $x_{jj} = 1$. Hence, the optimal value of the RPMP plus the constraint $x_{jj} = 1$ is less than or equal to the optimal value of the RPMP plus the constraint $x_{ij} = 1$. Therefore, we set $x_{jj} = 1$ instead of setting $x_{ij} = 1$. Sometimes, the optimal value of the RPMP plus the constraint $x_{tt} = 1$ is greater than or equal to $Z(X^{(0)}_{mat})$. Thus, we can fix the variables $x_{it} = 0$, $1 \leq i \leq n$, for further calculation, and discard the remaining unchosen values x^*_{ht} in column t of X^*_{mat} with $z^*_{ht} < \delta^{(0)}$. That is, we can reduce the computation time. Above all, we will consider the branching variable x_{jj}, if $0 < x^*_{jj} < 1$ in X^*_{mat} and some $x^*_{ij} = 0$ with the corresponding $z^*_{ij} < \delta^{(0)}$.

By Theorem 3.3.12, if all x_{jj} are integral in the optimal solution $X_{mat} = (x_{ij})$ to the RPMP, then there is an integral optimal solution $\bar{X}_{mat} = (\bar{x}_{ij})$ such that $\bar{x}_{jj} = x_{jj}$ for all j. Replace X_{mat} by $\bar{X}_{mat}$. To simplify the notation, we will say that X_{mat} is integral. This will not cause any confusion, since the p open facilities and the optimal values are the same. Thus, we only consider the variables x_{jj} for $1 \leq j \leq n$. In order to minimize the penalty cost, we choose the fractional variable x^*_{kk} first, if $z^*_{hk} = \min_{i,j}\{z^*_{ij} : x^*_{ij} = 0, 0 < x^*_{jj} < 1\}$. Set $x_{kk} = 1$ and $x_{kk} = 0$ respectively. Solve the RPMP again and get the new optimal solutions $X^{(k,1)}_{mat} = (x^{(k,1)}_{ij})$ and $X^{(k,0)}_{mat} = (x^{(k,0)}_{ij})$ respectively. There are three cases:

(i) $Z(X^{(k,q)}_{mat}) \geq Z(X^{(0)}_{mat})$, $q = 0$ or 1. Follow the branching strategy above, choose the fractional variable $x_{\ell\ell}$ if one of the $z^*_{h\ell} = \min_{i,j}\{z^*_{ij} : x^*_{ij} = 0, 0 < x^*_{jj} < 1\}$ in the remaining columns of X^*_{mat}. Repeat the entire process again.

(ii) $Z(X^{(k,q)}_{mat}) < Z(X^{(0)}_{mat})$ and $x^{(k,q)}_{jj}$ are integral for $1 \leq j \leq n$. By Theorem 3.3.12, $X^{(k,q)}_{mat}$ is integral. Replace $X^{(0)}_{mat}$ by $X^{(k,q)}_{mat}$. Choose the next candidate problem. If there is no candidate problem, terminate the algorithm.

(iii) $Z(X^{(k,q)}_{mat}) < Z(X^{(0)}_{mat})$ and $X^{(k,q)}_{mat}$ is still fractional. Follow the branching strategy above, choose another fractional variable $x^*_{\ell\ell}$, if one of the $z^*_{h\ell} = \min_{i,j}\{z^*_{ij} : x^*_{ij} = 0, 0 < x^*_{jj} < 1\}$ in the remaining columns of X^*_{mat}. Repeat the same process again.

This is the standard procedure for the branch and bound algorithm.

Example 3.4.1 (continued). We have an integral feasible solution $X^{(0)}_{mat} = (x^{(0)}_{ij})$ and the optimal fractional solution $X^*_{mat} = (x^*_{ij})$ to the RPMP as follows.

$$X^{(0)}_{mat} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad X^*_{mat} = \begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0 & 0.5 \end{pmatrix}.$$

$Z(X^{(0)}_{mat}) = 142$, $Z(X^*_{mat}) = 134.5$, $\delta^{(0)} = 7.5$, The optimal dual slack values are

$$z^*_{mat} = \begin{pmatrix} 0 & 0 & 0 & 28 & 14 & 0 \\ 0 & 0 & 0 & 7.5 & 25.5 & 34.5 \\ 1 & 0 & 0 & 0 & 1 & 16 \\ 43.5 & 10.5 & 0 & 0 & 10.5 & 21.5 \\ 10.5 & 19.5 & 0 & 1.5 & 12 & 0 \\ 0 & 40 & 30 & 20 & 0 & 0 \end{pmatrix}.$$

By Theorem 3.3.12, fix $x_{i5} = 0$ for $1 \le i \le 6$, since $z^*_{55} = 12 > \delta^{(0)} = 7.5$. Since $z^*_{13} = \min_{i,j}\{z^*_{ij} : x^*_{ij} = 0, 0 < x^*_{jj} < 1\} = 0$, set variable $x_{33} = 1$. Solving the RPMP again, we get an optimal solution

$$X^{(3,1)}_{mat}(2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$Z(X^{(3,1)}_{mat}(2)) = 142 = Z(X^{(0)}_{mat})$, which means that there is no improvement. Thus, we can fix variables $x_{i3} = 0$ for $1 \le i \le 6$. In the remaining columns of X^*_{mat} where x^*_{jj} are fractional, $z^*_{31} = 1$ is the minimum. Set variable $x_{11} = 1$. Solving the RPMP again, we get an optimal solution

$$X_{mat}^{(1,1)}(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2.1}$$

which is integral and $Z(X_{mat}^{(1,1)}(2)) = 136 < Z(X_{mat}^{(0)})$. Replace $X_{mat}^{(0)}$ by $X_{mat}^{(1,1)}(2)$. If we set $x_{11} = 0$, and solve the RPMP again, we get a higher optimal value than $Z(X_{mat}^{(1,1)}(2))$. Thus, $X_{mat}^{(1,1)}(2)$ is an integral optimal solution, i.e. $\{v_1, v_4\}$ is a 2-median of the network.

Second branching strategy

By Theorem 3.2.3, if $x_{jj}^* \neq 0$ in X_{mat}^*, then node v_j is a 1-median of the subgraph $<S_j>$, where $S_j = \{v_i : x_{ij}^* \neq 0\}$. By Corollary 2.2.5, X_p^* is a p-median of G if and only if $H(X_p^*)$ has the minimal value among all the $(1, \cdots, 1)$ stable types. Thus, we can choose the branching variable x_{kk} if $H(S_k, v_k) = \min_j\{H(S_j, v_j) : 0 < x_{jj}^* < 1\}$. Set $x_{kk} = 1$ and 0 respectively, solve the RPMP, and get a new optimal solution $X_{mat}^{**} = (x_{ij}^{**})$. There are three cases:

(i) $Z(X_{mat}^{**}) \geq Z(X_{mat}^{(0)})$. Follow the branching strategy above, choose the fractional variable $x_{\ell\ell}$ if $H(S_\ell, v_\ell) = \min_j\{H(S_j, v_j) : 0 < x_{jj}^* < 1\}$ in the remaining columns of X_{mat}^*. Repeat the entire process again.

(ii) $Z(X_{mat}^{**}) < Z(X_{mat}^{(0)})$ and x_{jj}^{**} are integral for $1 \leq j \leq n$. By Theorem 3.3.12, X_{mat}^{**} is integral. Replace $X_{mat}^{(0)}$ by X_{mat}^{**}. Choose the next candidate problem. If there is no candidate problem, terminate the algorithm.

(iii) $Z(X^{**}_{mat}) < Z(X^{(0)}_{mat})$ and X^{**}_{mat} is still fractional. Follow the branching strategy above, choose the branching variable x_{hh} if $H(S^{**}_h, v_h) = \min_j\{H(S^{**}_j, v_j) : 0 < x^{**}_{jj} < 1\}$ in the current optimal solution X^{**}_{mat}, where $S^{**}_j = \{v_i : x^{**} \neq 0\}$. Repeat the same process again.

It may happen that some variables $x^*_{jj} = 0$ in X^*_{mat} and $x^{**}_{jj} \neq 0$ in X^{**}_{mat}, or S_h and S^{**}_h are different with respect to X^*_{mat} and X^{**}_{mat}. We can follow the proof of Theorem 3.2.3 to prove that v_h is a 1-median of S^{**}_h under the condition that $x_{kk} = 1$ or 0. This means that the second branching strategy selects a 1-median node of a subgraph with the least distance-sum at each stage.

Example 3.4.1 (continued). From the optimal solution X^*_{mat} to the RPMP, we have $H(S_1, v_1) = 67$, $H(S_3, v_3) = 114$, $H(S_4, v_4) = 15$ and $H(S_6, v_6) = 73$. Hence, set $x_{44} = 1$. Solving the RPMP, we get the same optimal solution as (4.2.1). Set $x_{44} = 0$, and solve the RPMP again, we get an optimal solution with greater objective value than that of (4.2.1). Hence, $\{v_1, v_4\}$ is the 2-median.

4.3 A Complete Algorithm for the p-median Problem

To extend the previous two branching strategies to the SRPMP of Section 3.4, replace solving the RPMP with solving the SRPMP. However, at each iteration of solving the SRPMP, we must verify that the optimal solution to the SRPMP is feasible to the RPMP.

With the above preparation, we have the following complete algorithm for the p-median problem using the first branching strategy. Let $\mathbf{P}$ be the set of candidate problems, and let $P(X_{mat}^{(h,q)})$ be the subproblem with the optimal solution $X_{mat}^{(h,q)}$, $q = 0$ or 1. Let $\mathbf{F}$ be the set of constraints to the SRPMP. The input data is $C = (c_{ij})$, n and p.

Step 1. (Initialization): Use the greedy algorithm to find an integral feasible solution $X_{mat}^{(0)} = (x_{ij}^{(0)})$. For each i, let $L_i = \{i_1, i_2, \cdots, i_n\}$ be a sequence of indices such that if $i_j < i_k$, then $c_{ii_j} \leq c_{ii_k}$. If $c_{ii_j} = c_{ii_k}$, then c_{ii_j} precedes c_{ii_k} if $i_j < i_k$ in C. For each i, find the unique $j^{(0)}(i)$ such that $x_{ij}^{(0)} = 1$ with $j = j^{(0)}(i)$, if $j^{(0)}(i) = i_1$, set $j^{(0)}(i) = i_2$. $\mathbf{P} = \phi$.

Step 2. (Relaxation): Call the SRPMP procedure (step 9) with $X_{mat}^{(0)}$, get an optimal solution $X_{mat}^{*} = (x_{ij}^{*})$ to the RPMP. The current SRPMP is called the *initial SRPMP*, which will be solved repeatly later. If X_{mat}^{*} is integral go to step 5; else $\mathbf{P} = \mathbf{P} \cup \{P(X_{mat}^{*})\}$, go to step 3.

Step 3. (Fixing variables): Let $\delta^{(0)} = Z(X_{mat}^{(0)}) - Z(X_{mat}^{*})$. Fix $x_{ij} = 0$, $1 \leq i \leq n$, for those $x_{jj}^{*} = 0$ in X_{mat}^{*} that satisfy the condition of Theorem 3.3.10. Fix $x_{ij} = 0$ if $z_{ij}^{*} \geq \delta^{(0)}$.

Step 4. (Complementary Slackness): By the complementary slackness condition $y^T(Ax_{vec}^{*} - b) = 0$, solve the system of linear equations for the dual vector y, obtain the optimal dual values y^{*} and dual slack values z_{mat}^{*}. Go to step 8.

Step 5. (Termination test): If $\mathbf{P} = \phi$, $X^{(0)}_{mat}$ is the optimal solution. STOP.

Step 6. (Problem selection): Select and delete a problem $P(X^{(h,q)}_{mat})$ from $\mathbf{P}$. Call the SRPMP procedure (step 10) starting from the initial SRPMP with the additional constraint $x_{hh} = q$, get an optimal solution $X^{(h,q)}_{mat}$.

Step 7. (Pruning): (i) If $Z(X^{(h,q)}_{mat}) \geq Z(X^{(0)}_{mat})$, go to step 5.

(ii) If $Z(X^{(h,q)}_{mat}) < Z(X^{(0)}_{mat})$ and $x^{(h,q)}_{jj}$ are integral for $1 \leq j \leq n$, let $X^{(0)}_{mat} = X^{(h,q)}_{mat}$. Delete from $\mathbf{P}$ all problems with $Z(X^{(h,q)}_{mat}) \geq Z(X^{(0)}_{mat})$. Go to step 5.

(iii) If $Z(X^{(h,q)}_{mat}) < Z(X^{(0)}_{mat})$ and $X^{(h,q)}_{mat}$ is fractional, go to step 8.

Step 8. (Branching): Choose the branching variable x_{kk} if $z^*_{\ell k} = \min_{i,j}\{z^*_{ij} : x^*_{ij} = 0, 0 < x^*_{jj} < 1\}$ in the remaining columns of X^*_{mat}. Add problems $P(X^{(k,0)}_{mat})$ and $P(X^{(k,1)}_{mat})$ to $\mathbf{P}$. Go to step 5.

The SRPMP procedure is as follows:

Step 9. (Formulating the SRPMP): Let $\mathbf{F} = \{x_{11} + x_{22} + \cdots + x_{nn} = p\}$. For each $i = 1, \cdots, n$. Suppose $c_{ij^{(0)}(i)} = \cdots = c_{ii_{r(i)}} < c_{ii_{r(i)+1}}$, then

if $r(i) = 2$, then $\mathbf{F} = \mathbf{F} \cup \{x_{jj} - x_{ij} \geq 0 \text{ with } j = i_2\} \cup \{x_{ii_1} + x_{ii_2} + x_{ii_3} = 1\}$;

else $\mathbf{F} = \mathbf{F} \cup \{x_{jj} - x_{ij} \geq 0 \text{ for } j = i_2, \cdots, i_{r(i)-1}\} \cup \{x_{ii_1} + x_{ii_2} + \cdots + x_{ii_{r(i)}} = 1\}$.

The objective function is the function of those x_{ij} that appear in the above constraints.

Step 10. (Solving the SRPMP): Solve the SRPMP, get an optimal solution

$X_{SRPMP} = (x_{ij})$. If $x_{jj} - x_{ij} < 0$ for some $j = r(i)$, then if $r(i) < n$ set $j^{(0)}(i) = r(i) + 1$, go to step 9; if $r(i) = n$, add constraint $x_{jj} - x_{ij} \geq 0$ with $i = i_n$, go to step 9. If X_{SRPMP} is feasible to the RPMP, set missing variables $x_{ij} = 0$, return to the place that calls the SRPMP procedure with the optimal solution X_{SRPMP}.

Example 3.4.1 (continued). Start from the $X^{(0)}_{mat} = (x^{(0)}_{ij})$ with $x^{(0)}_{16} = x^{(0)}_{24} = x^{(0)}_{34} = x^{(0)}_{44} = x^{(0)}_{56} = x^{(0)}_{66} = 1; x^{(0)}_{ij} = 0$ otherwise, for $1 \leq i, j \leq 6$, we have the initial SRPMP as follows.

$$\min\ 21x_{12} + 28x_{16} + 27x_{21} + 36x_{23} + 54x_{24} + 20x_{32} + 15x_{34}$$

$$+66x_{42} + 33x_{43} + 45x_{53} + 54x_{54} + 45x_{56} + 40x_{61} + 50x_{65}$$

s.t.

$$\begin{aligned}
&x_{11} + x_{22} + x_{33} + x_{44} + x_{55} + x_{66} = 2\\
&x_{11} + x_{12} + x_{16} = 1\\
&x_{21} + x_{22} + x_{23} + x_{24} = 1\\
&x_{32} + x_{33} + x_{34} = 1\\
&x_{42} + x_{43} + x_{44} = 1\\
&x_{53} + x_{54} + x_{55} + x_{56} = 1\\
&x_{61} + x_{65} + x_{66} = 1\\
&x_{22} - x_{12} \geq 0\\
&x_{11} - x_{21} \geq 0\\
&x_{33} - x_{23} \geq 0\\
&x_{44} - x_{34} \geq 0\\
&x_{33} - x_{43} \geq 0\\
&x_{33} - x_{53} \geq 0\\
&x_{66} - x_{56} \geq 0\\
&x_{11} - x_{61} \geq 0\\
&x_{ij} \geq 0.
\end{aligned}$$

The branch and bound algorithm part of the initial SRPMP is the same as that of the RPMP in Chapter 3, because the optimal solutions are feasible to the RPMP by setting $x_{33} = 1$ and 0.

The algorithm can be improved in several aspects. There are many heuristic algorithms to find an integral feasible solution $X_{mat}^{(0)}$, which are mentioned previously. There are different algorithms to solve the SRPMP because of its special structure. The normal methods are the simplex method if the size of the problem is small, and the interior method if the size of the problem is large. The matrix A in (3.1.8) is a sparse matrix when $n > 10$. Sometimes, when we solve a SRPMP, we get an integral feasible solution which is better than $X_{mat}^{(0)}$, even though it is infeasible to the RPMP. Thus, we can save time for the remainder of the algorithm. For instance, in Example 3.4.1, when we solve the SRPMP (3.4.1), we get an integral optimal solution $X_{mat}^{(1)} = (x_{ij}^{(1)})$ with $x_{11}^{(1)} = x_{21}^{(1)} = x_{34}^{(1)} = x_{44}^{(1)} = x_{56}^{(1)} = x_{61}^{(1)} = 1; x_{ij}^{(1)} = 0$ otherwise, for $1 \leq i, j \leq 6$. Thus, we have a better integral feasible solution $\{v_1, v_4\}$ than $\{v_4, v_6\}$ with the objective value 136, although $X_{mat}^{(1)}$ is infeasible to the RPMP. Actually, $\{v_1, v_4\}$ is the optimal integral solution.

Chapter 5

The Uncapacitated Facility Location Problem

5.1 Introduction

Let $J = \{1, \cdots, n\}$ be a finite set of n possible places to establish new facilities. The facility location problem deals with the supply from a subset of these potential facility locations to a set $I = \{1, \cdots, m\}$ of m clients with prescribed demands. There are fixed costs for locating the facilities and transportation costs for shipping the supply between the facilities and the clients. We seek a minimum cost location and transportation plan to satisfy all clients demands. This problem has been extensively studied in the literature and is commonly referred to as the ***plant location problem*** or ***facility location problem.*** If each potential facility has a capacity, i.e. there is a maximum demand that it can supply, the problem is called the ***capacitated facility location problem.*** If each potential facility is assumed to have unlimited capacity, the problem is called the ***uncapacitated facility location problem*** (UFL

problem).

Let f_j be the given fixed cost of opening facility j, and let c_{ij} be the cost of serving client i from facility j. Typically, c_{ij} is a cost of production plus transportation. We assume that $f_j \geq 0$ for all $j \in J$ since if $f_j < 0$ every optimal solution contains facility j.

For any given set X of open facilities, it is optimal to serve client i from a facility j such that c_{ij} is minimal over all $j \in X$. Thus, given X, the cost is $Z(X) = \sum_{i \in I} \min_{j \in X} c_{ij} + \sum_{j \in X} f_j$. The UFL problem is to find a set X that gives the minimal cost Z, i.e. $Z = \min_{X \subseteq J} Z(X)$. This is a combinatorial optimization problem.

The integer programming formulation of the UFL problem is useful in the design of both heuristic and exact algorithms to solve it. An integer programming formulation is obtained by introducing the following variables. Let $x_j = 1$ if facility j is open and $x_j = 0$ otherwise; let $x_{ij} = 1$ if client i is served by facility j and $x_{ij} = 0$ otherwise. The integer programming problem is

$$Z = \min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} f_j x_j \tag{5.1.1}$$

s.t.

$$\sum_{j \in J} x_{ij} = 1, \text{ for } i \in I \tag{5.1.2}$$

$$x_j - x_{ij} \geq 0, \text{ for } i \in I,\ j \in J \tag{5.1.3}$$

$$x_j, x_{ij} \in \{0, 1\}, \text{ for } i \in I,\ j \in J. \tag{5.1.4}$$

The constraints (5.1.2) guarantee that every client is served, whereas (5.1.3)

guarantees that the clients are served only from open facilities.

Note that if we assume costs as negative profits, i.e. $c_{ij} = -p_{ij}$, where p_{ij} is the profit of serving client i from facility j, then the other integer programming formulation for the UFL problem is presented as

$$\max \sum_{i \in I} \sum_{j \in J} p_{ij} x_{ij} - \sum_{j \in J} f_j x_j \tag{5.1.1'}$$

subject to (5.1.2)-(5.1.4). These two formulations are mathematically equivalent since

$$\max \sum_{i \in I} \sum_{j \in J} p_{ij} x_{ij} - \sum_{j \in J} f_j x_j = -\min(\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} f_j x_j).$$

If c_{ij} is replaced by $c'_{ij} = c_{ij} + c_i$ for all $j \in J$, then, by (5.1.2), the cost of each feasible solution is changed by c_i, whereas the set of optimal solutions remains the same. Hence, we can add or subtract a constant to any row of the cost matrix $C = (c_{ij})$ and not affect the optimal solution.

We can relax the constraints (5.1.4) to

$$x_j \geq 0, x_{ij} \geq 0, \text{ for } i \in I, j \in J \tag{5.1.5}$$

The problem (5.1.1), (5.1.2), (5.1.3) and (5.1.5) is called the ***relaxed UFL problem*** (RUFL problem), and the problem (5.1.1), (5.1.2) and (5.1.5) is called the ***basic relaxed UFL problem*** (BRUFL problem).

There are many published papers in the literature on the UFL problem. Krarup

and Pruzan(1983), Tansel, Francis and Lowe(1983) gave an up-to-date survey, Cornuejols, Nemhauser and Wolsey (in Mirchandani and Francis, 1990) discussed some improved and new methods. Most algorithms mentioned in solving the p-median problem can be used to solve the UFL problem. We will show the relationship between the p-median problem and the UFL problem in the next Section. Among these algorithms, Erlenkotter's(1978) DUALOC appears to outperform all the others. His algorithm is so effective that in 45 out of 48 problems that he tested, optimality was reached at the first node of the branch and bound algorithm. Körkel(1989) modified a primal-dual version of Erlenkotter's algorithm and obtained a significantly improved procedure for the larger problems.

5.2 The p-median Problem and the UFL Problem

For a given UFL problem, we can construct a network $G = (V, E)$ as follows: $V = \{v_i : i \in I\} \cup \{v_j : j \in J\}$, an edge $e = v_i v_j \in E$ if c_{ij} exists and $c_{ij} < \infty$. The fixed cost $f_i = \infty$ for $i \in I$. For convenience, we will discuss both the p-median problem and the UFL problem on the network $G = (V, E)$ with $|V| = n$, $C = (c_{ij})_{n \times n}$ and $f = (f_1, \cdots, f_n)$. c_{ij} is the cost if client i is served by facility j. If client i is never served by facility j, then set $c_{ij} = \infty$. f_j is the fixed cost if facility j is open. If a facility is never open at j, then set $f_j = \infty$. Thus, in the integer programming formulation (5.1.1)-(5.1.4) of the UFL problem, $I = J$, where I and J are the index sets of V.

Two features distinguish the UFL problem from the p-median problem :

(i) A nonnegative *fixed cost* is associated with each potential facility location.

(ii) The number of facilities to be located is *not* prespecified.

Cornuejols, Fisher and Nemhauser(1977b) have the following result. If $X_{UFL} = (x_j, x_{ij})$ is a fractional optimal solution to the RUFL problem, let $J_1 = \{j: 0 < x_j < 1\}$ and $I_1 = \{i: x_{ij} = 0 \text{ or } x_j \text{ for } 1 \leq j \leq n, \text{ and } x_{ij} \text{ is fractional for at least one } j\}$. Let $a_{ij} = 1$ if $x_{ij} > 0$ and $a_{ij} = 0$ otherwise, and denote by A the $|I_1| \times |J_1|$ matrix whose entries are a_{ij} for $i \in I_1$, $j \in J_1$.

Proposition 5.2.1. (Cornuejols, Fisher and Nemhauser, 1977b) *$X_{UFL} = (x_j, x_{ij})$ is a fractional optimal solution to the RUFL problem if and only if*

(i) $x_j = \max_i x_{ij}$ *for all* $j \in J_1$.

(ii) For each i, there is at most one j, with $0 < x_{ij} < x_j$.

(iii) The rank of A equals $|J_1|$.

In the case where $f_j = 0$, $c_{ij} = w(v_i)d(v_i, v_j)$ for all i, j, and $\sum_{j=1}^{n} x_j = p$, then the UFL problem becomes the p-median problem. If $x_j \neq 0$ in the optimal solution to the UFL problem, then client j is served by itself, since $c_{jj} = w(v_j)d(v_j, v_j) = 0$, i.e. $x_j = x_{jj}$ always for all j. Thus, $x_j - x_{ij} \geq 0$ becomes $x_{jj} - x_{ij} \geq 0$ and $\sum_{j=1}^{n} x_j = p$ becomes $\sum_{j=1}^{n} x_{jj} = p$. This is precisely the integer programming formulation for the p-median problem. However, this is not our primary interest here.

It is clear that $x_j = x_{jj}$ in the optimal solution to the UFL problem, provided

that $c_{jj} = \min_{1 \le i \le n} c_{ij}$, for all j. A problem is called an *extended p-median problem* if it is a p-median problem and there is a fixed cost f_j if a facility is open at j. The integer programming formulation for the extended p-median problem is the same as (3.1.1)-(3.1.5). However, c_{jj} will be equal to f_j, instead of $c_{jj} = 0$ in the p-median problem. Virtually all the results of Chapter 3 are still correct for the extended p-median problem since we do not assume $c_{jj} = 0$ in the proof of these results. The exception occurs in Section 3.4 and Chapter 4 in constructing SRPMP from an integral feasible solution $X^{(0)}_{mat}$. At that time $c_{jj} = 0$, so $j_1 = j$ for all $L_j = \{j_1, \cdots, j_n\}$. We now can fix $j_1 = j$ even if $c_{jj} \neq 0$. The rest of the sequence of L_j is the same as before. Then all the results of Section 3.4 and algorithms in Chapter 4 remain true for the extended p-median problem.

In the general case of the UFL problem, c_{ij} could be any nonnegative number for all i, j. We do not claim that if $x_j \neq 0$ in the optimal solution to the RUFL problem, then v_j is a 1-median of $<S_j>$, where $S_j = \{v_i : x_{ij} \neq 0\}$. Sometimes, $x_{jj} = 0$ even if $x_j \neq 0$, which means $v_j \notin S_j$. Sometimes, $<S_j>$ is just the union of nodes v_i with no edges between them. However, some results for the p-median problem in Chapter 3 can be adapted for the UFL problem.

Given a network $G = (V, E)$, X^*_p is called a *p-UFL* of G, if

$$Z(X^*_p) = \min_{X_p \subseteq V} \left(\sum_{i=1}^{n} \min_{j \in X_p} c_{ij} + \sum_{j \in X_p} f_j \right).$$

The problem is called a *p-UFL problem* if it is a UFL problem and the number of open facilities is exactly p. Nemhauser and Wolsey's(1988) definition (page 402) is

slightly different.

We start from the BRUFL problem and then move to the RUFL problem.

Theorem 5.2.2. *If $X_{UFL} = (x_j, x_{ij})$ is an optimal solution to the BRUFL problem, $S_j = \{v_i : x_{ij} \neq 0\}$ and $x_j \neq 0$, then v_j is a 1-UFL of the subset S_j.*

Proof. Suppose $v_k \neq v_j$ is a 1-UFL of the subset S_j, i.e.

$$Z(S_j, v_k) < Z(S_j, v_j)$$

where $Z(S,U) = \sum_{v_i \in S} \min_{v_j \in U} c_{ij} + \sum_{v_j \in U} f_j$, $S, U \subseteq V$. By the assumption, $x_j \neq 0$. Let

$$\alpha = \min\{ \min_{v_i \in S_j} \{x_{ij} : x_{ij}, \neq 0\}, x_j \} > 0.$$

Construct a new solution to the BRUFL problem as follows:

$$x'_t = \begin{cases} x_t - \alpha, & \text{if } t = j, \\ x_t + \alpha, & \text{if } t = k, \\ x_t, & \text{otherwise.} \end{cases}$$

$$x'_{st} = \begin{cases} x_{st} - \alpha, & \text{if } t = j \text{ and } v_s \in S_j, \\ x_{st} + \alpha, & \text{if } t = k \text{ and } v_s \in S_j, \\ x_{st}, & \text{otherwise.} \end{cases}$$

It is easy to verify that $X'_{UFL} = (x'_t, x'_{st})$ is a feasible solution to the BRUFL problem. By the assumption $x_j \neq 0$, hence $x'_j = x_j - \alpha \geq 0$. For each $s, 1 \leq s \leq n$,

if $v_s \in S_j$, then

$$\sum_{t=1}^{n} x'_{st} = \sum_{\substack{1 \le t \le n \\ t \ne j \\ t \ne k}} x_{st} + (x_{sj} - \alpha) + (x_{sk} + \alpha)$$
$$= \sum_{t=1}^{n} x_{st} = 1$$

If $v_s \notin S_j$, then $x'_{st} = x_{st}$ for $1 \le t \le n$. So $\sum_{t=1}^{n} x'_{st} = \sum_{t=1}^{n} x_{st} = 1$. Thus, the constraints of (5.1.2) hold for X'_{UFL}. However,

$$Z_{BRUFL}(X'_{UFL}) - Z_{BRUFL}(X_{UFL}) = \alpha \cdot (Z(S_j, v_k) - Z(S_j, v_j)) < 0,$$

which contradicts the assumption that X_{UFL} is an optimal solution to the BRUFL problem. ∎

Let $Z_{BRUFL}(X_{UFL})$ be the objective value of X_{UFL} with respect to the BRUFL problem, and let $I_j = \{i : x_{ij} \ne 0\}$ for all j, then we have

Corollary 5.2.3. *If $X_{UFL} = (x_j, x_{ij})$ is an optimal solution to the BRUFL problem, $x_{jk} \ne 0$ and I_{jk} are pairwise disjoint for $1 \le k \le q$, then the set of q nodes $X_q = \{v_{j_1}, \cdots, v_{j_q}\}$ is a q-UFL of the subset $S = \{v_i : i \in \bigcup_{1 \le k \le q} I_{jk}\}$.* ∎

Following the proof of Theorem 3.2.2, it is easy to prove this Corollary with the same construction of $X'_{UFL} = (x'_t, x'_{st})$ as in the proof of Theorem 5.2.2. The same argument applies to the corollaries below.

For the RUFL problem, we do not need the conditions $x_{jk} \ne 0$ as in the BRUFL problem.

Corollary 5.2.4. *If $X_{UFL} = (x_j, x_{ij})$ is an optimal solution to the RUFL problem, I_{j_k} are pairwise disjoint for $1 \le k \le q$, then $X_q = \{v_{j_1}, \cdots, v_{j_q}\}$ is a q-UFL of the subset $S = \{v_i : i \in \bigcup_{1 \le k \le q} I_{j_k}\}$.* ∎

Corollary 5.2.5. *If some of the constraints of (5.1.3) are added to the BRUFL problem, then the results of Theorem 5.2.2 and Corollary 5.2.3 are still true.* ∎

By Proposition 5.2.1, if $X_{UFL} = (x_j, x_{ij})$ is an optimal solution to the RUFL problem, then for each i, there is at most one j, such that $0 < x_{ij} < x_j$. However, if x_j is either 0 or 1 in the X_{UFL} for all j, then we have

Theorem 5.2.6. *If the fractional optimal solution $X_{UFL} = (x_j, x_{ij})$ to the RUFL problem is of the form: x_j is either 0 or 1 for all j and some of the x_{ij} are in (0, 1), then there is an integral feasible solution to the RUFL problem with the same optimal value.*

Proof. Without loss of generality, suppose $x_1 = \cdots = x_q = 1$, $x_{q+1} = \cdots = x_n = 0$ in the X_{UFL} and $0 < x_{k1} < 1$ for some k. Then, some other x_{kj} are also fractional with $j = 2, \cdots, q$ since $x_{k1} + \cdots + x_{kq} = 1$ and $x_{k,q+1} = \cdots = x_{kn} = 0$. If $x_{k2} \neq 0$, say, then $c_{k1} = c_{k2}$, because if $c_{k1} < c_{k2}$, let $x'_{k1} = x_{k1} + x_{k2}, x'_{k2} = 0$; the remaining $x'_{ij} = x_{ij}$ and $x'_j = x_j$ for all j. We then get a feasible solution to the RUFL problem with strictly lower objective value. Similarly, we cannot have $c_{k1} > c_{k2}$. Thus, if some other x_{kj} are in (0, 1), then $c_{k1} = c_{kj}$. Let $X_{UFL} = (\bar{x}_j, \bar{x}_{ij})$ with $\bar{x}_j = x_j$ for all j and for each i, if $c_{ij^*} = \min_{x_j=1}\{c_{ij}\}$, then set one $\bar{x}_{ij^*} = 1$. Set the remaining $\bar{x}_{ij} = 0$. Thus, $\bar{X}_{UFL}$ is an integral feasible solution to the RUFL

problem with the same optimal value. ∎

By Theorem 5.2.6, we know that if the x_j are either 0 or 1 for all j in the optimal solution $X_{UFL} = (x_j, x_{ij})$ to the RUFL problem, then we have an integral optimal solution in hand. This is useful in the next Section.

5.3 An Algorithm for the UFL Problem

Given an integral feasible solution $X_{UFL}^{(0)}$ to the UFL problem, we construct a simplified relaxed UFL problem (SRUFL problem) from the RUFL problem. This is similar to our results for the p-median problem.

We order c_{ij} as follows: for each $i = 1, \cdots, n$, let $L_i = \{i_1, i_2, \cdots, i_n\}$ be a sequence of indices such that if $i_j < i_k$, then $c_{ii_j} \leq c_{ii_k}$. If $c_{ii_j} = c_{ii_k}$, then c_{ii_j} precedes c_{ii_k} if $i_j < i_k$ in the existing order. By Proposition 5.2.1, if $c_{ii_1} < c_{ii_2}$ for some i, then $x_{ii_1} = x_{i_1}$, since if $x_{i_1} \neq 0$, then $x_{ii_1} = x_{i_1}$ in the optimal solution. However, if $c_{ii_1} = c_{ii_2}$, we could not say that $x_{ii_1} = x_{i_1}$ and $x_{ii_2} = x_{i_2}$ in the optimal solution. Suppose $X_{UFL}^{(0)} = (x_j^{(0)}, x_{ij}^{(0)})$ is an integral feasible solution to the RUFL problem. Then, for each i, there is a unique j such that $x_{ij}^{(0)} = 1$. This j is denoted by $j^{(0)}(i)$. If $j^{(0)}(i) = i_1$, set $j^{(0)}(i) = i_2$. We construct a linear programming problem from the RUFL problem as follows.

For each $i = 1, \cdots, n$, if $c_{ii_1} < c_{ii_2}$, then set $x_{ii_1} = x_{i_1}$, which reduces the number of variables. If $c_{ij^{(0)}(i)} = \cdots = c_{ii_{r(i)}} < c_{ii_{r(i)+1}}$, add constraints $x_j - x_{ij} \geq 0$

for $j = i_2, \cdots, i_{r(i)}$, and $x_{i_1} + x_{ii_2} + \cdots + x_{ii_{r(i)+1}} = 1$. If $c_{ii_1} = c_{ii_2}$, then if $c_{ij^{(0)}(i)} = \cdots = c_{ii_{r(i)}} < c_{ii_{r(i)+1}}$, add constraints $x_j - x_{ij} \geq 0$ for $j = i_1, i_2, \cdots, i_{r(i)}$, and $x_{ii_1} + x_{ii_2} + \cdots + x_{ii_{r(i)+1}} = 1$. If $r(i) = n$ for some i, then we add constraint $x_{ii_1} + \cdots + x_{ii_{r(i)}} = 1$ instead of $x_{ii_1} + x_{ii_2} + \cdots + x_{ii_{r(i)+1}} = 1$. Finally, we add the constraint $x_1 + \cdots + x_n \geq 1$.

The objective function is a function of those x_{ij} that appear in the above constraints and all the x_j. For those x_{ii_1} that are replaced by x_{i_1}, we add the coefficient c_{ii_1} to the coefficient of x_{i_1}. The above linear programming problem is called the ***simplified relaxed UFL problem*** (SRUFL problem). Like SRPMP, we do not add constraints $x_j - x_{ij} \geq 0$ for $j = i_{r(i)+1}$, $1 \leq i \leq n$, to the SRUFL problem.

Let $X_{SRUFL} = (x_j, x_{ij})$ be an optimal solution to the SRUFL. $Z(X_{SRUFL})$ denotes by the optimal value of the SRUFL problem. It also gives the objective value of X_{SRUFL} for the RUFL problem by setting the missing variables $x_{ij} = 0$. These two values are equal even though X_{SRUFL} may not be feasible to the RUFL problem. Let $X^*_{UFL} = (x^*_j, x^*_{ij})$ be an optimal solution to the RUFL problem. Then,

Theorem 5.3.1. $Z(X_{SRUFL}) \leq Z(X^*_{UFL})$.

The proof is the same as that of Theorem 3.4.2, where RPMP and SRPMP in the proof of Theorem 3.4.2 are replaced by RUFL problem and SRUFL problem respectively.

If the optimal solution to the SRUFL problem violates constraints $x_j - x_{ij} \geq 0$,

for some $j = r(i)+1$, set $j^{(0)}(i) = r(i)+1$, for those violated constraints. Following the procedure above, we construct a new SRUFL problem and solve the new SRUFL problem again until all the constraints $x_j - x_{ij} \geq 0$ for the current $j = r(i)+1$, $1 \leq i \leq n$, are satisfied. The number of such iterations is at most $n^2 - 2n$, because there are only n^2 constraints in the RUFL problem and at least $2n$ constraints in the first formulation of the SRUFL problem. In each iteration of constructing a new SRUFL problem, at least one new constraint is added.

Corollary 5.3.2. *The optimal value of the new SRUFL problem is greater than or equal to that of the old SRUFL problem.*

The proof is the same as that of Corollary 3.4.3, where SRUFL problem replaces SRPMP in the proof of Corollary 3.4.3.

Example 5.3.1. The example is from Erlenkotter(1978), which was presented by Khumawala(1973). It has five potential facility locations and eight clients, i.e. $J = \{1, \cdots, 5\}$, $I = \{1, \cdots, 8\}$. The cost matrix $C = (c_{ij})$ is

$$C = \begin{pmatrix} 120 & 210 & 180 & 210 & 170 \\ 180 & +\infty & 190 & 190 & 150 \\ 100 & 150 & 110 & 150 & 110 \\ +\infty & 240 & 195 & 180 & 150 \\ 60 & 55 & 50 & 65 & 70 \\ +\infty & 210 & +\infty & 120 & 195 \\ 180 & 110 & +\infty & 160 & 200 \\ +\infty & 165 & 195 & 120 & +\infty \end{pmatrix}.$$

The fixed cost vector is $(f_1, f_2, f_3, f_4, f_5) = (200, 200, 200, 400, 300)$. Suppose that facilities located at j_2 and j_5 give an initial integral feasible solution. That is,

$X^{(0)}_{UFL} = (x^{(0)}_j, x^{(0)}_{ij})$ with $x^{(0)}_2 = x^{(0)}_5 = 1$ and the other $x^{(0)}_j = 0$; $x^{(0)}_{15} = x^{(0)}_{25} = x^{(0)}_{35} = x^{(0)}_{45} = x^{(0)}_{52} = x^{(0)}_{65} = x^{(0)}_{72} = x^{(0)}_{82} = 1$, and the remaining $x^{(0)}_{ij} = 0$. From the $X^{(0)}_{UFL}$, we have the following SRUFL problem.

$$
\begin{aligned}
\min\ & 420x_1 + 310x_2 + 250x_3 + 640x_4 + 600x_5 + 180x_{13} + 170x_{15} \\
& + 180x_{21} + 190x_{23} + 150x_{32} + 110x_{33} + 110x_{35} + 195x_{43} \\
& + 180x_{44} + 60x_{51} + 55x_{52} + 210x_{62} + 195x_{65} + 180x_{71} \\
& + 160x_{74} + 165x_{82} + 195x_{83}
\end{aligned}
\tag{5.3.1}
$$

s.t.

$$
\begin{aligned}
& x_1 + x_2 + x_3 + x_4 + x_5 \geq 1 \\
& x_1 + x_{15} + x_{13} = 1 \\
& x_5 + x_{21} + x_{23} = 1 \\
& x_1 + x_{33} + x_{35} + x_{32} = 1 \\
& x_5 + x_{44} + x_{43} = 1 \\
& x_3 + x_{52} + x_{51} = 1 \\
& x_4 + x_{65} + x_{62} = 1 \\
& x_2 + x_{74} + x_{71} = 1 \\
& x_4 + x_{82} + x_{83} = 1 \\
& x_5 - x_{15} \geq 0 \\
& x_1 - x_{21} \geq 0 \\
& x_3 - x_{33} \geq 0 \\
& x_5 - x_{35} \geq 0 \\
& x_4 - x_{44} \geq 0 \\
& x_2 - x_{52} \geq 0 \\
& x_5 - x_{65} \geq 0 \\
& x_4 - x_{74} \geq 0 \\
& x_2 - x_{82} \geq 0 \\
& x_{ij} \geq 0.
\end{aligned}
$$

In the problem (5.3.1), we have $x_{11} = x_{31} = x_1$, $x_{72} = x_2$, $x_{53} = x_3$, $x_{64} = x_{84} = x_4$ and $x_{25} = x_{45} = x_5$. The optimal solution of the problem (5.3.1) is $X^{(1)}_{UFL} = (x^{(1)}_j, x^{(1)}_{ij})$ with $x^{(1)}_1 = 1$ and the rest of the $x^{(1)}_j = 0$; $x^{(1)}_{21} = x^{(1)}_{43} = x^{(1)}_{51} = x^{(1)}_{62} = x^{(1)}_{71} = x^{(1)}_{83} = 1$, and the rest of the $x^{(1)}_{ij} = 0$. There are three violations :

$x_3 - x_{43} \geq 0$, $x_2 - x_{62} \geq 0$ and $x_3 - x_{43} \geq 0$. Add these three constraints and new variable x_{42} to the problem (5.3.1).

The new optimal solution to the current SRUFL problem is $X_{UFL}^{(2)} = (x_j^{(2)}, x_{ij}^{(2)})$ with $x_2^{(2)} = 1$ and the rest of the $x_j^{(2)} = 0$; $x_{13}^{(2)} = x_{23}^{(2)} = x_{32}^{(2)} = x_{42}^{(2)} = x_{52}^{(2)} = x_{62}^{(2)} = x_{82}^{(2)} = 1$, and the rest of the $x_{ij}^{(2)} = 0$. There are two violations $x_3 - x_{13} \geq 0$ and $x_3 - x_{23} \geq 0$. Add these two constraints and the two new variables x_{12} and x_{24} to the current SRUFL problem.

The new optimal solution to the current SRUFL problem is $X_{UFL}^{(3)} = (x_j^{(3)}, x_{ij}^{(3)})$ with $x_2^{(3)} = 1$ and the rest of the $x_j^{(3)} = 0$; $x_{12}^{(3)} = x_{24}^{(3)} = x_{32}^{(3)} = x_{42}^{(3)} = x_{52}^{(3)} = x_{62}^{(3)} = x_{82}^{(3)} = 1$, and the rest of the $x_{ij}^{(3)} = 0$. There is one violation $x_4 - x_{24} \geq 0$. Add this constraint to the current SRUFL problem.

The new optimal solution to the current SRUFL problem is $X_{UFL}^{(4)} = (x_j^{(4)}, x_{ij}^{(4)})$ with $x_2^{(4)} = x_3^{(4)} = x_5^{(4)} = \frac{1}{2}$ and the rest of the $x_j^{(4)} = 0$; $x_{13}^{(4)} = x_{15}^{(4)} = x_{23}^{(4)} = x_{33}^{(4)} = x_{35}^{(4)} = x_{43}^{(4)} = x_{52}^{(4)} = x_{62}^{(4)} = x_{65}^{(4)} = x_{71}^{(4)} = x_{82}^{(4)} = x_{83}^{(4)} = \frac{1}{2}$, and the rest of the $x_{ij}^{(4)} = 0$. There is one violation $x_1 - x_{71} \geq 0$. Add this constraint and new variable x_{75} to the current SRUFL problem.

Finally, the new optimal solution to the current SRUFL problem is $X_{UFL}^{(5)} = (x_j^{(5)}, x_{ij}^{(5)})$ with $x_1^{(5)} = x_3^{(5)} = x_5^{(5)} = \frac{1}{3}$, $x_2^{(5)} = \frac{2}{3}$ and $x_4^{(5)} = 0$, $x_{13}^{(5)} = x_{15}^{(5)} = x_{21}^{(5)} = x_{23}^{(5)} = x_{33}^{(5)} = x_{35}^{(5)} = x_{42}^{(5)} = x_{43}^{(5)} = x_{65}^{(5)} = x_{71}^{(5)} = x_{83}^{(5)} = \frac{1}{3}$, $x_{52}^{(5)} = x_{62}^{(5)} = x_{82}^{(5)} = \frac{2}{3}$ and the rest of the $x_{ij}^{(5)} = 0$. This is also the optimal solution to the RUFL problem.

In the final SRUFL problem, there are 26 variables and 25 constraints. There

are 38 variables and 41 constraints in the RUFL problem.

If the optimal solution to the RUFL is fractional, we use the branch and bound algorithm with the second branching strategy that is used for the p-median problem. That is, we choose the branching variable x_k with the smallest value $Z(S_k, v_k)$ over all the fractional variables x_j in the current optimal solution. Thus, we have the following algorithm for the UFL problem. Let $\mathbf{P}$ be the set of candidate problems, and let $P(X_{UFL}^{(h)})$ be the subproblem with optimal solution $X_{UFL}^{(h)}$. Let $\mathbf{F}$ be the set of constraints to the SRUFL problem. The input data is the cost matrix $C = (c_{ij})$, the fixed cost vector $f = (f_j)$, the number of clients m and the number of potential facilities n.

Step 1. (Initialization): Use the DUALOC algorithm to find an integral feasible solution $X_{UFL}^{(0)} = (x_j^{(0)}, x_{ij}^{(0)})$. Let $L_i = \{i_1, i_2, \cdots, i_n\}$ be a sequence of indices such that if $i_j < i_k$, then $c_{ii_j} \leq c_{ii_k}$. If $c_{ii_j} = c_{ii_k}$, then c_{ii_j} precedes c_{ii_k} if $i_j < i_k$ in C. Find $j^{(0)}(i)$ such that $x_{ij}^{(0)} = 1$ with $j = j^{(0)}(i)$, for $1 \leq i \leq n$. If $j^{(0)}(i) = i_1$, set $j^{(0)}(i) = i_2$. $\mathbf{P} = \phi$.

Step 2. (Relaxation): Call the SRUFL procedure (step 7) with $X_{UFL}^{(0)}$, and find an optimal solution $X_{UFL}^{*} = (x_j^{*}, x_{ij}^{*})$ to the RUFL problem. The current SRUFL problem is called the *initial SRUFL problem*, and it will be solved repeatedly later. If X_{UFL}^{*} is integral go to step 5; else set $\mathbf{P} = \mathbf{P} \cup \{P(X_{UFL}^{*})\}$, go to step 3.

Step 3. (Termination test): If $\mathbf{P} = \phi$, $X_{UFL}^{(0)}$ is the optimal solution. STOP.

Step 4. (Problem selection): Select and delete a problem $P(X_{UFL}^{(h)})$ from $\mathbf{P}$. Call the SRUFL procedure (step 8) starting from the initial SRUFL problem, get an optimal solution $X_{UFL}^{(h)}$.

Step 5. (Pruning): (i) If $Z(X_{UFL}^{(h)}) \geq Z(X_{UFL}^{(0)})$, go to step 3.

(ii) If $Z(X_{UFL}^{(h)}) < Z(X_{UFL}^{(0)})$ and $x_j^{(h)}$ are integral for $1 \leq j \leq n$, let $X_{UFL}^{(0)} = X_{UFL}^{(h)}$. Delete from $\mathbf{P}$ all problems with $Z(X_{UFL}^{(h)}) \geq Z(X_{UFL}^{(0)})$. Go to step 3.

(iii) If $Z(X_{UFL}^{(h)}) < Z(X_{UFL}^{(0)})$ and $X_{UFL}^{(h)}$ is fractional, go to step 6.

Step 6. (Branching): Choose the branching variable x_k if the corresponding $Z(S_k, c_k) = \min_j\{Z(S_j, c_j) : 0 < x_j^{(h)} < 1\}$ in $X_{UFL}^{(h)}$. Add problems $P(X_{UFL}^{(k_0)})$ and $P(X_{UFL}^{(k_1)})$ by setting $x_k = 0$ and $x_k = 1$ respectively to the $\mathbf{P}$. Go to step 3.

The SRUFL procedure is as follows:

Step 7. (Formulating the SRUFL): Let $\mathbf{F} = \{x_1 + x_2 + \cdots + x_n \geq 1\}$. For each $i = 1, \cdots, n$, if $c_{ii_1} < c_{ii_2}$, then set $x_{ii_1} = x_{i_1}$, if $c_{ij^{(0)}(i)} = \cdots = c_{ii_{r(i)}} < c_{ii_{r(i)+1}}$, then $\mathbf{F} = \mathbf{F} \cup \{x_j - x_{ij} \geq 0, \text{ for } j = i_2, \cdots, i_{r(i)}\} \cup \{x_{i_1} + x_{ii_2} + \cdots + x_{ii_{r(i)+1}} \geq 1\}$. If $c_{ii_1} = c_{ii_2}$, then if $c_{ij^{(0)}(i)} = \cdots = c_{ii_{r(i)}} < c_{ii_{r(i)+1}}$, then $\mathbf{F} = \mathbf{F} \cup \{x_j - x_{ij} \geq 0, \text{ for } j = i_1, \cdots, i_{r(i)}\} \cup \{x_{ii_1} + x_{ii_2} + \cdots + x_{ii_{r(i)+1}} \geq 1\}$.

The objective function is a linear combination of all x_j and those x_{ij} that appear in the above constraints. If some x_{ii_1} are replaced by x_{i_1}, replace these variables in the objective function also.

Step 8. (Solving the SRUFL problem): Solve the SRUFL problem, obtain an

optimal solution $X_{SRUFL} = (x_j, x_{ij})$.

If $x_j - x_{ij} < 0$ for some $j = r(i) + 1$, then

if $r(i) + 1 = n$, add constraint $x_{jj} - x_{ij} \geq 0$ with $j = i_n$.

else set $j^{(0)}(i) = r(i) + 1$;

go to step 7.

Else X_{SRUFL} is feasible to the RUFL problem, then set missing variables $x_j = 0$ and $x_{ij} = 0$, return to the calling place with the optimal solution X_{SRUFL}.

Example 5.3.1 (continued). For the optimal solution $X^{(5)}_{UFL} = (x^{(5)}_j, x^{(5)}_{ij})$, we have $Z(S_1, v_1) = 780$, $Z(S_2, v_2) = 980$, $Z(S_3, v_3) = 1120$ and $Z(S_5, v_5) = 1075$. Thus, we choose the branching variable x_1. The new optimal solution to the current SRUFL problem plus the constraint $x_1 = 1$ is $X^{(6)}_{UFL} = (x^{(6)}_j, x^{(6)}_{ij})$ with $x^{(6)}_1 = x^{(6)}_2 = 1$, the rest of the $x_j = 0$; $x^{(6)}_{11} = x^{(6)}_{21} = x^{(6)}_{31} = x^{(6)}_{42} = x^{(6)}_{52} = x^{(6)}_{62} = x^{(6)}_{72} = x^{(6)}_{82} = 1$, the rest of the $x_{ij} = 0$. The objective value is 1580. This is an integral solution. If we set $x_1 = 0$, we get a greater optimal value than that of $X^{(6)}_{UFL}$. Hence, $X^{(6)}_{UFL}$ is the optimal integral solution, which is the same as Erlenkotter's(1978) result.

Chapter 6

Final Remarks

The p-median problem and the UFL problem have many applications. The p-median problem was proposed by Hakimi(1964, 1965) for the problem of locating switching centers in a communication network. It can be applied to locating computer facilities, programs or data files in a computer network to minimize annual storage and transmission costs. Mirzain(1985) applied the UFL problem to the star star concentrator location problem, an important computer communication network design problem.

The UFL problem can also be used to study the bank account location problem (Cornuejols, Fisher and Nemhauser,1977a) and portfolio management problem (Beck and Mulvey, 1982). The p median problem and the UFL problem are useful models for location decision problems in emergency service systems such as medical services, accident rescue, fire and police stations. They are used in warehouse location problems as well. Anderson and Chelst(1986) reported an application of the p-median problem to the retail network in the automotive industry. In their decision

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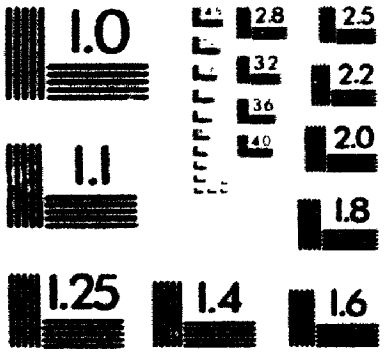

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support system, it is easy to find the effect of moving a dealership to a new location or establishing p new locations and so on. They also reported the actual use of their system to reconfigure networks of bank branches in existing networks and to design new networks in new markets. In statistics, the p-median problem is called cluster analysis. The p-median problem and the UFL problem are subproblems for other problems such as covering problem (see Mirchandani and Francis, 1990).

In Chapter 3, we have established the connection between the p selected facilities and the optimal solution of the relaxed 0-1 integer programming problem. The distance-sum of the p selected facilities is the summation of the optimal value of the relaxed integer programming problem plus the corresponding optimal dual values and optimal dual slack values. We also construct a simplified integer programming problem for the p-median problem and the UFL problem. The optimal value of the relaxed simplified integer programming problem is the lower bound of these two problems. After solving a sequence of these simplified integer programming problems, we finally get the same optimal solution of the relaxed integer programming problems of these two problems. We plan to use these ideas for other combinatorial optimization problems such as quadratic assignment problem, set covering problem and so on. We also intend to compare the two branching strategies for the p-median problem with the other algorithms of the p-median problem. We are interested in comparing our algorithm for the UFL problem with other known algorithms as well.

We note that Erlenkotter's(1978) algorithm is used in the SONET Toolkit (Cosares, Deutsch, Saniee and Wasem,1995). SONET, which stands for Synchro-

nous Optical NETwork, is a new transmission standard that is rapidly being introduced by telecommunication carriers. The cost of the "information superhighway" for the United States of America will be approximately $5 trillion (Government Computer News, 1993), for which SONET will provide the backbone network. We hope that our results may be of some use to these designers.

Appendix

The 1-median Problem on the Grid

The easiest case for the p-median problem is the 1-median problem. Goldman(1971) has a linear time algorithm for 1-median on the tree. In this appendix, we use the Goldman's algorithm to solve the 1-median problem on the n-dimensional grid. We proceed first with a 3-dimensional grid.

Let $a_{i_1 i_2 i_3}$ be a node of an $m_1 \times m_2 \times m_3$ 3-dimensional grid, with weight $w(a_{i_1 i_2 i_3})$. Let $a_j^{(1)} (1 \leq j \leq m_1)$ be the node of the line L_1 with the weight

$$w^{(1)}(a_j^{(1)}) = \sum_{i_2=1}^{m_2} \sum_{i_3=1}^{m_3} w(a_{j i_2 i_3}),$$

the sum of weight of nodes on the j-th plane along the first subscript. The line L_1 is a 1 dimensional projection of the graph onto the first coordinate. Similarly, the lines L_2, L_3 are defined respectively, i.e. let $a_j^{(2)} (1 \leq j \leq m_2)$ be the node of the line L_2 with the weight

$$w^{(2)}(a_j^{(2)}) = \sum_{i_1=1}^{m_1} \sum_{i_3=1}^{m_3} w(a_{i_1 j i_3}),$$

and let $a_j^{(3)}(1 \leq j \leq m_3)$ be the node of the line L_3 with the weight

$$w^{(3)}(a_j^{(3)}) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} w(a_{i_1 i_2 j}).$$

With the above notations, we have the following Theorem.

Theorem 1. *Let G be an $m_1 \times m_2 \times m_3$ 3-dimensional grid, then $a_{i_1}^{(1)}$, $a_{i_2}^{(2)}$, $a_{i_3}^{(3)}$ is the 1-median of the line L_1, L_2 and L_3 respectively if and only if $a_{i_1 i_2 i_3}$ is the 1-median of the grid G.*

Proof. Since

$$H(a_{i_1 i_2 i_3}) = \sum_{j=1}^{3} H(a_{i_j}^{(j)}) \tag{1}$$

where

$$H(a_{i_j}^{(j)}) = \sum_{l=1}^{m_j} w^{(j)}(a_l^{(j)}) d^{(j)}(l, i_j)$$

here $d^{(j)}(l, i_j)$ is the distance between the node l and node i_j in the line L_j, for any node $a_{i_1 i_2 i_3}$ in G.

If the left side of (1) is the minimum, then each term of the right side must be the minimum, as $a_{i_1}^{(1)}$, $a_{i_2}^{(2)}$, $a_{i_3}^{(3)}$ are choosen independently, and vice versa. ∎

If there are several 1-medians on the lines L_1, L_2 and L_3, we have the following:

Corollary 2. *If G is an $m_1 \times m_2 \times m_3$ 3-dimensional grid, and $a_{i_1}^{(1)}, \cdots, a_{i_r}^{(1)}$; $a_{j_1}^{(2)}, \ldots, a_{j_s}^{(2)}$; $a_{k_1}^{(3)}, \cdots, a_{k_t}^{(3)}$ are 1-medians of lines L_1, L_2, L_3 respectively, then a_{ijk} with $i \in \{i_1, \cdots, i_r\}$, $j \in \{j_1, \cdots, j_s\}$, $k \in \{k_1, \cdots, k_t\}$ is a 1-median of G.* ∎

For the n-dimensional grid, we have the following two corollaries.

Corollary 3. *Let G be an $m_1 \times m_2 \times \cdots \times m_n$ n-dimensional grid, then $a^{(1)}_{i_1}, a^{(2)}_{i_2}, \cdots, a^{(n)}_{i_n}$ is a 1-median of the line $L_1, L_2, \cdots, L_n$ respectively if and only if $a_{i_1 i_2 \cdots i_n}$ is a 1-median of the grid G.*

Corollary 4. *If G is an $m_1 \times m_2 \times \cdots \times m_n$ n-dimensional grid, and $a^{(1)}_{i_1}, \cdots, a^{(1)}_{i_r}$; $a^{(2)}_{j_1}, \cdots, a^{(2)}_{j_s}$; $a^{(n)}_{k_1}, \cdots, a^{(n)}_{k_t}$ are 1-medians of the line $L_1, L_2, \cdots, L_n$ respectively, then $a_{ij,\cdots,k}$ with $i \in \{i_1, \cdots, i_r\}$, $j \in \{j_1, \cdots, j_s\}$, $k \in \{k_1, \cdots, k_t\}$ is a 1-median of G.*

From Theorem 1, we see that to find a 1-median of the grid is equivalent to find 1-medians on the lines. To present the algorithm for the 1-median problem on the grid, we need Goldman's algorithm to find the 1-median on the tree.

Let $T = (V(T), E(T))$ be a tree, the weight $w(v) \geq 0$ for any $v \in V(T)$, then we have:

Algorithm 1.(Goldman, 1971)

Step 1: If $V(T)$ consists of a single node, then that node is the 1-median.

Step 2: Search for a leaf v_i, if $w(v_i) \geq \frac{1}{2} \sum_{v \in V'(T)} w(v)$, where $V'(T) = V(T) - \{v_i\}$, go to step 4, otherwise go to step 3.

Step 3: Let v_j be the adjacent node of v_i, update T by $V(T) \leftarrow V(T) - \{v_i\}$, $E(T) \leftarrow E(T) - \{(i,j)\}$, and $w(v_j) \leftarrow w(v_j) + w(v_i)$. Return to step 1.

Step 4: STOP. v_i is an 1-median.

Using Goldman's Algorithm as a subroutine, we have the following algorithm to find the 1-median of the grid. Let G be an $m_1 \times m_2 \times \cdots \times m_n$ n-dimensional grid with the set of nodes $V(G) = \{a_{i_1 i_2 \cdots i_n} : 1 \leq i_j \leq m_j, j = 1, 2, \cdots, n\}$ and with the weights $w(a_{i_1 i_2 \cdots i_n})$.

Algorithm 2.

Step 1: Form the lines $L_1, L_2, \cdots, L_n$ with the set of nodes $V(L_i) = \{a_j^{(i)} : 1 \leq j \leq m_i\}$ $i = 1, 2, \cdots, n$, and

$$w^{(1)}(a_j^{(1)}) = \sum_{i_2=1}^{m_2} \sum_{i_3=1}^{m_3} \cdots \sum_{i_n=1}^{m_n} w(a_{j i_2 i_3, \cdots, i_n}),$$

$$w^{(2)}(a_j^{(2)}) = \sum_{i_1=1}^{m_1} \sum_{i_3=1}^{m_3} \cdots \sum_{i_n=1}^{m_n} w(a_{i_1 j i_3, \cdots, i_n}),$$

$$\vdots$$

$$w^{(n)}(a_j^{(n)}) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} w(a_{i_1 i_2 \cdots, i_{n-1}}).$$

Step 2: Use Algorithm 1 to find the 1-medians $a_{i_1}^{(1)}, a_{i_2}^{(2)}, \cdots, a_{i_n}^{(n)}$ respectively of lines $L_1, L_2, \cdots, L_n$. Then $a_{i_1 i_2, \cdots, i_n}$ is a 1-median of G.

The complexity of Algorithm 2 is $O(\prod_{i=1}^{n} m_i)$ since there are $\prod_{i=1}^{n} m_i$ nodes and the complexity of Goldman's Algorithm is $O(|V(T)|)$.

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