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**QUASI-HEREDITARY ALGEBRAS AND
THEIR BOREL SUBALGEBRAS**

by

František Marko

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics
Carleton University
Ottawa, Ontario
June 18, 1996

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BOREL SUBALGEBRAS OF
QUASI-HEREDITARY ALGEBRAS

submitted by
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in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

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June, 1996

ABSTRACT

Quasi-hereditary algebras were introduced by Cline-Parshall Scott (see [CPS1] or [PS]) to describe highest weight categories which occur in the study of semi simple complex Lie algebras and algebraic groups. Since then, quasi-hereditary algebras were found to play an important role in various other applications. Exact Borel subalgebras (for short, Borel subalgebras) were defined by König (see [K1]) to capture another distinctive feature of quasi-hereditary algebras appearing in the applications which is related to Cartan decomposition of Lie algebras.

The objective of this thesis is to study properties of the quasi-hereditary algebras and their Borel subalgebras. We concentrate on homological properties of Borel subalgebras and their relationship to homological properties of quasi-hereditary algebras.

For two important classes of quasi-hereditary algebras, namely, endomorphism algebras of a semilocal module over a selfinjective local algebra and endomorphism algebras of Auslander type, we establish the existence of Borel subalgebras and the relationship between their global dimension and the global dimensions of their Borel subalgebras.

Later, we investigate the extension algebra of standard modules for quasi hereditary algebras with Borel subalgebras and directed standard modules and describe the bimodule structure of this algebra over its two subalgebras, the subalgebra of morphisms between standard modules and the subalgebra isomorphic to a Yoneda algebra of a Borel subalgebra of the original algebra. This result is analogous to Cartan decomposition of Lie algebras. We also relate the species of the extension algebra of standard modules and the species of its above mentioned subalgebras.

Finally, we characterize Borel subalgebras of separable quasi-hereditary algebra which are images of monomial subalgebras of the corresponding tensor algebra. These results are then used in a canonical construction of quasi-hereditary algebras from split standard system.

ACKNOWLEDGEMENTS

I would like to thank, first of all, my supervisor, Dr. Vlastimil Dlab, who introduced me to the representation theory of associative algebras. His support and insight were most helpful and his hard work was always an inspiration for me.

I would also like to thank several other colleagues: Dr. Istvan Agoston, Dr. Ibrahim Assem and Dr. Andrew Dean, conversations with whom helped me better understand the challenges of representation theory; many members and staff of the Department of Mathematics and Statistics, who showed interest in my work and life; and students, who provided much needed distraction from my work.

I also appreciate financial support from the Ontario Graduate Scholarship, the David and Rachel Epstein Foundation Scholarship and the Dewan Chand and Ratna Devi Marwah Memorial Scholarship, which have made it easier for me to concentrate on my thesis.

Finally, I wish to thank my family and friends, especially my wife Iveta and sons Tomas and Lukas, who have endured with me and provided a peaceful and joyful place for my work and life.

This thesis is dedicated to the memory of my father and to my mother, in honour of their understanding and love.

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INTRODUCTION

Quasi-hereditary algebras were introduced by Cline, Parshall and Scott in [CPS1] to deal with highest weight categories which occur in the study of semisimple complex Lie algebras and algebraic groups. Since then, quasi-hereditary algebras established themselves in many areas of mathematics including Lie theory, algebraic geometry, automorphic forms, finite group representation theory and mathematical physics. Soon after their introduction it was realized that it is advisable to restrict the class of quasi-hereditary algebras, because algebras appearing in the applications possess some additional properties. This motivated the definitions of lean algebras in [ADL1] and exact Borel subalgebras of quasi-hereditary algebras in [K1]. Both these concepts turned out to be very useful.

In the presented thesis we investigate quasi-hereditary algebras possessing exact Borel subalgebras (we called them shortly Borel subalgebras). Our main goal is to answer the following question: How does the existence of a Borel subalgebra influence the structure and properties of a quasi-hereditary algebra?

In Chapter 1, we introduce the notions of a standard module and a quasi-hereditary algebra. We characterize quasi-hereditary algebra A in terms of the dimension of A and the dimensions of its left and right standard modules over a central field K . In particular, we describe the structure of A as a $S - S$ -bimodule

over a maximal semisimple subalgebra S of A .

Chapter 2 starts with definition and characterization of a (left or right) Borel subalgebra. It is shown how the existence of a Borel subalgebra B describes a bimodule structure of A over B and S . Later, we concentrate on homological properties of Borel subalgebras. In the case when A is lean, has a left Borel subalgebra B° and the direct sum of its right standard modules is a factoralgebra of A we derive an important relationship between minimal projective resolution of simple modules over B° and A , namely, we prove that a minimal projective resolution over B° can be "topply" embedded into a minimal projective resolution over A .

In Chapter 3, we first review results of our joint paper [DHM], characterizing quasi-hereditary algebras A given as endomorphism algebras of direct sum of local ideals of a commutative self-injective algebra. The module category of these algebras possesses a duality functor fixing simple modules. The standard modules of these algebras are of a special type, e.g., all their socles are simple modules which are mutually isomorphic. It is proved that these algebras possess both left Borel subalgebra B° and right Borel subalgebra B . Finally, applying results from Chapter 2, we express the global dimension of A as a sum of the global dimensions of its Borel subalgebras B° and B , for the most important class of these algebras. This class is described solely in terms of a structure of a lattice of inclusions between local ideals defining A .

The results of Chapter 4 are formally similar to those of Chapter 3. This time we investigate a different kind of endomorphism algebras, which is a generalization of algebras introduced by Auslander in [A]. However, the structure of these algebras is significantly different from the algebras of Chapter 3, and we use different methods to derive the similar results. In the case when the modules appearing in the definition of these algebras are graded and their socle and radical filtrations coincide, we infer that these algebras have left and right Borel subalgebras and the global dimension of A is a sum of the global dimensions of its Borel subalgebras.

Chapter 5 is devoted to the study of extension algebras. The importance of extension algebras in our context lies in the fact that algebra A , corresponding to regular blocks of category $\mathcal{O}$ of representations of a semisimple complex Lie algebra, is homologically selfdual, i.e., the Yoneda Ext-algebra of A is isomorphic to A . The Yoneda Ext-algebra were studied within the framework of quasi-hereditary algebras by [CPS2], [ADL3] and [GM]. The importance of the species of the Yoneda Ext-algebra follows from their results, as this is related to the concepts of a linear resolutions, quasi-Koszul algebras and top embeddings of syzygies into the projective resolutions of simple modules. We introduce a concept of a standard extension algebra and investigate them for quasi-hereditary algebra A with directed standard modules and a Borel subalgebra B° . For such algebra A , we find a suitable method of computation of its standard extension algebra A^δ . As a main result, we determine a bimodule structure of A^δ over its subalgebras $B^{\circ\ast}$ (the Yoneda exten-

sion algebra of B'') and A_0^δ (extensions of degree 0). Finally, we infer a sufficient condition relating a species of A^δ to species of B'' and A_0^δ .

In Chapter 6, we investigate Borel subalgebras of a separable quasi-hereditary algebra A , which is represented as a factoralgebra of a tensor algebra T . We show that Borel subalgebras of a lean algebra A can be obtained as images of monomial subalgebras $\mathcal{R}$ of T . This leads to a definition of a $\mathcal{R}$ -Borel subalgebra of A . We investigate an important necessary condition of the existence of $\mathcal{R}$ -Borel subalgebras and provide a list of equivalent conditions for their existence. Finally, we investigate algebras with a $\mathcal{R}''$ -Borel subalgebra and a $\mathcal{R}$ -Borel subalgebra, generalizing Cartan decomposition from Lie theory and we also investigate algebras with split standard systems.

Using the results of Chapter 6, in Chapter 7 we characterize algebras with $\Delta_T^\circ - \Delta_T$ -Cartan decomposition. Based on this, we construct a quasi-hereditary algebra (called a canonical level-split algebra) from a given split standard system. Such an algebra possesses $\Delta_T^\circ - \Delta_T$ -Cartan decomposition. An important class of quasi-hereditary algebras (which includes algebras appearing in the applications) is obtained via a deformation of canonical level-split algebras which fixes standard modules. To conclude the thesis, we state an interesting property of an extension algebra of standard modules of a canonical level-split algebra.

Unless there is a reference to literature, the results presented are my own.

CHAPTER 1

QUASI-HEREDITARY ALGEBRAS

We start by introducing various notations and a concept of a quasi-hereditary algebra. Then we characterize quasi-hereditary algebra A in terms of dimensions of A and its left and right standard modules over a central field K .

1.1 DEFINITIONS

Let A be a finite dimensional connected basic (associative) algebra with a unit 1_A over a (commutative) field K .

For a (right) A -module M , the radical of M , i.e., the intersection of all maximal (right) submodules of M , will be denoted by $radM$ and the socle of M , i.e., the submodule of M generated by the set of all (right) simple submodules of M , will be denoted by $socM$.

Definition 1.1.1. A (two-sided) ideal J of A is a *heredity ideal* if $J^2 = J$, $J(radA)J = 0$ and J , considered as a right A -module, is projective. A is *quasi-hereditary* if there is a chain

$$0 = J_{m+1} \subset J_m \subset \cdots \subset J_{i+1} \subset J_i \subset \cdots \subset J_1 = A$$

of (two-sided) ideals of A such that for $1 \leq i \leq m$, J_i/J_{i+1} is a heredity ideal of A/J_{i+1} . Such a chain of idempotent ideals is called a *heredity chain*.

For another equivalent definition of quasi-hereditary algebras we will need the following notation.

An element e of A is called idempotent if $e^2 = e$. Two idempotents e and f are said to be orthogonal if $ef = fe = 0$. An idempotent e is primitive if it can not be decomposed into a sum of (non-zero) orthogonal idempotents. A set $\{e_1, \dots, e_n\}$ of primitive orthogonal idempotents is called complete if $e_1 + e_2 + \dots + e_n = 1_A$. For $i = 1, \dots, n$, we denote the (right) projective modules $e_i A$ by $P(i)$ and (right) simple modules $P(i)/\text{rad } P(i)$ by $S(i)$. Then the (right) regular module A_A decomposes as $A_A = \oplus_{i=1}^n P(i)$. We will consider a natural linear order $e = (e_1, \dots, e_n)$ on the set $\{e_1, \dots, e_n\}$ and write $\varepsilon_i = e_i + e_{i+1} + \dots + e_n$ for $1 \leq i \leq n$, $\varepsilon_{n+1} = 0$.

Definition 1.1.2. For $i = 1, \dots, n$, the (right) standard modules $\Delta(i)$ of A with respect to linear order e are given by $\Delta(i) = e_i A / e_i A \varepsilon_{i+1} A$.

Thus, $\Delta(i)$ is a maximal factormodule of $P(i)$ whose composition factors are isomorphic to $S(j)$, for $j \leq i$. It is important to realize that standard modules depend on the order $e = (e_1, \dots, e_n)$. Different orders of idempotents lead to different standard modules.

We denote by $p_i : P(i) \rightarrow \Delta(i)$ the canonical projection and write $\Delta = \oplus_{i=1}^n \Delta(i)$ and $\Delta^o = \oplus_{i=1}^n \Delta^o(i)$.

A (right) A -module M is said to have a Δ -filtration if there is a filtration

$$0 = M_{m+1} \subset M_m \subset \dots \subset M_{t+1} \subset M_t \subset \dots \subset M_1 = M$$

with $M_t/M_{t+1} \simeq \Delta(i)$ for some $i \in \{1, \dots, n\}$. The full subcategory of all (right)

A -modules with Δ -filtration is denoted by $\mathcal{F}(\Delta)$.

We call a standard module $\Delta(i)$ *Schurian* if its endomorphism algebra $\text{End}_A(\Delta(i))$ is a division algebra.

Definition 1.1.3. *An algebra A is quasi-hereditary if there is a linear order e such that*

- (1) $A_A \in \mathcal{F}(\Delta)$ and
- (2) every standard module $\Delta(i)$ for $1 \leq i \leq n$ is Schurian.

The equivalence of Definitions 1.1.1 and 1.1.3 is well known, see, e.g., [D].

If e is a linear order such that conditions (1) and (2) above are satisfied, we say that (A, e) is a quasi-hereditary algebra.

From now on, we will always consider an algebra A together with an ordered complete set $e = (e_1, \dots, e_n)$ of its primitive orthogonal idempotents. When necessary, we will even write (A, e) instead of A .

Note that A can be quasi-hereditary with respect to various orders on a complete set $\{e_1, \dots, e_n\}$ of its primitive orthogonal idempotents.

Previously we have defined the symbols $P(i), S(i), \Delta(i)$ for right indecomposable projective, simple and standard modules. Correspondingly, we will use symbols $P^o(i), S^o(i), \Delta^o(i)$ for left indecomposable projective, simple and standard modules. If we would like to stress that these are objects corresponding to an algebra A , we will use subscript A , like $P_A(i), S_A(i), \Delta_A(i)$ etc.

One can think about another definition of quasi-hereditary algebra by requiring

that a heredity ideal J from Definition 1.1.1 is projective when considered as left A -module. Alternatively, we can replace right standard modules by left standard modules in Definition 1.1.3.

Fortunately, these “left” definitions are equivalent with the “right” definitions, as we will see in the next section.

1.2 NUMERICAL CHARACTERIZATION OF QUASI-HEREDITARY ALGEBRAS

The aim of this section is to give characterization of quasi-hereditary algebras solely in terms of K -dimensions of A and its left and right standard modules. The dimensions of various classes of quasi-hereditary algebras were studied in [DR2].

Assume that all (right) standard modules of the algebra A are Schurian and denote $D_i = \text{End}_A \Delta(i) (= \text{End}_A S(i))$ and $d_i = \dim_K D_i$, for $i = 1, \dots, n$. For any (right) A -module M , denote by $[M : S(i)]$ the multiplicity of the (right) simple module $S(i)$ in the composition series for M . Then $[M : S(i)] \dim_K D_i = \dim_K M e_i$.

Proposition 1.2.1. *Let (A, e) be a finite dimensional algebra such that all its (right) standard modules are Schurian. Then*

$$\dim_K A \leq \sum_{i=1}^n \frac{1}{d_i} \dim_K \Delta_A^\circ(i) \dim_K \Delta_A(i)$$

and

$$\dim_K A = \sum_{i=1}^n \frac{1}{d_i} \dim_K \Delta_A^\circ(i) \dim_K \Delta_A(i)$$

if and only if (A, e) is quasi-hereditary.

Proof. The equality

$$[A_A : S(n)]d_n = \dim_K Ae_n = \sum_{i=1}^n [Ae_n : S(i)]d_i = \dim_K \Delta_A^\circ(n)$$

implies $[A_A : S(n)] = \frac{1}{d_n} \dim_K \Delta_A^\circ(n)$. Also,

$$\dim_K Ae_n A \leq [A_A : S(n)] \dim_K \Delta_A(n) = \frac{1}{d_n} \dim_K \Delta_A^\circ(n) \dim_K \Delta_A(n)$$

and $\dim_K Ae_n A = \frac{1}{d_n} \dim_K \Delta_A^\circ(n) \dim_K \Delta_A(n)$ if and only if $Ae_n A \simeq \bigoplus \Delta_A(n)$ as right A -modules (if and only if $Ae_n A \simeq \bigoplus \Delta_A^\circ(n)$ as left A -modules).

Similarly,

$$\dim_K Ae_i A / Ae_{i+1} A \leq \frac{1}{d_i} \dim_K \Delta_A^\circ(i) \dim_K \Delta_A(i)$$

and

$$\dim_K Ae_i A / Ae_{i+1} A = \frac{1}{d_i} \dim_K \Delta_A^\circ(i) \dim_K \Delta_A(i)$$

if and only if $Ae_i A / Ae_{i+1} A \simeq \bigoplus \Delta_A(i)$ as right A -modules (if and only if $Ae_i A / Ae_{i+1} A \simeq \bigoplus \Delta_A^\circ(i)$ as left A -modules), for every $i = 1, \dots, n$, using the fact that the standard module $\Delta_A(i)$ is right projective and the standard module $\Delta_A^\circ(i)$ is left projective in the algebra $A/Ae_{i+1}A$.

Since $A \simeq \bigoplus_{i=1}^n Ae_i A / Ae_{i+1} A$ as K -spaces, the statement of the lemma follows. $\square$

As a consequence of the above proposition, we deduce that the concept of a quasi-hereditary algebra is truly a two-sided concept; thus there is no need to

introduce separate terminology for right quasi-hereditary algebras and left quasi-hereditary algebras, as these two concepts coincide.

We say that A is of separable type if $A/\text{rad}A$ is a separable algebra.

In this case, the semisimple algebra $A/\text{rad}A$ can be identified with a subalgebra of A . This is given by a choice of an algebra homomorphism $\varepsilon : A/\text{rad}A \rightarrow A$, which splits the canonical map $A \rightarrow A/\text{rad}A$.

A lifting of $A/\text{rad}A$ gives a $A/\text{rad}A$ -bimodule structure on A . By Wedderburn-Malcev theorem, the dependence of this bimodule structure of A on the choice of the lifting of $A/\text{rad}A$ is unessential, as any two liftings are unipotently conjugate.

Denote by S an image of $A/\text{rad}A$ under ε . Then S is a maximal semisimple subalgebra of A and any right (left) A -module also assumes a structure of a right (left) S -module.

Corollary 1.2.2. *Let A be a finite dimensional algebra containing semisimple subalgebra S which is isomorphic to $A/\text{rad}A$ (which is the case if A is of separable type). Let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of S and $e = (e_1, \dots, e_n)$ be a linear order on this set such that all (right) standard modules of A are Schurian. Then (A, e) is quasi-hereditary if and only if $A \simeq \Delta_A^\circ \otimes_S \Delta_A$ as $S - S$ bimodules.*

Proof. If $A \simeq \Delta_A^\circ \otimes_S \Delta_A$, then $\dim_K A = \sum_{i=1}^n \frac{1}{d_i} \dim_K \Delta_K^\circ(i) \dim_K \hat{\Delta}_A(i)$, hence (A, e) is quasi-hereditary, by Proposition 1.2.1.

Conversely, if (A, e) is quasi-hereditary algebra, then by refining arguments

in the proof of the above proposition, we get $\dim_K e_j A \varepsilon_i A e_k / e_j A \varepsilon_{i+1} A e_k = \frac{1}{d_i} \dim_K e_j \Delta_A^\circ(i) \dim_K \Delta_A(i) e_k = \dim_K e_j \Delta_A^\circ(i) \otimes_{e, S e,} \Delta_A(i) e_k$, for $i, j, k = 1, \dots, n$.

Since $A \simeq \bigoplus_{i,j,k=1}^n e_j A \varepsilon_i A e_k / e_j A \varepsilon_{i+1} A e_k$ and $\Delta_A^\circ \otimes_S \Delta_A \simeq \bigoplus_{i,j,k=1}^n e_j \Delta_A^\circ(i) \otimes \Delta_A(i) e_k$ as $S - S$ -bimodules, the statement follows. $\square$

We will return to this decomposition in the next chapter.

In [ADL1], various types of quasi-hereditary algebra were introduced, like shallow, deep, replete, left and right medial algebras, each type accompanied by a canonical construction. With the help of the previous formula, we can compute dimensions of these quasi-hereditary algebras over various species. To illustrate this, we compute a dimension of a shallow quasi-hereditary algebra.

The *species* $\mathcal{S}(A)$ of (A, e) is the system $(D_i, i = 1, \dots, n; {}_i M_j, i, j = 1, \dots, n)$ of division algebras D_i and $D_i - D_j$ -bimodules ${}_i M_j$, where $D_i = e_i(A/\text{rad} A)e_i$ and ${}_i M_j = e_i(\text{rad} A/\text{rad}^2 A)e_j$. If all $D_i = K$, then ${}_i M_j = \oplus K$ and the species is given by the *quiver* $\Gamma(A)$, i.e., oriented graph having $\{1, \dots, n\}$ as vertices and $\dim_K {}_i M_j$ arrows from i to j .

An algebra A is called shallow if the radicals of all its left and right standard modules are semisimple modules.

Corollary 1.2.3. *Let (A, e) be a finite dimensional algebra with species $\mathcal{S} = (D_1, \dots, D_n; {}_i M_j, 1 \leq i, j \leq n)$ without loops (i.e. ${}_i M_i = 0$). Let $\dim_K D_i = d_i$, $\dim_{D_i} {}_i M_j = v_{ij}$ and $\dim_{D_j} {}_i M_j = u_{ij}$ for $1 \leq i \neq j \leq n$ and $v_{ii} = u_{ii} = 1$, for*

$i = 1, \dots, n$. Then (A, e) is shallow quasi-hereditary algebra over $\mathcal{S}$ if and only if

$$\dim_K(A) = \sum_{\substack{i,j,k \\ k \leq i \\ j \leq i}} d_i v_{ij} u_{ki}.$$

Proof. Straightforward counting argument. $\square$

CHAPTER II

BOREL SUBALGEBRAS OF QUASI-HEREDITARY ALGEBRAS

We introduce the notion of a Borel subalgebra B of a quasi-hereditary algebra A , provide examples and derive basic properties of Borel subalgebras.

Then we relate minimal projective resolutions of simple modules over Borel subalgebra B and minimal projective resolutions of corresponding standard modules over algebra A .

For lean algebra A , we derive a relationship between minimal projective resolutions of simple modules over A and B . This will be applied in chapters 3 and 4 to obtain results about global dimensions of special quasi-hereditary algebras and their Borel subalgebras.

2.1 DEFINITIONS

The following definition is motivated by [K1], although the notation there is different.

Definition 2.1.1. *Let A be an algebra, S be a semisimple subalgebra of A isomorphic to $A/\text{rad}A$, $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of S , and B be a subalgebra of A containing S . Assume A is quasi-hereditary with respect to a linear order $e = (e_1, \dots, e_n)$. Then B is called a right Borel subalgebra of A if A is projective as a right B -module and $A \otimes_B S_B^o(i) \simeq \Delta_A^o(i)$ as left*

A -modules, for $i = 1, \dots, n$. Correspondingly, B° containing S is called a left Borel subalgebra of A if A is projective as a left B° -module and $S_{B^\circ}(i) \otimes_{B^\circ} A \simeq \Delta_A(i)$ as right A -modules, for $i = 1, \dots, n$.

Borel algebras defined above are strong exact Borel subalgebras in the sense of [K1]. Since we will not consider any other subalgebras, there will be no confusion with other objects and therefore we will call these subalgebras simply Borel subalgebras.

The next proposition provides a simple characterization of Borel subalgebras.

Proposition 2.1.2. [K2] *Let A be an algebra, S be a semisimple subalgebra of A isomorphic to $A/\text{rad}A$, $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of S , and B be a subalgebra of A containing S . Assume A is quasi-hereditary with respect to a linear order $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. Then a subalgebra B of A containing S is a right Borel subalgebra of A if and only if the canonical surjective map $p_i : P_A(i) \rightarrow \Delta_A(i)$ restricts to an isomorphism of B -modules: image of $p_i|_{P_B(i)} \simeq \Delta_A(i)$, for each $i = 1, \dots, n$. Correspondingly, a subalgebra B° of A containing S is a left Borel subalgebra of A if and only if the canonical surjective map $p_i^\circ : P_A^\circ(i) \rightarrow \Delta_A^\circ(i)$ restricts to an isomorphism of B° -modules: image of $p_i^\circ|_{P_{B^\circ}^\circ(i)} \simeq \Delta_A^\circ(i)$, for each $i = 1, \dots, n$.*

Let us note again that the notation in [K2] is a little more complicated. While reading the proof in [K2], it is important to realize that $P_B(i) \otimes_B A \simeq P_A(i)$.

Borel subalgebras were introduced in order to replace standard modules, which

are factormodules of indecomposable projective modules, by suitable preimages under the canonical projections $p_i : P_A(i) \rightarrow \Delta_A(i)$. It is evident that one can not hope that these preimages will form a submodule of A , but in many important applications it turns out that these preimages can form a subalgebra of A which is then a Borel subalgebra of A .

It turns out that a right Borel subalgebra not only describes left standard modules, but it also describes the structure of a filtration of A by its right standard modules (which exists because A is assumed to be quasi-hereditary). Moreover, the existence of a Borel subalgebra restricts significantly the structure of extensions of standard modules, thus the way A is “glued” together out of its standard modules (see also chapters 6 and 7).

2.2. BASIC PROPERTIES

In what follows, we formulate the results only for right Borel subalgebras but analogous results are valid also for left Borel subalgebras. Usually we refrain from formulating these “opposite” statements.

From the previous description, it is obvious that a (right) Borel subalgebra B of A must be (right) directed, that is, $e_i B e_j = 0$ for $i < j$. In other words, the right standard B -modules are projective and left standard B -modules are simple.

When we are given a quasi-hereditary algebra A together with its semisimple subalgebra S and/or Borel subalgebra B or B^o , simple S -modules can and will be identified with simple A -modules and/or simple B -modules or B^o -modules.

It is interesting to note that the species of B , $\mathcal{S}(B) = (D_i; {}_iX_j, i, j = 1, \dots, n)$, can have a significantly different structure from the species of A , $\mathcal{S}(A) = (D_i; {}_iM_j, i, j = 1, \dots, n)$. In general, if $i > j$, then ${}_iX_j$ can be larger than ${}_iM_j$.

Example 2.2.1. Consider the algebra A given by a quiver

$$\begin{array}{ccc} 4 & \xrightarrow{\alpha} & 2 \\ \beta \downarrow & & \downarrow \gamma \\ 1 & \xrightarrow{\delta} & 3 \end{array}$$

and relation $\alpha\gamma = \beta\delta$. We write the right regular representation of A as follows:

$$A_A = \begin{array}{ccccccc} 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\ \downarrow & & \downarrow & & & & \\ 3 & & 3 & & 1 & & 2 \\ & & & & \searrow & & \swarrow \\ & & & & 3 & & \end{array}$$

Then A is quasi-hereditary and it has a right Borel subalgebra B such that

$$B_A = \begin{array}{ccccccc} 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\ & & & & & & \\ & & & & 1 & & 2 \\ & & & & \searrow & & \swarrow \\ & & & & 3 & & \end{array}$$

Algebra B itself is given by a quiver with arrows $4 \rightarrow 1, 4 \rightarrow 2$ and $4 \rightarrow 3$. The arrow $4 \rightarrow 3$ corresponds to the path $(4 \rightarrow 2 \rightarrow 3) = (4 \rightarrow 1 \rightarrow 3)$ in the algebra

A . The right regular representation of B looks like this:

$$B_B = \begin{array}{ccccccc} 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\ & & & & & & \\ 2 & & & & \downarrow & & 1 \\ & & & & 3 & & \end{array}$$

Example 2.2.2. The algebra A , given by its right regular representation

$$A_A = \begin{array}{ccccccc} 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\ \downarrow & & \downarrow & & & & \downarrow \\ 1 & & 2 & & 4 & & 1 \\ & & \searrow & & \swarrow & & \\ & & 1 & & \end{array}$$

does not possess a right Borel subalgebra.

Perhaps the most important property of a right Borel subalgebra B of a quasi-hereditary algebra A is derived directly from the fact that A is a right B -projective module. It implies that the functor $A \otimes_B -$ is exact. We will use this property in the next section to derive homological properties of Borel subalgebras.

We have seen previously that quasi-hereditary algebra A containing a semisimple subalgebra S isomorphic to $A/\text{rad}A$ satisfies $A \simeq \Delta_A^\circ \otimes_S \Delta_A$ as $S - S$ -bimodules. If A possesses a right Borel subalgebra, then by Proposition 2.1.2, we have $A \simeq \Delta_A^\circ \otimes_S B$ as $S - S$ -bimodules. The next proposition gives an even stronger statement.

Proposition 2.2.3. *Let A be an algebra, S be a semisimple subalgebra of A isomorphic to $A/\text{rad}A$. Assume A is quasi-hereditary with respect to a linear order $e = (e_1, \dots, e_n)$ on a complete set of primitive orthogonal idempotents of S . Let B and B° be subalgebras of A containing S .*

Then B is a right Borel subalgebra of A if and only if $A \simeq \Delta_A^\circ \otimes_S B (\simeq \Delta_A^\circ \otimes_S \Delta_A)$ as $S - B$ -bimodules. Correspondingly, B° is a left Borel subalgebra of A if and only if $A \simeq B^\circ \otimes_S \Delta_A (\simeq \Delta_A^\circ \otimes_S \Delta_A)$ as $B^\circ - S$ -bimodules.

Moreover, B is a right Borel subalgebra of A and B° is a left Borel subalgebra of A if and only if $A \simeq B^\circ \otimes_S B (\simeq \Delta_A^\circ \otimes_S \Delta_A)$ as $B^\circ - B$ -bimodules. In this case, we say that A has a Cartan decomposition.

Proof. Assume that B is a right Borel subalgebra of A , or B° is a left Borel

subalgebra of A , or both B and B° are right and left Borel subalgebras of A , respectively.

Note that $\Delta_A^\circ(i) \otimes_{e_i S e_i} \Delta_A(i) \simeq A e_i A / A e_{i+1} A$ as $A / A e_{i+1} A = A / A e_{i+1} A$ -bimodules, where the isomorphism is provided by the multiplication map, according to Statement 7 of [DR1]. In particular, this describes the actions of $S e_j$ and $e_j B$ or $B^\circ e_j$ and $e_j S$ or $B^\circ e_j$ and $e_j B$, for $j \leq i$, respectively, on both bimodules. Since the actions of $e_j S$ and $e_j B$ on the right and the actions of $B^\circ e_j$ and $S e_j$ on the left, for $j > i$, on both bimodules are trivial, we conclude that $\Delta_A^\circ(i) \otimes_{e_i S e_i} \Delta_A(i) \simeq A e_i A / A e_{i+1} A$ as $S - B$ -bimodules or $B^\circ - S$ -bimodules or $B^\circ - B$ -bimodules, respectively.

Hence $A e_i A \simeq (\Delta_A^\circ(i) \otimes_{e_i S e_i} \Delta_A(i)) \oplus A e_{i+1} A$ and $A \simeq \Delta_A^\circ \otimes_S B$ as $S - B$ -bimodules or $A \simeq B^\circ \otimes_S \Delta_A$ as $B^\circ - S$ -bimodules or $A \simeq B^\circ \otimes_S B$ as $B^\circ - B$ -bimodules, respectively.

Conversely, if $A \simeq \Delta_A^\circ \otimes_S B$ as $S - B$ -bimodules, then A is a right projective B -module and $A \otimes_B S \simeq (\Delta_A^\circ \otimes_S B) \otimes_B S \simeq \Delta_A^\circ \otimes_S (B \otimes_B S) \simeq \Delta_A^\circ \otimes_S S \simeq \Delta_A^\circ$ as $S - S$ -bimodules. However, we can check that the above isomorphism induces an isomorphism of left A -modules $A \otimes_B S e_i \simeq \Delta_A^\circ(i)$. For this, it is enough to check the left action of A for generators of $A \otimes_B S e_i$. These generators are of the form $a \otimes e_i$, where $a \in A$. Since an element $a_1(a \otimes e_i) = a_1 a \otimes e_i$ is sent to the class $a_1 a e_i + A e_{i+1} A e_i = a_1(a e_i + A e_{i+1} A e_i)$, the induced map is a morphism of left A -modules. Hence B is a right Borel subalgebra of A . The remaining implications

follow analogously. $\square$

Since the isomorphism $A \simeq B^\circ \otimes_S B$ is given by multiplication, we conclude that $A \simeq B^\circ B$ as $B^\circ - B$ -bimodules, where as usual $B^\circ B = \{\sum_{j=1}^k b_j^\circ b_j | b_j^\circ \in B^\circ, b_j \in B, k \text{ is positive integer}\}$.

Another interesting property of a right Borel subalgebra B of A is described in the next proposition.

Proposition 2.2.4. *If B is a right Borel subalgebra of an algebra A , then $\text{Hom}_A(P_A(i), \Delta_A(j)) \simeq \text{Hom}_B(\Delta_A(i), \Delta_A(j)) \simeq e_j B e_i$.*

Proof. Clearly, every right A -morphism from $P_A(i)$ to $\Delta_A(j)$ restricts to $P_B(i)$ and gives a right B -morphism from $P_B(i)$ to $\Delta_A(j)$. If we compose an isomorphism $\Delta_A(i) \simeq P_B(i)$ of right B -modules with this map, we get an element of $\text{Hom}_B(\Delta_A(i), \Delta_A(j))$. Using Proposition 2.1.2, we identify this homomorphism with an element of $e_j B e_i$. On the other hand, every element $b \in e_j B e_i$ induces a right A -morphism from $P_A(i)$ to $e_j B$, which sends e_i to b . By composing this map with the canonical map p_j , we get a right A -morphism from $P_A(i)$ to $\Delta_A(j)$. This gives the required bijective correspondence. $\square$

2.3. HOMOLOGICAL PROPERTIES OF BOREL SUBALGEBRAS

If A is quasi-hereditary, then the global dimension of A is finite. Therefore, the global dimensions of its Borel subalgebras are also finite.

Let B be a right Borel subalgebra of a quasi-hereditary algebra A . Consider a

minimal (left) projective resolution

$$0 \rightarrow P_B^{om} \rightarrow \cdots \rightarrow P_B^{o1} \rightarrow P_B^{o0} \rightarrow S_B^o(i) \rightarrow 0$$

of a (left) simple module $S_B^o(i) = S\epsilon_i$ over B . Since the functor $A \otimes_B -$ is exact, it induces an exact sequence

$$0 \rightarrow A \otimes_B P_B^{om} \rightarrow \cdots \rightarrow A \otimes_B P_B^{o1} \rightarrow A \otimes_B P_B^{o0} \rightarrow A \otimes_B S_B^o(i) \simeq \Delta_A^o(i) \rightarrow 0,$$

which is a (left) projective resolution of the left standard module $\Delta_A^o(i)$ over A .

By Eckmann-Shapiro lemma, we have $Ext_A^i(\Delta_A^o(i), -) \simeq Ext_B^i(S_B^o(i), -)$ as K -spaces, which implies that the above projective resolution of $\Delta_A^o(i)$ over A is minimal.

On the other hand, if

$$0 \rightarrow P_A^r \rightarrow \cdots \rightarrow P_A^1 \rightarrow P_A^0 \rightarrow S_A(i) \rightarrow 0$$

is a minimal (right) projective resolution of a simple module $S_A(i) = \epsilon_i S$ over A , then this resolution viewed as a complex of (right) B -modules gives a (right) B -projective resolution of a simple B -module $S_B(i) = \epsilon_i S$, because all modules $P_A(j)$ are projective B -modules. This resolution is not minimal, though.

Nevertheless, we can use the previous observation to derive the following result which links the global dimension of an algebra A with Cartan decomposition and the global dimensions of its Borel subalgebras.

Recall that a global dimension of an algebra A is a minimal number m such that $Ext_A^m(X, Y) = 0$ for all (right) A -modules X, Y . It is well known that the

global dimension of A equals the maximum of projective dimensions of its (right) simple modules. The projective dimension of a simple module can be recognized from its minimal projective resolution, namely, if

$$0 \rightarrow P_A^r \rightarrow \cdots \rightarrow P_A^1 \rightarrow P_A^0 \rightarrow S_A(i) \rightarrow 0$$

is a minimal projective resolution of $S_A(i)$, then the projective dimension of a simple module $S_A(i)$ equals r .

Proposition 2.3.1. *Let A be a quasi-hereditary algebra with Cartan decomposition, B be its right Borel subalgebra, and B° be its left Borel subalgebra. Then*

$$\text{global dimension } A \leq \text{global dimension } B + \text{global dimension } B^\circ.$$

Proof. Since B° is a left Borel subalgebra of A , we conclude that A is a left projective B° -module. Then any minimal A -projective resolution of the simple left module $S^\circ(i)$ is also a projective resolution of $S^\circ(i)$ over B° , not necessarily minimal. Denote by $\Omega_{B^\circ}(i)$ the first syzygy in this resolution which is projective as left B° -module. Then $\Omega_{B^\circ}(i)$ is filtered by left standard modules of A .

Since A is quasi-hereditary and it has also right Borel subalgebra B , the projective A -dimension of a left standard module $\Delta_A^\circ(j)$ is the same as the projective B° -dimension of a right simple module $S(j)$. Therefore, the projective A -dimension of $\Omega_{B^\circ}(i)$ does not exceed the global dimension of the algebra B° .

Thus, the global dimension of A is not bigger than the sum of the global dimensions of B° and B . $\square$

Next, we will define a class of algebras which is important in many applications of quasi-hereditary algebras, like highest weight categories, Schur algebras, Yoneda and Koszul algebras etc., and which is connected to the so-called Kazhdan-Lusztig theory.

Definition 2.3.2. *Let A be a finite dimensional algebra and $\{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents of A . Then A is called right lean if and only if $e_i \text{rad}^2 A e_j = e_i \text{rad} A e_j \text{rad} A e_j$, for each $i \geq j$; A is called left lean if and only if $e_i \text{rad}^2 A e_j = e_i \text{rad} A e_i \text{rad} A e_j$, for each $i \leq j$; finally, A is called lean if it is both right and left lean.*

For a more conceptual approach to lean algebras, we will use the notion of a top embedding, see [ADL1]. We will adjust the definition presented there to our situation.

Definition 2.3.3. *Let A be an algebra, B be its subalgebra, X be a right B -module and Y be a right A -module such that $X \subset Y$. Then the inclusion of X into Y is called a top embedding if $\text{rad} X_B = X_B \cap \text{rad} Y_A$, where $\text{rad} Y_A$ is viewed as a right B -module. In this case, we write $X_B \subset^t Y_A$ or simply $X \subset^t Y$ if $A = B$.*

For each $i = 1, \dots, n$, denote by $V_A(i)$ the kernel of a projective cover $P_A(i) \rightarrow \Delta_A(i)$ and by $V_A^o(i)$ the kernel of a projective cover $P_A^o(i) \rightarrow \Delta_A^o(i)$. It is proved in [ADL1] that A is lean if and only if $V_A(i) \subset^t \text{rad} P_A(i)$ and $V_A^o(i) \subset^t \text{rad} P_A^o(i)$. In the same vein, we can prove that A is right lean if and only if $V_A^o(i) \subset^t \text{rad} P_A^o(i)$ (or equivalently the trace filtration of $\text{rad} \Delta_A(i)$ is top filtration) and A is left lean

if and only if $V_A(i) \subset {}^t \text{rad} P_A(i)$ (or equivalently the trace filtration of $\text{rad} \Delta_A^\circ(i)$ is top filtration), for every $i = 1, \dots, n$. For homological characterization, look at [ADL2].

We would like to mention that leanness of an algebra A can be recognized from its standard modules; thus, it does not depend on the particular way these standard modules are “glued” together to form the algebra A .

Algebras A for which the right standard module $\Delta_A = \bigoplus_{i=1}^n \Delta_A(i)$ is a factoralgebra of A , i.e., for which the right ideal $\bigoplus_{i=1}^n e_i A \varepsilon_{i+1} A = \bigoplus_{i=1}^n V_A(i)$ is also a left ideal of A , form a special class of right lean algebras. Really, $\bigoplus_{i=1}^n e_i A \varepsilon_{i+1} A$ is a left ideal of A if and only if $e_j A e_i A \varepsilon_{i+1} A \subset e_j A \varepsilon_{j+1} A$, for every $i, j = 1, \dots, n$, which implies that A is right lean.

For quasi-hereditary algebra A for which Δ_A is a factoralgebra of A , and B is a right Borel subalgebra of A , we will construct a minimal projective A -resolution of a right simple module $S(t)$, which is closely related to a minimal projective B -resolution of that simple module.

Until the end of this section, we will assume that A is a quasi-hereditary algebra such that Δ_A is its factoralgebra and B is its right Borel subalgebra.

First, let us fix the following notation:

Since A is quasi-hereditary and it has a right Borel subalgebra B , every element $x \in A$ can be written in the form $x = hd(x) + tl(x)$, where the head of x , $hd(x) \in B$ and the tail of x , $tl(x) \in \bigoplus_{i=1}^n e_i A \varepsilon_{i+1} A$.

A minimal projective resolution of a right simple module $S(t)$ over A , which we are going to construct, will be denoted by

$$\dots \xrightarrow{f_{i+1}} P^i \xrightarrow{f_i} P^{i-1} \xrightarrow{f_{i-1}} \dots \xrightarrow{f_1} P^0 \xrightarrow{f_0} S(t) \longrightarrow 0,$$

the syzygies of this resolution by Ω^i , the inclusion of $\Omega^i \rightarrow P^{i-1}$ by ι_i and the minimal projective cover $P^i \rightarrow \Omega^i$ by g_i . Then $f_i = \iota_i g_i$ and we would have exact sequences

$$0 \longrightarrow \Omega^1 \xrightarrow{\iota_1} P^0 \xrightarrow{f_0} S(t) \longrightarrow 0$$

and

$$0 \longrightarrow \Omega^{i+1} \xrightarrow{\iota_{i+1}} P^i \xrightarrow{g_i} \Omega^i \longrightarrow 0,$$

for each $i > 0$.

We start by taking $f_0 : P^0 = e_t A \rightarrow S(t)$ a minimal projective cover of $S(t)$ over A and $\iota_1 : \Omega^1 \rightarrow P^0$ an inclusion of its kernel to P^0 . Then $\Omega^1 = \text{rad} P^0$ decomposes as $\Omega^1 = (\Omega^1 \cap e_t B) \oplus (\Omega^1 \cap e_t A \varepsilon_{t+1} A)$, where the second summand is a right A -module while the first summand is only a right B -module. Based on this decomposition, we can construct a minimal projective cover $g_1 : P^1 = P_+^1 \oplus P_-^1 \rightarrow \Omega^1$ over A such that if $P_+^1 = \bigoplus_{j[1]} e_{j[1]} A$, then $g_1(e_{j[1]}) \in e_t B e_{j[1]}$ for each $e_{j[1]}$ and $g_1(P_-^1) \subset e_t A \varepsilon_{t+1} A$.

We continue further by induction:

Assume that we have a map $f_i : P^i \rightarrow P^{i-1}$ and decompositions $P^i = P_+^i \oplus P_-^i$, $P^{i-1} = P_+^{i-1} \oplus P_-^{i-1}$, $P_+^i = \bigoplus_{j[i]} e_{j[i]} A$ and $P_+^{i-1} = \bigoplus_{j[i-1]} e_{j[i-1]} A$

such that $f_i(P_+^i) \subset P_+^{i-1}$, $f_i(P_-^i) \subset P_-^{i-1} \oplus \bigoplus_{j[i-1]} e_{j[i-1]} A e_{j[i-1]+1} A$ and each $f_i(e_{j[i]}) \in \bigoplus_{j[i-1]} e_{j[i-1]} B e_{j[i]}$.

We will refer to these conditions as (C_i) .

First note that conditions (C_1) are satisfied if we put $P^0 = P_+^0$ and $P_-^0 = 0$ in the previous construction.

Define $P_B^i = \bigoplus_{j[i]} e_{j[i]} B$. Then P_B^i represents the “top portion” of P_+^i .

Lemma 2.3.4. *Under the above notation, the condition (C_i) implies that $\Omega^{i+1} = \ker f_i$ splits as $\Omega^{i+1} = \Omega_+^{i+1} \oplus \Omega_-^{i+1}$, where $\Omega_+^{i+1} = \Omega^{i+1} \cap P_B^i$ and $\Omega_-^{i+1} = \Omega^{i+1} \cap (P_-^i \oplus \bigoplus_{j[i]} e_{j[i]} A e_{j[i]+1} A)$. Here, Ω_+^{i+1} is a right B -module and Ω_-^{i+1} is a right A -module.*

Proof. Assume $y \in \Omega^{i+1}$ and $y = y_+ + y_-$, where $y_+ \in P_+^i$, $y_- \in P_-^i$ and we identify Ω^{i+1} with its image $\iota_{i+1} \Omega^{i+1}$ in P^i . By (C_i) , we know that $f_i(y_+) \in P_+^{i-1}$ and $f_i(y_-) \in P_-^{i-1} \oplus \bigoplus_{j[i-1]} e_{j[i-1]} A e_{j[i-1]+1} A$. Split $y_+ = hd(y_+) + tl(y_+)$. Then (C_i) implies $f_i(hd(y_+)) \in P_B^{i-1}$.

Since $tl(y_+) \in \bigoplus_{j[i]} e_{j[i]} A e_{j[i]+1} A$ and Δ_A is a factoralgebra of A , we obtain $f_i(tl(y_+)) \in \bigoplus_{j[i-1]} e_{j[i-1]} A e_{j[i-1]+1} A$.

Thus, $f_i(hd(y_+)) = -f_i(tl(y_+) + y_-) \in P_-^{i-1} \oplus \bigoplus_{j[i-1]} e_{j[i-1]} A e_{j[i-1]+1} A$, which is possible only if $f_i(hd(y_+)) = 0$. Therefore, both $hd(y_+)$ and $tl(y_+) + y_-$ belong to $\ker f_i = \Omega^{i+1}$. Since $hd(y_+) \in \Omega_+^{i+1}$ and $tl(y_+) + y_- \in \Omega_-^{i+1}$, this proves the claim. $\square$

Returning back to the induction, using the previous proposition we are able to construct a minimal projective cover $g_{i+1} : P^{i+1} \longrightarrow \Omega^{i+1}$ such that all the

conditions in (C_{i+1}) are satisfied. Here, P_+^{i+1} will be a minimal projective cover of the right A -module $\langle \Omega_+^{i+1} \rangle_A$ generated by Ω_+^{i+1} such that $g_{i+1}(e_{j[i+1]}) \in B$ for each $e_{j[i+1]}$. Split a projection $\Omega_-^{i+1} \rightarrow \Omega_-^{i+1} / \Omega_-^{i+1} \cap \langle \Omega_+^{i+1} \rangle_A$ by a linear map s such that $\text{top}(\Omega_-^{i+1} / \Omega_-^{i+1} \cap \langle \Omega_+^{i+1} \rangle_A) \simeq \text{top}(\langle R_{i+1} \rangle_A)$, where $\langle R_{i+1} \rangle_A$ is the right A -module generated by the image R_{i+1} of s in Ω_+^{i+1} . Define P_-^{i+1} to be a minimal projective cover of the module $\langle R_{i+1} \rangle_A \subset \Omega_-^{i+1}$.

Then all the conditions (C_{i+1}) are satisfied, hence by induction we get a minimal projective A -resolution of a simple right module S which satisfies properties (C_i) for every i .

The significance of this projective resolution lies in the fact that it induces a minimal projective resolution of the simple module $S(t)$ over B .

Proposition 2.3.5. *Let A be a quasi-hereditary algebra such that Δ_A is a factoragebra of A and B be a right Borel subalgebra of A . Then, there is a minimal projective resolution*

$$0 \longrightarrow P^m \xrightarrow{f_m} \dots P^j \xrightarrow{f_j} P^{j-1} \xrightarrow{f_{j-1}} \dots P^0 \xrightarrow{f_0} S(t) \longrightarrow 0$$

of simple module $S(t)$ over A and a minimal projective resolution

$$0 \longrightarrow P_B^r \xrightarrow{\tilde{f}_r} \dots P_B^i \xrightarrow{\tilde{f}_i} P_B^{i-1} \xrightarrow{\tilde{f}_{i-1}} \dots P_B^0 \xrightarrow{\tilde{f}_0} S(t) \longrightarrow 0$$

of the simple module $S(t)$ over B such that the diagram

$$\begin{array}{ccccccccccc} P^r & \xrightarrow{f_r} & \dots & \xrightarrow{f_{i+1}} & P^i & \xrightarrow{f_i} & P^{i-1} & \xrightarrow{f_{i-1}} & \dots & \xrightarrow{f_1} & P^0 & \xrightarrow{f_0} & S(t) \\ \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow = \\ P_B^r & \xrightarrow{\tilde{f}_r} & \dots & \xrightarrow{\tilde{f}_{i+1}} & P_B^i & \xrightarrow{\tilde{f}_i} & P_B^{i-1} & \xrightarrow{\tilde{f}_{i-1}} & \dots & \xrightarrow{\tilde{f}_1} & P_B^0 & \xrightarrow{\tilde{f}_0} & S(t) \end{array}$$

commutes, $P_B^i \rightarrow P_A^i$ is a top embedding and $\tilde{f}_i$ is induced by the restriction of the map f_i on P_B^i , for every $i = 1, \dots, r$.

Proof. By the previous construction, the map g_i sends P_+^i to the A -module generated by Ω_+^i . Thus $g_i|_{P_B^i}$ factors through Ω_+^i and induces a map $\tilde{g}_i : P_B^i \rightarrow \Omega_+^i$.

Moreover, from the definition of f_i , we infer that $\tilde{g}_i$ is a surjective map.

Since $f_i|_{P_+^i}$ is given by multiplication by elements from B , we see that $f_i(P_B^i) \subset P_B^{i-1}$, hence, it induces a map $\tilde{f}_i : P_B^i \rightarrow P_B^{i-1}$. Since $f_i = \iota_i g_i$, we get $\tilde{f}_i = \tilde{\iota}_i \tilde{g}_i$ for a map $\tilde{\iota}_i$ induced by ι_i . Since $\tilde{g}_i$ is surjective, we conclude that $\iota_i|_{\Omega_+^i}$ factors through P_B^i and induces a map $\tilde{\iota}_i : \Omega_+^i \rightarrow P_B^i$. Clearly, $\tilde{\iota}_i$ is an injective map.

In order to prove the proposition, it is enough to show that the induced sequences

$$0 \longrightarrow \Omega_+^{i+1} \xrightarrow{\iota_{i+1}} P_B^i \xrightarrow{\tilde{g}_i} \Omega_+^i \longrightarrow 0$$

are exact.

We have seen that $\tilde{\iota}_{i+1}$ is injective and $\tilde{g}_i$ is surjective. Exactness in the middle term follows from $\tilde{\iota}_{i+1}\Omega_+^{i+1} = \ker f_i \cap P_B^i = \ker \tilde{f}_i = \ker \tilde{g}_i$. $\square$

This result will be used later in chapters 3 and 4.

Remark 2.3.6. It is worth mentioning that we can not drop the assumption that Δ_A is factorialgebra of A in the above proposition. If A is the algebra from example 2.2.1, then projective resolutions of simple module $S(4)$ over A and B do not behave in the way described in the previous proposition. Thus, Δ_A being a factorialgebra of A is crucial for achieving the nice correspondence between projective

resolutions of simple modules over A and B .

CHAPTER III

ENDOMORPHISM ALGEBRAS OF A SEMILOCAL MODULE

First, we introduce a special class of endomorphism algebras A of a semilocal module over a commutative self-injective algebra. Then we state our previous result, which completely describes those endomorphism algebras which are quasi-hereditary.

It is proved that these algebras possess both left Borel subalgebra B° and right Borel subalgebra B . For an important special class of such algebras, we prove that the global dimension of A is the sum of global dimensions of its Borel subalgebras B° and B .

3.1 QUASI-HEREDITY

Investigation of endomorphism algebras of semilocal modules over a commutative selfinjective algebra was motivated by algebras associated with regular blocks of Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of representations of complex semisimple Lie algebras. These algebras A , which possess many interesting properties to be discussed later, are given as endomorphism algebras and are quasi-hereditary with respect to the partial order on the set of weights of the corresponding Lie algebra. To explain this, we need another definition. For any modules X, Y , we denote by $\text{trace}(X \rightarrow Y)$ the submodule of Y generated by all homomorphic images of X in

Y .

Definition 3.3.1. Let A be an algebra, $\{e_\lambda; \lambda \in \Lambda\}$ be a complete set of primitive orthogonal idempotents of A , and $\succ$ be a partial order on the set Λ . The algebra A is called quasi-hereditary with respect to the partial order $\succ$ if there is a linear order $e : e_{\lambda_1} < e_{\lambda_2} < \cdots < e_{\lambda_n}$ corresponding to a refinement of the above partial order $\succ$ on Λ such that (A, e) is quasi-hereditary and

$$\Delta(\lambda_i) = P(\lambda_i) / \text{trace}(\bigoplus_{j \succ i} P(\lambda_j) \rightarrow P(\lambda_i)).$$

In what follows, we will also write e_i instead of e_{λ_i} .

Note that the last condition in the above definition is equivalent to the fact that the modules

$$\text{trace}(\bigoplus_{j=k}^n P(\lambda_j) \rightarrow P(\lambda_i)) / \text{trace}(\bigoplus_{j=k+1}^n P(\lambda_j) \rightarrow P(\lambda_i))$$

are direct sums of $\Delta(\lambda_k)$'s ($i \leq k \leq n$), see [D].

In this chapter, let R be a commutative selfinjective K -algebra over a splitting field K such that R is finite dimensional over K . We will consider a set $\mathcal{X} = \{X(\lambda) | \lambda \in \Lambda\}$ of pairwise non-isomorphic local right ideals of R indexed by a n -element set Λ such that $R = X(\lambda_1)$ belongs to $\mathcal{X}$.

Since each ideal $X(\lambda)$ is a colocal module, there is an embedding of $X(\lambda)$ into R . We will therefore view the modules $X(\lambda)$ as submodules of R . Observe that every R -homomorphism $f : X(\lambda) \rightarrow X(\kappa)$ is induced by multiplication by an element

$r \in R$: Given f , there is an extension $\bar{f} : R \rightarrow R$ and every endomorphism of R_R is given by multiplication m_r by r ,

$$\begin{array}{ccccc} 0 & \longrightarrow & X(\lambda) & \longrightarrow & R_R \\ & & f \downarrow & & \downarrow \bar{f} = m_r \\ 0 & \longrightarrow & X(\kappa) & \longrightarrow & R_R \end{array}$$

Thus, in particular the image $Im f$ is isomorphic to a submodule of $X(\lambda)$.

Therefore, there is a monomorphism from $X(\lambda)$ to $X(\kappa)$ if and only if $X(\lambda) \subset X(\kappa)$ and the images of any two monomorphisms from $X(\lambda)$ to $X(\kappa)$ coincide.

We are interested in an endomorphism algebra $A = End_R(\oplus_{\lambda \in \Lambda} X(\lambda))$. Write, for each λ , $e_\lambda = \iota_\lambda \pi_\lambda$, where $\pi_\lambda : X \rightarrow X(\lambda)$ and $\iota_\lambda : X(\lambda) \rightarrow X$ are the canonical projection and embedding, respectively. Let S be a K -span of the set $\{e_\lambda; \lambda \in \Lambda\}$. Then S is a maximal semisimple subalgebra of A . Its right simple modules $e_\lambda S$ will be denoted by $S(\lambda)$ and left simple modules $S e_\lambda$ by $S^\circ(\lambda)$.

The first order of business is to determine when is A quasi-hereditary with respect to a partial order $\succ$ on Λ , which is given by inclusions between modules $X(i)$'s, i.e., we require that $X(\lambda') \subset X(\lambda'')$ if and only if $\lambda' \succ \lambda''$. The above partial order corresponds to the partial order appearing in the context of algebras from category $\mathcal{O}$.

The question of quasi-heredity of A was completely solved in [DHM], namely by the following description.

Theorem 3.1.2 [DHM]. *Let R be a commutative local selfinjective K -algebra over a splitting field K , $\dim_K R < \infty$. Let $\mathcal{X} = \{X(\lambda) | \lambda \in \Lambda\}$ be a set of local*

right ideals of R indexed by a finite set Λ such that $R = X(\lambda_1)$ belongs to $\mathcal{X}$, and let $\succ$ be a partial order on Λ defined by $X(\lambda') \subset X(\lambda'')$ if and only if $\lambda' \succ \lambda''$. Then $A = \text{End}_R(\oplus_{\lambda \in \Lambda} X(\lambda))$ is a quasi-hereditary algebra with respect to the partial order $\succ$ if and only if

(i) $\text{card}(\Lambda) = \dim_K R$ and

(ii) $\text{rad}X(\lambda) = \sum_{\kappa \succ \lambda} X(\kappa)$, for all $\lambda \in \Lambda$.

Note that if A is quasi-hereditary, then λ_n is the least index with respect to partial order $\succ$ and $X(\lambda_n) \simeq \text{soc}R$.

It was possible to write a Ph.D. thesis based solely on this result (see [H]). We will not comment about the proof of this theorem here. Instead, we start from this result and further describe quasi-hereditary algebras A of this form. We show the existence of Borel subalgebras B°, B of A , and in the most important special case, we relate the global dimension of A to the global dimension of its Borel subalgebras.

3.2 BOREL SUBALGEBRAS

It was proved in [K1], that algebras associated with regular blocks of the category $\mathcal{O}$ possess left and right Borel subalgebras.

In our situation, we show first the existence of a left Borel subalgebra B° of A .

Define B° as a set of all elements of A which corresponds to R -morphisms $X(\lambda) \rightarrow X(\kappa)$ given by multiplication by elements $a \in K$. Clearly, the set B° is

a subalgebra of A . Note that if $\alpha \neq 0$, then the induced map is injective.

Proposition 3.2.1. *The algebra B° , consisting of morphisms between modules $X(\lambda) \rightarrow X(\kappa)$ given by multiplications by elements from K , is a left Borel subalgebra of A .*

Proof. First, we show that the restriction of a canonical projection $p_\lambda^\circ : Ae_\lambda \rightarrow \Delta_A^\circ(\lambda)$ to $B^\circ e_\lambda$ is injective.

If $a \in e_\lambda Ae_\lambda$ is mapped to 0 via p_λ° , then the corresponding map $X(\lambda) \rightarrow X(\kappa)$ factors through $\bigoplus_{\rho \neq \lambda} X(\rho)$, hence is not injective. For $a \in B^\circ$, this implies $a = 0$.

Next, we show that $p_\lambda^\circ(B^\circ e_\lambda) = \Delta_A^\circ(\lambda)$. Assume that a map f is given by multiplication by element $r \in R$. Since K is a splitting field of R and R is local, we can write $r = \alpha + s$, where $\alpha \in K \simeq R/\text{rad}R$ and $s \in \text{rad}R$.

The multiplication by α induces a map $g : X(\lambda) \rightarrow X(\mu)$, which belongs to B° , and multiplication by s induces a map which factors through $\text{rad}X(\lambda)$, hence by Lemma 2 of [DHM] it factors through $\bigoplus_{\kappa \neq \lambda} X(\kappa)$. This shows that f and g represent the same class in $\Delta^\circ(\lambda)$. $\square$

Remark 3.2.2. Let us point out that Proposition 3.2.1 describes the structure of the left standard A -modules: If we denote by $m_{\kappa\lambda} : X(\lambda) \rightarrow X(\kappa)$ the inclusion $X(\lambda) \subset X(\kappa)$ given by multiplication by 1, then the factorizations $m_{\kappa\lambda} = m_{\kappa\rho}m_{\rho\lambda}$ correspond to the embeddings $X(\lambda) \subset X(\rho) \subset X(\kappa)$. In particular, every left standard module $\Delta_A^\circ(\lambda)$ has a simple socle generated by $m_{\lambda_1\lambda}$, and hence is isomorphic to $S(\lambda_1)$. This fact will be used in the next section.

The existence of a right Borel subalgebra follows from the existence of an involution on A preserving idempotents e_λ .

Each $X(\lambda)$ is a factor module of R and as such has a natural structure of a local commutative selfinjective K -algebra; thus, $\text{Hom}_K(X(\lambda), K) \simeq X(\lambda)$. As a consequence, $A = \text{End}_R X$ is an algebra with involution $*$: $A \rightarrow A$ defined for

$$f : X \xrightarrow{\pi_\lambda} X(\lambda) \xrightarrow{f_{\lambda\kappa}} X(\kappa) \xrightarrow{\iota_\kappa} X$$

by

$$f^* : X \xrightarrow{\pi_\kappa} X(\kappa) \simeq \text{Hom}_K(X(\kappa), K) \xrightarrow{\text{Hom}(f_{\lambda\kappa}, K)} \text{Hom}_K(X(\lambda), K) \simeq X(\lambda) \xrightarrow{\iota_\lambda} X.$$

In addition to the relations $(ab)^* = b^*a^*$ and $(a^*)^* = a$, we have also $e_\lambda^* = e_\lambda$ for all $\lambda \in \Lambda$.

Let us note that we get a duality functor F from the category of left A modules to the category of right A -modules if, for every left A -module ${}_A Y$, we define the right module $F(Y)_A$ by putting $F(Y) = Y$ and $ga = a^*y$. Moreover, F preserves the simple modules, i.e., $F(S^o(\lambda)) = S(\lambda)$ for each $\lambda \in \Lambda$. In a similar way, we get a duality functor G from the category of right A -modules to the category of left A -modules if, for a right A -module Y_A , we define the left module ${}_A G(Y)$ by putting $G(Y) = Y$ and $ay = ga^*$. Then G also preserves the simple modules, i.e., $G(S(\lambda)) = S^o(\lambda)$ for each $\lambda \in \Lambda$.

Proposition 3.2.3. *The image of a left Borel subalgebra B^o of A under involution $*$ is a right Borel subalgebra B of A . Moreover, B is isomorphic to the opposite algebra of B^o .*

Proof. First, if $b_1, b_2 \in B^\circ$, then $(b_1 b_2)^* = b_2^* b_1^*$. Therefore, $(B^\circ)^*$ is a subalgebra of A isomorphic to the opposite algebra of B° .

Since involution $*$ fixes idempotents e_λ , we have $(P^\circ(\lambda))^* = P(\lambda)$, $(S^\circ(\lambda))^* \simeq S(\lambda)$; consequently, every $(B^\circ e_\lambda)^* = e_\lambda B$ is filtered only by simple modules $S(\kappa)$ for $\lambda \succ \kappa$. Since the K -dimensions of $e_\lambda B$ and $\Delta_A(\lambda)$ are the same, we have $e_\lambda B \simeq \Delta_A(\lambda)$ as right B -modules. $\square$

It was mentioned previously that the structure of the left Borel subalgebra B° (and left standard A -modules) is given completely by the lattice of inclusions $X(\lambda) \subset X(\kappa)$ between modules $X(\lambda), \lambda \in \Lambda$. Dually, the structure of the right Borel subalgebra B (and right standard A -modules) is completely determined by surjective R -morphisms $X(\kappa) \rightarrow X(\lambda)$ between modules $X(\lambda), \lambda \in \Lambda$.

To further illustrate this, consider the following. Let R_1 and R_2 be two local self-injective commutative algebras and $X_1(\lambda), X_2(\lambda)$, for $\lambda \in \Lambda$, be local ideals of R_1 and R_2 , respectively, $A_1 = \text{End}_{R_1}(\oplus_{\lambda \in \Lambda} X_1(\lambda))$ and $A_2 = \text{End}_{R_2}(\oplus_{\lambda \in \Lambda} X_2(\lambda))$ are quasi-hereditary algebras and $B_1^\circ, B_2^\circ, B_1$ and B_2 their Borel subalgebras. If the lattice of inclusions between $X_1(\lambda)$'s is isomorphic to the lattice of inclusions between $X_2(\lambda)$'s, then $B_1^\circ \simeq B_2^\circ$ and $B_1 \simeq B_2$ as algebras. However, the algebras A_1 and A_2 need not be isomorphic because their structures depend on the particular gluing of their standard modules. On the other hand, we always have $\dim_K A_1 = \dim_K A_2$, by Proposition 1.2.1.

Having described the structure of Borel subalgebras of A , we conclude easily

that Δ_A and Δ_A° are factoralgebras of A , hence, in particular, A is lean.

Proposition 3.2.4. *The standard modules Δ_A, Δ_A° are factoralgebras of A .*

Proof. If $b \in e_k A e_j A e_i \cap B^\circ e_i$, where $i < j$ and $j < k$, then b corresponds to a map $f : X(\lambda_k) \rightarrow X(\lambda_i)$ which factors through $X(\lambda_j)$. Since $j < k$, any map from $X(\lambda_k)$ to $X(\lambda_j)$ factors through $\text{rad} X(\lambda_j)$; consequently, f is not injective. Since $b \in B^\circ e_i$, we get $b = 0$ showing that $A e_{j+1} A e_j A e_i \subset A e_{i+1} A e_i$. Dually, we get $e_i A e_j A e_{j+1} A \subset e_i A e_{i+1} A$; hence A is lean. $\square$

3.3 GLOBAL DIMENSIONS

The relationship between the global dimension of certain types of quasi-hereditary algebras and the global dimensions of their Borel subalgebras was investigated in [K6] and [K7]. These papers provide motivation for our next results.

Recall that the Lloewy length of a module M over an algebra D , denoted by $l(M)$, is the length of its D -radical filtration, i.e., it is the minimal number l such that $\text{rad}_D^{l+1} M = 0$.

Using some of the properties of an endomorphism algebra A established in the previous section, we will prove the following theorem.

Theorem 3.3.1. *Let A be a quasi-hereditary endomorphism algebra defined in section 3.1, B° be its left Borel subalgebra. Assume that the global dimension of B° equals the Lloewy length $l(B^\circ)$ of B° . Then the global dimension of A equals twice the global dimension of B° .*

Since $B \simeq (B^\circ)^{\text{op}}$, the corresponding statement is true if we replace B° by B .

Let us first comment about the condition that the global dimension of B° equals the Lloewy length of B° . First of all, for any directed algebra D , the global dimension of D is smaller than or equal to the Lloewy length of D . For, consider the projective resolution of a semisimple module $D/\text{rad}D$. If Ω_i is a i -th syzygy in this resolution, then $\cdots < l(\Omega_{i+1}) < l(\Omega_i) < \cdots < l(\Omega_0) < l(D)$. Therefore, the condition in question is an extreme case. Important examples of such algebras are Borel subalgebras of algebras associated with regular blocks of category $\mathcal{O}$. Moreover, the projective resolution of simple modules over a Borel subalgebra is given by the Bernstein-Gelfand-Gelfand resolution (i.e., the finite resolution by finite direct sum of Verma modules) of corresponding simple modules over a semisimple Lie algebra, for details see [K5].

Proof of Theorem 3.3.1. Since B° is given by inclusions between $X(\lambda)$'s and $X(\lambda_n) \subset X(\lambda)$, for each $\lambda \in \Lambda$, we get that $l(B^\circ) = l(B^\circ e_n) > l(B^\circ e_j)$, for $1 \leq j < n$. Therefore the assumption that the global dimension of B° equals $l(B^\circ)$ is equivalent to the condition that the projective B° -dimension of the simple module $S^\circ(\lambda_n)$ equals $l(B^\circ)$, which is equivalent to the condition that the projective B -dimension of the simple module $S(\lambda_n)$ equals $l(B) = l(B^\circ) = d$.

In this case, the d -th syzygy Ω_B^d of the B -projective resolution of $S(\lambda_n)$ is a nonzero direct sum of simple modules $S(\lambda_1)$.

Another equivalent condition for the global dimension of B° to be equal $l(B^\circ)$ is

that the projective B^o -dimension of the simple module $S(\lambda_1)$ equals $l(B^o)$, which is equivalent to the condition that the projective B -dimension of the simple module $S^o(\lambda_1)$ equals $l(B) = l(B^o) = d$.

Since B is a Borel subalgebra of A , by Proposition 3.2.3, and Δ_A is factoralgebra of A , by Proposition 3.2.4, we can apply Proposition 2.2.5 to get a minimal projective resolution

$$\dots \xrightarrow{f_{i+1}} P^i \xrightarrow{f_i} P^{i-1} \xrightarrow{f_{i-1}} \dots \xrightarrow{f_1} P^0 \xrightarrow{f_0} S(\lambda_n) \longrightarrow 0$$

of the simple module $S(\lambda_n)$ over A , which induces a minimal projective resolution

$$\dots \xrightarrow{\tilde{f}_{i+1}} P_B^i \xrightarrow{\tilde{f}_i} P_B^{i-1} \xrightarrow{\tilde{f}_{i-1}} \dots \xrightarrow{\tilde{f}_1} P_B^0 \xrightarrow{\tilde{f}_0} S(\lambda_n) \longrightarrow 0$$

of the simple module $S(\lambda_n)$ over B .

Then P_+^d is the direct summand of P^d , generated by P_B^d . The assumption that the global dimension of B^o equals $l(B^o)$ implies that $\Omega_B^d = \oplus S(\lambda_1)$ and $P_+^d = \oplus P(\lambda_1)$. Since $S(\lambda_1) \simeq P_B(\lambda_1)$ and the above B -projective resolution is minimal, the module P_+^{d-1} has no direct summands isomorphic to $P(\lambda_1)$. We will look closer at the restriction of the map f_d to one fixed direct summand $P(\lambda_1)$ of P_+^d . Firstly, $f_d(P(\lambda_1)) \subset P_+^{d-1}$.

Since $\Omega_B^d \subset {}^t\Omega_+^d$, the image $g_d(P(\lambda_1))$ is not included in $\text{rad}\Omega^d$. We will show that $g_d(P(\lambda_1)) \simeq S(\lambda_1)$. Note that $\text{rad}P(\lambda_1) = \sum_{\kappa} \text{trace}(P(\kappa) \rightarrow P(\lambda_1))$, where $\kappa \neq \lambda_1$ are such that $\kappa \succ \lambda$ implies $\lambda = \lambda_1$. If $\text{trace}(P(\kappa) \rightarrow g_d(P(\lambda_1))) \neq 0$ for κ as above, then $\Delta_A(\kappa)$ is a direct summand of $g_d(P(\lambda_1))$ as a right B -module

because the last module is B -projective. Since $f_d(P(\lambda_1)) \subset \text{rad}P_+^{d-1}$, due to the structure of Δ -filtration of A given by Borel subalgebra B° , this is possible only if P_+^{d-1} contains a direct summand isomorphic to $P(\lambda_1)$, which is a contradiction with minimality of the projective resolution over B . Therefore, $\text{trace}(P(\kappa) \rightarrow g_d(P(\lambda_1))) = 0$ for every $\kappa \neq \lambda_1$, showing that $g_d(P(\lambda_1)) \simeq S(\lambda_1)$.

This implies that $\text{rad}P(\lambda_1)$ is a direct summand of Ω^{d+1} . Therefore, the global dimension of A is not smaller than d plus projective A -dimension of $\text{rad}P(\lambda_1)$ plus one. The last quantity equals d plus projective A -dimension of $S(\lambda_1)$. The projective A -dimension of $S(\lambda_1)$ is not smaller than the projective B° -dimension of $S(\lambda_1)$, which equals d . Thus, the global dimension of A is at least $2d$. The opposite inequality follows from Proposition 2.3.1. $\square$

From the above proof we deduce that the next generalization of the previous theorem is true.

Proposition 3.3.2. *Assume A is a quasi-hereditary algebra, B is its right Borel subalgebra, Δ_A° is a factoralgebra of A , and B° is a left Borel subalgebra of A such that the global dimension of $B^\circ = l(B^\circ)$, every $\text{soc}B^\circ(i) = \oplus S^\circ(\lambda_\bullet)$ for a fixed simple module $S(\lambda_\bullet)$, and global dimension of B equals the projective dimension of $S(\lambda_\bullet)$. Then the global dimension of A is the sum of the global dimensions of its Borel subalgebras B and B° .*

Hence, we need not be restricted to special endomorphism algebras investigated in [DHM].

Many examples of algebras studied in this chapter can be found in [DHM]. However, we could not resist giving the following example, which is related to algebras associated with the category $\mathcal{O}$ for a Lie algebra $sl_3(C)$.

Example 3.3.3. Take $R = K[x, y]/\langle x^2 - y^2, y^3 \rangle$ and $X(1) = R$, $X(2) = \langle x \rangle_R$, $X(3) = \langle y \rangle_R$, $X(4) = \langle x^2 \rangle_R$, $X(5) = \langle xy \rangle_R$ and $X(6) = \langle x^3 \rangle_R$. Then the largest indecomposable projective A -module $P_A(1)$ is given as

and the other indecomposable projectives modules $P_A(i)$ are isomorphic to the traces of simple modules $S(i)$ in $P_A(1)$. The reader might wish to compute the global dimension of A directly or at least a minimal projective resolution of simple module $S(6)$ to appreciate the power of the previous results about the global dimension of A .

CHAPTER IV

ENDOMORPHISM ALGEBRAS OF AUSLANDER TYPE

We investigate the properties of endomorphism algebras A which are generalizations of quasi-hereditary algebras introduced in [A]. We prove the existence of right Borel subalgebras for these algebras.

We derive also the existence of a left Borel subalgebra provided certain conditions about radical and socle filtrations and graded structure of modules defining A are satisfied. Under these assumptions, we again relate the global dimension of A and its Borel subalgebras.

4.1 NOTATION

In this section, let R be a finite dimensional algebra over a central field K , $\{e_1, \dots, e_l\}$ be a complete primitive orthogonal set of idempotents of R , and S_R be a maximal semisimple subalgebra of R containing the set $\{e_1, \dots, e_l\}$. Without loss of generality, we can assume that R is basic, i.e., projective modules $e_1 R, \dots, e_l R$ are pairwise isomorphic. Then $e_i S_R e_i$ is a division algebra which will be denoted by D_i or D_{e_i} .

Let $\text{rad } R = J$ be the (Jacobson) radical of R and $Y = \bigoplus_{j=1}^m Y_j$ be a direct sum of S_R - R -bimodules Y_j which are local as right R -modules and of the Lloewy length n_j . For $k = 1, \dots, n_j$, denote by $X_{jk} = Y_j / Y_j J^k$ the factorbimodule of Y_j , which is

local as right R -module and of the Lloewy length k and put $X = \bigoplus_{j=1}^m \bigoplus_{k=1}^{n_j} X_{jk}$ and $\tilde{A} = \text{End}_R(X)$.

Note that if K is a splitting field of R , then the situation is much simpler: $S_R \simeq \bigoplus_{i=1}^n K$ and any right R -module is automatically a $S_R - R$ -bimodule.

Algebra $\tilde{A}$ is not usually basic because different modules X_{jk} and X_{qs} can be isomorphic. Therefore, instead of $\tilde{A}$, we will investigate the basic algebra A which is Morita equivalent to $\tilde{A}$. Morita equivalence is an equivalence between module categories of $\tilde{A}$ and A ; hence it preserves quasi-heredity and global dimensions. Here is a description of A :

If $X_{jk} \simeq X_{fg}$ and $X_{qs} \simeq X_{uv}$, then $\text{Hom}_R(X_{jk}, X_{qs}) \simeq \text{Hom}_R(X_{fg}, X_{uv})$. Therefore, if we denote by $\tilde{X}_{jk}$ the class of those direct summands of X which are isomorphic to X_{jk} , we can define $\text{Hom}_R(\tilde{X}_{jk}, \tilde{X}_{qs}) = \text{Hom}_R(X_{jk}, X_{qs})$. Extending this, we put $\tilde{X} = \bigoplus \tilde{X}_{jk}$, where the summation is over all classes $\tilde{X}_{jk}$ and $A = \text{Hom}_R(\tilde{X})$.

Hence, the only difference between $\tilde{X}$ and X is that we put only one copy of X_{jk} as a direct summand of X for each class $\tilde{X}_{jk}$.

Following [LX], we define a partial order $\succ$ on these isomorphism classes $\tilde{X}_{jk}$ as follows: $\tilde{X}_{jk} \succ \tilde{X}_{qs}$ if and only if either $X_{jk} \simeq X_{qs}$ or $k < s$ or ($k = s$ and there exists a surjective R -morphism from X_{jk} to X_{qs}). It is clear that this definition does not depend on the choices of modules X_{jk} and X_{qs} within their isomorphism classes.

Denote by $e_{j,k}$ the idempotent of A given by the composition of a natural projection and inclusion: $\tilde{X} \longrightarrow \tilde{X}_{j,k} \longrightarrow \tilde{X}$. Be aware that we will use different notation for the same idempotent; if $X_{i,k} \simeq X_{j,k}$, then we identify $e_{i,k}$ and $e_{j,k}$ and use the most suitable notation, depending on the circumstances. When needed, we might also identify $e_{j,k}$ with the map $X \longrightarrow X_{j,k} \longrightarrow X$ given by the composition of a natural projection and inclusion. The indecomposable right projective A -modules are $e_{j,k}A \simeq \text{Hom}_R(\tilde{X}, \tilde{X}_{j,k}) \simeq \text{Hom}_R(X, X_{j,k})$ and the indecomposable left projective A -modules are $Ae_{j,k} \simeq \text{Hom}_R(\tilde{X}_{j,k}, \tilde{X}) \simeq \text{Hom}_R(X_{j,k}, X)$. We have also $e_{q,s}Ae_{j,k} \simeq \text{Hom}_R(\tilde{X}_{j,k}, \tilde{X}_{q,s}) \simeq \text{Hom}_R(X_{j,k}, X_{q,s})$. Because of this, we will regard the R -morphisms from $X_{j,k}$ to $X_{q,s}$ as elements of A .

The idempotents $e_{j,k}$, corresponding to the isomorphism classes $\tilde{X}_{j,k}$, form a complete set of primitive orthogonal idempotents of A . We choose a linear order e on this set which corresponds to a refinement of the previously defined partial order $\succ$ on the isomorphism classes $\tilde{X}_{j,k}$.

The case $Y = R$ was introduced by Auslander in [A]. It was shown by Dlab and Ringel [DR3] that (A, e) is quasi-hereditary in this case. That result was extended by Lin and Xi [LX] to semilocal modules Y , i.e., for modules Y which are direct sums of local modules.

In the following sections, we will discuss the existence of left and right Borel subalgebras of A and the relation between the global dimension of A and its Borel subalgebras.

4.2 SPECIES OF A

Let us start with the description of simple A -modules:

Since each module X_{jk} is local, it is a factormodule of an indecomposable R -projective module, say $e_i R$. To indicate that e_i corresponds to X_{jk} , we will write $e_i = e_{X_{jk}}$. Then $\text{End}_R(X_{jk})/\text{rad End}_R(X_{jk}) \simeq e_{X_{jk}} S_R e_{X_{jk}} \simeq D_{e_{X_{jk}}}$, which will be also denoted by D_{jk} .

For $d \in e_{jk} S_R e_{jk}$, we denote by $f_d : X_{jk} \rightarrow X_{jk}$ the map induced by left multiplication by d . Since X_{jk} is a $S_R - R$ -bimodule, this map is a morphism of right R -modules. Such morphisms form a maximal subalgebra $S = S_A$ of A . This subalgebra decomposes as $S \simeq \bigoplus_{\tilde{X}_{jk}} D_{jk}$; hence, it need not be basic (and usually is not) despite the fact that S_R is basic.

The next simple observation will help us to understand the species of A :

Lemma 4.2.1. Assume $f : X_{jk} \rightarrow X_{qs}$ does not belong to $\text{rad}^2 A$.

If $k \geq s$, then f is surjective and $k = s$ or $k = s + 1$. If $k = s + 1$, then $f = g c_{j,s+1,s}$, where g is an isomorphism from X_{js} onto X_{qs} .

If $k < s$, then the Lloewy length of $f(X_{jk})$ is k .

Proof. First assume $k \geq s$. If f is not surjective, then $f(X_{jk}) \subset X_{qs} J$ because X_{qs} is local. Therefore, the Lloewy length of $f(X_{jk})$ is less than or equal to $s - 1$, $f(X_{jk} J^{s-1}) = 0$ and f factors through $c_{j,k,s-1}$, which gives $f \in \text{rad}^2 A$, a contradiction.

Correspondingly, if $k > s + 1$, then $f(X_{jk} J^{k-1}) = 0$ and f factors through

$c_{j,k,k-1}$, hence $f \in \text{rad}^2 A$, a contradiction.

Next assume $k = s + 1$. Then $f(X_{jk}J^s) = 0$ implies that f factors through $c_{j,s+1,s}$ as $f = gc_{j,s+1,s}$, where $g : X_{js} \rightarrow X_{qs}$ is surjective. If g is not injective, then $f \in \text{rad}^2 A$, a contradiction.

If $k < s$ and the Lloewy length of $f(X_{jk})$ is less than k , then $f(X_{jk}J^{k-1}) = 0$, f factors through $c_{j,k,k-1}$ and $f \in \text{rad}^2 A$, a contradiction. $\square$

Note that if $f : X_{jk} \rightarrow X_{qs}$ does not belong to $\text{rad}^2 A$, then the Lloewy length of $f(X_{jk})$ is either k or s .

Lemma 4.2.2. *The standard module Δ_A is a factoralgebra of $A = \text{End}_R(\tilde{X})$. If radical and socle filtrations of each Y_j coincide, then Δ_A° is a factoralgebra of A .*

Proof. Start with the observation that $X_{jk} > X_{qs}$ implies that there is no surjective morphism from X_{qs} to X_{jk} . Let $e_u = e_{qs}$ and $e_v = e_{jk}$. First, consider $f \in e_u A e_{u+1} A$ and $g : X_{qs} \rightarrow X_{jk}$. Since $X_{jk} > X_{qs}$, the map g is not surjective. Since X_{jk} is local module, $gf(X) \subset \text{rad} X_{jk}$, hence gf factors through $\bigoplus_l \bigoplus_{i < k} X_{li}$. Therefore, $e_v A e_u A e_{u+1} A \subset e_v A e_{v+1} A$.

Next assume that socle and radical filtrations of each Y_j coincide. If $f \in A e_{u+1} A e_u$, then $f = \sum_i \beta_i \alpha_i$, where $\alpha_i : X_{qs} \rightarrow X_{u_i v_i}$, $\beta_i : X_{u_i v_i} \rightarrow X$ and $X_{u_i v_i} > X_{qs}$. Then each α_i is not surjective, hence $\alpha_i(X_{qs}) \subset \text{rad} X_{u_i v_i}$ because $X_{u_i v_i}$ is a local module.

Therefore, $\text{soc} X_{qs} \subset \ker f$ and if $g : X_{jk} \rightarrow X_{qs}$, then $\text{soc} X_{jk} \subset \ker fg$. Since $\text{soc} X_{jk} = \text{rad}^{k-1} X_{jk}$, fg factors through $c_{jk(k-1)}$, which shows $A e_{u+1} A e_u A e_v \subset$

$A\varepsilon_{v+1}Ac_v$. $\square$

4.3 RIGHT BOREL SUBALGEBRA B

We denote by $c_{j,k,s}$ the canonical projection $c_{j,k,s} : X_{jk} \longrightarrow X_{js}$, provided $k \geq s$.

Let B be a subalgebra of A containing S and all maps $c_{j,k,s}$ for $k > s$. Then the simple B -modules are isomorphic to simple S -modules and $\text{rad}B/\text{rad}^2B$ is generated by maps $c_{j,k,k-1}$.

Moreover, B is a hereditary algebra, i.e., the radicals of projective B -modules are again projective, and each indecomposable projective B -module is uniserial, i.e., it has a unique composition series, and the multiplicity of each simple module in an indecomposable (right or left) B -projective module is at most one.

However, right projective B -modules viewed as right A -modules can contain commutativity relations, because $c_{j,k,s}$ for $X_{js} \simeq X_{qs}$ can factor as

$$\begin{array}{ccc} X_{jk} & \xrightarrow{g} & X_{qk} \\ c_{j,k,s} \downarrow & & \downarrow c_{q,k,s} \\ X_{js} & \xrightarrow{\cong} & X_{qs} \end{array} ,$$

hence $c_{j,k,s} = c_{q,k,s}g$, where $g : X_{jk} \longrightarrow X_{qk}$ is surjective.

Proposition 4.3.1. B is a right Borel subalgebra of A .

Proof. First, we show that the restriction of canonical projection $p_{qs} : e_{qs}A \longrightarrow$

$\Delta_A(e_{qs})$ on $e_{qs}B$ is injective.

Take $0 \neq f \in e_{qs}Be_{jk}$. Then f is a composition of $c_{j,k,s}$ and an isomorphism $g : X_{js} \rightarrow X_{qs}$, in particular, f is surjective. If $p_{qs}(f) = 0$, then f factors through $\bigoplus_{X_{uv} > X_{qs}} X_{uv}$ as $f = \sum_i \alpha_i \beta_i$, where $\beta_i : X_{jk} \rightarrow X_{uv}$, and $\alpha_i : X_{uv} \rightarrow X_{qs}$. Since f is surjective and X_{qs} is a local module, there exists an index i such that $\alpha_i \beta_i$ is surjective. Then α_i is surjective, which implies $v_i = s$. Since $X_{u_i,s}$ is a local module, β_i must be surjective, too. We have $\beta_i = \gamma_i c_{j,k,s}$, where $\gamma_i : X_{js} \rightarrow X_{u_i,s}$ is surjective. Since $X_{js} \simeq X_{qs}$, there is a surjective morphism from X_{qs} to $X_{u_i,s}$, which contradicts $X_{u_i,s} > X_{qs}$.

Next, we show that $p_{qs}|e_{qs}B$ is surjective. If $g : X_{jk} \rightarrow X_{qs}$ is such that $k < s$, then $p_{qs}(g) = 0$. If $k \geq s$ and g is not surjective, then it factors through $c_{j,k,s-1}$, hence, $p_{qs}(g) = 0$. If g is surjective, then it factors as $g = hc_{j,k,s}$, where $h : X_{js} \rightarrow X_{qs}$ is surjective. If $X_{js} \not\simeq X_{qs}$, then $X_{js} > X_{qs}$ and again $p_{qs}(g) = 0$. If $X_{js} \simeq X_{qs}$, then h is isomorphism of local modules, which is induced by the left multiplication by $r \in e_{X_{qs}}Re_{X_{qs}}$.

We can write $r = d + t$, where $d \in e_{X_{qs}}Se_{X_{qs}} \simeq D_{qs}$ and $t \in \text{rad}R$. Then the left multiplication by d induces an automorphism f_d of X_{qs} and the left multiplication by t induces an endomorphism $f_t = h - f_d$ of X_{qs} , which factors through $c_{q,s,s-1}$. Therefore, $p_{qs}(g) = p_{qs}(f_d c_{j,k,s})$. But the map $f_d c_{j,k,s}$ belongs to $e_{qs}B$ because it belongs to a simple A -module generated by $c_{j,k,s}$. $\square$

It is also true that the algebra $\hat{A}$ has a right Borel subalgebra, but we will not require this in the future.

4.4 LEFT BOREL SUBALGEBRA B^o

In order to guarantee the existence of a left Borel subalgebra of A , we will impose additional conditions on the algebra R and bimodule Y .

We start by replacing the algebra R by an algebra T which is graded by powers of its radical. The purpose of doing this is twofold. Firstly, we will set $R = T$ at the beginning and using T instead of R will remind us that the assumptions on R are stronger. Secondly, at the end of this section we will relax assumptions imposed on R and it will be convenient that all the previous conditions are expressed in term of an algebra T instead of R .

Let us consider algebra T together with its radical filtration $T = \text{rad}^0 T \supset \text{rad}^1 T \supset \text{rad}^2 T \cdots \supset \text{rad}^n T \supset \text{rad}^{n+1} T = 0$. We say that algebra T is *graded by radicals* if T is isomorphic to its graded algebra $G(T) = \bigoplus_{i=0}^n \text{rad}^i T / \text{rad}^{i+1} T$. The image of $\text{rad}^i T / \text{rad}^{i+1} T$ under this isomorphism will be denoted by T_i .

Corresponding to the previous situation, we assume that T is basic and contains a maximal semisimple subalgebra $S_T \simeq T_0 \simeq T / \text{rad} T$ and $e_1^T, \dots, e_t^T$ is a complete set of primitive orthogonal idempotents of S_T .

Furthermore, we will require that the module Y is a graded right T module. Then each module $X_{j,k}$ is a graded right local T -module, which is a factormodule of an indecomposable right projective T -module $e_{X_{j,k}}^T T$ by an ideal $I_{j,k}$. The fact that $X_{j,k}$ is graded means that $I_{j,k} = \bigoplus_{i=1}^n (I_{j,k} \cap T_i)$.

Finally, we will assume that *radical and socle T -filtrations of each module Y_j*

coincide. This immediately implies that the same is true for each X_{jk} and for X .

If $f \in e_{qs} A e_{jk}$, then $f : X_{jk} \rightarrow X_{qs}$ is induced by left multiplication by an element $t = e_{X_{qs}}^T t e_{X_{jk}}^T$. Then $t I_{jk} \subset I_{qs}$.

Express t in the form $t = t_0 + t_1 + \dots + t_n$, where $t_i \in T_i$. Then $t(I_{jk} \cap T_v) \subset I_{qs}$ for all $1 \leq v \leq n$. Since $t_i(I_{jk} \cap T_v) \subset T_{v+1}$ for all i and I_{qs} is graded right T -module, we get $t_i(I_{jk} \cap T_v) \subset T_{v+1} \cap I_{qs}$. Hence, $t_i I_{jk} \subset I_{qs}$ because I_{jk} is graded, too.

Hence, if the left multiplication by t induces morphism from X_{jk} to X_{qs} , then also the left multiplication by each component t_i of t induces morphism from X_{jk} to X_{qs} . This observation will be crucial for the proof that A possesses a left Borel subalgebra.

We will assume first that $R = T$.

Define B^0 to be the smallest subalgebra of A containing the maximal semisimple subalgebra S of A and all morphisms $f_r : X_{jk} \rightarrow X_{qs}$ for $s > k$ induced by the left multiplication by $\tau \in e_{X_{qs}}^T T_{s-k} e_{X_{jk}}^T$ (for which $\tau I_{jk} \subset I_{qs}$).

Since $f_{r_1} f_{r_2} = f_{r_1 r_2}$, we see that every element in $e_{qs} B^0 e_{jk}$ is of the form f_r if $s > k$ or f_d , where f_d is induced by the left multiplication by $d \in e_{X_{qs}}^T T e_{X_{jk}}^T \simeq e_{X_{qs}} S T e_{X_{jk}} \simeq D_{jk}$, in the case $e_{qs} = e_{jk}$.

Moreover, B^0 is an algebra graded by radicals because $\text{rad}^i B^0 / \text{rad}^{i+1} B^0$ is isomorphic to the set of all maps $f_r : X_{jk} \rightarrow X_{q(k+i)}$ induced by the left multiplication by $\tau \in e_{X_{q(k+i)}}^T T_i e_{X_{jk}}^T$ (for which $\tau I_{jk} \subset I_{q(k+i)}$).

Proposition 4.4.1. Assume $R = T$ is graded by radicals, $Y = \bigoplus_{j=1}^m Y_j$ is a graded right T -module, and the radical and socle filtrations of each Y_j coincide. Then B^0 is a left Borel subalgebra of A .

Proof. First, we show that the restriction of canonical projection $p_{jk}^0 : Ae_{jk} \rightarrow \Delta_A^0(e_{jk})$ on $B^0 e_{jk}$ is injective. If $f \in e_{qs} Ae_{jk}$ is such that $p_{jk}^0(f) = 0$, then write $f = \sum_i \beta_i \alpha_i$, where $\alpha_i : X_{jk} \rightarrow X_{u_i v_i}$, $\beta_i : X_{u_i v_i} \rightarrow X_{qs}$ and $X_{u_i v_i} > X_{jk}$.

We claim that $\alpha_i(X_{jk}) \subset \text{soc}^{k-1} X_{u_i v_i}$, for each i . This is clear if $v_i < k$. If $v_i = k$, then there is a surjective map from $X_{u_i v_i}$ onto X_{jk} . Then α_i is not surjective because $X_{jk} \not\subset X_{u_i v_i}$. Since $X_{u_i v_i}$ is a local module of the Loewy length $v_i = k$, the claim follows.

Then $\beta_i \alpha_i(X_{jk}) \subset \text{soc}^{k-1} X_{qs}$, for each i , hence $f(X_{jk}) \subset \text{soc}^{k-1} X_{qs}$.

We will show that if $f \in e_{qs} B^0 e_{jk}$ and $f \neq 0$, then $f(X_{jk}) \not\subset \text{soc}^{k-1} X_{qs}$, which would imply that $p_{jk}^0|_{B^0 e_{jk}}$ is injective. If $f \neq 0$ is induced by the left multiplication by $d \in e_{X_{qs}}^T T e_{X_{jk}}^T$, then the Loewy length of $f(X_{jk})$ equals k , hence $f(X_{jk}) \not\subset \text{soc}^{k-1} X_{qs}$. If $f \neq 0$ is induced by the left multiplication by $\tau \in e_{X_{qs}}^T T_{s-k} e_{X_{jk}}^T$, then $\tau \notin I_{qs}$. Since X_{qs} is graded T -module, we have $f_\tau(e_{X_{jk}}^T + I_{jk}) = \tau + I_{qs} \notin \text{rad}^{s-k+1} X_{qs}$. Since radical and socle filtrations of X_{qs} coincide, we get $f_\tau(e_{X_{jk}}^T + I_{jk}) \not\subset \text{soc}^{k-1} X_{qs}$, therefore $f(X_{jk}) \not\subset \text{soc}^{k-1} X_{qs}$. Thus $p_{jk}^0|_{B^0 e_{jk}}$ is injective.

Next, we show that $p_{jk}^0(B^0 e_{jk}) = \Delta_A^0(e_{jk})$. If $f \in e_{qs} Ae_{jk}$, then $f : X_{jk} \rightarrow X_{qs}$ is induced by the left multiplication by some $t \in e_{X_{qs}}^T T e_{X_{jk}}^T$, where $t =$

$t_{s-k} + \dots + t_n$ and each $t_i \in T_i$. We have seen previously that the left multiplication by t_{s-k} induces morphism $f' : X_{jk} \rightarrow X_{qs}$, which belongs to $B^\circ e_{jk}$. Then $f - f'$ is induced by the left multiplication by $t - t_{s-k}$ and factors through $X_{j(k-1)}$, because $t - t_{s-k} \in \text{rad}^{s-k+1}T$. Therefore, $p_{jk}^\circ(f) = p_{jk}^\circ(f')$ and $p_{jk}^\circ(B^\circ e_{jk}) = \Delta_A(e_{jk})$. $\square$

Next, we will relax our assumptions about R .

Proposition 4.4.2. *Let T be a basic K -algebra graded by radicals containing a maximal semisimple subalgebra $S_T \simeq T_0 \simeq T/\text{rad}T$. Assume that an algebra R is an epimorphic image of an algebra T via a map $\varphi : T \rightarrow R$, which maps a complete set of primitive orthogonal idempotents $e_1^T, \dots, e_l^T$ of S_T to a complete set of primitive orthogonal idempotents $e_1, \dots, e_l$ of a maximal semisimple subalgebra S_R of R . Let $Y = \bigoplus_{j=1}^m Y_j$ be a S_R - R -bimodule, semilocal and graded as a right T -module such that the radical and socle filtrations of each local module Y_j coincide. Define the algebras A and B° as before.*

Then B° is a left Borel subalgebra of A .

Proof. Since $X_{jk}\text{rad}^i R = X_{jk}\text{rad}^i T$ and $\text{soc}_R^i(X_{jk}) = \text{soc}_T^i(X_{jk})$, we can speak simply about radical and socle filtrations of X_{jk} .

Moreover, $\text{Hom}_R(X_{jk}, X_{qs}) \simeq \text{Hom}_T(X_{jk}, X_{qs})$ and any such morphism is given by left multiplication by an element $t \in T$.

Therefore, the arguments of the proof of Proposition 4.4.1 are applicable in this situation. $\square$

Note that R itself need not be graded by its radicals (it need not be graded at

all, see example 4.6.3).

The algebra $\tilde{A}$ has also a left Borel subalgebra, but we will not need this.

4.5 GLOBAL DIMENSIONS

Now, we can state the main result about endomorphism algebras of Auslander type.

Theorem 4.5.1. *Let T be a basic algebra over a central field K graded by radicals containing a maximal semisimple subalgebra $S_T \simeq T/\text{rad}T$. Assume that an algebra R is an epimorphic image of an algebra T via a map $\varphi : T \rightarrow R$, which maps a complete set of primitive orthogonal idempotents $e_1^T, \dots, e_l^T$ of S_T to a complete set of primitive orthogonal idempotents $e_1, \dots, e_l$ of a maximal semisimple subalgebra S_R of R . Let $Y = \bigoplus_j^m Y_j$ is a $S_R - R$ -bimodule, which is semilocal right R -module and graded right T -module such that radical and socle filtrations of each Y_j coincide. Define $\tilde{A} = \text{End}_R(X)$, where $X = \bigoplus_{j=1}^m \bigoplus_{k=1}^{n_j} X_{j,k}$ and $X_{j,k} = Y_j/Y_j \text{rad}R^k$. Let A be an algebra Morita equivalent to $\tilde{A}$.*

Then A is quasi-hereditary and possesses both right Borel subalgebra B and left Borel subalgebra B° and global dimension $\tilde{A} = \text{global dimension } A = \text{global dimension } B^\circ + \text{global dimension } B$.

Proof. The first part of the statement follows from section 4.1 and Proposition 4.3.1 and 4.4.2. It remains to prove only the statement regarding the global dimension of A .

If X is semisimple R -module, then A is semisimple algebra and $A = S = B = B^o$. Then the global dimension of A equals 0, which is also the sum of global dimensions of B^o and B .

Next, assume X is not semisimple R -module. Then A and B are not semisimple algebras. Since B is a hereditary algebra, the global dimension of B equals 1.

If B^o is semisimple (that is the global dimension of B^o is 0), then, by Proposition 2.3.1, the global dimension of A is 1, which is the same as the sum of the global dimensions of B^o and B .

If B^o is not semisimple, then global dimension of $B^o = z \geq 1$. Since Δ_A is a factoralgebra of A by Proposition 4.2.2, we can apply Proposition 2.3.5 to obtain a minimal projective resolution

$$\dots \xrightarrow{f_{i+1}} P^{i0} \xrightarrow{f_i} P^{(i-1)0} \xrightarrow{f_{i-1}} \dots \xrightarrow{f_1} P^{00} \xrightarrow{f_0} Se_{jk} \longrightarrow 0$$

of a left simple module Se_{jk} over A , which induces a minimal projective resolution of Se_{jk} over B^o for simple module Se_{jk} of (the highest) projective B^o -dimension z .

Let $Ae_{uv} \simeq \text{Hom}_R(X_{uv}, X)$ be a direct summand of P^{z0} , $P_+^{(z-1)0} = \bigoplus_i Ae_{r_i s_i} \simeq \bigoplus_i \text{Hom}_R(X_{r_i s_i}, X)$ and $p_j : \bigoplus_i Ae_{r_i s_i} \longrightarrow Ae_{r_j s_j}$ be a natural projection. From the induced B^o -projective resolution, we get $p_j f_z(e_{uv}) \in \text{rad} B^o e_{r_j s_j}$. Moreover, if $p_j f_z|Ae_{uv} \neq 0$, then $v > s_j$ because algebra B^o is graded by radicals. If $v > s_j$, then any composition of a map $X_{r_j s_j} \longrightarrow X_{uv}$ with a canonical projection $c_{u,v,1}$ is zero. Therefore, the map $c_{u,v,1} \in \ker f_z = \Omega^{z+1}$, showing that $\Omega^{z+1} \neq 0$.

Therefore, the global dimension of A is bounded below by the global dimension

of B^0 increased by 1, which equals to the sum of the global dimensions of B^0 and B . The opposite inequality follows from Proposition 2.3.1 $\square$

Using the ideas of the above proof and the proof of Proposition 3, we easily infer that if X is not semisimple $\tilde{R}$ -module, then the projective A -dimension of any left simple module Se_{jk} , which is not projective as B^0 -module, equals the projective B^0 -dimension of Se_{jk} increased by one. Thus, if the projective A -dimension of Se_{jk} equals the projective B^0 -dimension of Se_{jk} , then Se_{jk} is projective as left B^0 -module and as left A -module. This happens if and only if $k = 1$ and the multiplicity of simple R -module X_{j1} in $rad Y$ equals 0. This is the only case when the projective A -dimension of Se_{jk} is equaled by the projective B^0 -dimension of Se_{jk} . In every other case, the projective A -dimension of Se_{jk} equals the projective B^0 -dimension of Se_{jk} increased by one.

4.6 EXAMPLES

We start with an easy example to illustrate the results of this chapter.

Example 4.6.1. Assume algebra R is given by the quiver with two vertices $\{1, 2\}$, two arrows $\alpha : 1 \rightarrow 2$, $\beta : 2 \rightarrow 1$ and relation $\beta\alpha = 0$. Then the right regular representation of R is given as

$$R_R = \begin{array}{ccc} 1 & \psi & 2 \\ \downarrow & & \downarrow \\ 2 & & 1 \\ \downarrow & & \\ 1 & & \end{array}$$

Further, put $Y = R$. Then $X_{13} = e_1 R$, $X_{12} = e_1 R / e_1 \text{soc} R$, $X_{22} = e_2 R$, $X_{11} = e_1 R / e_1 \text{rad} R$, $X_{21} = e_2 R / e_2 \text{rad} R$. The left regular representation of an algebra A is given as

$$\begin{array}{ccccccccc}
 {}_A A = & 1 & \oplus & 2 & \oplus & & 3 & & \oplus & 4 & \oplus & 5 \\
 & \downarrow & & \downarrow & & & \swarrow & & \searrow & & \downarrow & & \downarrow \\
 & 2 & & 4 & & 5 & & & & 1 & & 3 & & 2 \\
 & \downarrow & & \downarrow & & & \searrow & & \swarrow & & \downarrow & & \\
 & 4 & & 3 & & & & & & & & 1 & & \\
 & \downarrow & & \downarrow & & & & & & & & & & \\
 & 3 & & 1 & & & & & & & & & & \\
 & \downarrow & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & &
 \end{array}$$

Here, the simple module $S(1)$ corresponds to the module X_{13} , $S(2)$ to X_{12} , $S(3)$ to X_{22} , $S(4)$ to X_{11} and $S(5)$ to X_{21} .

The left regular representation of a left Borel subalgebra B^o of A is given by

$$\begin{array}{ccccccccc}
 {}_{B^o} B^o = & 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 & \oplus & 5 \\
 & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & 1 & & 3 & & 2 \\
 & & & & & & & \downarrow & & \\
 & & & & & & & 1 & &
 \end{array}$$

and the right regular representation of a right Borel subalgebra B of A is given by

$$\begin{array}{ccccccccc}
 B_B = & 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 & \oplus & 5 \\
 & & & \uparrow & & & & \uparrow & & \uparrow \\
 & & & 1 & & & & 2 & & 3 \\
 & & & & & & & \uparrow & & \\
 & & & & & & & 1 & &
 \end{array}$$

The global dimension of both B^o and B is one and the global dimension of A is two.

Another example illustrating previous results is given below.

Example 4.6.2 Put $R = K[x, y] / \langle xy, x^2 - y^2 \rangle$, $Y_1 = R$, $Y_2 = \langle x \rangle$ and $Y_3 = \langle y \rangle$. Then $X_{13} = R$, $X_{12} = R / \text{soc} R$, $X_{22} = \langle x \rangle$, $X_{32} = \langle y \rangle$, $X_{11} \simeq$

$X_{21} \simeq X_{31} \simeq R/\text{rad}R$ and the left regular representation of an algebra A is given as ${}_A A =$

Here, the simple module $S(1)$ corresponds to the module X_{13} , $S(2)$ to X_{22} , $S(3)$ to X_{32} , $S(4)$ to X_{12} and $S(5)$ to X_{11} .

The left regular representation of a left Borel subalgebra B^o of A is given by

and the right regular representation of a right Borel subalgebra B of A is given by

Here, the last projective indecomposable summand of B viewed as a right A

module looks like

This shows in particular that the species of B are not induced from the species of A (The path 45 in B equals $425 = 435$ in A).

The global dimension of B is one, the global dimension of B^o is two and the global dimension of A is three.

It is easy to give an example of an algebra which is graded by radicals such that its radical and socle filtrations of its indecomposable projective modules do not coincide. On the other hand, if socle and radical filtrations of indecomposable projective R -modules coincide and R is monomial, then it is graded by radicals. Surprisingly this is not true in general. The next example shows a local algebra R for which the radical and the socle filtrations coincide but which is not graded by radicals – in fact not graded at all.

Example 4.6.3. The algebra R given as $R = K[x, y, z]/\langle x, y, z \rangle^4 + \langle xy - z^3, yz - x^3, zx - y^3 \rangle$ is a 10-dimensional algebra. Since $\text{rad}^0 R = \text{soc}^3 R = \langle 1 \rangle$, $\text{rad} R = \text{soc}^2 R = \langle x, y, z \rangle$, $\text{rad}^2 R = \text{soc}^1 R = \langle x^2, y^2, z^2, x^3, y^3, z^3 \rangle$ and $\text{rad}^3 R = \text{soc}^0 R = 0$, the radical and socle filtrations of R coincide. However, R is not graded by radicals because $x \in \text{rad} R \setminus \text{rad}^2 R$ but $x^3 \in \text{rad}^2 R$.

The fact that R is not graded is more difficult to see and it is proved in [S].

Our final example gives an algebra R graded by radicals, module Y which is a graded R -module such that its socle and radical filtrations coincide but which is not a semilocal R -module. The quiver of the endomorphism algebra A contains no loops and A is not quasi-hereditary.

Example 4.6.4. Put $R = k[x, y] / \langle x^2, y^2, xy \rangle$, $Y_1 = (k[x_1, y_1] \oplus k[x_2, y_2] \oplus k[x_3, y_3]) / I_1$, where $I_1 = \langle y_1 - x_1, y_2 - x_3, y_3, \text{all polynomials of degree 2 or more} \rangle$ and $Y_2 = (k[x_1, y_1] \oplus k[x_2, y_2]) / \langle y_1 - x_2, y_2, \text{all polynomials of degree 2 or more} \rangle$.

CHAPTER V

EXTENSION ALGEBRAS

In this chapter, we introduce the notion of a standard extension algebra. This notion is related to that of a Yoneda Ext-algebra. These algebras are important in the representation theory of complex semisimple algebras, namely, an (associative) algebra A corresponding to a regular block of category $\mathcal{O}$ is homologically selfdual, i.e., the Yoneda Ext-algebra of A is isomorphic to A . For this reason, the Yoneda Ext-algebra were studied outside and within the framework of quasi-hereditary algebras by [CPS2], [ADL3] and [GM]. It has turned out that the species of the Yoneda Ext-algebra is the most interesting invariant of this algebra.

Later, we investigate standard extension algebra for quasi-hereditary algebra A with directed standard modules and a Borel subalgebra B^o . For such algebra A , we find a suitable method of computation of its standard extension algebra A^δ . Then we determine a bimodule structure of A^δ over its subalgebras B^{o*} (the Yoneda extension algebra of B^o) and A_0^δ (extension of degree 0).

Finally, we infer a sufficient condition which relates a species of A^δ to species of B^o and A_0^δ .

5.1 THE YONEDA EXT-ALGEBRA

We start with formal definitions.

Definition 5.1.1. Let A be a basic algebra, $S = \bigoplus_{i=1}^n S(i)$ be a direct sum of its (right) simple modules. The Yoneda Ext-algebra A^* is an algebra whose underlying vector space is $\bigoplus_{k \geq 0} \text{Ext}_A^k(S, S) \simeq \bigoplus_{k \geq 0} \bigoplus_{i,j=1}^n \text{Ext}_A^k(S(j), S(i))$ with the usual operation of addition and multiplication given by the Yoneda product of extensions.

We recall (see [M]) that for (right) A -modules M, N , an element of $\text{Ext}_A^k(N, M)$ corresponds to a class of a k -fold exact sequences $0 \rightarrow M \rightarrow X_{k-1} \rightarrow X_{k-2} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow N \rightarrow 0$ of (right) A -modules and A -morphisms under the equivalence relation given as follows: Two k -fold exact sequences E and E' starting at M and ending at N are equivalent if and only if there is a sequence of morphisms of k -fold exact sequences $E = E_0 \rightarrow E_1 \leftarrow E_2 \rightarrow \dots \leftarrow E_{2l-2} \rightarrow E_{2l-1} \leftarrow E_{2l} = E'$, running alternately to the left and to the right, all starting with 1_M and ending with 1_N . Such a morphism between k -fold exact sequences $0 \rightarrow M \rightarrow X_{k-1} \rightarrow X_{k-2} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow N \rightarrow 0$ and $0 \rightarrow M \rightarrow Y_{k-1} \rightarrow Y_{k-2} \rightarrow \dots \rightarrow Y_1 \rightarrow Y_0 \rightarrow N \rightarrow 0$ is given by A -morphisms such that the diagram

$$\begin{array}{ccccccccccc} M & \longrightarrow & X_{k-1} & \longrightarrow & X_{k-2} & \longrightarrow & \dots & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & N \\ \downarrow = & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & Y_{k-1} & \longrightarrow & Y_{k-2} & \longrightarrow & \dots & \longrightarrow & Y_1 & \longrightarrow & Y_0 & \longrightarrow & N \end{array}$$

commutes.

There is another characterization of $\text{Ext}_A^k(N, M)$ using minimal projective resolution of the module N . In what follows, we denote by

$$\dots \xrightarrow{d_{k+1}} P^k(N) \xrightarrow{d_k} P^{k-1}(N) \dots \xrightarrow{d_1} P^0(N) \xrightarrow{d_0} N \rightarrow 0$$

a minimal A -projective resolution of N and by $\Omega^k(N)$ the k -th syzygy of this resolution.

Then we have the exact sequences

$$0 \longrightarrow \Omega^1(N) \xrightarrow{\iota_1} P^0(N) \xrightarrow{d_0} N \longrightarrow 0$$

and

$$0 \longrightarrow \Omega^k(N) \xrightarrow{\iota_k} P^{k-1}(N) \xrightarrow{d_{k-1}} \Omega^{k-1}(N) \longrightarrow 0,$$

for $k \geq 0$.

Applying functor $\text{Hom}_A(-, M)$ to the above A -projective resolution of N gives a cochain complex $\text{Hom}_A(P^*(N), M)$. The k -th cohomology of this complex, $\text{Hom}_A(\Omega^k(N), M) / \iota_k^* \text{Hom}_A(P^{k-1}(N), M)$, is isomorphic to $\text{Ext}_A^k(N, M)$. More specifically, the cohomology class of $f \in \text{Hom}_A(\Omega^k(N), M)$ corresponds to the equivalence class of a k -fold exact sequence $0 \rightarrow M \rightarrow X_{k-1} \rightarrow \cdots \rightarrow X_0 \rightarrow N \rightarrow 0$ such that there is a commutative diagram of A -morphisms

$$\begin{array}{ccccccccccc} \Omega^k(N) & \xrightarrow{\iota_k} & P^{k-1}(N) & \xrightarrow{d_{k-1}} & \cdots & \xrightarrow{d_1} & P^0(N) & \xrightarrow{d_0} & N & \longrightarrow & 0 \\ \downarrow f & & \downarrow & & & & \downarrow & & \downarrow = & & \\ M & \longrightarrow & X_{k-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

On the other side, any k -fold exact sequence $0 \rightarrow M \rightarrow X_{k-1} \rightarrow \cdots \rightarrow X_0 \rightarrow N \rightarrow 0$ corresponds to some $f \in \text{Hom}_A(\Omega^k(N), M)$, which can be found by lifting the map id_N along the above projective resolution of N .

The addition of k -fold exact sequences representing elements from $\text{Ext}_A^k(N, M)$ is induced by addition of corresponding linear maps from $\text{Hom}_A(\Omega^k(N), M)$.

The product of two 0-extensions is given by composition of morphisms.

If $E : 0 \rightarrow M \rightarrow X \rightarrow Y \rightarrow 0$ represents an element from $Ext^1(N, M)$ and $\alpha : M \rightarrow Q$ and $\beta : L \rightarrow Y$ are A -morphisms, then the product $\alpha E : 0 \rightarrow Q \rightarrow Y \rightarrow Y \rightarrow 0$ is given by pushout of E and α , and the product $E\beta : 0 \rightarrow M \rightarrow Z \rightarrow L \rightarrow 0$ is given by pullback of β and E .

If $k, m \geq 1$ and a k -fold exact sequence $0 \rightarrow M \rightarrow X_{k-1} \rightarrow X_{k-2} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow N \rightarrow 0$ represents an element from $Ext_A^k(N, M)$ and a m -fold exact sequence $0 \rightarrow Q \rightarrow Y_{m-1} \rightarrow Y_{m-2} \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 \rightarrow M \rightarrow 0$ represents an element from $Ext_A^m(M, Q)$, then the Yoneda product of these two elements lies in $Ext_A^{m+k}(N, Q)$ and is represented by a $(m+k)$ -fold exact sequence $0 \rightarrow Q \rightarrow Y_{m-1} \rightarrow \cdots \rightarrow Y_0 \rightarrow X_{k-1} \rightarrow \cdots \rightarrow X_0 \rightarrow N \rightarrow 0$.

Since the Yoneda multiplication is associative, A^* is an algebra. For more details see [M].

In the case of algebra A^* , we have a characterization of elements from $Ext_A^k(S(j), S(i))$ which is simpler than the previous general characterization. Namely, since $S(i)$ is simple A -module and the image of the map $d_k : P^k(S(j)) \rightarrow P^{k-1}(S(j))$ lies in the radical of $P^{k-1}(S(j))$, all the coboundaries of the cochain complex $Hom_A(P^*(S(j)), S(i))$ are zeroes, hence the k -th cohomology of this complex coincides with $Hom_A(\Omega^k(S(j)), S(i))$. Therefore, $Ext_A^k(S(j), S(i)) \simeq Hom_A(\Omega^k(S(j)), S(i))$.

The elements $f_i = id_{S(i)} \in Hom_A(S(i), S(i)) \subset A^*$, for $i = 1, \dots, n$, form a

complete set of primitive orthogonal idempotents of A^* . Hence, the indecomposable left projective A^* -modules are $P^{*o}(i) = A^* f_i \simeq \text{Ext}_A^*(S(i), S)$.

The decomposition $A^* = \bigoplus_{k \geq 0} \text{Ext}_A^k(S, S)$ defines a graded algebra structure on A^* . Often the filtration obtained from this grading does not coincide with the radical filtration of A^* . The case, when they do coincide, is important and justifies the following definition.

Definition 5.1.2. *An algebra A is called Koszul if the radical filtration of its Yoneda Ext-algebra A^* coincides with the filtration obtained from grading $A^* = \bigoplus_{k \geq 0} \text{Ext}_A^k(S, S)$.*

Koszul algebras were investigated in [GM] and [ADL3]. We would like to caution that the terminology of these papers is slightly different from ours. The characterization of Koszul algebras is given in the next proposition.

If $S = (D_i; {}_iM_j, i, j = 1, \dots, n)$ is a species then the dual species DS is defined by $DS = (D_i; {}_i\tilde{M}_j, i, j = 1, \dots, n)$, where ${}_i\tilde{M}_j$ is the dual of ${}_jM_i$.

Proposition 5.1.3[ADL3]. *Let A be an algebra. Then the following conditions are equivalent:*

- (i) A is Koszul
- (ii) $S(A^*) = DS(A)$
- (iii) For every simple module $S(i)$ and every $k \geq 0$, we have $\Omega^k(S(i)) \subset \text{rad} P^{k-1}(S(i))$.

We will use this characterization of Koszul algebras to prove the following state-

ment.

Proposition 5.1.4. *Let A be a quasi-hereditary Koszul algebra such that Δ_A is a factorialgebra of A and B is a right Borel subalgebra of A . Then algebra B is Koszul.*

Proof. We can use Proposition 2.3.5 to relate minimal projective resolutions of simple modules over algebra A and B . In the commutative diagram

$$\begin{array}{ccc} \Omega^k(S(i)) & \longrightarrow & \text{rad}P^{k-1}(S(i)) \\ \uparrow & & \uparrow \\ \Omega_B^k(S(i)) & \longrightarrow & \text{rad}P_B^{k-1}(S(i)) \end{array}$$

of embeddings both vertical maps are top embeddings, by Proposition 2.3.5, and the first horizontal map is top embedding, by Proposition 5.1.3, because A is a Koszul algebra. Therefore, the second horizontal map is a top embedding. Since this is true for every k , the algebra B is Koszul, again by Proposition 5.1.3. $\square$

5.2 STANDARD EXT-ALGEBRA

For an algebra A , we define a standard Ext-algebra of A as an extension algebra of standard modules of A .

Definition 5.2.1. *Let A be a basic algebra, $\Delta = \bigoplus_{i=1}^n \Delta_A(i)$ be a direct sum of its (right) standard modules. The standard Ext-algebra A^b is an algebra whose underlying vector space is $\bigoplus_{k \geq 0} \text{Ext}_A^k(\Delta, \Delta) \simeq \bigoplus_{k \geq 0} \bigoplus_{i,j=1}^n \text{Ext}_A^k(\Delta(j), \Delta(i))$ with the usual operation of addition and multiplication given by the Yoneda product of extensions.*

It is important to realize that there is a corresponding definition of a left standard extension algebra when we start with left standard A -modules. Of course, everything that will be said for standard extension algebras applies correspondingly to left standard extension algebras, but we will refrain from formulating the corresponding statements.

The decomposition $A^\delta \simeq \bigoplus_{k \geq 0} \text{Ext}_A^k(\Delta, \Delta)$ defines a graded algebra structure on A^δ . The elements $c_i = \text{id}_{\Delta_A(i)} \in \text{Hom}_A(\Delta(i), \Delta(i)) \subset A^\delta$, for $i = 1, \dots, n$, form a complete set of primitive orthogonal idempotents of A^δ and the indecomposable left projective A^δ -modules are $P^{\delta o}(i) = A^\delta c_i \simeq \text{Ext}_A^*(\Delta(i), \Delta)$.

As before, elements of $\text{Ext}_A^k(\Delta(j), \Delta(i))$ are represented by equivalence classes of k -fold exact sequences starting at $\Delta(i)$ and ending at $\Delta(j)$, and we have $\text{Ext}_A^k(\Delta(j), \Delta(i)) \simeq \text{Hom}_A(\Omega^k(\Delta(j)), \Delta(i)) / \iota_{\Delta(i)}^* \text{Hom}_A(P^{k-1}(\Delta(j)), \Delta(i))$.

Since our primary interest lies in quasi-hereditary algebras, we will investigate standard extension algebra of quasi-hereditary algebra A . The important role played by a left Borel subalgebra B^o of A can be seen in the following proposition.

Proposition 5.2.2. *Let A be a quasi-hereditary algebra and B^o be its left Borel subalgebra of the global dimension d . Then B^{c*} is isomorphic to a graded subalgebra of A^δ and $\text{Ext}_A^k(\Delta, \Delta) = 0$ for $k > d$.*

Proof. As mentioned in section 2.3, a (right) minimal projective resolution

$$0 \rightarrow P_{B^o}^m \rightarrow \dots \rightarrow P_{B^o}^1 \rightarrow P_{B^o}^0 \rightarrow S(j) \rightarrow 0$$

of a (right) simple module $S(j)$ over B^o induces a minimal (right) projective

resolution

$$0 \rightarrow P_{B^\circ}^m \otimes_{B^\circ} A \rightarrow \cdots \rightarrow P_{B^\circ}^1 \otimes_{B^\circ} A \rightarrow P_{B^\circ}^0 \otimes_{B^\circ} A \rightarrow \Delta_A(j) \rightarrow 0$$

of the (right) standard module $\Delta_A(j)$ over A . Therefore, the projective B° -dimension of $S(j)$ is the same as the projective A -dimension of $\Delta_A(j)$ and $\text{Ext}_A^k(\Delta, \Delta) = 0$ if $k > d$.

Also $\text{Ext}_{B^\circ}^k(S(j), S(i)) \simeq \text{Hom}_{B^\circ}(\Omega_{B^\circ}^k(S(j)), S(i))$. To every morphism $f \in \text{Hom}_{B^\circ}(\Omega_{B^\circ}^k(S(j)), S(i))$ we assign a map $f \otimes id_A \in \text{Hom}_A(\Omega_A^k(\Delta(j)), \Delta(i))$. If f is nonzero, then both f and $f \otimes id_A$ are surjective and the cohomology class of $f \otimes id_A$ in $\text{Hom}_A(\Omega_A^k(\Delta(j)), \Delta(i)) / \iota_k^* \text{Hom}_A(P^{k-1}(\Delta(j)), \Delta(i))$ is nontrivial. This gives a linear map from B^{o*} to A^b which is injective.

If $0 \rightarrow S(i) \rightarrow X_{k-1} \rightarrow \cdots \rightarrow X_0 \rightarrow S(j) \rightarrow 0$ represents an element from $\text{Ext}_{B^\circ}^k(S(j), S(i))$ and $0 \rightarrow S(l) \rightarrow Y_{m-1} \rightarrow \cdots \rightarrow Y_0 \rightarrow S(i) \rightarrow 0$ represents an element from $\text{Ext}_{B^\circ}^m(S(i), S(l))$, then their product is represented by $0 \rightarrow S(l) \rightarrow Y_{m-1} \rightarrow \cdots \rightarrow Y_0 \rightarrow X_{k-1} \rightarrow \cdots \rightarrow X_0 \rightarrow S(j) \rightarrow 0$. If we apply the functor $- \otimes_{B^\circ} A$, we get exact sequences

$$0 \rightarrow \Delta(i) \rightarrow X_{k-1} \otimes_{B^\circ} A \rightarrow \cdots \rightarrow X_0 \otimes_{B^\circ} A \rightarrow \Delta(j) \rightarrow 0,$$

$$0 \rightarrow \Delta(l) \rightarrow Y_{m-1} \otimes_{B^\circ} A \rightarrow \cdots \rightarrow Y_0 \otimes_{B^\circ} A \rightarrow \Delta(i) \rightarrow 0$$

which represent the images of the corresponding two elements of B^{o*} in A^b . Their product is represented by $0 \rightarrow \Delta(l) \rightarrow Y_{m-1} \otimes_{B^\circ} A \rightarrow \cdots \rightarrow Y_0 \otimes_{B^\circ} A \rightarrow X_{k-1} \otimes_{B^\circ} A \rightarrow \cdots \rightarrow X_0 \otimes_{B^\circ} A \rightarrow \Delta(j) \rightarrow 0$, which coincides with the image of

$0 \rightarrow S(l) \rightarrow Y_{m-1} \rightarrow \cdots \rightarrow Y_0 \rightarrow X_{k-1} \rightarrow \cdots \rightarrow X_0 \rightarrow S(j) \rightarrow 0$. Therefore, the image of B^{o*} is a subalgebra of A^5 such that its graded structure is induced by the graded structure on A^6 . $\square$

In what follows, we will identify B^{o*} with its isomorphic image in A^6 .

Let us note that since $\Delta(i)$ are local modules, the image of B^{o*} in A^6 covers all the classes of surjective maps from $\text{Hom}_A(\Omega^k(\Delta(j)), \Delta(i))$. Therefore, all remaining elements of A^6 correspond to the maps from $\text{Hom}_A(\Omega^k(\Delta(j)), \Delta(i))$, which are not surjective.

Our next task will be to identify a class of algebras for which the computation of extension of standard modules does not depend on the entire cohomology class given above but only on its cocycle part $\text{Hom}_A(\Omega^k(\Delta(j)), \Delta(i))$ (as is the case for algebra A^*).

In section 2.2, we have mentioned the notion of a directed algebra. Now, we will generalize this to a notion of the directed module.

Definition 5.2.3. *Let A be an algebra, $e = (e_1, \dots, e_n)$ be an ordered complete set of primitive orthogonal idempotents of A , and M be a (right) A -module. The module M is called directed (with respect to A and e) if every local submodule N of M is a surjective image of a (right) standard module of A .*

In other words, a module M is directed if every local submodule N of M satisfies the property that the simple module $\text{top}N = N/\text{rad}N$ is the highest among all composition factors of N with respect to the linear order on simple A -modules

induced by ϵ .

As a special case, if Δ_A is a factorialgebra of A , then the module Δ_A is directed.

Note that an algebra A is directed if and only if all its projective indecomposable modules are directed. In this case all A -modules are directed. Also if Δ_A is a factorialgebra of A , then Δ_A is directed as a right A -module.

Now we can formulate the following result.

Proposition 5.2.4. *Let (A, ϵ) be a quasi-hereditary algebra which has a left Borel subalgebra and all right standard A -modules $\Delta(i)$ are directed (with respect to (A, ϵ)). Then $\text{Ext}_A^k(\Delta(j), \Delta(i)) \simeq \text{Hom}_A(\Omega^k(\Delta(j)), \Delta(i))$.*

Proof. Consider a minimal (right) projective resolution

$$0 \rightarrow P_{B^0}^m \otimes_{B^0} A \rightarrow \cdots \rightarrow P_{B^0}^1 \otimes_{B^0} A \rightarrow P_{B^0}^0 \otimes_{B^0} A \rightarrow \Delta(j) \rightarrow 0$$

of the right standard module $\Delta(j)$ over A induced from a minimal (right) projective resolution

$$0 \rightarrow P_{B^0}^m \rightarrow \cdots \rightarrow P_{B^0}^1 \rightarrow P_{B^0}^0 \rightarrow S(j) \rightarrow 0$$

of a (right) simple module $S(i)$ over B^0 .

Since $\Omega_A^k(\Delta(j)) = \Omega_{B^0}^k \otimes_{B^0} A$, it is enough to show that $\iota_k^*(\Omega_{B^0}^k(S(j)) \otimes_{B^0} A, \Delta(i)) = 0$.

Decompose $P_{B^0}^{k-1} = \bigoplus_s e_{[k-1, s]} B^0$ into a direct sum of right indecomposable B^0 projective modules. Then $P_{B^0}^{k-1} \otimes_{B^0} A \simeq \bigoplus_s e_{[k-1, s]} A$ and the natural projection from this module onto $e_{[k-1, s]} A$ will be denoted by p_s . Take $g \in \text{Hom}_A(P_{B^0}^{k-1} \otimes_{B^0} A, \Delta(i))$ and write $g = \sum_s g_s p_s$, where $g_s = g|e_{[k-1, s]} A$.

Since the map $\iota_k : \Omega_A^k(\Delta(j)) \rightarrow P^{k-1}(\Delta(j))$ is induced by tensoring from the (right) projective resolution over B^o , its image $\iota_k(\Omega_A^k(\Delta(j)))$ is included in $(\text{rad} P_{B^o}^{k-1}) \otimes_{B^o} A$. If $e_{[k-1,s]}A \simeq e_q A$, then $p_s \iota_k(\Omega_A^k(\Delta(j))) \subset e_q A \varepsilon_{q+1} A$ because B^o is a left directed algebra. Since $\Delta(i)$ is a directed A -module, any morphism from $e_q A$ to $\Delta(i)$ factors through $\Delta(q)$. Therefore, $g_s p_s \iota_k(\Omega_A^k(\Delta(j))) = 0$ for every s , hence $g \iota_k(\Omega_A^k(\Delta(j))) = 0$. $\square$

The next two examples show that both assumptions, about the directness of right standard A -modules and about the existence of left Borel subalgebra of A , are necessary.

Example 5.2.5. If A is a hereditary algebra given by the quiver $3 \rightarrow 1 \rightarrow 2$, then A has a left Borel subalgebra, $\Delta(3)$ is not directed as A -module, $\text{Ext}_A^1(\Delta(1), \Delta(3)) = 0$ but $\text{Hom}_A(\Omega^1(\Delta(1)), \Delta(3)) \neq 0$.

Example 5.2.6. Let A be given by its right regular representation

Then all right standard A -modules are directed, A has no left Borel subalgebra and $\text{Ext}_A^2(\Delta(1), \Delta(4)) = 0$ but $\text{Hom}_A(\Omega^2(\Delta(1)), \Delta(4)) \neq 0$.

5.3 DECOMPOSITION OF STANDARD EXT-ALGEBRA

It follows from the definition that the set $A_0^\delta = \text{Hom}_A(\Delta, \Delta)$ of elements of A^δ of degree 0 is a graded subalgebra of A^δ . If A has a left Borel subalgebra B° , then $B^{\circ*}$ is isomorphic to a graded subalgebra of A^δ , by Proposition 5.2.2. If, in addition to this, the right standard A -modules are directed, then Proposition 5.2.4 provides an effective method of computing A^δ . We can say even more.

Proposition 5.3.1. *Let (A, e) be a quasi-hereditary algebra over K which has a left Borel subalgebra B° and all right standard A -modules $\Delta(i)$ are directed (with respect to (A, e)). Then A^δ is generated by A_0^δ and $B^{\circ*}$ and $A^\delta \simeq A_0^\delta \oplus_S B^{\circ*}$ as left A_0^δ -modules.*

Proof. Firstly, left indecomposable projective A_0^δ -modules are given as $A_0^\delta c_i \simeq \text{Hom}_A(\Delta(i), \Delta)$ and $c_j B^{\circ*} c_s \simeq \oplus_k \text{Ext}_{B^\circ}^k(S(s), S(j)) \otimes_{B^\circ} A$, i.e., they are represented by extensions of $\Delta(s)$ by $\Delta(j)$ which are induced from extensions of $S(s)$ by $S(j)$.

Next, consider a left indecomposable projective A^δ -module $A^\delta c_j \simeq \oplus_k \text{Ext}_A^k(\Delta(j), \Delta)$. We will show that each direct summand $\text{Ext}_A^k(\Delta(j), \Delta)$ is a left projective A_0^δ -module. As before, $\text{Ext}_A^k(\Delta(j), \Delta) \simeq \text{Hom}_A(\Omega_{B^\circ}^k(S(j)) \otimes_{B^\circ} A, \Delta)$.

Since Δ_A is a directed A -module, the module $\text{rad} \Omega_{B^\circ}^k(S(j)) \otimes_{B^\circ} A$ is in the kernel of every morphism $f : \Omega_{B^\circ}^k(S(j)) \otimes_{B^\circ} A \rightarrow \Delta_A$. Therefore, every such f factors through the projection $p : \Omega_{B^\circ}^k(S(j)) \otimes_{B^\circ} A \rightarrow (\Omega_{B^\circ}^k(S(j)) / \text{rad} \Omega_{B^\circ}^k(S(j))) \otimes_{B^\circ} A \simeq \oplus_s \Delta(i_s)$. Denote by p_s the canonical projection from $\oplus_s \Delta(i_s)$ to $\Delta(i_s)$. Then

we have a commutative diagram

$$\begin{array}{ccc} (\Omega_{B^{\circ}}^k(S(j))/\text{rad}\Omega_{B^{\circ}}^k(S(j))) \otimes_{B^{\circ}} A & \xleftarrow{p} & \Omega_A^k(\Delta(j)) \\ \downarrow \oplus_s p_s & & \downarrow f \\ \oplus_s \Delta(i_s) & \xrightarrow{\sum_s h_s} & \Delta \end{array},$$

where $h_s : \Delta(i_s) \rightarrow \Delta$ are suitable morphisms. Thus, $f = \sum_s h_s p_s p$, where $p_s p : \Omega_A^k(\Delta(j)) \rightarrow \Delta(i_s)$ are surjective maps which correspond to extensions from $B^{\circ\circ}$ and $h_s \in \text{Hom}_A(\Delta(i_s), \Delta)$. This shows that A^δ is generated by $B^{\circ\circ}$ and A_0^δ . Moreover, $\text{Ext}_A^k(\Delta(j), \Delta) \simeq \oplus_s \text{Hom}_A(\Delta(i_s), \Delta) = \oplus_s A_0^\delta c_{i_s}$ is a left projective A_0^δ -module.

Since $\Omega_{B^{\circ}}^k(S(j))/\text{rad}\Omega_{B^{\circ}}^k(S(j)) \simeq \text{Hom}_{B^{\circ}}(\Omega_{B^{\circ}}^k(S(j)), S) \simeq \oplus_s S(i_s)$ corresponds to elements of $\oplus_s c_{i_s} B^{\circ\circ} c_j$ of degree k , we get $A^\delta c_j = \oplus_k \text{Ext}_A^k(\Delta(j), \Delta) \simeq A_0^\delta \otimes_S B^{\circ\circ} c_j$, hence $A^\delta \simeq A_0^\delta \otimes_S B^{\circ\circ}$ as left A_0^δ -modules. $\square$

Corollary 5.3.2. *Let A be as in Proposition 5.3.1. If $t_l = \dim_K \text{End}_A(S(l))$, then*

$$\begin{aligned} \dim_K \text{Ext}_A^k(\Delta(j), \Delta(i)) &= \sum_{l=1}^n \dim_K \text{Ext}_{B^{\circ}}^k(S(j), S(l)) [\Delta(i) : S(l)] = \\ &= \sum_{l=1}^n t_l [(\Omega_{B^{\circ}}(S(j))/\text{rad}\Omega_{B^{\circ}}(S(j))) : S(l)] [\Delta(i) : S(l)]. \end{aligned}$$

Proof. According to Proposition 5.2.4, the K -dimension of $\text{Ext}_A^k(\Delta(j), \Delta(i))$ equals the K -dimension of $\text{Hom}_A(\Omega^k(\Delta(j)), \Delta(i))$.

Since $\Delta(i)$ is directed as A -module, every A -morphism from $P(l) \rightarrow \Delta(i)$ factors through $\Delta(l)$. Therefore $\dim_{\text{End}_A S(l)} \text{Hom}_A(\Delta(l), \Delta(i)) = [\Delta(i) : S(l)]$.

Using previous considerations, we evaluate $\dim_K(\text{Hom}_A(\Omega^k(\Delta(j)), \Delta(i)))$ as

$$\sum_l t_l \dim_{\text{End}_A(S(l))} \text{Hom}_{B^\bullet}(\Omega_{B^\bullet}(S(j)), S(l)) \dim_{\text{End}_A(S(l))} \text{Hom}_A(\Delta(l), \Delta(i)) =$$

$$\sum_l t_l [(\Omega_{B^\bullet}(S(j))/\text{rad} \Omega_{B^\bullet}(S(j))) : S(l)] [\Delta(i) : S(l)].$$

The claim follows. $\square$

Our present position is similar to the one after section 1.2, where we determined the dimension formula for quasi-hereditary algebra A and a $S - S$ -bimodule decomposition $A \simeq \Delta_A^o \otimes_S \Delta_A$. Later, in Proposition 2.2.2, we have showed that in the presence of Borel subalgebras B^o and B of A , this decomposition is a decomposition of $B^o - B$ -bimodules.

Thus, the following natural question arises: Can we achieve a similar decomposition for standard Ext-algebras? A positive answer to this question is provided in the next proposition.

Instead of left and right standard modules for quasi-hereditary algebra A , the building blocks of A^δ are now A_0^δ and B^{o*} , by Propositions 5.2.2 and 5.3.1.

Theorem 5.3.3. *Let (A, e) be a quasi-hereditary algebra which has a left Borel subalgebra B^o and all right standard A -modules $\Delta(i)$ are directed (with respect to (A, e)). Then A^δ is projective as a right B^{o*} -module and $A^\delta \simeq A_0^\delta \otimes_S B^{o*}$ as $A_0^\delta - B^{o*}$ -bimodules.*

Proof. Due to the associativity of the Yoneda product and Proposition 5.3.2, it is enough to show that A^δ is projective as a right B^{o*} -module.

Any map $f \in \text{Hom}_A(\Omega_A^k(\Delta(j)), \Delta(i))$ is a sum of maps of the form $h_s(p_s p)$, where $h_s : \Delta(i_s) \rightarrow \Delta(i)$ and $p_s p : \Omega_A^k(\Delta(j)) \rightarrow \Delta(i_s)$ is a canonical projection (see the proof of Proposition 5.3.1), which is given as $\tilde{p}_s \otimes_{B^\bullet} id_A$ for some $\tilde{p}_s \in \text{Hom}_{B^\bullet}(\Omega_{B^\bullet}^k(S(j)), S(i_s))$.

Since $\text{Ext}_A^k(\Delta(j), \Delta(i)) \simeq \text{Hom}_A(\Omega_A^k(\Delta(j)), \Delta(i))$, any extension from $\text{Ext}_A^k(\Delta(j), \Delta(i))$ is a sum of extensions of the form $h_s E_s$, where $h_s \in \text{Hom}_A(\Delta(i_s), \Delta(i))$ and the extension $E_s \in \text{Ext}_A^k(\Delta(j), \Delta(i_s))$ corresponds to $p_s p$ as above and is induced by tensoring an extension $\tilde{E}_s \in \text{Ext}_{B^\bullet}^k(S(j), S(i_s))$ with A over $B^\bullet$.

Since $c_i A^\delta = \oplus_k \text{Ext}_A^k(\Delta, \Delta(i))$, every right $B^{\bullet\bullet}$ -module $h_s c_i B^{\bullet\bullet}$ is a direct summand of $c_i A^\delta$. Therefore, $c_i A^\delta \simeq \oplus_{t=1}^n \oplus c_t B^{\bullet\bullet}$, where the summand $c_t B^{\bullet\bullet}$ appears in this sum exactly $\dim_{\text{End}_A S(t)} \text{Hom}_A(\Delta(t), \Delta(i))$ -times. Hence A is a right projective $B^{\bullet\bullet}$ -module. $\square$

The following example illustrates the previous theorem.

Example 5.3.4. Consider a quasi-hereditary algebra A given by its right regular representation

$$A_A = \begin{array}{ccccccc} 1 & \oplus & 2 & & \oplus & 3 & \oplus & 4 \\ \downarrow & & \swarrow & \searrow & & \swarrow & \searrow & \downarrow \\ 2 & & 1 & & 3 & & 2 & 4 \\ \downarrow & & \searrow & \swarrow & & \searrow & \swarrow & 3 \\ 1 & & & 2 & & & 3 & \end{array}$$

Then the left regular representation of its left Borel subalgebra $B^\bullet$ is

$${}_{B^\bullet} B^\bullet = \begin{array}{ccccccc} 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 1 & & 2 & & 3 \end{array}$$

the species of B^{**} is $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$ and the right regular representation of B^{**} is given by

$$B_{B^{**}}^{**} = \begin{array}{cccc} 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\ & & \Downarrow & & \Downarrow & & \Downarrow \\ & & 1 & & 2 & & 3 \\ & & & & \Downarrow & & \Downarrow \\ & & & & 1 & & 2 \\ & & & & & & \Downarrow \\ & & & & & & 1 \end{array}$$

The species of A_0^δ is $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$ and the left regular representation of A_0^δ is

$$A_0^\delta A_0^\delta = \begin{array}{cccc} 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\ \uparrow & & \uparrow & & \uparrow & & \\ 2 & & 3 & & 4 & & \end{array}$$

The left regular representation of A^δ is

$$A^\delta A^\delta = \begin{array}{cccc} 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\ \uparrow \searrow & & \uparrow \searrow & & \uparrow \searrow & & \\ 2 & & 2 & & 3 & & 4 \\ \uparrow \searrow & \uparrow & \uparrow \searrow & \uparrow & & & \\ 3 & & 3 & & 4 & & \\ \uparrow \searrow & \uparrow & & & & & \\ 4 & & 4 & & & & \end{array}$$

We have showed that A^δ decomposes as $A^\delta \simeq A_0^\delta \oplus_S B^{**}$ as $A_0^\delta - B^{**}$ bimodules.

Since both A_0^δ and B^{**} are subalgebras of A^δ , it would be interesting to know when $A^\delta \simeq A_0^\delta \oplus_S B^{**}$ as a tensor product of algebras. Examples of such algebras A will be provided by quasi-hereditary algebras obtained by canonical construction in chapter 7.

5.4 SPECIES OF STANDARD EXT-ALGEBRA

Motivated by the importance of the species for the Yoneda Ext-algebras A^* , we investigate the species of the algebra A^δ .

We will need the following lemma.

Lemma 5.4.1. *Let A be a lean algebra and let*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_A^1(\Delta_A(j)) & \longrightarrow & P_A(j) & \longrightarrow & \Delta_A(j) \longrightarrow 0 \\ & & \downarrow \beta'' & & \downarrow \beta' & & \downarrow \beta \\ 0 & \longrightarrow & \Omega_A^1(\Delta_A(s)) & \longrightarrow & P_A(s) & \longrightarrow & \Delta_A(s) \longrightarrow 0 \end{array}$$

be a commutative diagram, where the image of a map $\beta : \Delta_A(j) \rightarrow \Delta_A(s)$ is not surjective. Then the image of a map $\beta'' : \Omega_A^1(\Delta_A(j)) \rightarrow \Omega_A^1(\Delta_A(s))$ lies in $\text{rad}\Omega_A^1(\Delta_A(s))$.

Proof. Since $\Delta_A(s)$ is a local module, the image of β lies in the $\text{rad}\Delta_A(s)$. Since $P_A(s) \rightarrow \Delta_A(s)$ is a projective cover of $\Delta_A(s)$, we have $\beta'(P_A(j)) \subset \text{rad}P_A(s)$, hence, $\beta'(\text{rad}P_A(j)) \subset \text{rad}^2P_A(s)$. Assume there is an element $x \in \Omega_A^1(\Delta_A(j))$ such that $\beta''(x) \notin \text{rad}\Omega_A^1(\Delta_A(s))$. Because A is lean, the inclusion $\Omega_A^1(\Delta_A(s)) \rightarrow \text{rad}P_A(s)$ is a top embedding and $\beta'(x) \notin \text{rad}^2P_A(s)$. Since $x \in \text{rad}P_A(j)$, we have reached a contradiction. We have thus proved that $\beta''(\Omega_A^1(\Delta_A(j))) \subset \text{rad}\Omega_A^1(\Delta_A(s))$. $\square$

Proposition 5.4.2. *Let A be a lean quasi-hereditary algebra with directed right standard modules and B° be its left Borel subalgebra, which is Koszul. Let $\mathcal{S}(A) = (D_i; {}_iM_j)$, $\mathcal{S}(B^\circ) = (D_i; {}_iN_j)$, $\mathcal{S}(B^{\circ*}) = (D_i; {}_i\hat{N}_j)$ and $\mathcal{S}(A_0^\delta) = (D_i; {}_iH_j)$, where $i, j = 1, \dots, n$ and ${}_iN_j = D_jN_i$ are the species of $A^\delta, B^\circ, B^{\circ*}$ and A_0^δ , respectively. Then ${}_iM_j \simeq {}_iH_j \oplus {}_i\hat{N}_j$, for every $i, j = 1, \dots, n$.*

Proof. Since $\text{rad}A^\delta = \text{rad}A_0^\delta \oplus \bigoplus_{k \geq 1} \text{Ext}_A^k(\Delta, \Delta)$, there is an embedding of ${}_iH_j$ into ${}_iM_j$. Moreover, to prove the assertion of the proposition, it is enough to show that ${}_i\hat{N}_j$ is also embedded into ${}_iM_j$.

We will show first that the extension E_f from $Ext_A^1(\Delta(j), \Delta(i))$, corresponding to a map $f \in Hom_A(\Omega_A^1(\Delta(j)), \Delta(i))$, belongs to $rad^2 A^\delta$ if and only if the image of f lies in $rad \Delta(i)$. If the image of f lies in $rad \Delta(i)$, then $f = \sum_s h_s(p_s p)$ and E_f is a sum of products of elements from $rad A_0^\delta$ and 1-extensions of standard modules, hence it lies in $rad^2 A^\delta$. Conversely, if a 1-extension $E_f \in rad^2 A^\delta$, then E_f is a sum of extensions of the form αE , where $\alpha : \Delta(s) \rightarrow \Delta(i)$ is not surjective and $E \in Ext_A^1(\Delta(j), \Delta(s))$, or of the form $E\beta$, where $\beta : \Delta(j) \rightarrow \Delta(s)$ and $E \in Ext_A^1(\Delta(s), \Delta(i))$.

An extension of the form αE with nonsurjective α corresponds to a map from $\Omega_A^1(\Delta(j))$ to $\Delta(i)$ whose image lies in the $rad \Delta(i)$, because it is a composition of a map from $\Omega_A^1(\Delta(j))$ to $\Delta_A(s)$ with α . Therefore, every such extension belongs to $rad^2 A^\delta$.

For an extension of the form $E\beta$, write $E = \gamma F$, where F_s is a 1-extension given by $0 \rightarrow \Omega_A^1(\Delta(s)) \rightarrow P_A(s) \rightarrow \Delta(s) \rightarrow 0$ and $\gamma : \Omega_A^1(\Delta(s)) \rightarrow \Delta(i)$ is the corresponding map. If F_j is a 1-extension given by $0 \rightarrow \Omega_A^1(\Delta(j)) \rightarrow P_A(j) \rightarrow \Delta(j) \rightarrow 0$, then $E\beta = \gamma F_s \beta = \gamma \beta'' F_j$. If β is not surjective, then by Proposition 5.4.1, the image of β'' lies in $rad \Omega_A^1(\Delta(s))$, hence, the map $\gamma \beta''$ which corresponds to the extension $E\beta$ is not surjective and $E\beta \in rad^2 A^\delta$.

Since B° is a Koszul algebra, we have

$$D_j N_i \simeq {}_i \tilde{N}_j \simeq Ext_{B^\circ}^1(S(j), S(i)) \simeq Hom_{B^\circ}(\Omega_{B^\circ}^1(S(j)), S(i)).$$

The embedding of $\text{Hom}_{B^\bullet}(\Omega_{B^\bullet}^1(S(j)), S(i))$ into

$$\text{Hom}_A(\Omega_A^1(\Delta(j)), \Delta(i)) / \psi^* \text{Hom}_A(\Omega_A^1(\Delta(j)), \text{rad} \Delta(i)),$$

where $\psi : \text{rad} \Delta(i) \rightarrow \Delta(i)$ is the natural inclusion, given via $- \otimes_{B^\bullet} A$, provides an embedding of ${}_i \hat{N}_j$ into ${}_i M_j$. $\square$

Note that while $B^\bullet$ Koszul means that the species of $B^{\bullet*}$ is not bigger than that of $B^\bullet$, the previous proposition describes the situation where the species of $A^\flat$ is not too small.

The next example shows that the species of $A^\flat$ need not include a species of $B^{\bullet*}$, where $B^\bullet$ is a left Borel subalgebra of A , if the assumptions of the last proposition are not satisfied.

Example 5.4.9. Consider a quasi-hereditary algebra A given by its right regular representation

$$A_A = \begin{array}{cccc} 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\ \downarrow & & \downarrow & & & & \downarrow \\ 3 & & 1 & & & & 3 \\ & & \downarrow & & & & \\ & & 3 & & & & \end{array}$$

Then the left regular representation of its left Borel subalgebra $B^\bullet$ is

$${}_B B^\bullet = \begin{array}{cccc} 1 & \oplus & 2 & \oplus & 3 & & \oplus & 4 \\ & & & & \downarrow & \searrow & & \\ & & & & 1 & & 2 & \end{array},$$

the species of $B^{\bullet*}$ is $(1 \leftarrow 3 \Rightarrow 2 \rightarrow 4)$ and the right regular representation of $B^{\bullet*}$ is given by

$$B_{B^{\bullet*}}^{\bullet*} = \begin{array}{cccc} 1 & \oplus & 2 & \oplus & 3 & & \oplus & 4 \\ & & & & \Downarrow & \searrow & & \\ & & & & 2 & & 1 & \end{array}.$$

The species of A_0^δ is $(1 \leftarrow 2 \quad 3 \leftarrow 4)$ and the left regular representation of A_0^δ is

$$A_0^\delta A_0^\delta = \begin{array}{cccc} 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\ \uparrow & & & & \uparrow & & \\ 2 & & & & 4 & & \end{array}.$$

However the species of A^δ is $1 \leftarrow 2 \Leftarrow 3 \leftarrow 4$ and the left regular representation of A^δ is

$$A^\delta A^\delta = \begin{array}{cccc} 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\ \uparrow & & \uparrow & & \uparrow & & \\ 2 & & 3 & & 4 & & \\ \uparrow & & \uparrow & & & & \\ 3 & & 4 & & & & \\ \uparrow & & & & & & \\ 4 & & & & & & \end{array}.$$

In this example the arrow $1 \leftarrow 3$ from the species of B^{o*} decomposes into the product of the arrow $1 \leftarrow 2$ from the species of A_0^δ and the arrow $2 \Leftarrow 3$ from the species of B^{o*} .

In fact,

$$B^{o*} \epsilon_3 \simeq \begin{array}{c} 3 \\ \uparrow \\ 1 \\ \uparrow \\ 2 \end{array}$$

as a left A -module and the right A -module 213 represents the 1-extension of $\Delta(2)$ by $\Delta(3)$. The 1-extension of $\Delta(1)$ by $\Delta(3)$ is represented by a module 13 . The commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & 3 & \longrightarrow & 13 & \longrightarrow & 1 \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 3 & \longrightarrow & 213 & \longrightarrow & 21 \longrightarrow 0 \end{array}$$

shows that the 1-extension 213 of $\Delta(2)$ by $\Delta(3)$ does not belong to the species of A^δ .

CHAPTER VI

BOREL SUBALGEBRAS OF
ALGEBRAS OF SEPARABLE TYPE

The purpose of this chapter is to investigate Borel subalgebras of a quasi-hereditary algebra A of separable type. Since A is of separable type, it can be represented as a factoralgebra of a tensor algebra T . We show that Borel subalgebras of a lean algebra A can be obtained as images of monomial subalgebras $\mathcal{R}$ of T . This leads to a definition of a $\mathcal{R}$ -Borel subalgebra of A .

We investigate an important necessary condition of the existence of $\mathcal{R}$ -Borel subalgebras, and later provide a list of equivalent conditions for their existence.

Finally, we investigate algebras with $\mathcal{R}^o$ - $\mathcal{R}$ -Cartan decomposition and algebras with split standard systems. These results will be used in chapter 7.

6.1 BOREL SUBALGEBRAS REVISITED

The next proposition shows that if a lean algebra A has a Borel subalgebra, then the species of this Borel subalgebra is included in the species of the algebra A .

Proposition 6.1.1. *Let A be a quasi-hereditary algebra with species $\mathcal{S}(A) = (D_i; {}_iM_j, i, j = 1, \dots, n)$ and B be its right Borel subalgebra with species $\mathcal{S}(B) = (D_i; {}_iN_j, i, j = 1, \dots, n)$. Then $\mathcal{S}(B) \subset \mathcal{S}(A)$ if and only if A is right lean. In this case, we have also ${}_iN_j \simeq {}_iM_j$ for all $i > j$.*

Proof. It follows from the definition of B that $S \simeq A/\text{rad}A \simeq B/\text{rad}B = \bigoplus_i D_i$, where S is a maximal semisimple subalgebra of A with a complete set of primitive orthogonal idempotents $\{e_1, \dots, e_n\}$.

The first part of the statement is a reformulation of Lemma 6.4 from [K4].

To finish the proof, we will show that $e_i \text{rad} A_{Ae_j} = e_i \text{rad} B_{Be_j} + e_i \text{rad}^2 A_{Ae_j}$, provided $i > j$. Take $x \in e_i \text{rad} A_{Ae_j}$ and write $x = hd(x) + tl(x)$. Then $hd(x) \in e_i Be_j \subset e_i \text{rad} B_{Be_j}$ and $tl(x) \in e_i A e_{i+1} A e_j \subset e_i \text{rad}^2 A_{Ae_j}$ because $i + 1 > i > j$. $\square$

From now on, we assume that quasi-hereditary algebra A is of separable type, which means that $A/\text{rad}A$ is a separable algebra.

Obviously, any Borel subalgebra B of such an algebra A is also of separable type.

It is well known that any algebra of separable type is a quotient algebra of the tensor algebra over its species.

Definition 6.1.2. Let $\mathcal{S} = (D_i; {}_i M_j, i, j = 1, \dots, n)$ be a species. Then the tensor algebra $T(\mathcal{S})$ over the species $\mathcal{S}$ is defined as

$$T(\mathcal{S}) = \Lambda \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \dots,$$

where $\Lambda = D_1 \times \dots \times D_n$, $M = \bigoplus_{i,j=1}^n {}_i M_j$ is a Λ - Λ -bimodule and the multiplication is induced via the tensor product over Λ .

Note that T need not be a finite dimensional algebra.

The decomposition of Λ into a direct product of division algebras gives a complete set of primitive orthogonal idempotents $e^T = \{e_1^T, \dots, e_n^T\}$ of T .

Each K -space of the form D_i or ${}_i M_{i_2} \otimes \dots \otimes {}_{i_{n-1}} M_{i_n}$, for some finite sequence $(i_1, \dots, i_n)$, is called a monomial of T . A subalgebra $\mathcal{R}$ of T is called monomial if its underlying space is a sum of monomials of T . In particular, the (right) standard module Δ_T can be identified with a monomial algebra whose underlying space is a sum of all monomials D_i and ${}_i M_{i_2} \otimes \dots \otimes {}_{i_{n-1}} M_{i_n}$, where $i_1 = \max\{i_1, \dots, i_n\}$. That monomial algebra will be also denoted by Δ_T .

If a (finite dimensional) algebra A is of separable type and $T_A = T(\mathcal{S}(A))$ is the tensor algebra over the species of A , then there is a surjective algebra morphism $\varphi : T_A \longrightarrow A$ which sends Λ to S (and in particular each $e_i^{T_A}$ to e_i) and its kernel I is contained in $\text{rad}^2 T_A$. Any such algebra morphism will be called *admissible*. The image of M under such φ is isomorphic to $\text{rad} T_A / \text{rad}^2 T_A$. Moreover, any admissible morphism φ is completely determined by the image of M . The ideal $I \supset \text{rad}^k T_A$ for some integer k because A is finite dimensional over K . Using φ , we can linearly order the idempotents $e_i^{T_A}$ in the same way the idempotents e_i are ordered.

Proposition 6.1.3. *Assume A is a quasi-hereditary algebra of separable type which is right lean and/or left lean. Let B be a right Borel subalgebra of A and/or B° be a left Borel subalgebra of A . Then there is a finite dimensional monomial subalgebra $\mathcal{R}$ and/or $\mathcal{R}^\circ$ of T_A and an admissible algebra homomorphism $\varphi :$*

$T_A \rightarrow A$ such that $\varphi(\mathcal{R}) = B$ and/or $\varphi(\mathcal{R}^o) = B^o$.

Proof. Since B and/or B^o are subalgebras of algebra A of separable type, they are also of separable type. Therefore, there is an admissible algebra morphism $\varphi_B : T_B \rightarrow B$ and/or an admissible algebra morphism $\varphi_{B^o} : T_{B^o} \rightarrow B^o$. If A is right lean and/or left lean, then by Proposition 6.1.1 and/or its "left version" the maps φ_B and/or φ^o can be extended to an admissible algebra morphism $\varphi_A : T_A \rightarrow A$ such that the following diagrams

$$\begin{array}{ccc} T_A & \xrightarrow{\varphi_A} & A \\ \uparrow & & \uparrow \\ T_B & \xrightarrow{\varphi_B} & B \end{array}$$

and/or

$$\begin{array}{ccc} T_A & \xrightarrow{\varphi_A} & A \\ \uparrow & & \uparrow \\ T_{B^o} & \xrightarrow{\varphi_{B^o}} & B^o \end{array}$$

commute, where all the vertical maps are inclusions.

The algebras T_B and/or T_{B^o} are finite dimensional and can be identified with a maximal right directed monomial subalgebra $\mathcal{R}$ and/or maximal left directed monomial subalgebra $\mathcal{R}^o$ of T_A .

Then $\varphi_A(\mathcal{R}) = B$ and/or $\varphi_A(\mathcal{R}^o) = B^o$. $\square$

This provides a motivation for an investigation of images of monomial subalgebras $\mathcal{R}$ under algebra morphisms $\varphi_A : T_A \rightarrow A$.

It turns out that even in the case when the species of a Borel subalgebra B is not included in the species of A it is often possible to find some monomial subalgebra

$\mathcal{R}$ of T which is mapped to B under some admissible algebra morphism $\varphi : T \rightarrow A$.

In the next sections, we will look at this situation from a different angle. Assume A is a quasi-hereditary algebra of separable type and we have no knowledge about its Borel subalgebras. We consider an admissible algebra morphism $\varphi : T \rightarrow A$ and a monomial subalgebra $\mathcal{R}$ of T , look at the image $\varphi(\mathcal{R})$ and verify whether it is a Borel subalgebra of A . This way, we can construct Borel subalgebras as images of a monomial subalgebra of T .

6.2. $\mathcal{R}$ -MODULE STRUCTURE OF STANDARD MODULES

In what follows, we will assume that an admissible algebra morphism $\varphi = \varphi_A : T = T_A \rightarrow A$ is fixed. Let $p = p_A$ be the canonical projection $p : A \rightarrow \Delta_A$, and $\psi = \psi_A$ be the composition of φ_A with p_A .

Let R_A be a $\Lambda - \Lambda$ -bimodule which is a sum of all monomials of T which has nonzero image under ψ , and $\mathcal{R}_A$ be a monomial subalgebra of T (actually of Δ_T) generated by R_A . Since right standard A -modules are Schurian, the algebra $\mathcal{R}_A$ is finite dimensional.

If it is possible to construct some Borel subalgebra of A as the image of a monomial subalgebra $\mathcal{R}$ of T under the map φ , then we can do it for some $\mathcal{R} \subset \mathcal{R}_A$.

The next example shows that we might need a proper monomial subalgebra $\mathcal{R}$ of $\mathcal{R}_A$ for this purpose.

Example 6.2.1. Consider the quasi-hereditary algebra A given by the quiver with vertices $1, 2, 3, 4, 5$, arrows $41, 42, 45, 13, 23, 53$ and relation $453 = 413 + 423$. Then

$\mathcal{R}_A$ is spanned by paths 1, 2, 3, 4, 5, 41, 413, 42, 423 and 53. The image of $\mathcal{R}_A$ in A is not a Borel subalgebra of A but the image of $\mathcal{R}$ spanned by paths 1, 2, 3, 4, 5, 41, 413, 42 and 53 is a Borel subalgebra of A .

If $\mathcal{R}$ is a monomial algebra such that $\varphi(\mathcal{R}) = B$ is a Borel subalgebra of A , then $\mathcal{R} \cap \mathcal{R}_A$ is a monomial subalgebra of $\mathcal{R}$ and $\varphi(\mathcal{R} \cap \mathcal{R}_A) = \varphi(\mathcal{R})$. Moreover, every monomial of $\mathcal{R}$ not in $\mathcal{R}_A$ is included in the kernel of the map φ . Therefore, from the point of view of the description of the Borel subalgebra B , there is no difference between taking monomial algebra $\mathcal{R}$ or $\mathcal{R} \cap \mathcal{R}_A$. However, there is a difference in the description of the structure of the algebra A resulting from this. If a Borel subalgebra $B = \varphi(\mathcal{R}_1) = \varphi(\mathcal{R}_2)$ for monomial algebras $\mathcal{R}_1 \subsetneq \mathcal{R}_2$, then the fact that $B = \varphi(\mathcal{R}_2)$ is more valuable than $B = \varphi(\mathcal{R}_1)$ because it gives more information about the kernel of the map φ and consequently about the structure of the algebra A . Therefore, it is desirable to make a monomial algebra $\mathcal{R}$ as big as possible. The case $\mathcal{R} = \Delta_T$ will be of special interest later.

Definition 6.2.2. Let A be a quasi-hereditary algebra, $\varphi : T \rightarrow A$ an admissible algebra morphism from a tensor algebra T to A , and $\mathcal{R}, \mathcal{R}^\circ$ be monomial subalgebras of T . If $\varphi(\mathcal{R}) = B_{\mathcal{R}}$ is a right Borel subalgebra of A , then it is called a right $\mathcal{R}$ -Borel subalgebra of A and if $\varphi(\mathcal{R}^\circ) = B_{\mathcal{R}^\circ}$ is a left Borel subalgebra of A , then it is called a left $\mathcal{R}^\circ$ -Borel subalgebra of A .

By definition, $B_{\mathcal{R}} = \varphi(\mathcal{R})$ is a factoralgebra of $\mathcal{R}$, but in general it is neither a subalgebra nor a factoralgebra of A .

First, we will consider necessary conditions for $B_{\mathcal{R}}$ to be a Borel subalgebra of A which depend solely on the $\mathcal{R}$ -module structure of the standard module Δ_A .

A restriction of $\psi : T \rightarrow \Delta_A$ to $\mathcal{R}$ gives a $\mathcal{R}$ -morphism which will also be denoted by ψ and its kernel by $Z_{\mathcal{R}}$.

A trivial condition for $B_{\mathcal{R}}$ to be a Borel subalgebra of A is that $\psi : \mathcal{R} \rightarrow \Delta_A$ is surjective map. Then $\mathcal{R}/Z_{\mathcal{R}} \simeq \Delta_A$ as right $\mathcal{R}$ -modules. From now on, we will only consider monomial subalgebras $\mathcal{R}$ satisfying this condition.

If $B = B_{\mathcal{R}}$ is a Borel subalgebra of A , then $B \simeq \mathcal{R}/Z_{\mathcal{R}}$ is a factoralgebra of R and $B \simeq \Delta_A$ as $S - \mathcal{R}$ -bimodules. Therefore, $End_{\mathcal{R}}(\Delta_A) \simeq End_{\mathcal{R}}(B) \simeq B \simeq \Delta_A$ as right S -modules. Conditions equivalent to this are collected in the next proposition.

The exact sequence of $\mathcal{R}$ -modules

$$0 \rightarrow Z_{\mathcal{R}} \rightarrow \mathcal{R} \xrightarrow{\psi} \Delta_A \rightarrow 0$$

induces the following exact sequence (+):

$$\begin{aligned} 0 \rightarrow Hom_{\mathcal{R}}(\Delta_A, \Delta_A) &\xrightarrow{\psi^*} Hom_{\mathcal{R}}(\mathcal{R}, \Delta_A) \simeq \Delta_A \rightarrow \\ &\rightarrow Hom_{\mathcal{R}}(Z_{\mathcal{R}}, \Delta_A) \rightarrow Ext_{\mathcal{R}}^1(\Delta_A, \Delta_A) \rightarrow 0. \end{aligned}$$

Proposition 6.2.3. *Let A , $\mathcal{R}$, ψ and $Z_{\mathcal{R}}$ be as before. Then the following conditions are equivalent:*

- (i) $End_{\mathcal{R}}(\Delta_A) \simeq \Delta_A$ as right S -modules.

(ii) The map $\psi^* : \text{End}_{\mathcal{R}}(\Delta_A) \longrightarrow \text{Hom}_{\mathcal{R}}(\mathcal{R}, \Delta_A) \simeq \Delta_A$, which sends g to $g\psi$, is surjective.

(iii) $\text{Hom}_{\mathcal{R}}(Z_{\mathcal{R}}, \Delta_A) \simeq \text{Ext}_{\mathcal{R}}^1(\Delta_A, \Delta_A)$.

(iv) $Z_{\mathcal{R}}$ is a left ideal of $\mathcal{R}$.

(v) $\Delta_A \simeq \mathcal{R}/Z_{\mathcal{R}}$ is a factoralgebra of $\mathcal{R}$.

(vi) $\text{End}_{\mathcal{R}}(\Delta_A) \simeq \Delta_A \simeq \mathcal{R}/Z_{\mathcal{R}}$ as algebras.

(vii) $\text{End}_{\mathcal{R}}(\Delta_A) \simeq \Delta_A$ as right $\mathcal{R}$ -modules.

(viii) $\text{Hom}_{\mathcal{R}}(Z_{\mathcal{R}}, \Delta_A) = 0$.

Proof. (i) $\implies$ (ii) : If $\text{End}_{\mathcal{R}}(\Delta_A) \simeq \Delta_A$ as S -modules, then ψ^* is an isomorphism, hence is a surjective map.

(ii) $\iff$ (iii) : It follows from the exact sequence (+)

(ii) $\implies$ (iv) : If $r \in \mathcal{R}$, then the left multiplication m_r by r is an endomorphism of $\mathcal{R}$. Therefore, $f = \psi m_r \in \text{Hom}_{\mathcal{R}}(\mathcal{R}, \Delta_A)$. By (ii), the map f factors through ψ , hence $0 = f(Z_{\mathcal{R}}) = \psi(rZ_{\mathcal{R}})$. As $rZ_{\mathcal{R}} \subset \mathcal{R}$, we have $rZ_{\mathcal{R}} \subset Z_{\mathcal{R}}$ for each $r \in \mathcal{R}$, hence, $Z_{\mathcal{R}}$ is a left ideal of $\mathcal{R}$.

(iv) $\implies$ (v) : If $Z_{\mathcal{R}}$ is a left ideal of $\mathcal{R}$, then it is a two sided ideal of $\mathcal{R}$ and $\Delta_A \simeq \mathcal{R}/Z_{\mathcal{R}}$ is a factoralgebra of $\mathcal{R}$.

(v) $\implies$ (vi) : If $\Delta_A \simeq \mathcal{R}/Z_{\mathcal{R}}$ is a factoralgebra of $\mathcal{R}$, then $\text{End}_{\mathcal{R}}(\Delta_A) \simeq \text{End}_{\mathcal{R}}(\mathcal{R}/Z_{\mathcal{R}}) \simeq \mathcal{R}/Z_{\mathcal{R}} \simeq \Delta_A$ as algebras.

(vi) $\implies$ (vii) : Since $\Delta_A \simeq \mathcal{R}/Z_{\mathcal{R}}$ as algebras, Δ_A is a left $\mathcal{R}$ module, $\text{End}_{\mathcal{R}}(\Delta_A)$ has a natural structure of a right $\mathcal{R}$ -module and $\text{End}_{\mathcal{R}}(\Delta_A) \simeq \Delta_A$ as

right $\mathcal{R}$ -modules.

(vii) $\implies$ (i) : Clear.

(iii) $\implies$ (viii) : Since (iii) implies (v), $\Delta_A \simeq \mathcal{R}/Z_{\mathcal{R}}$ is a factoralgebra of $\mathcal{R}$ and $\text{Ext}_{\mathcal{R}}^1(\Delta_A, \Delta_A) = 0$. Using exact sequence (+), we infer that (iii) implies $\text{Hom}_{\mathcal{R}}(Z_{\mathcal{R}}, \Delta_A) = 0$.

(viii) $\implies$ (ii) : It follows from exact sequence (+). $\square$

If the previous conditions are satisfied, then $\text{Ext}_{\mathcal{R}}^1(\Delta_A, \Delta_A) = 0$. However this condition is not equivalent to conditions of Proposition 6.2.3 as shown by the following example.

Example 6.2.4. Consider an algebra A given by its right regular representation $A_A =$

and a monomial algebra $\mathcal{R}$ generated by paths $\{1, 2, 3, 4, 5, 31, 42, 53, 531, 534, 5342\}$. Then $\text{End}_{\mathcal{R}}(\Delta_A) \simeq \Delta_A$ as right $\mathcal{R}$ -modules but $\text{End}_{\Delta_T}(\Delta_A) \not\simeq \Delta_A$ as right Δ_T -modules because there is no Δ_T -morphism between right standard modules $\Delta_A(1)$ and $\Delta_A(3)$, sending 1 to 31. Also, $\text{Ext}_{\Delta_T}^1(\Delta_A, \Delta_A) = 0$.

Note that if A_1 and A_2 are two factoralgebras of T such that $\Delta_{A_1} \simeq \Delta_{A_2}$ as right $\mathcal{R}$ -modules and $\psi_2 : \mathcal{R} \rightarrow \Delta_{A_2}$ is a composition of $\psi_1 : \mathcal{R} \rightarrow \Delta_{A_1}$ and this

isomorphism, then A_1 satisfies the conditions of Proposition 6.2.3 if and only if A_2 satisfies them. Thus, the conditions of Proposition 6.2.3 depend only on the $\mathcal{R}$ -module structure of Δ_A and not on the particular structure of A .

The above proposition is especially valuable in the case when A is right lean because we can take $\mathcal{R} = \Delta_T$ and using discussion in section 6.1 we get the following sufficient condition for the existence of a Borel subalgebra of A : If A is right lean algebra possessing a Borel subalgebra, then any Δ_T -morphism from $\Delta_T(i)$ to $\Delta_A(j)$ factors through $\Delta_A(i)$. This is related to canonical construction of quasi-hereditary algebras to be presented in chapter 7, compare also with [DM].

Remark 6.2.5. Proposition 6.2.3 for $\mathcal{R} = T$ provides characterization of algebras A for which Δ_A is a factoralgebra of T (or equivalently Δ_A is a factoralgebra of A). Even if A is not of separable type, we can characterize when Δ_A is a factoralgebra of A by substituting $\mathcal{R}$ by A , φ by $p : A \rightarrow \Delta_A$ and $Z_{\mathcal{R}}$ by $\ker p$ and making obvious modifications in the statement and the proof of Proposition 6.2.3. Since we will not need it in what follows, we leave it as an easy exercise.

6.3. $\mathcal{R}$ -BOREL SUBALGEBRAS

Proposition 6.3.1. *Let $A = T/I$, $\mathcal{R}$, φ , ψ , $B_{\mathcal{R}}$ and $Z_{\mathcal{R}}$ be as before. Then the following conditions are equivalent:*

- (i) $B_{\mathcal{R}} \simeq \Delta_A$ as right S -modules.
- (ii) $Z_{\mathcal{R}} \subset I$.
- (iii) The map $\psi^* : \text{Hom}_{\mathcal{R}}(\Delta_A, A) \rightarrow \text{Hom}_{\mathcal{R}}(\mathcal{R}, A) \simeq A$, which sends g to $g\psi$,

is surjective.

(iv) The canonical projection $p : A \longrightarrow \Delta_A$, viewed as a $\mathcal{R}$ -morphism, splits by a map s which satisfies $\varphi = s\psi$.

(v) $A \simeq \Delta_A^o \otimes_S \Delta_A$ as right $\mathcal{R}$ -modules.

(vi) $\text{Hom}_{\mathcal{R}}(Z_{\mathcal{R}}, A) \simeq \text{Ext}_{\mathcal{R}}^1(\Delta_A, A)$.

(vii) $B_{\mathcal{R}} \simeq \text{End}_{\mathcal{R}}(\Delta_A) \simeq \Delta_A$ as right $\mathcal{R}$ -modules.

(viii) $B_{\mathcal{R}}$ is a right $\mathcal{R}$ -Borel subalgebra of A .

(ix) $B_{\mathcal{R}} \simeq \mathcal{R}/Z_{\mathcal{R}} \simeq \text{End}_A(\Delta_A)$ as algebras.

Proof. (i) $\Rightarrow$ (ii) : If $B_{\mathcal{R}} \simeq \Delta_A$ as right S -modules, then the kernel of $\varphi : \mathcal{R} \longrightarrow A$ is $Z_{\mathcal{R}}$.

(ii) $\Rightarrow$ (iii) : Since A assumes the structure of $\mathcal{R}$ -module via the map $\varphi : \mathcal{R} \rightarrow A$, the module $Z_{\mathcal{R}}$ lies in the kernel of every map from $\text{Hom}_{\mathcal{R}}(\mathcal{R}, A)$. Since $\mathcal{R}/Z_{\mathcal{R}} \simeq \Delta_A$ as right $\mathcal{R}$ -modules, we conclude that the map $\psi^* : \text{Hom}_{\mathcal{R}}(\Delta_A, A) \rightarrow \text{Hom}_{\mathcal{R}}(\mathcal{R}, A)$ is surjective.

(iii) $\Rightarrow$ (iv) : Take $\varphi \in \text{Hom}_{\mathcal{R}}(\mathcal{R}, A)$ in (iii). The corresponding map $s : \Delta_A \rightarrow A$ satisfies $\varphi = s\psi$. Hence, $\psi = p\varphi = ps\psi$. Since ψ is surjective, we get $ps = id_{\Delta_A}$, thus, s splits p as $\mathcal{R}$ -morphism.

(iv) $\Rightarrow$ (v) : It is enough to prove that $A\varepsilon_i A \simeq \bigoplus \Delta_A(i) \oplus A\varepsilon_{i+1} A$ as right $\mathcal{R}$ -modules, for each $i = 1, \dots, n$. The canonical projection $h : A\varepsilon_i A \longrightarrow A\varepsilon_i A/A\varepsilon_{i+1} A$ is a surjective morphism of right $\mathcal{R}$ -modules and its image is isomorphic to $\bigoplus \Delta_A(i)$ as right $\mathcal{R}$ -modules. By (iv), there is a splitting $\tilde{s}$ of a canonical

map $\tilde{p}: \bigoplus e_i A \rightarrow \bigoplus \Delta_A(v)$. Since $\tilde{p}$ factors through h as $\tilde{p} = hq$ for some q , the map $g\tilde{s}$ provides a splitting of h .

(v) $\Rightarrow$ (iv): Clear.

(iv) $\Rightarrow$ (i): Since $\varphi = s\psi$, we have $\varphi(Z_R) = s\psi(Z_R) = 0$, hence, $Z_R \in I$.

(iii) $\Leftrightarrow$ (vi): It follows from the exact sequence

$$\begin{aligned} 0 \rightarrow \operatorname{Hom}_R(\Delta_A, A) &\xrightarrow{\psi^*} \operatorname{Hom}_R(R, A) \simeq A \rightarrow \\ &\rightarrow \operatorname{Hom}_R(Z_R, A) \rightarrow \operatorname{Ext}_R^1(\Delta_A, A) \rightarrow 0. \end{aligned}$$

(ii) $\Rightarrow$ (vi): The assumption $Z_R \in I$ implies $RZ_R \in I$, hence, by Proposition 6.2.3 $\operatorname{End}_R(\Delta_A) \simeq \Delta_A$ as right R -modules. Since $\varphi(R) = B_R$ and $Z_R \in I$, the restriction of φ to R factors through R/Z_R . Since the restriction of ψ to E , which is the composition of φ and p , is surjective, $R/Z_R \simeq \Delta_A$ implies that it gives an isomorphism $B_R \simeq \Delta_A$ of right R -modules.

(vi) $\Rightarrow$ (vii): By Proposition 6.2.3, R/Z_R is a factoralgebra of R . Since $B_R \simeq \Delta_A \simeq R/Z_R$ as right R -modules, we get that algebra morphism $\varphi: R \rightarrow A$ factors through canonical projection from R to its factoralgebra R/Z_R . The induced map $\bar{\varphi}: R/Z_R \rightarrow B_R$ is an algebra isomorphism, too. Hence, $B_R = \bar{\varphi}(R/Z_R)$ is a subalgebra of A . Since $B_R \simeq R/Z_R$, then $B_R \simeq \Delta_A$ as right R -modules. Hence, B_R is a right R -Borel subalgebra.

(vii) $\Rightarrow$ (ix): If B_R is a R -Borel subalgebra, then $B_R \simeq R/Z_R$ as algebras and B_R satisfies conditions of Proposition 6.2.3, hence, $R/Z_R \simeq \operatorname{End}_R(\Delta_A)$ as algebras.

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(ix) $\implies$ (i) : Comparing dimensions over K , we see $\dim_K \text{End}_{\mathcal{R}}(\Delta_A) \leq \dim_K \Delta_A \leq \dim_K B_{\mathcal{R}}$. Hence, $B_{\mathcal{R}} \simeq \text{End}_{\mathcal{R}}(\Delta_A)$ implies $B_{\mathcal{R}} \simeq \text{End}_{\mathcal{R}}(\Delta_A) \simeq \Delta_A$ as right S -modules. $\square$

Let us show an example of an algebra A possessing a $\mathcal{R}$ -Borel subalgebra.

Example 6.3.2. Consider the algebra A given as

$$A_A = \begin{array}{c} 1 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array} \oplus \begin{array}{ccc} & 2 & \\ \swarrow & & \searrow \\ 1 & & 3 \\ \searrow & & \swarrow \\ & 2 & \end{array} \oplus \begin{array}{c} 3 \\ \downarrow \\ 2 \end{array}$$

and a monomial subalgebra $\mathcal{R} = \langle 1, 2, 21, 3, 32 \rangle$. Then $B_{\mathcal{R}}$ is a $\mathcal{R}$ -Borel subalgebra of A .

We have seen that if A satisfies the conditions of Proposition 6.3.1, then Δ_A is a factoralgebra of $\mathcal{R}$, hence, by Proposition 6.2.3, we get $\text{End}_{\mathcal{R}}(\Delta_A) \simeq \Delta_A$ as algebras and right $\mathcal{R}$ -modules.

The conditions of Proposition 6.2.3 are weaker than the conditions of Proposition 6.3.1 as the following algebra and a choice $\mathcal{R} = \mathcal{R}_A$ shows:

Example 6.3.3.

$$A_A = \begin{array}{c} 1 \\ \downarrow \\ 1 \end{array} \oplus \begin{array}{ccc} & 2 & \\ \swarrow & & \searrow \\ 2 & & 4 \\ \searrow & & \swarrow \\ & 1 & \end{array} \oplus \begin{array}{c} 3 \\ \downarrow \\ 1 \end{array}$$

6.4. CARTAN DECOMPOSITION

Algebras associated with regular blocks of the category $\mathcal{O}$ and classical Schur algebras [K1], endomorphism algebras of [DHM], dual extensions of algebras of

$[X]$, $[DX]$ and double incidence algebras of [K5] possess both left and right Borel subalgebras, hence they have a Cartan decomposition.

For monomial algebras $\mathcal{R}$ and $\mathcal{R}^\circ$, we can define $\mathcal{R}^\circ - \mathcal{R}$ -decomposition as follows.

Definition 6.4.1. A quasi-hereditary algebra A is said to have $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition if $A \simeq \text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \bigotimes_S \text{End}_{\mathcal{R}}(\Delta_A)$ as $\mathcal{R}^\circ - \mathcal{R}$ -bimodules.

A few explanatory words are in place. Since $\dim_K \text{End}_{\mathcal{R}}(\Delta_A) \leq \dim_K \Delta_A$ and $\dim_K \text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \leq \dim_K \Delta_A^\circ$, if A has $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition, then $\text{End}_{\mathcal{R}}(\Delta_A) \simeq \Delta_A$ as right S -modules and $\text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \simeq \Delta_A^\circ$ as left S -modules, by Proposition 1.2.1. Using decompositions $\text{End}_{\mathcal{R}}(\Delta_A) \simeq \bigoplus_{i=1}^n \text{Hom}_{\mathcal{R}}(\Delta_A, \Delta_A(i)) \simeq \bigoplus_{i=1}^n \Delta_A(i)$, $\text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \simeq \bigoplus_{i=1}^n \text{Hom}_{\mathcal{R}^\circ}(\Delta_A^\circ(i), \Delta_A) \simeq \bigoplus_{i=1}^n \Delta_A^\circ(i)$ and the fact that all modules $\Delta_A(i)$ and $\Delta_A^\circ(i)$ are local, we check that the tensor product $\text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \otimes_S \text{End}_{\mathcal{R}}(\Delta_A)$ is well defined. Finally, by Proposition 6.2.3, $\text{End}_{\mathcal{R}}(\Delta_A) \simeq \mathcal{R}/Z_{\mathcal{R}}$ has a structure of a right $\mathcal{R}$ -module and similarly $\text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \simeq \mathcal{R}^\circ/Z_{\mathcal{R}^\circ}$ has a structure of a left $\mathcal{R}^\circ$ -module.

Example 6.4.2. An algebra A given by

has a $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition, where $\mathcal{R}^\circ = (1, 2, 3, 13, 4, 24)$ and $\mathcal{R} =$

(1, 2, 21, 3, 31, 4, 43, 431). In this case, $A \simeq \mathcal{R}^\circ \otimes \mathcal{R} \simeq \text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \otimes \text{End}_{\mathcal{R}}(\Delta_A)$ as $\mathcal{R}^\circ - \mathcal{R}$ -bimodules.

If an algebra A has a $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition, then $A \otimes_{\mathcal{R}} S^\circ(i) \simeq \Delta_A^\circ(i)$ as left $\mathcal{R}^\circ$ -modules and similarly $S(i) \otimes_{\mathcal{R}^\circ} A \simeq \Delta_A(i)$ as right $\mathcal{R}$ -modules. Really, since $A \simeq \text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \otimes_S \text{End}_{\mathcal{R}}(\Delta_A)$ as $\mathcal{R}^\circ - \mathcal{R}$ -bimodules, we get $A \otimes_{\mathcal{R}} S^\circ(i) \simeq (\text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \otimes_S \text{End}_{\mathcal{R}}(\Delta_A)) \otimes_{\mathcal{R}} S^\circ(i) \simeq \text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \otimes_S (\text{End}_{\mathcal{R}}(\Delta_A) \otimes_{\mathcal{R}} S^\circ(i)) \simeq \text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \otimes_S S^\circ(i) \simeq \text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \epsilon_i \simeq \Delta_A^\circ(i)$ as left $\mathcal{R}^\circ$ -modules.

This property is not surprising in view of the following result.

Proposition 6.4.3. *Assume A is a quasi-hereditary algebra of separable type, $\mathcal{R}$ and $\mathcal{R}^\circ$ are monomial subalgebras of T_A . Then the following are equivalent:*

- (i) *A has a $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition*
- (ii) *A has a right $\mathcal{R}$ -Borel subalgebra $B_{\mathcal{R}}$ and a left $\mathcal{R}^\circ$ -Borel subalgebra $B_{\mathcal{R}^\circ}$.*

Proof. (i) $\implies$ (ii) : We have already explained that if A has a $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition, then A satisfies conditions of Proposition 6.2.3, hence $\text{End}_{\mathcal{R}}(\Delta_A) \sim \mathcal{R}/Z_{\mathcal{R}}$ as right $\mathcal{R}$ -modules and $\text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \simeq \mathcal{R}^\circ/Z_{\mathcal{R}^\circ}$ as left $\mathcal{R}^\circ$ -modules. Since $A \simeq \text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \otimes_S \text{End}_{\mathcal{R}}(\Delta_A)$ as $\mathcal{R}^\circ - \mathcal{R}$ -bimodules, this implies that both $Z_{\mathcal{R}}$ and $Z_{\mathcal{R}^\circ}$ are included in the ideal $I = \ker \varphi$. Thus, by Proposition 6.3.1, $B_{\mathcal{R}}$ is a right $\mathcal{R}$ -Borel subalgebra of A and similarly $B_{\mathcal{R}^\circ}$ is a left $\mathcal{R}^\circ$ -Borel subalgebra of A .

(ii) $\implies$ (i) : If $B_{\mathcal{R}}$ is a right $\mathcal{R}$ -Borel subalgebra of A and $B_{\mathcal{R}^\circ}$ is a left $\mathcal{R}^\circ$ -Borel subalgebra of A , then by Proposition 6.3.1, $A \simeq \Delta_A^\circ \otimes_S \Delta_A$ as right $\mathcal{R}$ -modules

and left $\mathcal{R}^\circ$ -modules. Using Proposition 6.2.3, we have also $\text{End}_{\mathcal{R}}(\Delta_A) \simeq \Delta_A$ as right $\mathcal{R}$ -modules and $\text{End}_{\mathcal{R}^\circ}(\Delta_A^\circ) \simeq \Delta_A^\circ$ as left $\mathcal{R}^\circ$ -modules. Since the structure of A as a right $\mathcal{R}$ -module and a left $\mathcal{R}^\circ$ -module is given via the map $\varphi : T \rightarrow A$, the claim follows. $\square$

Note that the condition (ii) can be expressed as $Z_{\mathcal{R}^\circ} \cup Z_{\mathcal{R}} \subset I$.

To conclude this section, let us make the following observation. Assume an algebra A is such that Δ_A is its factoralgebra and A contains a right Borel subalgebra B . Then A is right lean and, by Proposition 6.1.3, we conclude that B is a right Δ_T -Borel subalgebra of A . Moreover, since right standard A -module Δ_A is directed, all monomials from T not included in Δ_T lie in the kernel of the map ψ . Therefore, $B \simeq \Delta_A$ as right A -modules.

On the other hand, if A is right lean and has a Borel subalgebra, then by proposition 6.1.3, we get that B is a factoralgebra of Δ_T , hence all the conditions of Proposition 6.2.3 are satisfied for $\mathcal{R} = \Delta_T$. This again underlines the importance of the special case $\mathcal{R} = \Delta_T$.

Our main interest in the next chapter will be to describe algebras with right Δ_T -Borel subalgebra B and left Δ_T° -Borel subalgebra B° for which $B \simeq \Delta_A$ as right A -modules and $B^\circ \simeq \Delta_A^\circ$ as left A -modules.

Definition 6.4.4. *Let A be an algebra of separable type and Δ_A and Δ_A° be its standard modules. Then we say that the standard system $(\Delta_A^\circ, \Delta_A)$ is split if the following conditions (*) are satisfied:*

- (i) Δ_A^0 is a factoralgebra of Δ_T^0 and Δ_A is a factoralgebra of Δ_T ,
- (ii) Δ_A^0 is a left Δ_T -module and Δ_A is a right Δ_T^0 -module,
- (iii) every morphism of right Δ_T^0 -modules from Δ_T^0 to Δ_A factors through $\psi^0 : \Delta_T^0 \rightarrow \Delta_A^0$ and every morphism of left Δ_T -modules from Δ_T to Δ_A^0 factors through $\psi : \Delta_T \rightarrow \Delta_A$.

Note that (i) implies that the map $\psi^0 : \Delta_T^0 \rightarrow \Delta_A^0$ is a morphism of right Δ_T^0 -modules and the map $\psi : \Delta_T \rightarrow \Delta_A$ is a morphism of left Δ_T -modules.

Using previous notation we can reformulate the above conditions as follows: $Z_{\Delta_T^0}$ is a right Δ_T^0 -module, Z_{Δ_T} is a left Δ_T -module and $Z_{\Delta_T^0} \cup Z_{\Delta_T}$ is a $\Delta_T - \Delta_T^0$ -bimodule.

This can be also formulated in the language of vector spaces, for $i, j \in \{1, \dots, n\}$, as

- (i) $Z_{\Delta_T^0} e_i \Delta_T^0 e_j \subset Z_{\Delta_T^0} e_j, e_j \Delta_T e_i Z_{\Delta_T} \subset e_j Z_{\Delta_T},$
- (ii) $e_j \Delta_T Z_{\Delta_T^0} e_i \subset Z_{\Delta_T^0} e_i, e_i Z_{\Delta_T} \Delta_T^0 e_j \subset e_i Z_{\Delta_T}$ if $j \leq i$ and
- (iii) $e_j \Delta_T Z_{\Delta_T^0} e_i \subset e_j Z_{\Delta_T}, e_i Z_{\Delta_T} \Delta_T^0 e_j \subset Z_{\Delta_T^0} e_j$ if $j \geq i.$

The concept of a split standard system describes the mutual relationship between right, left and bimodule structures in an algebra A with $\Delta_T^0 - \Delta_T$ -Cartan decomposition.

Proposition 6.4.5. *If A has a $\Delta_T^0 - \Delta_T$ -Cartan decomposition, then the standard system (Δ_A^0, Δ_A) is split.*

Proof. The property (i) of Definition 6.4.4 follows from Propositions 6.2.3, 6.3.1

and their left versions. For the properties (ii) and (iii), take into account that the actions of Δ_T and Δ_T^o on the algebra A are induced by the map $\varphi : T \rightarrow A$. If $x \in e_i Z_{\Delta_T}$ and $y \in \Delta_T^o e_j$, then $xy \in e_i \Delta_T$ if $i \geq j$ and $xy \in \Delta_T^o e_j$ otherwise. Since $\varphi(x) = 0$ and $\varphi(xy) = \varphi(x)\varphi(y) = 0$, we conclude that $xy \in e_i Z_{\Delta_T}$ or $xy \in Z_{\Delta_T^o} e_j$. Correspondingly, we can prove that $e_j \Delta_T Z_{\Delta_T^o} e_i \subset Z_{\Delta_T^o} e_i$ if $i \geq j$ and $e_j \Delta_T Z_{\Delta_T^o} e_i \subset e_j Z_{\Delta_T}$ otherwise. Hence, properties (ii) and (iii) follow. $\square$

Remark 6.4.6. In contrast to Proposition 6.4.5, it is possible for an algebra A that condition (i) of Definition 6.4.4 is satisfied but the standard system (Δ_A^o, Δ_A) is not split. To see this, take algebra A given below.

Define a morphism f of right Δ_T -modules from Δ_T to $\Delta_A^o(5)$ by $f(1) = f(21) = f(41) = f(321) = 15$, $f(2) = f(32) = 215$, $f(3) = 3215$, $f(4) = 415$ and $f(5) = 5$. Then f does not factor through Δ_A .

Note that no algebra with a $\Delta_T^o - \Delta_T$ -Cartan decomposition can have standard modules isomorphic to standard modules of this algebra.

If the standard system (Δ_A^0, Δ_A) is split, then $J = \Delta_T^0 Z_{\Delta_T} + Z_{\Delta_T} \Delta_T$ is a two-sided T -ideal. In the next chapter, we will see that T/J has a $\Delta_T^0 = \Delta_T$ -Cartan decomposition and algebras A and T/J are related via the $K[x]$ deformations fixing standard modules (hence also the species).

CHAPTER VII

CANONICAL LEVEL-SPLIT ALGEBRAS

We start this last chapter by characterizing algebras with $\Delta_T^o - \Delta_T$ -Cartan decomposition. Following this, we explain the construction of a canonical level-split algebra from a given split standard system.

Then we show how an important class of quasi-hereditary algebras (which includes algebras appearing in the applications) is obtained via a deformation of canonical level-split algebras which fixes standard modules.

Finally, we state an interesting property of an extension algebra of standard modules of a canonical level-split algebra.

7.1 LEVEL-SPLIT ALGEBRAS

To start, we introduce level-split algebras and show that they coincide with algebras having a $\Delta_T^o - \Delta_T$ -Cartan decomposition.

Definition 7.1.1. *Define the level of a monomial $j_1 M_{j_2} \otimes_{j_2} M_{j_3} \otimes \cdots \otimes_{j_{k-1}} M_{j_k}$ to be $i = \max(j_1, \dots, j_k)$ and the level of D_i to be i . Also, for every $i = 1, \dots, n$, denote by L_i the sum of all monomials of level i in T .*

Note that $L_i = \Delta_T^o \Delta_T$. Clearly, L_i is a direct complement to $T\varepsilon_{i+1}T$ in $T\varepsilon_i T$. Obviously, each element $t \in T$ can be uniquely written in the form $t = \sum_{i=1}^n t_i$, where $t_i \in L_i$.

For $1 \leq k \leq n$, let $\mathcal{S}_k$ be the restriction of the species $\mathcal{S}$ given by $\mathcal{S}_k = (D_i, {}_iM_j; 1 \leq i, j \leq k)$ and $T_k = T(\mathcal{S}_k)$ be the corresponding tensor algebra. Then T_k is embedded naturally in T and its image coincides with the $\Lambda - \Lambda$ -bimodule $\sum_{i=1}^k L_i$, which is the sum of all monomials of level at most k . Thus, the bimodules L_i appear as factors in the natural filtration of T : $0 \subset T_1 \subset T_2 \subset \cdots \subset T_n = T$.

Define by V_A the kernel of the map $\psi : T \rightarrow \Delta_A$, by V_A^o the kernel of the map $\psi^o : T \rightarrow \Delta_A^o$ and put $\epsilon_i V_A = V_A(i)$, $V_A^o(i) = V_A^o \epsilon_i$, $Z_{\Delta_T} = Z_A$, $\epsilon_i Z_{\Delta_T} = Z_A(i)$, $Z_{\Delta_T^o} = Z_A^o$ and $Z_A^o(i) = Z_A^o \epsilon_i$, for every $i = 1, \dots, n$.

Definition 7.1.2. *A two sided (admissible) ideal I of T is called level-split if and only if $I = \sum_{i=1}^n (I \cap L_i)$. If I is level-split, then the factoralgebra $A = T/I$ is called T -level-split.*

Note that I is level-split if and only if for each $t \in I$ each component t_i of level i in the decomposition $t = \sum_{i=1}^n t_i$, $t_i \in L_i$ also belongs to I . Intuitively, we regard level-split algebras as factoralgebras of monomial algebras obtained by “gluing” together some elements of the same level.

Proposition 7.1.3. *Let $A = T/I$. Then the following conditions are equivalent:*

- (i) *A has a $\Delta_T^o - \Delta_T$ -Cartan decomposition.*
- (ii) *The standard system (Δ_A^o, Δ_A) is split and $I = Z_A^o \Delta_T + \Delta_T^o Z_A$ is a level split ideal of T .*

Proof. (i) $\implies$ (ii) : If $A \simeq \Delta_A^o \otimes_S \Delta_A$ as $\Delta_T^o - \Delta_T$ -bimodules, then (Δ_A^o, Δ_A) is split, by Proposition 6.4.5, and $\tilde{I} = Z_A^o \Delta_T + \Delta_T^o Z_A \subset I$. Since (Δ_A^o, Δ_A) is split,

the set $\tilde{I}$ is a two-sided ideal of T , which is clearly level-split.

We will prove that the algebra $\tilde{A} = T/\tilde{I}$ satisfies $\Delta_{\tilde{A}} \simeq \Delta_A$ as right T -modules, $\Delta_{\tilde{A}}^o \simeq \Delta_A^o$ as left T -modules and $\tilde{A}\varepsilon_i\tilde{A}/\tilde{A}\varepsilon_{i+1}\tilde{A} \simeq \Delta_A^o(i) \bigotimes_S \Delta_A(i)$ as $T - T$ -bimodules. Then, by comparing the dimensions over K , we will conclude that $I = \tilde{I}$ and $A = \tilde{A}$. Since $V_{\tilde{A}}(i) = \epsilon_i \Delta_T^o Z_A + \epsilon_i Z_A^o \Delta_T + \epsilon_i T \varepsilon_{i+1} T$, $\epsilon_i \Delta_T^o Z_A \subset Z_A(i) + \epsilon_i T \varepsilon_{i+1} T$ and $\epsilon_i Z_A^o \Delta_T \subset \epsilon_i T \varepsilon_{i+1} T$, we conclude that $V_{\tilde{A}}(i) \subset V_A(i)$.

On the other hand, $V_A(i) \subset \epsilon_i Z_A + \epsilon_i T \varepsilon_{i+1} T \subset V_{\tilde{A}}(i)$. Thus, $V_{\tilde{A}}(i) = V_A(i)$ and similarly $V_{\tilde{A}}^o(i) = V_A^o(i)$, for every $i = 1, \dots, n$. Therefore, $\Delta_{\tilde{A}}(i) \simeq \Delta_A(i)$ as right T -modules and $\Delta_{\tilde{A}}^o(i) \simeq \Delta_A^o(i)$ as left T -modules.

Since $T\epsilon_i T \cap \tilde{I} = Z_A^o(i) \Delta_T + \Delta_T^o Z_A(i)$, we have

$$\tilde{A}\varepsilon_i\tilde{A} \simeq T\varepsilon_i T / \left(\sum_{j=1}^n Z_A^o(j) \Delta_T + \Delta_T^o Z_A(j) \right)$$

and

$$\begin{aligned} \tilde{A}\varepsilon_i\tilde{A}/\tilde{A}\varepsilon_{i+1}\tilde{A} &\simeq T\varepsilon_i T / (T\varepsilon_{i+1} T \div (Z_A^o(i) \Delta_T + \Delta_T^o Z_A(i))) \simeq \\ &\Delta_T^o \epsilon_i \Delta_T / (Z_A^o(i) \Delta_T + \Delta_T^o Z_A(i)) \end{aligned}$$

as $T - T$ -bimodules.

However,

$$\begin{aligned} \Delta_T^o \epsilon_i \Delta_T / (Z_A^o(i) \Delta_T + \Delta_T^o Z_A(i)) &\simeq (\Delta_T^o(i) / Z_A^o(i)) \bigotimes_S (\Delta_T(i) / Z_A(i)) \simeq \\ &\Delta_A^o(i) \bigotimes_S \Delta_A(i) \end{aligned}$$

as $T - T$ -bimodules, where the first isomorphism is given by tensor product multiplication.

(ii) $\implies$ (i) : Since (Δ_A^o, Δ_A) is split, the argument in the first part of the proof gives that $A \cong \Delta_A^o \otimes_S \Delta_A$. Since $Z_A^o \cup Z_A \subset I$, A has a $\Delta_T^o = \Delta_T$ Cartan decomposition, by Proposition 6.4.3. $\square$

Note that if for an algebra $A \cong T/I$ the standard system (Δ_A^o, Δ_A) is split, then A need not be level-split, as seen from Example 6.3.2.

7.2 SPLIT STANDARD SYSTEMS

Next, we will try to express the properties of standard systems of an algebra which distinguish them from other modules. We have seen that standard systems of algebras with $\Delta_T^o = \Delta_A$ Cartan decomposition satisfy some additional properties, namely, they are split.

In this section, we assume that $\mathcal{S}$ is a species and $T = T(\mathcal{S})$ is the corresponding tensor algebra.

Definition 7.2.1. An ordered pair $(D^o, D) = (\bigoplus_{i=1}^n D^o(i), \bigoplus_{i=1}^n D(i))$ of left T module D^o and right T -module D (finite dimensional over K) is called standard T -system if

- for each $i = 1, \dots, n$, there exists an exact sequence

$$0 \rightarrow V(i) \rightarrow e_i T \rightarrow D(i) \rightarrow 0,$$

where $e_i \text{rad}^2 T \supset V(i)$ and $V(i) \supset e_i T e_{i+1} T$.

- for each $i = 1, \dots, n$, there exists an exact sequence

$$0 \rightarrow V^o(i) \rightarrow T e_i \rightarrow D^o(i) \rightarrow 0,$$

where $\text{rad}^2 T e_i \supset V^o(i)$ and $V^o(i) \supset T e_{i+1} T e_i$.

The standard T -system (D^o, D) is called *Schurian* if all $\text{End}_T D(i), \text{End}_T D^o(i)$ are division algebras or equivalently if $V(i) \supset e_i \text{rad } T e_i$ and $V^o(i) \supset e_i \text{rad } T e_i$.

We say that the standard T -system (D^o, D) is *split* if the following conditions are satisfied:

- (i) D^o is a factoralgebra of Δ_T^o and D is a factoralgebra of Δ_T ,
- (ii) D^o is a left Δ_T -module and D is a right Δ_T^o -module,
- (iii) every morphism of right Δ_T^o -modules from Δ_T^o to D factors through the projection from Δ_T^o to D^o , and every morphism of left Δ_T -modules from Δ_T to D^o factors through the projection from Δ_T to D .

For each $i = 1, \dots, n$, we define $Z(i) = V(i) \cap L_1$, $Z^o(i) = V^o(i) \cap L_1$, $Z = \bigoplus_{i=1}^n Z(i)$ and $Z^o = \bigoplus_{i=1}^n Z^o(i)$.

Let us note again that the condition (i) in the definition of a split standard system implies that the projection from Δ_T^o to D^o is a morphism of right Δ_T^o -modules and the projection from Δ_T to D is a morphism of left Δ_T -modules.

If (D^o, D) is a standard T -system, then for each $i = 1, \dots, n$, there exists an exact sequence

$$0 \rightarrow Z(i) \rightarrow e_i \Delta_T \rightarrow D(i) \rightarrow 0,$$

where $e_i \text{rad}^2 T \supset V(i)$, and for each $i = 1, \dots, n$, there exists an exact sequence

$$0 \rightarrow Z^o(i) \rightarrow \Delta_T^o e_i \rightarrow D^o(i) \rightarrow 0,$$

where $\text{rad}^2 T e_i \supset V^o(i)$.

Also a standard T -system (D°, D) is split if and only if Z° is a $\Delta_T^\circ - \Delta_T^\circ$ -bimodule, Z is a $\Delta_T - \Delta_T$ -bimodule and $Z^\circ \cup Z$ is a $\Delta_T^\circ - \Delta_T$ -bimodule. In other words,

$$Z^\circ(i)\Delta_T^\circ e_j \subset Z^\circ(j), e_j \Delta_T Z(i) \subset Z(j),$$

$$e_j \Delta_T Z^\circ(i) \subset Z^\circ(i) \text{ and } Z(i)\Delta_T^\circ e_j \subset Z(i) \text{ if } j < i \text{ and}$$

$$e_j \Delta_T Z^\circ(i) \subset Z(j) \text{ and } Z(i)\Delta_T^\circ e_j \subset Z^\circ(j) \text{ if } j > i.$$

Proposition 7.2.2. *Let (D°, D) be a standard T -system such that both D° and D are factoralgebras of T . Then (D°, D) is split standard system. Moreover, every morphism of right Δ_T° -modules from Δ_T° to D and every morphism of left Δ_T -modules from Δ_T to D° factors through the semisimple subalgebra S of T .*

Proof. From the definition of a standard T -system and the assumption that D° and D are factoralgebras of T , we get immediately that conditions (i) and (ii) of the definition of a split standard system are satisfied. The condition (iii) and the rest of the claim follows from inclusions $e_j \Delta_T \text{rad} \Delta_T^\circ e_i \subset Z(j)$ and $e_i \text{rad} \Delta_T \Delta_T^\circ e_j \subset Z^\circ(j)$ for $j \geq i$. $\square$

The modules $D(i)$ (and $D^\circ(i)$, respectively) in the standard system (D°, D) are factormodules of $\Delta_T(i)$ (and $\Delta_T^\circ(i)$, respectively). They serve as models for standard modules $\Delta_A(i)$ (and $\Delta_A^\circ(i)$, respectively) of a suitable factoralgebra A of T . In this connection, the following natural questions arise: What conditions on a standard T -system (D°, D) are necessary in order that there exists an algebra $A = T/I$ such that $\Delta_A(i) \simeq D(i)$ as right T -modules, $\Delta_A^\circ(i) \simeq D^\circ(i)$ as left T

modules, for $i = 1, \dots, n$, and $A \simeq \Delta_A^\circ \bigotimes_S \Delta_A$? What conditions on (D°, D) are necessary in order that the algebra A as above is quasi-hereditary? Obviously, (D°, D) must be Schurian in this case.

We will show that for any split standard system (D°, D) , there exists a unique algebra which has a $\Delta_T^\circ - \Delta_T$ -Cartan decomposition and its standard system is isomorphic to (D°, D) .

Theorem 7.2.3. *Let S be a species, $T = T(S)$ be the respective tensor algebra and I be an admissible ideal of T . If $A = T/I$ is level-split algebra, then the standard T -system $(D_A^\circ, D_A) = (\Delta_A^\circ, \Delta_A)$ is split. Conversely, if (D°, D) is a split standard T -system, then there exists a unique two-sided ideal $J(D^\circ, D)$ of T such that $J(D^\circ, D)$ is level-split and for $A(D^\circ, D) = T/J(D^\circ, D)$ and every $i = 1, \dots, n$: $\Delta_{A(D^\circ, D)}(i) \simeq D(i)$ as right T -modules, $\Delta_{A(D^\circ, D)}^\circ(i) \simeq D^\circ(i)$ as left T -modules and $A(D^\circ, D) \simeq D^\circ \bigotimes_S D$ as $\Delta_T^\circ - \Delta_T$ -bimodules.*

Moreover, $J(D^\circ, D) = \Delta_T^\circ Z + Z^\circ \Delta_T$ and if a T -level-split algebra C satisfies: $\Delta_C(i) \simeq D(i)$ as right T -modules and $\Delta_C^\circ(i) \simeq D^\circ(i)$ as left T -modules, for each $i = 1, \dots, n$, then C is a factoralgebra of $A(D^\circ, D)$. In particular, if (D°, D) is Schurian finite dimensional split standard T -system, then there exists a unique quasi-hereditary algebra $A = A(D^\circ, D)$ such that $A \simeq D^\circ \bigotimes_S D$ as $\Delta_T^\circ - \Delta_T$ -bimodules.

Proof. First assume that $A = T/I$ is a level-split algebra. We will prove that $(\Delta_A^\circ, \Delta_A)$ is split. Let us prove first that $Z_A \subset I$. Since $V_A(i) = e_i I + e_i T e_{i+1} T$,

each $x \in V_A(i) \cap L_i$ can be written as $x = y + z$, $y \in e_i I$, $z \in e_i T e_{i+1} T$. Write $y = \sum_{l=1}^n y_l$ and $z = \sum_{l=i+1}^n z_l$, where $y_l, z_l \in L_l$. Since I is level-split, each $y_l \in I$. But $x \in L_i$ forces $y_l + z_l = 0$, for $l > i$. Hence, $z \in I$ and $x = y + z \in I$. Therefore, $e_j \Delta_T Z_A(i) \subset e_j \Delta_T I \subset Z_A(j)$ and $Z_A(i) \Delta_T^o e_j \subset I \cap \Delta_T^o e_j \subset Z_A^o(j)$ if $j \geq i$ and $Z_A(i) \Delta_T^o e_j \subset Z_A(i)$ if $j \leq i$. The other two inclusions follow analogously.

Since the standard system (D^o, D) is split, the set $J(D^o, D) = \Delta_T^o Z + Z^o \Delta_T$ is a two-sided ideal of T (generated by Z and Z^o), which is obviously level-split. By a similar argument to the proof of Proposition 7.1.3, we can prove that $\Delta_A^o(i) \simeq D^o(i)$ as left T -modules, $\Delta_A(i) \simeq D(i)$ as right T -modules and $A \simeq D^o \otimes_S D$ as $\Delta_T^o - \Delta_T$ -bimodules.

Finally, assume that a level-split algebra $C = T/I$ is such that $\Delta_C(i) \simeq D(i)$ as right T -modules and $\Delta_C^o(i) \simeq D^o(i)$ as left T -modules. Then $Z_C(i) = Z(i)$ and $Z_C^o(i) = Z^o(i)$, for every $i = 1, \dots, n$. Since C is level-split, we have $Z, Z^o \subset I$. Therefore, $\Delta_T^o Z + Z^o \Delta_T \subset I$ because I is a two-sided ideal of T . Thus every level-split algebra with a standard system (D^o, D) is isomorphic to a factoralgebra of $A(D^o, D)$. $\square$

The algebra $A(D^o, D)$ of the above theorem will be called the *canonical level-split algebra*.

We can consider a two-sided ideal J of T generated by Z and Z^o in the general case, but only if the standard system (D^o, D) is split, the factoralgebra $A = T/J$ will be quasi-hereditary with standard modules Δ_A^o and Δ_A isomorphic to D^o and

D , respectively. If the standard system (D°, D) is not split, then the ideal J is too big. The special form $J(D^\circ, D) = \Delta_T^\circ Z + Z^\circ \Delta_T$ represents a minimality condition for the ideal J , which was important in the above proof.

Corollary 7.2.4. *Let (D°, D) be a Schurian standard T -system of factoralgebras of T . Then there is a quasi-hereditary algebra $A(D^\circ, D)$ such that $D^\circ \simeq \Delta_{A(D^\circ, D)}^\circ$ as left T -modules and $D \simeq \Delta_{A(D^\circ, D)}$ as right T -modules.*

Proof. It follows from Propositions 7.2.2 and 7.2.3. $\square$

Let us note that the previous corollary extends the results of [DM] because lean homogeneous standard systems defined there satisfy the assumptions of the above corollary. It is interesting to point out that the result of [DM] cannot be extended to lean standard systems as seen from the next example.

Example 7.2.5. Assume that a standard system (D°, D) over the tensor algebra T with induced species is given as follows

$$(D^\circ, D) = (\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & & & & \swarrow \searrow \\ 3 & 2 & & & & 5 \\ \downarrow & \downarrow & & & & \swarrow \searrow \\ 3 & 4 & 1 & 2 & & \\ & & 3 & & & \end{array} , \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & & & \\ 1 & 2 & 1 & & & \end{array}) .$$

Then the standard system (D°, D) is not split, but it is lean in the sense of [DM]. There is a lean quasi-hereditary algebra which has (D°, D) as its standard system (to construct it, look at $D^\circ(6)$), but this algebra does not have $\Delta_T^\circ - \Delta_T$ -Cartan decomposition.

Finally, we present an example of the canonical level-split algebra.

Example 7.2.6. Assume that a standard system (D°, D) over the tensor algebra T with induced species is given as

$$(D^\circ, D) = (\begin{array}{c} 1 \oplus 2 \oplus 3 \oplus 4 \oplus 5 \\ \downarrow \quad \downarrow \quad \swarrow \quad \searrow \\ 1 \quad 3 \quad 4 \quad 2 \\ \quad \quad \quad \searrow \quad \swarrow \\ \quad \quad \quad 3 \end{array} , \begin{array}{c} 1 \oplus 2 \oplus 3 \oplus 4 \oplus 5 \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ 1 \quad 1 \quad 1 \quad 2 \quad 2 \\ \quad \quad \quad \swarrow \quad \searrow \\ \quad \quad \quad 3 \quad 1 \\ \quad \quad \quad \swarrow \quad \searrow \\ \quad \quad \quad 2 \quad 1 \end{array} \oplus \begin{array}{c} 4 \\ \swarrow \quad \searrow \\ 3 \quad 2 \\ \swarrow \quad \searrow \\ 2 \quad 1 \end{array} \oplus \begin{array}{c} 5 \\ \swarrow \quad \searrow \\ 3 \quad 4 \\ \swarrow \quad \searrow \\ 2 \quad 3 \\ \swarrow \quad \searrow \\ 1 \end{array})$$

Then $Z(1) = Z(2) = 0$, $Z(3) = \langle 321, 313 \rangle_{\Delta_T}$, $Z(4) = \langle 431 - 421, 4321, 4313, 434 \rangle_{\Delta_T}$ and $Z(5) = \langle 5343 - 513, 53421 - 5131, 5134, 5132, 51313, 532, 531, 5345, 53425 \rangle_{\Delta_T}$. Also $Z^\circ(1) = Z^\circ(2) = 0$, $Z^\circ(3) = \Delta_T \langle 313, 213 \rangle$, $Z^\circ(4) = \Delta_T \langle 434, 134 \rangle$ and $Z^\circ(5) = \Delta_T \langle 345 - 325, 5345, 1345, 4345, 425 \rangle$.

Then $J(D^\circ, D) = {}_T \langle 321, 313, 213, 431 - 421, 4321, 434, 143, 5343 - 513, 5132, 532, 531, 5345, 345 - 325, 425 \rangle_T$ and the right regular representation of the canonical level-split algebra $A = A(D^\circ, D)$ is given by:

$$A_A = \begin{array}{c} 1 \\ \downarrow \\ 3 \\ \downarrow \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 1 \quad 5 \\ \quad \downarrow \quad \downarrow \\ \quad 3 \quad 1 \\ \quad \swarrow \quad \searrow \\ 2 \quad 4 \\ \quad \swarrow \quad \searrow \\ \quad 1 \quad 3 \\ \quad \swarrow \quad \searrow \\ \quad 2 \quad 1 \end{array} \oplus \begin{array}{c} 3 \\ \swarrow \quad \downarrow \quad \searrow \\ 4 \quad 1 \quad 2 \\ \swarrow \quad \downarrow \quad \searrow \\ 3 \quad 2 \quad 5 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad 1 \quad 3 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad 4 \quad 1 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad 1 \quad 3 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad 1 \end{array} \oplus \begin{array}{c} 4 \\ \swarrow \quad \downarrow \quad \searrow \\ 3 \quad 2 \quad 5 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad 1 \quad 3 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad 4 \quad 1 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad 1 \quad 3 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad 1 \end{array} \oplus \begin{array}{c} 5 \\ \swarrow \quad \downarrow \quad \searrow \\ 3 \quad 4 \quad 1 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad 3 \quad 1 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad 1 \quad 3 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad 1 \end{array}$$

7.3 DEFORMATIONS OF CANONICAL LEVEL SPLIT ALGEBRAS

In this section, we consider a quasi-hereditary algebra A with $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition and split standard system. Those algebras can be obtained by

deformation of canonical level-split algebras. In general, this deformation does not preserve the species. However if $\mathcal{R}^\circ \supset \mathcal{R}_A^\circ$ and $\mathcal{R} \supset \mathcal{R}_A$, then these algebras can be obtained from canonical level-split algebra under a deformation which preserves not only the species but also the standard modules.

As mentioned previously, most of algebras appearing in applications possess Cartan decomposition, but this decomposition is not $\Delta_T^\circ - \Delta_T$ -Cartan decomposition but only a $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition for smaller monomial subalgebras $\mathcal{R} \supset R_A$ and $\mathcal{R}^\circ \supset R_A^\circ$. On the other hand, the standard system of these algebras is split. We will formulate conditions on monomial subalgebras $\mathcal{R}^\circ$ and $\mathcal{R}$ and standard A -modules in order that the standard system $(\Delta_A^\circ, \Delta_A)$ is split.

Proposition 7.3.1. *Let $A = T/I$ be a quasi-hereditary algebra with $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition, where $\mathcal{R}^\circ \supset \mathcal{R}_A$ and $\mathcal{R} \supset \mathcal{R}_A$. Then Δ_A° is a left Δ_T -module and Δ_A is a right Δ_T° -module. Moreover the standard system $(\Delta_A^\circ, \Delta_A)$ is split if and only if the following conditions are satisfied:*

- (i) *If M_1, M_2 are monomials in Δ_T and $M_1 M_2 \subset R_A$, then both $M_1, M_2 \subset R_A$.*
- (i[°]) *If M_1, M_2 are monomials in Δ_T° and $M_1 M_2 \subset R_A^\circ$, then both $M_1, M_2 \subset R_A^\circ$.*
- (ii) *If M_1 is a monomial in Δ_T , M_2 is a monomial in Δ_T° and $M_1 M_2 \subset R_A$, then $M_1 \subset R_A$ and $M_2 \subset R_A^\circ$.*
- (ii)[°] *If M_1 is a monomial in Δ_T° , M_2 is a monomial in Δ_T and $M_1 M_2 \subset R_A^\circ$, then $M_1 \subset R_A^\circ$ and $M_2 \subset R_A$.*

Proof. Write $\Delta_T = \mathcal{R} \oplus Y_{\mathcal{R}}$ and $\Delta_T^\circ = \mathcal{R}^\circ \oplus Y_{\mathcal{R}^\circ}$, where $Y_{\mathcal{R}}$ and $Y_{\mathcal{R}^\circ}$ are sums

of monomials of T . Since $\mathcal{R} \supset R_A$ and $\mathcal{R}^\circ \supset R_A^\circ$, we have $\psi(Y_{\mathcal{R}}) = 0$ and $\psi^\circ(Y_{\mathcal{R}^\circ}) = 0$, hence $Z_A = Z_{\mathcal{R}} \oplus Y_{\mathcal{R}}$ and $Z_A^\circ = Z_{\mathcal{R}^\circ} \oplus Y_{\mathcal{R}^\circ}$.

To show the first part of the proposition, it is enough to see that $Z_A(i)\Delta_T^\circ e_j \subset Z_A(i)$ and $e_j\Delta_T Z_A^\circ(i) \subset Z_A^\circ(i)$ whenever $j \leq i$. We will verify the first condition only; the second will follow by symmetry. If $j \leq i$, then $e_i Z_{\mathcal{R}} \Delta_T^\circ e_j \subset e_i \Delta_T \cap I \subset Z_A(i)$. On the other hand, since ψ is a right T -morphism and $\psi(Y_{\mathcal{R}}) = 0$, we have $\psi(e_i Y_{\mathcal{R}} \Delta_T^\circ e_j) = 0$. Therefore, $e_i Y_{\mathcal{R}} \Delta_T^\circ e_j \subset Z_A(i)$ and the claim follows from $Z_A = Z_{\mathcal{R}} \oplus Y_{\mathcal{R}}$.

By Proposition 6.4.3, $B_{\mathcal{R}}$ and $B_{\mathcal{R}^\circ}$ are Borel subalgebras of A and $Z_{\mathcal{R}} \cup Z_{\mathcal{R}^\circ} \subset I$, by Proposition 6.3.1. Next, Δ_A is a factoralgebra of Δ_T if and only if $e_j \Delta_T Z_A(i) \subset Z_A(j)$, for all $i, j = 1, \dots, n$. Since $Z_{\mathcal{R}} \subset I$, we have $\Delta_T Z_{\mathcal{R}} \subset Z_A$. Since ψ is a right T -morphism and $\psi(Y_{\mathcal{R}}) = 0$, we infer that $Y_{\mathcal{R}} Z_A \subset Z_A$. Finally, $Z_A = Z_{\mathcal{R}} \oplus Y_{\mathcal{R}}$ implies that $e_j \Delta_T Z_A(i) \subset Z_A(j)$ is equivalent with $e_j \mathcal{R} Y_{\mathcal{R}} e_i \subset Z_A(j)$ which is in turn equivalent to condition (i) of this proposition.

The last condition for standard system we consider is: $e_j \Delta_T Z_A^\circ(i) \subset Z_A(j)$ for $j \geq i$. Similarly as before, $e_j \Delta_T Z_{\mathcal{R}^\circ} \subset Z_A(j)$ and $e_j Y_{\mathcal{R}} Z_A^\circ(i) \subset Z_A(j)$. Therefore, the last condition is equivalent to $e_j \mathcal{R} Y_{\mathcal{R}^\circ} e_i \subset Z_A(j)$ for $j \geq i$, which is equivalent to condition (ii) of this proposition.

By symmetry, we conclude that the standard system $(\Delta_A^\circ, \Delta_A)$ is split if and only if conditions (i), (i^o), (ii) and (ii^o) are satisfied. $\square$

In order to relate canonical level-split algebras and algebras with $\mathcal{R}^\circ - \mathcal{R}$ -

Cartan decomposition via deformation, we will use a concept of an integral quasi-hereditary algebra, see [CPS3] and [K3].

Definition 7.3.2. *Let $A[t]$ be an algebra over $K[t]$, which is free of finite rank over $K[t]$. Let $e_1, \dots, e_n$ be an ordered finite set of primitive orthogonal idempotents of $A[t]$ such that these idempotents give rise to a complete set of primitive orthogonal idempotents of $A[t_0]$, for each $t_0 \in K$. Then $A[t]$ is called integral quasi-hereditary (over $K[t]$ with respect to the given order on $e_1, \dots, e_n$) if and only if $A[t_0]$ (with induced order) is quasi-hereditary, for each $t_0 \in K$.*

By a deformation of an algebra A to an algebra A' we mean integral quasi-hereditary algebra $A[t]$ such that $A[1] \simeq A$ and $A[0] \simeq A'$.

Every quasi-hereditary algebra with $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition can be obtained by deformation of a canonical level-split algebra. This result is an easy generalization of [K3], Theorem 6.1 (the terminology there is different but we trust that the reader, who has fought his way through this thesis, should be able to figure this out easily). The deformation constructed there has one serious disadvantage, namely, it does not preserve the species of an algebra. This disadvantage cannot be corrected within a construction of quasi-hereditary algebras presented there. However, results of part 7.2 enable us to show that every algebra, with $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition, where $\mathcal{R} \supset \mathcal{R}_A$ and $\mathcal{R}^\circ \supset \mathcal{R}_A^\circ$ and split standard system, is a deformation of a canonical level-split algebra, where this deformation not only preserves the species of particular algebras but also fixes all their standard

modules.

Proposition 7.3.3. *Let $A = T/I$ be an algebra with split standard system and $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition, for some $\mathcal{R}^\circ \supset \mathcal{R}_A^\circ$ and $\mathcal{R} \supset \mathcal{R}_A$. Then there is an integral quasi-hereditary algebra $A[t]$ such that $A[1] \simeq A$, $A[0]$ is a canonical level split algebra, every algebra $A[t_0]$ has a $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition, for each $t_0 \in K$, all left Borel subalgebras $B_{\mathcal{R}^\circ}[t_0]$ of $A[t_0]$ are isomorphic as algebras and left T -modules and all right Borel subalgebras $B_{\mathcal{R}}[t_0]$ are isomorphic as algebras and right T -modules.*

Proof. First, we replace the tensor algebra T over the species $\mathcal{S} = (D_i, {}_iM_j, i, j = 1, \dots, n)$ by a tensor algebra $T[t]$ over the a species $\mathcal{S}[t] = (D_i[t], {}_iM_j[t], i, j = 1, \dots, n)$, where each $D_i[t_0] = D_i$ and every ${}_iM_j[t_0] = {}_iM_j$. Any element $x \in T\varepsilon_i T \cap I$ decomposes as $x = x_i + (x - x_i)$, where $x_i \in L_i$ and $(x - x_i) \in T\varepsilon_{i+1}T$. Let $I[t]$ be an ideal of $T[t]$ generated by elements of the form $x(t) = x_i + t(x - x_i)$ for each $x \in T\varepsilon_i T \cap I$ and $A[t] = T[t]/I[t]$. Then $A[t]$ is a free $K[t]$ -module of rank $\dim_K A$, $A[1] \simeq A$. $B_{\mathcal{R}^\circ}[t]$ and $B_{\mathcal{R}}[t]$ are subalgebras of $A[t]$ such that $B_{\mathcal{R}^\circ}[t_0] \simeq B_{\mathcal{R}^\circ}$ is a left Borel subalgebra of $A[t_0]$, and $B_{\mathcal{R}}[t_0] \simeq B_{\mathcal{R}}$ is a right Borel subalgebra of $A[t_0]$, for each $t_0 \in K$. Therefore, $A[t_0]$ has a $\mathcal{R}^\circ - \mathcal{R}$ -Cartan decomposition and $A[t]$ is an integral quasi-hereditary algebra. The fact, that $B_{\mathcal{R}}[t_0] \simeq B_{\mathcal{R}}$ as right T -modules and $B_{\mathcal{R}^\circ}[t_0] \simeq B_{\mathcal{R}^\circ}$ as left T -modules, follows from $\mathcal{R} \supset \mathcal{R}_A$ and $\mathcal{R}^\circ \supset \mathcal{R}_A^\circ$. Finally, algebra $A[0]$ is a factoralgebra of T by an ideal generated by elements from various L_i , hence, $A[0]$ is a level-split algebra.

Since $A[0]$ is quasi-hereditary, it is a canonical level-split algebra. $\square$

7.4 EXTENSION ALGEBRA OF STANDARD

MODULES OF A CANONICAL LEVEL-SPLIT ALGEBRA

We finish with an observation relating the structures of some canonical level-split algebras and extension algebras of their standard modules.

Proposition 7.4.1. *Let A be a canonical level split algebra such that its standard modules are directed. Then A_0^δ and B^{o*} are subalgebras of A^δ , $A^\delta \simeq A_0^\delta \otimes_S B^{o*}$ as $A_0^\delta - B^{o*}$ -bimodules and $\text{rad} B^{o*} \text{rad} A_0^\delta = 0$.*

Proof. Since A has both left and right Borel subalgebras B^o and B , the first part of the claim follows from results in sections 5.2 and 5.3. It remains to show that $\text{rad} B^{o*} \text{rad} A_0^\delta = 0$. The assumptions on A imply that $\text{rad} B \text{rad} B^o = 0$. The projection $p : P(i) \rightarrow \Delta_A(i)$ induces a map $p|_{\text{rad} P(i)} : \text{rad} P(i) \rightarrow \text{rad} \Delta_A(i)$. The condition $\text{rad} B \text{rad} B^o = 0$ implies that the last map splits as a right A -morphism. Since Yoneda multiplication is associative and each $E \in \text{rad} B^{o*} c_i$ is of the form $E_1 E_0$, where E_0 is represented by the exact sequence

$$0 \rightarrow \Omega_1(\Delta_A(i)) \rightarrow P(i) \rightarrow \Delta_A(i) \rightarrow 0,$$

in order to show that $\text{rad} B^{o*} \text{rad} A_0^\delta = 0$ it is enough to check that $E_0 \alpha = 0$ for every map $\alpha : \Delta_A(j) \rightarrow \Delta_A(i)$ and $j \neq i$. Consider a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_1(\Delta_A(i)) & \longrightarrow & X & \longrightarrow & \Delta_A(j) \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \alpha \\ 0 & \longrightarrow & \Omega_1(\Delta_A(i)) & \longrightarrow & P(i) & \longrightarrow & \Delta_A(i) \longrightarrow 0 \end{array},$$

where $j \neq i$. The upper exact sequence represents $E_0\alpha$. Since the image of α lies in $\text{rad}\Delta_A(i)$ and $p|_{\text{rad}P(i)} : \text{rad}P(i) \rightarrow \text{rad}\Delta_A(i)$ splits as a right A -morphism, there is a map $g : \Delta_A(j) \rightarrow P(i)$ such that $pg = \alpha$. But this means that $E_0\alpha = 0$. $\square$

Let us note the remarkable symmetry of this situation. We have started with algebra A having decomposition $A \simeq B^o \rtimes B$ as $B^o - B$ -bimodules, where B^o, B are subalgebras of A such that $\text{rad}B\text{rad}B^o = 0$. Then the extension algebra A^δ of (right) standard modules of A has a decomposition $A^\delta \simeq A_0^\delta \rtimes B^{o*}$ as $A_0^\delta - B^{o*}$ bimodules, where A_0^δ, B^{o*} are subalgebras of A^δ such that $\text{rad}B^{o*}\text{rad}A_0^\delta = 0$.

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