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THE THEORY OF ARRANGEMENTS AND APPLICATIONS
TO THE CROSSING NUMBER OF $C_m \times C_n$

by
Jay Adamsson

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

School of Mathematics and Statistics

Carleton University

April 2000

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submitted by
Jay Adamsson

in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

for Dr. Kenneth S. Williams, Director, School of Mathematics and Statistics

DR. R.B. Richter, Thesis Supervisor

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April 2000

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Abstract

This thesis introduces and develops the theory of arrangements, then applies it to computing the crossing number of $C_m \times C_n$. This approach is an attempt to prove an outstanding conjecture that the crossing number of $C_m \times C_n$ with $n \geq m \geq 3$ is equal to $(m - 2)n$. The graph $C_m \times C_n$ contains mn vertices, labelled v_{ij} with $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. Two vertices v_{ij} and $v_{k\ell}$ are then adjacent if and only if either $i = k$ and $j \equiv \ell \pm 1 \pmod{n}$ or $i \equiv k \pm 1 \pmod{m}$ and $j = \ell$. Arrangements are a new approach to this question, providing the basis to deal with intersecting cycles within $C_m \times C_n$.

Chapter 1 gives an overview of the crossing number of $C_m \times C_n$. The history of the crossing number conjecture is summarized, and the current state of knowledge is reviewed.

Chapter 2 recounts two of the most recent approaches taken in this topic. First, the work of Gelasio Salazar is reviewed. He showed that the conjecture holds for drawings of $C_m \times C_n$ in which one of the two families of cycles are disjoint. Secondly, the work of Bryan Crapper is summarized, in which he attempted to show that the conjecture holds for restricted drawings of $C_m \times C_n$. A restricted drawing is a drawing of $C_m \times C_n$ where each cycle in the restricted family is allowed to intersect at most one other cycle in that family. Unfortunately, his analysis was flawed, and did not prove this claim.

Chapter 3 defines arrangements and proves the Fundamental Theorem of Arrangements. This theorem is the cornerstone for the rest of the thesis. It identifies a substructure that occurs in arrangements, and shows how a crossing occurs whenever this substructure exists. It is then shown that, within an (m, n) -arrangement, there are $(m - 2)n$ crossings.

Chapter 4 introduces circular arrangements. Circular arrangements are essentially a generalization of drawings of $C_m \times C_n$, and form the bridge between these graphs and arrangements. It is shown that circular arrangements with certain properties have $(m - 2)n$ crossings.

Chapter 5 applies these results to $C_m \times C_n$. The theorems of Salazar and Crapper, as reviewed in Chapter 2, are easily shown by this theory. It then introduces the notion of drawings with many components, and applies the theory of arrangements to these structures. Finally, it is shown that the crossing number of $C_7 \times C_n = 5n$, a previously unproven result.

Chapter 6 provides some discussion on the general aspects of the theory of arrangements, and suggestions for further research.

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ψ	18
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ν	20
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Chapter 1

Introduction

1.1 Crossing Numbers

A *drawing* of a finite graph G is a representation of G in a plane where each vertex in G is represented by a distinct point on the plane, and two vertices on the plane are connected by an arc in the drawing if and only if the two corresponding vertices are connected by an edge in G . An arc, representing an edge incident with the vertices u and v , begins at the point representing one of u and v , ends at the other, and does not cross any point representing a vertex. Obviously there are infinitely many drawings of a graph.

A *crossing* of a drawing of a graph is a point in the plane and a pair of arcs such that the point does not correspond to a vertex, and the two arcs pass through the point. The *crossing number* of a drawing of a graph is the number of crossings in the drawing. Finally, the *crossing number* $cr(G)$ of a graph G , is the minimum crossing number of all drawings of a graph. This will be a finite number.

In general, crossing numbers of graphs are difficult to compute. For complete and complete bipartite graphs, only partial results are known. For

n -dimensional cubes, only asymptotic estimates are known. There is very little theory on crossing numbers available.

Figure 1.1 shows a drawing of a graph with two crossings. However, redrawing this graph in Figure 1.2 gives a drawing with one crossing. In fact, this graph has crossing number one.

In 1930, Kuratowski [7] answered the question of which graphs have crossing number zero. He showed that the crossing number of a graph is zero if and only if the graph contains no subdivision of $K_{3,3}$ or K_5 . Since this result, investigation in this area has turned towards determining the crossing number of specific graphs or families of graphs.

1.2 Crossing Number of $C_m \times C_n$

A good review of the current state of knowledge of crossing numbers of families of graphs can be found in [12]. This section discusses the family of graphs $C_m \times C_n$.

The product graph $C_m \times C_n$ is formed by taking m copies of the cycle C_n , yielding m copies of each vertex in C_n . These copies are all connected cyclically. In other words, this construction gives mn vertices, labeled v_{ij} for $i = 1, \dots, m$ and $j = 1, \dots, n$. Then the two vertices v_{ij} and $v_{k\ell}$ are adjacent if and only if either $i = k$ and $j \equiv \ell \pm 1 \pmod{n}$ or $i \equiv k \pm 1 \pmod{m}$ and $j = \ell$. Figure 1.3 shows a drawing of $C_3 \times C_4$.

Figure 1.1

Figure 1.2

It has been conjectured by Harary et al [6] that the crossing number of $C_m \times C_n$ is $(m - 2)n$ where $3 \leq m \leq n$. This will hereafter be referred to as the HKS-conjecture. Note that the drawing of $C_3 \times C_4$ in Figure 1.3 has many more crossings than the conjectured crossing number of 4 for this graph. This conjecture has been proved for $m = 3$ in [9], $m = 4$ in [5] and [3], $m = 5$ in [10] and [8], $m = 6$ in [1] and [12], and $m = n = 7$ in [2]. The latest contribution to this question is in [13] where it is shown that the result holds for arbitrary m and n where one of the families of cycles is disjoint. Finally, this thesis shows that $C_7 \times C_n = 5n$.

The standard technique for proving the HKS-conjecture is to separately compute $cr(C_m \times C_m)$ and then induct on n to get the result for $C_m \times C_n$. In the induction step, it is shown that either at least one of the m -cycles is crossed at least $m - 2$ times, or that each of the n m -cycles has $m - 2$ crossings associated with it (called the *force* of the cycle) for a total of $(m - 2)n$ crossings. The difficult step in this process is when none of the m -cycles is crossed at least $m - 2$ times. This has led to the consideration of arrangements and circular arrangements in this thesis.

The crossing numbers of $C_3 \times C_3$ and $C_4 \times C_4$ were computed directly with straightforward, if somewhat involved, methods [6], [5]. For the proof that the crossing number of $C_5 \times C_5$ is 15, the idea of a *mesh* was introduced. Essentially, a mesh is a generalization of a drawing of $C_m \times C_n$.

Figure 1.3

An (m, n) -*mesh* consists of two families of closed curves, containing m and n members, where each member in the first family intersects each member in the second family, and vice versa. Thus, every drawing of $C_m \times C_n$ is a mesh with additional conditions. By showing any $(5, 5)$ -mesh has at least 40 intersections, the crossing number of $C_5 \times C_5$ was deduced to be at least 15 [10]. The same method was used to determine the crossing number of $C_6 \times C_6$ and $C_7 \times C_7$ [1], [2]. For the cases of $C_5 \times C_n$ and $C_6 \times C_n$, inductive arguments were then used to prove the conjecture in each case [8], [12].

In [13], the conjecture was shown for drawings of $C_m \times C_n$ where one of the families consists of disjoint cycles. This was accomplished by defining the force of a cycle. The force of a cycle is the number of crossings that are associated with a cycle. By showing the force of each cycle is large enough, and the associated crossings are disjoint from those associated with other cycles, the number of crossings is then shown to be at least equal to $(m - 2)n$ as conjectured.

This report extends these results by defining a circular arrangement. Essentially, a circular arrangement is a mesh with some additional conditions, but it is still a more general structure than $C_m \times C_n$. By using circular arrangements, a force can be defined, even for cycles that are not disjoint. This more general result is then applied to extend the results to drawings of graphs that can be described as a circular arrangement.

Chapter 2 gives an overview of the previous results in [13] and the flawed efforts in [4] to expand the knowledge of the crossing number of $C_m \times C_n$. Chapter 3 introduces the notion of an arrangement, and shows that the cross-

ing number of an (m, n) -arrangement is at least $(m - 2)n$. This is the central contribution of the thesis. Chapter 4 introduces circular arrangements and partitioned circular arrangements, and expands the results of Chapter 3 to show that the crossing number of a partitioned (m, n) -circular arrangement is at least $(m - 2)n$. This is the crucial link between the theory of arrangements and the HKS-conjecture. Chapter 5 applies these results to $C_m \times C_n$ and expands the results of [13] to a wider range of drawings. It also shows that the crossing number of restricted drawings of $C_m \times C_n$ from [4] really is at least $(m - 2)n$. It then proceeds to show that $cr(C_7 \times C_n) = 5n$, a new result matching the HKS-conjecture. Finally, Chapter 6 provides an overview of arrangements and a discussion of a few issues dealing with arrangements.

Chapter 2

Previous Results

2.1 Introduction

This chapter will review the principal result of [13] that the HKS-conjecture holds for drawings of $C_m \times C_n$ that have a disjoint family of cycles. It will also review an attempt to extend this result to the situation where pairs of cycles are permitted to intersect within a family of cycles [4].

2.2 Disjoint Family of Cycles

In [13], the n m -cycles are called red cycles, and the m n -cycles are called blue cycles. A drawing of $C_m \times C_n$ is a *red-disjoint (rd) drawing* if the red cycles are pairwise disjoint. A drawing is a *blue-disjoint (bd) drawing* if the blue cycles are pairwise disjoint. The minimum number of crossings in a rd-drawing of $C_m \times C_n$ is denoted by $cr_{rd}(C_m \times C_n)$, and the minimum number of crossings in a bd-drawing of $C_m \times C_n$ is denoted by $cr_{bd}(C_m \times C_n)$.

The main result in [13] is stated as follows:

Theorem 2.1 *Let m, n be such that $n \geq m \geq 3$. Then $cr_{rd}(C_m \times C_n) \geq (m-2)n$ and $cr_{bd}(C_m \times C_n) \geq (m-2)n$.*

We will later show how Theorem 2.1 follows from:

Lemma 2.2 *Suppose that $\min\{p, q\} \geq 3$. Let $\mathcal{D}$ be a drawing of $C_p \times C_q$ such that the q p -cycles are pairwise disjoint and no p -cycle separates two p -cycles. Then $cr(\mathcal{D}) \geq (p-2)q$.*

This lemma is proved by ordering the red p -cycles as $R_1, \dots, R_q$, and denoting by H_i the subgraph of $C_p \times C_q$ induced by the vertices of $R_{i-1} \cup R_i$ (all indices R_i are read modulo q). The *force* $f(H_i)$ is the total number of crossings of the form:

1. a crossing of a blue edge in H_i with an edge in $R_i \cup R_{i+1}$;
2. a crossing of a blue edge in H_{i+1} with an edge in H_i ;
3. a crossing of a blue edge in H_i with a blue edge in H_i ; and
4. a self-intersection of R_i .

Now $cr(\mathcal{D}) \geq \sum_{1 \leq i \leq q} f(H_i)$ since no crossing is counted twice. Thus, it suffices to show that $f(H_i) \geq p-2$.

Let I be the collection of edges of H_1 that intersect R_2 , and $f_I(H_1)$ the contribution of these crossings to $f(H_1)$, with $f_{\bar{I}}(H_1)$ the remainder of the crossings.

The final step is to show that if $|I| < p-2$, then $f_{\bar{I}}(H_1) \geq p-2-|I|$ (note that if $|I| \geq p-2$, then $f(H_1) \geq p-2$ as required). This is done

more generally in Chapter 3, as Corollary 3.5. The techniques of [13] are generalized there, specifically in Theorem 3.4, The Fundamental Theorem of Arrangements.

Theorem 2.1 can now be derived from Lemma 2.2 since Lemma 2.2 does not assume that $p > q$ or $p < q$. First, the base case of $C_3 \times C_n$ is assumed, as proved in [9]. Now if in the *bd* drawing $\mathcal{D}$ of $C_m \times C_n$ one of the n -cycles separates two of the others, it has at least n crossings (one from each of the m -cycles). Let $\mathcal{D}'$ be the drawing of $C_{m-1} \times C_n$ obtained by removing the separating n -cycle. By induction, $cr(\mathcal{D}') \geq (m-3)n$, so $cr(\mathcal{D}) \geq (m-3)n + n = (m-2)n$ as required. Otherwise, no n -cycle is separating, so Lemma 2.2 applies to give $cr(\mathcal{D}) \geq (n-2)m \geq (m-2)n$. The result for *rd* drawings is proved similarly.

Lemma 2.2 is also a direct consequence of Theorem 4.1, proved later in this thesis, since *rd*-drawings and *bd*-drawings without separating cycles are instances of partitioned circular arrangements.

2.3 Restricted Family of Cycles

An attempt was made to extend this result by Bryan Crapper in [4] to *restricted* families of cycles, where every cycle in one family intersects at most one other cycle in the same family. Using similar notation to section 2.2, drawings are referred to as *red-restricted* (*rr*) and *blue-restricted* (*br*) drawings. In this project, he attempted to show:

Theorem 2.3 *Let m, n be such that $n \geq m \geq 3$. Then every drawing of $C_m \times C_n$ such that either the n m -cycles or the m n -cycles are restricted has*

at least $(m - 2)n$ crossings.

Theorem 2.3 is a consequence of:

Lemma 2.4 *Suppose that $\min\{p, q\} \geq 3$ and $q \geq 6$. Let D be a drawing of $C_p \times C_q$ such that the family of q p -cycles is restricted and no p -cycle separates two p -cycles. Then $cr(D) \geq (p - 2)q$.*

As with lemma 2.2, the attempted proof for this lemma cyclically orders the red p -cycles as $R_1, R_2, \dots, R_q$, and defines H_i as the subgraph induced by the vertices in $R_i \cup R_{i-1}$ for each $i \in \{1, \dots, q\}$. The *force* of H_i , $f(H_i)$, is defined as follows:

For a blue edge $e_i \in E(H_i)$, we let e_{i+1} denote the blue edge in $E(H_{i+1})$ such that e_i and e_{i+1} are in the same blue cycle. Define the following sets of blue edges:

- $J_i = \{e_i \in E(H_i) \mid e_i \text{ crosses an edge in } R_{i+1}\};$
- $K_i = \{e_i \in E(H_i) \setminus J_i \mid e_i \text{ crosses an edge in } R_{i+2}\};$
- $L_i = \{e_i \in E(H_i) \mid e_{i+1} \text{ crosses an edge in } R_{i-1}\};$
- $M_i = \{e_i \in E(H_i) \setminus L_i \mid e_{i+1} \text{ crosses an edge in } R_{i-2}\}.$

Let $I_i = J_i \cup K_i \cup L_i \cup M_i$. Further, define the following types of crossings:

- Type (1) crossings:

1. a blue edge in H_{i+1} crossing an edge in R_i

- 2. a blue edge in H_{i+1} crossing a blue edge in H_i
- 3. a blue edge in H_i crossing an edge in R_i
- Type (2) crossings: An edge in R_{i-1} crosses an edge in R_i , and a blue edge in H_{i-1} crosses both an edge in R_i and at least one of:
 - 1. an edge in R_{i-1}
 - 2. a blue edge in H_i
 - 3. a blue edge in H_{i+1}
 - 4. an edge in R_{i+1}

(one of the crossings between the blue edge in H_{i-1} and the edge in the list is the crossing counted)

- Type (3) crossings: An edge in R_i crosses an edge in R_{i+1} , and a blue edge in H_{i+2} crosses both an edge in R_i and at least one of:
 - 1. an edge in R_{i+1}
 - 2. a blue edge in H_{i+1}
 - 3. a blue edge in H_i
 - 4. an edge in R_{i-1}

(one of the crossings between the blue edge in H_{i+1} and the edge in the list is the crossing counted)

Now define:

- $X_i = \{e_i \in E(H_i) \setminus I_i \mid e_i \text{ is associated with a Type (1) crossing}\};$

- $Y_i = \{e_i \in E(H_i) \setminus I_i \mid e_i \text{ is associated with a Type (2) crossing}\};$
- $Z_i = \{e_i \in E(H_i) \setminus I_i \mid e_i \text{ is associated with a Type (3) crossing}\};$

Let $W_i = X_i \cup Y_i \cup Z_i$. The *force* $f(H_i)$ is the number of crossings counted by the edges of $I_i \cup W_i$. There are two steps taken in proving the lemma. First, it is shown that there is no double-counting of crossings in the definition of the force. There are only a handful of types of crossings that could be double-counted, and it is not difficult to show that in these cases, the conditions on the countings above are sufficient to prevent double-counting.

As an example, let X be a Type (3,3) crossing counted towards $f(H_i)$ (i.e., a blue edge in H_{i+2} crossing a blue edge in H_i). Let X' be a Type (2,3) crossing counted towards $f(H_j)$ (i.e., a blue edge in H_{j-1} crossing a blue edge in H_{j+1}). If $i = j - 1$, then X and X' are the same crossing. However, since X' is a Type (2,3) crossing, the blue edge in H_{j-1} also crosses an edge in R_j which is a blue edge in H_i crossing an edge in R_{i+1} . This edge is an element of the set J_i in H_i and is therefore not counted as a Type (3,3) crossing. A similar procedure is followed for each pair of crossing types to show that all the counted crossings are distinct.

The final step required for the proof is to show that $f(H_i) \geq p - 2$ for all i . Let H'_i denote the subgraph $H_i - (I_i \cup W_i)$ with any vertices of degree two suppressed. There are four cases looked at:

1. The cycles R_{i-1} , R_i , and R_{i+1} are pairwise disjoint;
2. An edge in R_{i-1} crosses an edge in R_i ;

3. An edge in R_i crosses an edge in R_{i+1} ;
4. An edge in R_{i-1} crosses an edge in R_{i+1} ;

It is now sufficient to show that in each of these cases, H'_i has at most two blue edges, since this would imply that all of the other edges have an associated crossing in $f(H_i)$. In [4], each of these cases are dealt with individually by assuming there are three edges and showing this is a contradiction. However, there is an error in the consideration of case 2. It is stated:

"If there is exactly one vertex of R'_{i-1} that is inside R'_i and is not separated from R'_{i-2} by the edges of R'_{i-1} (type (2,1) crossing), R_{i+1} (type (2,4) crossing), and the blue edges of H'_i (type (2,2) crossing) and H_{i+1} (type (2,3) crossing), then one of the two vertices of R'_i , corresponding to the vertices of R'_{i-1} that are outside of R'_i , is separated from R_{i+1} by the edges of H'_i . Hence, the corresponding blue edge of H'_{i+1} crosses R'_i (type (1,1) crossing), a blue edge in H'_i (type (1,2) crossing), or R'_{i-1} (type L_i crossing) and the corresponding blue edge in H'_i cannot be an edge in H'_i ."

The drawing in Figure 2.1 shows that this is not the case. Here one vertex of R_{i-1} is inside R_i as required. This vertex is not separated from R_{i-2} by the edges of R_{i-1} , R_{i+1} , or the blue edges of H_i or H_{i+1} . Thus, it satisfies the conditions in the statement above. However, it does not satisfy the conclusion. It is claimed that one of the blue edges of H_{i-1} has a crossing. This is not true. A quick check of the definitions of the crossings associated with $f(H_i)$ also shows that none of these types of crossings appear in this drawing, so $f(H_i) = 0$, not 1 as required.

Furthermore, the next paragraph continues:

"If there are at least two vertices of R'_{i-1} that are inside R'_i , then one of the vertices inside R'_i is separated from R_{i-2} and the corresponding blue edge of H'_{i-1} must cross R'_i and at least one of R'_{i-1} (type (2,1) crossing), a blue edge of H'_i (type (2,2) crossing), a blue edge of H'_{i+1} (type (2,3) crossing), or R'_{i+1} (type (2,4) crossing). Hence the corresponding blue edge in H'_i cannot be an edge in H'_i . Therefore H_i has at most two blue edges."

The drawing in Figure 2.2 demonstrates that this also is not the case. Once again, there are three edges in this drawing, so by the lemma, there should be one crossing in the force. However, there are no crossings covered by the definition of force.

These two counter-examples show that the definition of force used in [4] is not sufficient to prove the HKS-conjecture for restricted drawings. However, it will be shown later in Theorem 5.8 that restricted drawings of $C_m \times C_n$ indeed have crossing number at least $(m - 2)n$.

Figure 2.1

Figure 2.2

Chapter 3

Arrangements

3.1 Introduction

This chapter introduces the notion of an *arrangement*. It is shown that an (m, n) -arrangement has at least $(m - 2)n$ crossings. This theorem is the central result of this thesis, and is used in later chapters to obtain results about circular arrangements and the crossing number of $C_m \times C_n$. In reference to the results discussed in Chapter 2, the result of [13] will be extended and the theorem of [4] will be shown.

3.2 Definitions

A *path* is a continuous function $f : [0, 1] \rightarrow \mathbb{R}^2$ and a *cycle* is a path f such that $f(0) = f(1)$.

An (m, n) -*arrangement* is a quintuple $(S, T, \mathcal{C}, \mathcal{P}, \psi)$, such that:

1. $\mathcal{C} = (C_1, C_2, \dots, C_n)$ is a sequence of cycles;
2. No cycle in $\mathcal{C}$ separates two other cycles in $\mathcal{C}$;

3. $\mathcal{P} = \{P_1, P_2, \dots, P_m\}$ is a set of paths;
4. S and T are cycles and $S[0, 1] \cup T[0, 1]$ is contained in one component of $\mathbf{R}^2 \setminus \bigcup_{i=1}^n C_i[0, 1]$;
5. $\psi : (\{S, T\} \cup \mathcal{C}) \times \mathcal{P} \rightarrow [0, 1] \times [0, 1]$ is a function such that:
 - for each $(C, P) \in (\{S, T\} \cup \mathcal{C}) \times \mathcal{P}$, if $\psi(C, P) = (s, t)$, then $C(s) = P(t)$;
 - for each $P \in \mathcal{P}$, $i = 1, \dots, n$, if $\psi(C_i, P) = (s_i, t_i)$, then $0 < t_1 < t_2 < \dots < t_n < 1$;
 - for each $P \in \mathcal{P}$, there exists $s \in [0, 1]$ such that $\psi(S, P) = (s, 0)$; and
 - for each $P \in \mathcal{P}$, there exists $s \in [0, 1]$ such that $\psi(T, P) = (s, 1)$.

Also, if $\psi(C, P) = (s, t)$, define $\psi_{\mathcal{P}}(C, P) = t$ and $\psi_{\mathcal{C}}(C, P) = s$. The definition of ψ can be extended to $\psi : \mathcal{P} \times (\{T, S\} \cup \mathcal{C}) \rightarrow [0, 1] \times [0, 1]$ by $\psi(P, C) = (t, s)$ if $\psi(C, P) = (s, t)$.

Although the definition is quite technical, the image of an arrangement is not very difficult. It is simply two cycles, labelled S and T , together with a sequence of cycles $(C_1, \dots, C_n)$ between them. Each path in $\{P_1, \dots, P_m\}$ starts at S , hits each of the C_i in turn, and ends at T . An example of an arrangement is shown in Figure 3.1. The definition as given above simply parameterizes this idea, and also provides a function ψ to distinguish vertices in the drawing from crossings.

Figure 3.1

A *vertex* in the arrangement $(S, T, \mathcal{C}, \mathcal{P}, \psi)$ is a quadruple (C, P, s, t) with $C \in \mathcal{C} \cup \{S, T\}$, $P \in \mathcal{P}$ such that $\psi(C, P) = (s, t)$. Define $\nu : \mathcal{C} \times \mathcal{P} \rightarrow \mathbf{R}^2$ by $\nu(C, P) = C(s)$ where $\psi(C, P) = (s, t)$.

If P is a path in $\mathcal{P}$, with $s, t \in [0, 1]$ and $s < t$, then $P[s, t]$ is the segment of P given by $\{P(x) : s \leq x \leq t\}$. Also, if $X, Y \in \mathcal{C} \cup \{S, T\}$, let $P[X, Y] = P[\psi_{\mathcal{P}}(X, P), \psi_{\mathcal{P}}(Y, P)]$ (i.e., the segment of P between the vertices of P on X and Y).

A *crossing* in the arrangement $(S, T, \mathcal{C}, \mathcal{P}, \psi)$ is a quadruple (X, Y, s, t) such that:

1. $X, Y \in \{S, T\} \cup \mathcal{C} \cup \mathcal{P}$, $s, t \in [0, 1]$, $X(s) = Y(t)$;
2. either $X \neq Y$ or $s \neq t$; and
3. if $\psi(X, Y)$ is defined, then $(s, t) \neq \psi(X, Y)$.

The *crossing number* $cr(\mathcal{A})$ of an (m, n) -arrangement $\mathcal{A}$ is the cardinality of the set of crossings of $\mathcal{A}$.

If (X, Y, s, t) is a crossing, then so is (Y, X, t, s) , but we shall regard these as the same crossing, and only count it once. Also, if $X \in \{S, T\} \cup \mathcal{C}$, then $(X, X, 0, 1)$ and $(X, X, 1, 0)$ are not to be considered crossings. More generally, the ordered pairs $(X, 0)$ and $(X, 1)$ will be considered the same throughout this work.

Thus, a crossing is a non-vertex intersection of a cycle or a path with a cycle or a path.

3.3 Preliminary Results

Before proceeding into The Fundamental Theorem of Arrangements, a few preliminary technical results are needed.

Theorem 3.1 *For every (m, n) , there is an (m, n) -arrangement with only finitely many crossings.*

Figure 3.2 shows an (m, n) -arrangement with finitely many crossings. In fact, this arrangement has exactly $(m - 2)n$ crossings.

For an (m, n) -arrangement $\mathcal{A} = (S, T, \mathcal{C}, \mathcal{P}, \psi)$ and a point $z \in \mathbf{R}^2$, let $\mathcal{A}(z) = \{(X, s) : X \in \{S, T\} \cup \mathcal{C} \cup \mathcal{P}, s \in [0, 1], X(s) = z\}$. The arrangement $\mathcal{A}$ is in *general position* if $\forall z \in \mathbf{R}^2, |\mathcal{A}(z)| \leq 2$. Note that $cr(\mathcal{A}) = (\sum_{z \in \mathbf{R}^2} \max\{0, |\mathcal{A}(z)| - 1\}) - mn$ if $\mathcal{A}$ is in general position.

Figure 3.2

Theorem 3.2 *For every (m, n) -arrangement $\mathcal{A}$, there is an (m, n) -arrangement $\mathcal{A}'$ in general position such that $cr(\mathcal{A}') \leq cr(\mathcal{A})$.*

Proof: We can assume $cr(\mathcal{A})$ is finite, since the arrangements exhibited in the proof of Theorem 3.1 are in general position. Thus, for every $z \in \mathbf{R}^2$, the number $n(z) = |\mathcal{A}(z)|$ is finite, and the number of z for which $n(z) > 1$ is finite. Let $N(\mathcal{A})$ denote the sum of the $n(z)$, over all z such that $n(z) > 2$.

We note that $N(\mathcal{A}) = 0$ if and only if $\mathcal{A}$ is in general position. If $N(\mathcal{A}) > 0$, then we shall exhibit an arrangement $\mathcal{A}'$ such that $cr(\mathcal{A}') \leq cr(\mathcal{A})$ and $N(\mathcal{A}') < N(\mathcal{A})$. To this end, let z be chosen such that $n(z) > 2$. Let (X, s) be chosen such that z is not the endpoint of X (note that this exists, since otherwise z is the endpoint of a path, and hence also lies on S or T). This proof will proceed by showing that a drawing exists with an identical or reduced number of crossings with a smaller $n(z)$.

The arrangement $\mathcal{A}$ may be viewed as a graph embedded in the plane, with the vertices being the multiple points and the segments between vertices being the edges. By Fary's Theorem, c.f. [11], there is an equivalent embedding in which all the edges are line segments. Thus, every vertex has a small neighbourhood $\mathcal{W}$ that consists of just the vertex and straight rays representing the edges incident with the vertex. Starting with one of the endpoints of X and traveling around the boundary of $\mathcal{W}$, let $(s_0, \dots, s_n)$ be the endpoints of the rays incident with z . Furthermore, assume s_a is the other endpoint of X . Let x_0 be a point on the segment of X between s_0 and z (but not equal to either) and x_1 a point on the segment of X between s_a and z (but again not equal to either).

- Assume z is the endpoint of some path P . Let s_i be the endpoint of P on the boundary of $\mathcal{W}$. There are at least $\frac{n-2}{2}$ other paths and cycles through z other than X , so P was involved in at least this many crossings.

Without loss of generality, we can assume $0 < i < a$. Redrawing P from s_i to x_0 gives $i - 1$ crossings, and redrawing P from s_i to x_1 gives $a - i - 1$ crossings (Figure 3.3). This totals $a - 2$ crossings, so one path has at most $\frac{a-2}{2}$ crossings. Since $n \geq a$, this path has at most $\frac{n-2}{2}$ crossings. Thus, moving this path either reduces or maintains the number of crossings, but decreases $n(z)$ by one.

- We can now assume that z is not the endpoint of some path P . Without loss of generality, we can assume that $a \neq 1$. Let Y be the path or cycle associated with s_1 , and let s_i be the other endpoint of Y . Since every path or cycle has two endpoints on the boundary of $\mathcal{W}$, n is odd, and therefore there are either $\frac{n-1}{2}$ or $\frac{n+1}{2}$ crossings associated with Y , depending on whether one of the intersections is a vertex.

- If $1 \leq i < a$, then there are two ways to redraw Y (Figure 3.4).

First, redraw Y by starting at s_1 , going to x_0 , then straight to s_i . This produces $i - 2$ crossings with other paths or cycles, plus the possible crossing at x_0 for a total of at most $i - 1$ crossings. The second way to redraw the path is to start at s_1 , go to x_0 , then proceed from x_0 to x_1 *on the other side of X from s_1* , then from x_1 to s_i . This path crosses X at x_0 , crosses $n - a$ paths or cycles,

crosses X again at x_1 , then crosses an additional $a - i - 1$ paths or cycles for a total of at most $n - i + 1$ crossings. These two paths produce a total of not more than n crossings, so one of the paths has at most $\frac{n}{2}$ crossings. Since n is odd, this is at most $\frac{n-1}{2}$ crossings, not more than the original number of crossings, while $n(z)$ is decreased by one.

- If $a < i \leq n$, then Y can be redrawn in two ways (Figure 3.5). First, starting at s_0 , proceeding to x_0 , then to s_i produces $n - i + 1$ crossings. Secondly, starting at s_0 , going to x_1 , then to s_i produces $i - 1$ crossings. These two paths again produce a total of n crossings, so as above, one path does not increase the number of crossings, while both reduce $n(z)$ by one.

Q.E.D.

Figure 3.3

Figure 3.4

Figure 3.5

An (m, n) -arrangement $\mathcal{A}$ is *optimal* if, for any (m, n) -arrangement $\mathcal{A}'$, $cr(\mathcal{A}) \leq cr(\mathcal{A}')$.

Proposition 3.3 *There is an optimal (m, n) -arrangement that is in general position. Furthermore, every optimal (m, n) -arrangement $\mathcal{A}$ has the following properties:*

- *$\mathcal{A}$ has only simple cycles (i.e., a cycle does not cross itself).*
- *the paths of $\mathcal{A}$ have distinct endpoints.*
- *If x is a tangential point of intersection in $\mathcal{A}$, then x is a vertex.*
- *For any path P and any $i \in \{1, \dots, n\}$, the portion of the path P between C_{i-1} and C_i , does not cross the portion of the path P between C_i and C_{i+1} .*
- *For any path P and any $i \in \{1, \dots, n\}$, the portion of the path P between C_{i-1} and C_i , does not cross itself.*

The first of these permits the following definition: let $\mathcal{A} = (S, T, \mathcal{C}, \mathcal{P}, \psi)$ be an optimal arrangement. Then, for each $C \in \mathcal{C}$, define E_C to be the component of $\mathbb{R}^2 \setminus C$ containing S and T . Define I_C to be the component of $\mathbb{R}^2 \setminus C$ not containing S and T . Also define $E_{\mathcal{C}} = \bigcap_{i=1, \dots, n} E_{C_i}$ and $I_{\mathcal{C}} = \bigcup_{i=1, \dots, n} I_{C_i}$.

The main purpose of this proposition is to note that the kinds of crossings listed above are easily removed, without affecting the other crossings in the arrangement. So we will sometimes note ‘a crossing can be removed, as in Proposition 3.3.’

3.4 Fundamental Theorem of Arrangements

Let $(S, T, \mathcal{C}, \mathcal{P}, \psi)$ be an arrangement. For $X, Y \in \mathcal{C} \cup \mathcal{P} \cup \{S, T\}$, let $X \cap Y = \{(s, t) \mid X(s) = Y(t); s, t \in [0, 1]\}$.

Theorem 3.4 The Fundamental Theorem of Arrangements

Let $(S, T, (C_1), \mathcal{P}, \psi)$ be a $(3, 1)$ -arrangement in general position with $S \cap T = \emptyset$. Then at least one of the following occurs:

1. *there is a path $P \in \mathcal{P}$ and a crossing of the form (C_1, P, s, t) ;*
2. *there is a path $P \in \mathcal{P}$ and a crossing of the form (S, P, s, t) , with $t > \psi_{\mathcal{P}}(C_1, P)$;*
3. *there is a path $P \in \mathcal{P}$ and a crossing of the form (T, P, s, t) , with $t < \psi_{\mathcal{P}}(C_1, P)$;*
4. *there are distinct paths $P, Q \in \mathcal{P}$ and a crossing of the form (P, Q, s, t) with $s < \psi_{\mathcal{P}}(C_1, P)$ and $t > \psi_{\mathcal{P}}(C_1, Q)$.*

Proof: For each $P \in \mathcal{P}$, let p_S denote the largest $s < \psi_{\mathcal{P}}(C_1, P)$ such that $P(s) \in S$. Similarly, let p_T denote the smallest $t > \psi_{\mathcal{P}}(C_1, P)$ such that $P(t) \in T$. Set $\mathcal{P}' = \{P : [p_S, p_T] \rightarrow \mathbb{R}^2 \mid P \in \mathcal{P}\}$. Now consider the arrangement $(S, T, (C_1), \mathcal{P}', \psi)$.

If there is a $P \in \mathcal{P}'$ such that $|P \cap C_1| > 1$, then we have (1).

If there is a $P \in \mathcal{P}'$ such that $|P \cap S| > 1$, then, by construction, any point in $P \cap S$ that is not the origin of P necessarily occurs after $\psi_{\mathcal{P}}(C_1, P)$ and we have (2).

Similarly, if there is a $P \in \mathcal{P}'$ such that $|P \cap T| > 1$, then we have (3).

Thus, we can assume that, for every $P \in \mathcal{P}'$, $P \cap S$, $P \cap C_1$ and $P \cap T$ all have a unique point in them. It follows that neither S nor T separates the other from C_1 . Therefore, one face of S contains $C_1 \cup T$ and the other is disjoint from every path in $\mathcal{P}$. The analogous statement holds for T .

Pick a point s (respectively, t) in the empty face of S (resp., T) and join s (resp., t) by paths, pairwise disjoint except for the common end s (resp., t) to the ends of the paths in $\mathcal{P}'$ on S (resp., T). For each $P \in \mathcal{P}'$, let v_P denote the point in $P \cap C_1$.

If, for every $P, Q \in \mathcal{P}'$, there are no crossings of P and Q , then we have an embedding of the graph $K_5 - st$ in the plane, where the vertices are s , t and the three points $\{v_P : P \in \mathcal{P}'\}$. The cycle C_1 joins the three vertices v_P .

In every embedding of $K_5 - st$ in the plane, the cycle C_1 separates s from t , which, in the embedding in question, implies that C_1 separates S from T , contradicting the fact that this is an arrangement.

Therefore, there is some $P, Q \in \mathcal{P}'$ such that $P \cap Q \neq \emptyset$. As in Proposition 3.3, we may assume that no path will cross itself. If these crossings are only of the form $P[S, C_1] \cap Q[S, C_1]$ or $P[C_1, T] \cap Q[C_1, T]$, then these crossings can be eliminated as in Proposition 3.3, returning us to the previous case. Therefore, there is at least one of the form $P[S, C_1] \cap Q[C_1, T]$. **Q.E.D.**

For the rest of this thesis, without loss of generality, we shall assume that S and T are disjoint, arrangements are in general position, and do not have tangential non-vertex intersections.

Corollary 3.5 *Let $\mathcal{A}$ be an $(m, 1)$ -arrangement. Then $cr(\mathcal{A}) \geq (m - 2)$.*

Proof: Let $\mathcal{A} = (S, T, (C), \mathcal{P}, \psi)$. If $m \leq 2$, then the theorem is true trivially since $m - 2 \leq 0$. So assume $m \geq 3$.

Let $\mathcal{Q}$ be the set of paths $P \in \mathcal{P}$ such that $|P \cap C| = 1$, and $\mathcal{Q}' = \mathcal{P} \setminus \mathcal{Q}$.

By the Fundamental Theorem of Arrangements, the set $\mathcal{Q}$ can be partitioned into three sets:

1. $\mathcal{Q}_a = \{P \in \mathcal{Q} : \text{with a crossing of the form } (S, P, s, t), \text{ with } t > \psi_{\mathcal{P}}(C_1, P)\};$
2. $\mathcal{Q}_b = \{P \in \mathcal{Q} \setminus \mathcal{Q}_a : \text{with a crossing of the form } (T, P, s, t), \text{ with } t < \psi_{\mathcal{P}}(C_1, P)\}; \text{ and}$
3. $\mathcal{Q}_c = \{P \in \mathcal{Q} \setminus (\mathcal{Q}_a \cup \mathcal{Q}_b)\}$

Each member of $\mathcal{Q}_a \cup \mathcal{Q}_b \cup \mathcal{Q}'$ contributes a crossing of P with either S or T or C .

If $|\mathcal{Q}_c| \leq 2$, then $|\mathcal{Q}_a| + |\mathcal{Q}_b| + |\mathcal{Q}'| \geq m - 2$ and we are done.

If $|\mathcal{Q}_c| > 2$, then by The Fundamental Theorem of Arrangements applied to $(S, T, (C), \mathcal{Q}_c, \psi)$, every three paths in $\mathcal{Q}_c$ contributes a crossing.

Each crossing occurs between a pair of paths, and this pair can occur in at most $|\mathcal{Q}_c| - 2$ triples. Thus, there are at least $\frac{\binom{|\mathcal{Q}_c|}{3}}{|\mathcal{Q}_c| - 2}$ crossings.

Therefore, $cr(\mathcal{A}) \geq |\mathcal{Q}'| + |\mathcal{Q}_a| + |\mathcal{Q}_b| + \frac{\binom{|\mathcal{Q}_c|}{3}}{|\mathcal{Q}_c| - 2}$. It remains to show:

$$\frac{\binom{|\mathcal{Q}_c|}{3}}{|\mathcal{Q}_c| - 2} = \frac{|\mathcal{Q}_c|(|\mathcal{Q}_c| - 1)}{6} \geq (|\mathcal{Q}_c| - 2)$$

$$|\mathcal{Q}_c|^2 - |\mathcal{Q}_c| \geq 6|\mathcal{Q}_c| - 12$$

$$|\mathcal{Q}_c|^2 - 7|\mathcal{Q}_c| + 12 \geq 0$$

$$(|Q_c| - 3)(|Q_c| - 4) \geq 0$$

This inequality is satisfied for all integers. Now since $m = |Q'| + |Q_a| + |Q_b| + |Q_c|$, $cr(\mathcal{A}) \geq (m - 2)$. **Q.E.D.**

3.5 (m, n) -Arrangements

This section will show that every (m, n) -arrangement has $(m - 2)n$ crossings. This is shown by assigning $m - 2$ crossings to each of the n m -cycles in the arrangement. This is done in such a way that no crossing is assigned to more than one cycle. Since there are n of these cycles, the result is the required $(m - 2)n$ crossings.

A path $P \in \mathcal{P}$ and a cycle $C \in \mathcal{C}$ are said to be *tangent* if $|P \cap C| = 1$. P and C are *semi-tangent* if there exists an open interval P' of P containing $\nu(C, P)$ such that P' is entirely on $E_C \cup \{\nu(C, P)\}$ and $|P[C, T] \cap C| > 1$.

Let $\mathcal{A} = (S, T, \mathcal{C}, \mathcal{P}, \psi)$ be an (m, n) -arrangement. For convenience, we will sometimes use $C_0 = S$ and $C_{n+1} = T$.

For each $j = 1, 2, \dots, n$, we partition $\mathcal{P}$ into sets as follows:

1. $st(j) = \{P \in \mathcal{P} : P \text{ is semitangent to } C_j\};$
2. $cross(j) = \{P \in \mathcal{P} \setminus st(j) : |P \cap C_j| > 1\};$
3. for $0 \leq i < j$, $tan(i, j) = \{P \in \mathcal{P} : |P \cap C_j| = 1; \nu(C_j, P) \in E_{C_i}; \forall k, i < k < j, \nu(C_j, P) \in I_{C_k}\};$
 - $aft(i, j) = \{P \in tan(i, j) : P[C_j, T] \cap C_i \neq \emptyset\};$
 - $last(i, j) = \{P \in tan(i, j) \setminus aft(i, j) : P[C_i, C_j] \cap T \neq \emptyset\};$

- $rest(i, j) = tan(i, j) \setminus (aft(i, j) \cup last(i, j))$.

Lemma 3.6 *Let $P \in \mathcal{P}$ and $i \in \{1, \dots, n\}$. If $|P \cap C_i| = 1$, then, for all $j \in \{0, \dots, n+1\} \setminus \{i\}$, $\nu(P, C_j) \in E_{C_i}$.*

Proof: This is because $S \cup T \subseteq E_{C_i}$. If some $\nu(P, C_j) \in I_{C_i}$ then P has to enter and leave I_{C_i} and therefore $P \cap C_i$ has at least two points. **Q.E.D.**

Lemma 3.7 *Let $j \in \{1, \dots, n\}$. Then for every $P \in \mathcal{P}$, $P \in st(j) \cup cross(j) \cup \bigcup_{0 \leq i < j} tan(i, j)$. Moreover, these sets are pairwise disjoint.*

Proof: Let $P \in \mathcal{P}$. If $P \notin st(j) \cup cross(j)$, then P is tangent to C_j . By definition, $C_0 = S$ is disjoint from C_j . Thus, $\nu(P, S) \in E_{C_j}$, so either $P \in \bigcup_{1 \leq i < j} tan(i, j)$ or $P \in tan(0, j)$. Thus, every P is in $st(j) \cup cross(j) \cup \bigcup_{0 \leq i < j} tan(i, j)$.

Also, these sets are obviously pairwise disjoint: $st(j)$ and $cross(j)$ are disjoint by the definition of $cross(j)$; $tan(i, j)$ is disjoint from $st(j) \cup cross(j)$ since a path in $tan(i, j)$ has only one intersection with C_j whereas a path in $st(j)$ or $cross(j)$ has at least two intersections with C_j ; and finally if $0 \leq i < i' < j$ and $P \in tan(i, j)$, then by definition $\nu(P, C_j) \in E_{C_i}$ and $\nu(P, C_j) \in I_{C_{i'}}$. But $P \in tan(i', j)$ implies $\nu(P, C_j) \in E_{C_{i'}}$, a contradiction. **Q.E.D.**

For each j and $0 \leq i < j$, we define the sets $X(j)$ and $X(i, j)$ of crossings as follows:

1. if $P \in st(j)$, put into $X(j)$ the second crossing of $P[C_j, T]$ with C_j ;

2. if $P \in \text{cross}(j)$, put into $X(j)$ the first crossing of P with C_j ;
3. if $P \in \text{aft}(i, j)$, put into $X(i, j)$ the first crossing of $P[C_j, T]$ with C_i ;
4. if $P \in \text{last}(i, j)$, put into $X(i, j)$ the last crossing of $P[C_i, C_j]$ with T ;
5. if $P, Q \in \text{rest}(i, j)$ are such that $P[C_x, C_{x+1}]$ and $Q[C_y, C_{y+1}]$ cross, with $i \leq x < j \leq y$, then put into $X(i, j)$ the first crossing of $Q[C_j, T]$ with $P[C_i, C_j]$.

Lemma 3.8 *The sets $X(j)$, $X(i, j)$ are pairwise disjoint. That is:*

1. if $i \neq j$, then $X(i) \cap X(j) = \emptyset$; and
2. for all i, j, k , $X(i) \cap X(j, k) = \emptyset$;
3. if $(i, j) \neq (k, \ell)$, then $X(i, j) \cap X(k, \ell) = \emptyset$.

In order to prove Lemma 3.8, the following simple fact will be used:

Lemma 3.9 *If $P \in \text{tan}(i, j)$ and $P[C_j, T]$ crosses C_i , then either $P \in \text{st}(i)$ or there is a crossing of $P[S, C_j]$ with C_i .*

Proof: If $P \notin \text{st}(i)$, then it is because there is no open interval P' containing $\nu(P, C_i)$ and contained in E_{C_i} . Therefore, there is a point $x \in [0, 1]$ near $\psi_P(C_i, P)$ such that $P(x)$ is in I_{C_i} . Since $\nu(P, C_j) \in E_{C_i}$, $P[x, \psi_P(C_j, P)]$ crosses C_i . **Q.E.D.**

Proof of Lemma 3.8: To prove (1), the crossings in $X(i)$ are all of paths with C_i , while those in $X(j)$ are with C_j , so if $i \neq j$, $X(i) \cap X(j) = \emptyset$.

To prove (2), suppose $x \in X(i) \cap X(j, k)$. Then, because $x \in X(i)$, x is a crossing of some path P with C_i for $i \in \{1, \dots, n\}$, by definition of $X(i)$. Since $x \in X(j, k)$, it follows that $P \in \text{aft}(j, k)$ (if $P \in \text{last}(j, k)$, then x is a crossing between a path and $C_{n+1} = T$, while $P \in \text{rest}(j, k)$ implies x is a crossing between two paths, both contradictions). Thus $i = j$ and, since $P \in \text{aft}(j, k)$, x is the first crossing of $P[C_k, T]$ with C_j .

If $P \in \text{st}(j)$, then x is the second crossing of $P[C_j, T]$ with C_j . But $\nu(P, C_k) \in E_{C_j}$ implies an even number of crossings of $P[C_j, C_k]$ with C_j , so the second crossing of $P[C_j, T]$ with C_j either occurs before C_k , or else there are no crossings before C_k , implying the second crossing of $P[C_k, T]$ with C_j is not the same as the first crossing of $P[C_k, T]$ with C_j .

If $P \in \text{cross}(j)$, then x is the first crossing of P with C_j . Since $P \notin \text{st}(j)$, and since there is a crossing of $P[C_k, T]$ with C_j , then by Lemma 3.9 there must be a crossing of $P[S, C_j]$ with C_j . Such a crossing is in $X(j)$ and is not x , again a contradiction. Therefore, (2) holds.

For (3), suppose $x \in X(i, j) \cap X(k, \ell)$. It is clear that x arises from a crossing involving a path $P \in \text{tan}(i, j) \cap \text{tan}(k, \ell)$. Without loss of generality, we can assume $j \leq \ell$.

If $j = \ell$, then $P \in \text{tan}(i, j) \cap \text{tan}(k, j)$ which implies $i = k$, a contradiction. Therefore, $j < \ell$. Since P is tangent to both C_j and C_ℓ , $\nu(C_\ell, P) \in E_{C_j}$ and, therefore, $j \leq k$ (else $P \in \text{tan}(j, \ell)$ and not $\text{tan}(k, \ell)$). Thus, we have $i < j \leq k < \ell$.

If x is a crossing of a path P with C_i , then $i = k$ is the only possibility, an immediate contradiction.

If x is a crossing between P and C_{n+1} , then x occurs on $P[C_i, C_j]$; x is the last crossing of C_{n+1} by $P[C_i, C_j]$. Similarly x occurs on $P[C_k, C_\ell]$. Obviously, for both of these to occur, the intervals $P[C_i, C_j]$ and $P[C_k, C_\ell]$ must overlap, a contradiction, since $i < j \leq k < \ell$.

The last possibility is that x is a crossing between two paths P and Q . So suppose x is the crossing on $P[C_r, C_{r+1}]$ and $Q[C_s, C_{s+1}]$. We can assume $r < s$. As this is in $X(i, j) \cap X(k, \ell)$, $i \leq r < j \leq s$ and $k \leq r < \ell \leq s$. But then $r < j \leq k \leq r$, a contradiction. **Q.E.D.**

Before showing that the crossing number of an (m, n) -arrangement is at least $(m - 2)n$, one final result must be established. Define an *intersecting arrangement* as a quintuple $(S, T, (C), \mathcal{P}, \psi)$ such that:

1. C is a cycle;
2. $C \cap S \neq \emptyset$;
3. $\mathcal{P} = \{P_1, \dots, P_m\}$ is a set of paths;
4. S and T are cycles and $T[0, 1]$ is contained in one component of $\mathbf{R}^2 \setminus (C[0, 1] \cup S[0, 1])$;
5. $\psi : (\{S, T, C\}) \times \mathcal{P} \rightarrow [0, 1] \times [0, 1]$ is a function such that:
 - for each $(X, P) \in \{S, T, C\} \times \mathcal{P}$, if $\psi(X, P) = (s, t)$, then $X(s) = P(t)$;
 - for each $P \in \mathcal{P}, C \in \mathcal{C}$, if $\psi(C, P) = (s, t)$, then $0 < t < 1$;
 - for each $P \in \mathcal{P}$, $\psi(S, P) = (s, 0)$ for some $s \in [0, 1]$; and

- for each $P \in \mathcal{P}$, $\psi(T, P) = (s, 1)$ for some $s \in [0, 1]$.
6. If $X \in \{S, C\}$ and $P \in \mathcal{P}$ with $\psi(C, P) = (s, t)$, then there exists an open neighbourhood U of t in $[0, 1]$ such that $P(u)$ is on the same face of X as T for all $u \in U$.

Essentially, this is an $(m, 1)$ -arrangement, but permitting S and C to intersect. The final condition ensures that every vertex on S is in the exterior of C and every vertex on C is in the exterior of S . Furthermore, we will also assume, without loss of generality, that intersecting arrangements are in general position. Then the following result holds:

Theorem 3.10 *Let $\mathcal{A} = (S, T, (C), \mathcal{P}, \psi)$ be an intersecting arrangement. Then $cr(\mathcal{A}) \geq |\mathcal{P}|$.*

Proof: Let $m = |\mathcal{P}|$ and let J be the boundary of $S \cup C$. Since S and C intersect, some segment of S must lie on J . Denote by S_i , $i = 1, \dots, k$, the disjoint segments of S on J . For each S_i , let $\mathcal{P}_i = \{P \in \mathcal{P} : \nu(S, P) \in S_i\}$ and $m_i = |\mathcal{P}_i|$.

For each S_i , find any arc R_i inside J , disjoint from C except for its ends, so that $D_i = S_i \cup R_i$ is a cycle.

For each i , $\mathcal{A}_i = (D_i, T, (C), \mathcal{P}_i, \psi)$ is an $(m_i, 1)$ -arrangement. By Corollary 3.5 it has at least $m_i - 2$ crossings. Summed over all i gives $\sum_{i=1}^k (m_i - 2) = (m - 2k)$. But in $\mathcal{A}$, the two endpoints of each S_i are in C , for an additional $2k$ crossing. Thus, there is a total of m crossings. **Q.E.D.**

Lemma 3.11 *For each j , $|X(j)| + \sum_{0 \leq i < j} (|X(i, j)| + |C_i \cap C_j|) \geq m - 2$.*

Proof: For every j , $\mathcal{P}$ is the disjoint union $st(j) \cup cross(j) \cup \bigcup_{0 \leq i < j} tan(i, j)$.

We see that $|st(j)| + |cross(j)| + \sum_{0 \leq i < j} (|tan(i, j)|) = m$.

Since every $P \in st(j) \cup cross(j)$ contributes one crossing to $X(j)$, it follows that $|X(j)| = |st(j)| + |cross(j)|$.

Let $a = \max\{k < j : C_k \cap C_j = \emptyset\}$. (Note that $C_0 \cap C_j = \emptyset$, so a exists and $0 \leq a < j$.) Now $\nu(C_j, P) \in E_{C_a}$, so $tan(i, j) = \emptyset$ for $i < a$. Moreover, $C_a \cap C_j = \emptyset$ and $C_i \cap C_j \neq \emptyset$ for $k < i < j$.

Let $\mathcal{P}'_i = tan(i, j)$ for $i \in \{a, \dots, j-1\}$. Since C_a is disjoint from C_j , $(C_a, T, \{C_j\}, \mathcal{P}'_a[C_a, T])$ is a $(|P'_a|, 1)$ -arrangement. (If C_j separates T and C_a , then all m paths cross C_j , so $\mathcal{P} = cross(j) \cup st(j)$.) By Corollary 3.5, $|X(a, j)| \geq |P'_a| - 2$.

Now consider $i \in \{a+1, \dots, j-1\}$. In this instance, $C_i \cap C_j \neq \emptyset$. By Theorem 3.10 applied to the intersecting arrangement $(C_i, T, (C_j), \mathcal{P}'_i[C_i, C_{n+1}])$, $|X(i, j)| + |C_i \cap C_j| \geq |P'_i|$.

Thus, summing over all $i < j$,

$$|X(j)| + \sum_{0 \leq i < j} (|X(i, j)| + |C_i \cap C_j|) \geq m - 2.$$

Q.E.D.

Theorem 3.12 *Let $\mathcal{A}$ be an (m, n) -arrangement. Then $cr(\mathcal{A}) \geq (m - 2)n$.*

Proof: By Lemma 3.11, for each $j \in \{1, \dots, n\}$,

$$|X(j)| + \sum_{0 \leq i < j} (|X(i, j)| + |C_i \cap C_j|) \geq m - 2.$$

By Lemma 3.8, none of the crossings in $X(j)$ or $X(i, j)$ is counted more than once. Finally, crossings between C_i and C_j are counted for at most one j .

Thus, over all j , the crossings counted are distinct. Since there are n cycles, there are at least $(m - 2)n$ crossings. **Q.E.D.**

Chapter 4

Circular Arrangements

4.1 Introduction

This chapter provides definitions and an important result for (m, n) -circular arrangements. The result, while interesting in its own right, will later be applied to $C_m \times C_n$ to improve the major result of [13], to prove the main theorem in [4], and to show the HKS-conjecture holds for $C_7 \times C_n$.

4.2 Definitions

An (m, n) -circular arrangement is a triple $(\mathcal{C}, \mathcal{P}, \psi)$ such that:

1. $\mathcal{C} = (C_1, \dots, C_n)$ is a cyclic sequence of cycles;
2. $\mathcal{P} = \{P_1, \dots, P_m\}$ is a set of cycles;
3. $\psi : \mathcal{C} \times \mathcal{P} \rightarrow [0, 1] \times [0, 1]$ is a function such that:
 - For each $(C, P) \in \mathcal{C} \times \mathcal{P}$, if $\psi(C, P) = (s, t)$, then $C(s) = P(t)$;
 - For all $P \in \mathcal{P}$, there exists $t \in [0, 1]$ such that $\psi(C_1, P) = (0, t)$;

- For all $P \in \mathcal{P}, C_i \in \mathcal{C}, C_j \in \mathcal{C}$, if $\psi(C_i, P) = (s_i, t_i)$ and $\psi(C_j, P) = (s_j, t_j)$, then $i < j$ if and only if $t_i < t_j$;

As with arrangements, it suffices to consider only circular arrangements in general position.

A circular arrangement $(\mathcal{C}, \mathcal{P}, \psi)$ is *non-separating* if there do not exist $C, C', C'' \in \mathcal{C}$ such that C' and C'' are in different components of $\mathbb{R}^2 \setminus C$.

A circular arrangement $(\mathcal{C}, \mathcal{P}, \psi)$ is *2-disjoint* if for every $C \in \mathcal{C}$, there exist cycles C', C'' such that C, C', C'' are pairwise disjoint.

A *vertex* in the circular arrangement $(\mathcal{C}, \mathcal{P}, \psi)$ is an ordered quadruple (C, P, s, t) such that $\psi(C, P) = (s, t)$. Let $\psi_{\mathcal{C}}(C, P) = s$ and $\psi_{\mathcal{P}}(C, P) = t$.

Define $\nu : \mathcal{C} \times \mathcal{P} \rightarrow \mathbb{R}^2$ by $\nu(C, P) = C(s)$ where $\psi(C, P) = (s, t)$.

A *crossing* in the circular arrangement $(\mathcal{C}, \mathcal{P}, \psi)$ is any ordered quadruple (X, Y, s, t) such that:

1. $X, Y \in \mathcal{C} \cup \mathcal{P}$;
2. $s, t \in [0, 1]$;
3. $X(s) = Y(t)$;
4. either $X \neq Y$ or $s \neq t$
5. if $\psi(X, Y)$ is defined, then $(s, t) \neq \psi(X, Y)$.

Note that $\mathcal{C} = (C_1, \dots, C_n)$ imposes a particular cyclic ordering on the sequence of cycles. For $i, j \in \{1, \dots, n\}$, let $[C_i, C_j]_{\mathcal{C}}$ denote the subsequence

$(C_i, C_{i+1}, \dots, C_j)$. Note that if $j < i$, this still makes sense as the sequence $(C_i, C_{i+1}, \dots, C_n, C_1, \dots, C_j)$. Denote by $\langle C_i, C_j \rangle_{\mathcal{C}}$ the sequence $[C_{i+1}, C_{j-1}]_{\mathcal{C}}$ where the indices are read modulo n .

A *partitioned (m, n) -circular arrangement with z parts* is an (m, n) -circular arrangement $(\mathcal{C}, \mathcal{P}, \psi)$ together with z arrangements $\mathcal{A}_i = (S_i, T_i, \mathcal{C}_i, \mathcal{P}_i, \psi)$, $i = 1, 2, \dots, z$ such that:

- for each i , if $\mathcal{C}_i = (C_{i_1}, \dots, C_{i_r})$, then $(S_i, C_{i_1}, \dots, C_{i_r}, T_i)$ is a subsequence of $\mathcal{C}$ and every cycle in $\mathcal{C}$ appears in exactly one $\mathcal{C}_i$;
- for each i , $\mathcal{P}_i = \{P[S_i, T_i] \mid P \in \mathcal{P}\}$;
- for $i, j \in \{1, \dots, z\}$, if $i \neq j$, then either $\langle S_i, T_i \rangle_{\mathcal{C}} \cap \mathcal{C}_j = \emptyset$ or $\langle S_j, T_j \rangle_{\mathcal{C}} \cap \mathcal{C}_i = \emptyset$;
- If $i \neq j$ and $\{S_i, T_i\}, \{S_j, T_j\}$ are disjoint, then (S_i, T_j, S_j, T_i) is not a cyclic subsequence of $\mathcal{C}$.

If $n_i = |\mathcal{C}_i|$, then it is easy to confirm that $\mathcal{A}_i = (S_i, T_i, \mathcal{C}_i, \mathcal{P}_i, \psi)$ is an (m, n_i) -arrangement. This is implied by the first two conditions. The third and fourth conditions are required to limit the amount of overlap between any two arrangements. As will be seen later, this is required to ensure that crossings in the force of each cycle are disjoint.

An example of an $(m, 7)$ -circular arrangement is shown in Figure 4.1. In this figure, the paths have been omitted, showing only the cycles. This circular arrangement may be partitioned into six parts:

Figure 4.1

$$\mathcal{A}_1 = (C_7, C_3, (C_1), \mathcal{P}, \psi)$$

$$\mathcal{A}_2 = (C_1, C_3, (C_2), \mathcal{P}, \psi)$$

$$\mathcal{A}_3 = (C_2, C_4, (C_3), \mathcal{P}, \psi)$$

$$\mathcal{A}_4 = (C_3, C_7, (C_4, C_5), \mathcal{P}, \psi)$$

$$\mathcal{A}_5 = (C_5, C_7, (C_6), \mathcal{P}, \psi)$$

$$\mathcal{A}_6 = (C_6, C_1, (C_7), \mathcal{P}, \psi)$$

It is not difficult to check that all of the conditions are satisfied for this circular arrangement.

4.3 Partitioned Circular Arrangements

The purpose of this section is to prove the following result:

Theorem 4.1 *Let $\mathcal{A}$ be a partitioned (m, n) -circular arrangement. Then $cr(\mathcal{A}) \geq (m - 2)n$.*

Let $\mathcal{A}$ be a partitioned (m, n) -circular arrangement with parts $\mathcal{A}_1, \dots, \mathcal{A}_z$. For $j \in \{1, 2, \dots, n\}$ and $k \in \{1, 2, \dots, z\}$, define $st_k(j) = \emptyset$ if $C_j \notin \mathcal{C}_k$ and to be the set of all paths $P \in \mathcal{P}$ for which the segment of $P[S_k, T_k]$ is semi-tangent to C_j when considering only the arrangement $\mathcal{A}_k$, i.e., there is an open interval around $\nu(C_j, P)$ contained in E_{C_j} and the segment $P[C_j, T_k]$ crosses C_j . Similar definitions are given to $cross_k(j)$, $tan_k(i, j)$, $aft_k(i, j)$, $last_k(i, j)$, and $rest_k(i, j)$. The sets of crossings $X_k(j)$ and $X_k(i, j)$ are defined in exactly the same way they were in Section 3.5 for arrangements:

1. if $P \in st_k(j)$, put into $X_k(j)$ the second crossing of $P[C_j, T_k]$ with C_j ;

2. if $P \in \text{cross}_k(j)$, put into $X_k(j)$ the first crossing of $P[S_k, T_k]$ with C_j ;
3. if $P \in \text{aft}_k(i, j)$, put into $X_k(i, j)$ the first crossing of $P[C_j, T_k]$ with C_i ;
4. if $P \in \text{last}_k(i, j)$, put into $X_k(i, j)$ the last crossing of $P[C_i, C_j]$ with T ;
5. if $P, Q \in \text{rest}_k(i, j)$ are such that $P[C_x, C_{x+1}]$ and $Q[C_y, C_{y+1}]$ cross, with $i \leq x < j \leq y$, then put into $X_k(i, j)$ the first crossing of $Q[C_j, T_k]$ with $P[C_i, C_j]$.

For an (m, n) -circular arrangement with z parts, for all $i, j \in \{1, \dots, n\}$, let $X(j) = \cup_{k \in \{1, \dots, z\}} X_k(j)$ and $X(i, j) = \cup_{k \in \{1, \dots, z\}} X_k(i, j)$.

Lemma 4.2 *Let $\mathcal{A}$ be a partitioned (m, n) -circular arrangement with z parts, and $a, b \in \{1, \dots, z\}$. Then:*

1. *if $i \neq j$, then $X_a(i) \cap X_b(j) = \emptyset$; and*
2. *for all i, j, k , $X_a(i) \cap X_b(j, k) = \emptyset$;*
3. *if $(i, j, a) \neq (k, \ell, b)$, then $X_a(i, j) \cap X_b(k, \ell) = \emptyset$.*

Proof: By Lemma 3.8, we can assume $a \neq b$.

For (1), assume $C_i \in \mathcal{C}_a$. The crossings in $X_a(i)$ are all crossings of paths with C_i , while those in $X_b(j)$ are with C_j , so if $i \neq j$, then $X_a(i) \cap X_b(j) = \emptyset$.

For (2), suppose $C_i \in \mathcal{C}_a$, $C_j \in \mathcal{C}_b$, $C_k \in \mathcal{C}_b$, and $x \in X_a(i) \cap X_b(j, k)$. Since C_i is a cycle in $\mathcal{A}_a$, it is not a cycle in $\mathcal{A}_b$. Thus, $C_i = S_b$ or $C_i = T_b$. Let P be the path such that x is a crossing between P and C_i .

- If $C_i = S_b$, then $P \in aft_b(j, k)$ and x is the first crossing of $P[C_k, T_b]$ with C_i . In particular, this crossing is from E_{C_i} to I_{C_i} .
 - If $P \in st_a(i)$, then x is the second crossing of $P[C_i, T_a]$ with $C_i = C_j$. This implies x is a crossing from I_{C_i} to E_{C_i} , a contradiction.
 - If $P \in cross_a(i)$, then x is the first crossing of $P[S_a, T_a]$ with C_i . Since $C_i = S_b$, it follows that $S_b \in \langle S_a, T_a \rangle_C$ and, by the definition of partitioned circular arrangements, $T_b \notin \langle S_a, S_b \rangle_C$. Thus, $P[S_a, C_i]$ is disjoint from $P[S_b, T_b]$, so the crossing occurs on $P[C_i, T_a]$. This implies that the crossing is from I_{C_i} to E_{C_i} , a contradiction.
- If $C_i = T_b$, then it follows that x is the last crossing of $P[S_b, C_k]$ with $C_i = T_b$. In particular, this crossing is from I_{C_i} to E_{C_i} .
 - $P \in st_a(i)$ implies x is a crossing of $P[C_i, T_a]$ with C_i . Thus, $T_a \in \langle S_b, T_b \rangle_C$, otherwise the two intervals are disjoint, so the crossing occurs between $P[S_b, T_a]$ and C_i . But $C_i = T_b$ and C_i a cycle in $\mathcal{A}_a$ implies $S_a \in \langle T_a, T_b \rangle_C$. This gives the subsequence (S_b, T_a, S_a, T_b) , a contradiction.
 - $P \in cross_a(i)$ implies x is the first crossing of $P[S_a, T_a]$ with C_i . But since x is also a crossing from I_{C_i} to E_{C_i} , x must be a crossing of $P[C_i, T_a]$ with C_i . Since $C_i = T_b$, it must follow that $T_a \in \langle S_b, T_b \rangle_C$, as otherwise the segments are disjoint. But $C_i = T_b$ now implies that $S_a \in \langle T_a, T_b \rangle_C$. This gives the subsequence (S_b, T_a, S_a, T_b) , a contradiction.

For (3), suppose $x \in X_a(i, j) \cap X_b(k, \ell)$. If x is a crossing of a path with a cycle C , then either $C \notin \mathcal{C}_a$ or $C \notin \mathcal{C}_b$. Without loss of generality, we assume $C \notin \mathcal{C}_b$. As $x \in X_b(k, \ell)$, $C \in \{S_b, T_b\}$. Therefore, there are three possibilities to consider: x is a crossing of a path P with S_b , x is a crossing of P with T_b , or x is a crossing of two paths.

- x crosses S_b . This implies $P \in aft_b(k, \ell)$ and $C_k = S_b$. In particular, x is the first crossing of $P[C_\ell, T_b]$ with C_k , and is a crossing from E_{C_k} to I_{C_k} .
 - $P \in aft_a(i, j)$ implies x is a crossing of $P[C_j, T_a]$ with C_i . Thus, both C_j and C_ℓ occur on the segment of P between $C_i = C_k$ and x , and both T_a and T_b occur after x . This implies that $C_j \in \langle S_b, T_b \rangle_C$ and $C_\ell \in \langle S_a, T_a \rangle_C$, a contradiction to the definition of a partitioned circular arrangement.
 - $P \in last_a(i, j)$ implies x is the last crossing of $P[S_a, C_j]$ with T_a , and hence is a crossing from I_{T_a} to E_{T_a} , a contradiction.
- x crosses T_b . This implies $P \in last_b(k, \ell)$. In particular, x is the last crossing of $P[C_k, C_\ell]$ with T_b , and is a crossing from I_{T_b} to E_{T_b} .
 - $P \in aft_a(i, j)$ implies x is the first crossing of $P[C_j, T_a]$ with C_i , and hence is a crossing from E_{C_i} to I_{C_i} , a contradiction.
 - $P \in last_a(i, j)$ implies $T_a = T_b$ and x is the last crossing of $P[S_a, C_j]$ with T_a . Thus, both C_j and C_ℓ occur on the segment of P from x to $T_a = T_b$. This implies that $C_j \in \langle S_b, T_b \rangle_C$ and

$C_\ell \in \langle S_a, T_a \rangle_c$, a contradiction to the definition of a partitioned circular arrangement.

- Finally, assume x is a crossing of paths $P, Q \in \text{rest}_a(i, j) \cap \text{rest}_b(k, \ell)$. We can assume that x is a crossing of P between C_u and C_{u+1} , and of Q between C_v and C_{v+1} . We can further assume without loss of generality that $C_u, C_{u+1} \in [C_i, C_j]_c$ and $C_v, C_{v+1} \in [C_j, T_a]_c$. Now there are two possibilities:

- $C_u, C_{u+1} \in [C_k, C_\ell]_c$ and $C_v, C_{v+1} \in [C_\ell, T_b]_c$. In this case, $C_j \in [C_{u+1}, T_b]_c$, so $C_a \cap \langle S_b, T_b \rangle_c \neq \emptyset$. Similarly, $C_\ell \in [C_{u+1}, T_a]_c$, so $C_b \cap \langle S_a, T_a \rangle_c \neq \emptyset$, a contradiction.
- $C_u, C_{u+1} \in [C_\ell, T_b]_c$ and $C_v, C_{v+1} \in [C_k, C_\ell]_c$. As $T_b \in [C_{u+1}, C_v]_c$, it follows that $T_b \in \langle S_a, T_a \rangle_c$. Also, $S_b \in [T_b, C_v]_c$, so $S_b \in \langle T_b, T_a \rangle_c$, yielding the sequence (S_a, T_b, S_b, T_a) , a contradiction to the definition of a partitioned circular arrangement. **Q.E.D.**

The following lemma is now proven in the same way as its counterpart Lemma 3.11.

Lemma 4.3 *For each j , $|X(j)| + \sum_{0 \leq i < j} (|X(i, j)| + |C_i \cap C_j|) \geq m - 2$.*

Theorem 4.1 follows immediately from this lemma. The following corollaries are direct consequences of Theorem 4.1.

Corollary 4.4 *Suppose $\mathcal{A}$ is a non-separating (m, n) -circular arrangement. If there are three cycles C, C', C'' of $\mathcal{A}$ such that each is disjoint from all the other $n - 1$ cycles of $\mathcal{A}$, then $cr(\mathcal{A}) \geq (m - 2)n$.*

Proof: Assume without loss of generality that in the cyclic ordering of $\mathcal{C}$, the three cycles appear in the order (C, C', C'') . Consider the arrangements $\mathcal{A}_1 = (C, C'', [C, C'] \setminus C, \mathcal{P}, \psi)$, $\mathcal{A}_2 = (C', C, [C', C''] \setminus C', \mathcal{P}, \psi)$, and $\mathcal{A}_3 = (C'', C', [C'', C] \setminus C'', \mathcal{P}, \psi)$. It is routine to verify that these arrangements partition $\mathcal{A}$. **Q.E.D.**

Corollary 4.5 *Suppose $\mathcal{A}$ is a non-separating (m, n) -circular arrangement with $n \geq 3$. If no two cycles of $\mathcal{A}$ intersect, then $cr(\mathcal{A}) \geq (m - 2)n$.*

Chapter 5

Crossing Number of $C_m \times C_n$

5.1 Introduction

The best result previously known for the crossing number of $C_m \times C_n$ comes from [13]. It was proved there that for disjoint cycles, the HKS-conjecture holds.

The theory of arrangements can expand that result to cases where the requirement is not for all cycles to be disjoint. However, there remains a condition requiring some cycles to be disjoint. This condition is more difficult to state than the full disjoint case, but it is also much more general, covering a much larger set of drawings.

5.2 Crossing Number of $C_3 \times C_n$

The theory of arrangements gives a simple proof of the following theorems:

Theorem 5.1 *The crossing number of $C_3 \times C_3$ is 3.*

Proof: Let G be a drawing of $C_3 \times C_3$ with the minimum number of crossings. Let (C_1, C_2, C_3) be the three cycles in one family. No cycle has more than

two crossings, otherwise the result is proved.

Assume C_1 intersects C_2 . Since we can assume that the graph is in general position, there are two points of intersection between C_1 and C_2 , thus none of the cycles in the other family can cross C_1 or C_2 . This further implies that the vertices on C_1 are in the exterior face of C_2 and vice versa (in this case, the exterior face of the cycle is defined as the face in which C_3 lies). The cycles C_1 and C_2 can therefore be replaced by two non-intersecting cycles C'_1 and C'_2 resulting in a drawing G' of $C_3 \times C_3$ with fewer crossings, a contradiction.

We can now assume the three cycles are pairwise disjoint. There are two ways this can occur. If one cycle separates the other two, then all of the cycles in the other family cross this one, yielding three crossings, and proving the result. If no cycle separates the other two, Corollary 4.5 applies. **Q.E.D.**

Theorem 5.2 *The crossing number of $C_3 \times C_n$ with $n \geq 3$ is n .*

Proof: This will be shown by induction. The result is known for $C_3 \times C_3$. Assume it is known that $cr(C_3 \times C_{n-1}) = n - 1$.

If any of the n 3-cycles has a crossing, then this cycle can be removed to give a $C_3 \times C_{n-1}$ graph. Since the crossing number of this graph is known to be $n - 1$, the result is shown. Therefore, none of these cycles can have a crossing, and in particular, they are all disjoint. If one of the 3-cycles separates two others, then all of the n -cycles must cross it, so it has a crossing. Hence, no cycle separates any others. Corollary 4.5 now applies. **Q.E.D.**

This proof generalizes to:

Theorem 5.3 *If $m \geq 3$, the crossing number of any $(3, m)$ -circular arrangement is at least m .*

Theorem 5.3 should be compared with the fact that, for a $(3, m)$ -mesh, the crossing number can be as small as, roughly, $\frac{m}{2}$ [10]. Figure 5.1 shows a $(3, 7)$ -mesh with six crossings. This is not a $(3, 7)$ -circular arrangement since the 7-cycles do not hit the 3-cycles in a consistent order.

Also, using the fact that any $(4, 4)$ -mesh has crossing number at least 8, it can be similarly proven:

Theorem 5.4 *If $m \geq 4$, then the crossing number of any $(4, m)$ -circular arrangement is at least $2m$.*

5.3 Crossing Number of $C_m \times C_n$

This section applies the theory of the previous chapter to the general graph $C_m \times C_n$. This extends the result of [13] past drawings of graphs with a disjoint family to drawings of graphs that are circular arrangements. In particular, a drawing of a graph with a disjoint family is a specific example of a circular arrangement. Thus, the result of [13] is a corollary of this theorem.

Figure 5.1

Theorem 5.5 *Let G be the graph $C_m \times C_n$ with $n \geq m \geq 3$. Then every drawing of G that is either a non-separating, partitioned (m, n) -circular arrangement or a non-separating, partitioned (n, m) -circular arrangement has at least $(m - 2)n$ crossings.*

Proof: This result follows almost immediately from the previous theorems. If G is drawn as an (m, n) -circular arrangement, then, by Theorem 4.1, there are $(m - 2)n$ crossings as required. If G is drawn as an (n, m) -circular arrangement, then there are $(n - 2)m$ crossings, but $(n - 2)m \geq (m - 2)n$ since $n \geq m \geq 3$. Thus, there are at least $(m - 2)n$ crossings. **Q.E.D.**

This implies the main result of [13]:

Corollary 5.6 *Let $n \geq m \geq 3$. Then $cr_{rd}(C_m \times C_n) \geq (m - 2)n$ and $cr_{bd}(C_m \times C_n) \geq (m - 2)n$.*

Proof: This will be shown by induction.

First, note that if $m = 3$, the result is known.

Assume that $cr_{bd}(C_{m-1} \times C_n) \geq (m - 3)n$. If there is an n -cycle C that separates two other n -cycles C', C'' , then there are at least n crossings of C . Let $\mathcal{D}'$ be the drawing of $\mathcal{D}$ removing C . This is now a blue-disjoint drawing of $C_{m-1} \times C_n$, and by induction there are at least $(m - 3)n$ crossings of $\mathcal{D}'$. Thus, adding in the crossings of C , there are at least $(m - 3)n + n = (m - 2)n$ crossings in $\mathcal{D}$. If there is no n -cycle separating two others, then this is a non-separating, partitioned (n, m) -circular arrangement, and $cr(\mathcal{D}) \geq (n - 2)m \geq (m - 2)n$ as required.

Let $\mathcal{D}$ be a red-disjoint drawing of $C_m \times C_n$. If $m = n$, then this is also a blue-disjoint drawing of $C_n \times C_m$. By the previous paragraph, $cr_{rd}(C_m \times C_m) \geq (m-2)m$. For $n > m$, assume $cr_{rd}(C_m \times C_{n-1}) \geq (m-2)(n-1)$. If some m -cycle C separates two other m -cycles C', C'' , then there are at least m crossings of C . If $\mathcal{D}'$ is the drawing obtained by removing C from $\mathcal{D}$, then $\mathcal{D}'$ is a red-disjoint drawing of $C_m \times C_{n-1}$ and thus $cr(\mathcal{D}') \geq (m-2)(n-1)$. Adding in the crossings of C gives $(m-2)(n-1) + m > (m-2)n$ crossings. If there is no separating cycle, then $\mathcal{D}$ is a non-separating, partitioned (m, n) -circular arrangement, and $cr(\mathcal{D}) \geq (m-2)n$ as required. **Q.E.D.**

5.4 Drawings With At Least Three Components

Let $\mathcal{A} = (\mathcal{C}, \mathcal{P}, \psi)$ be an (m, n) -circular arrangement. A *pre-component* H of $\mathcal{C}$ is a set of cycles such that if $C \in H$ and $C' \notin H$, then $C \cap C' = \emptyset$. A pre-component H is a *component* if no proper subset of H is a pre-component of $\mathcal{C}$. A circular arrangement is *k-restricted* if the largest component contains no more than k cycles.

Let $\mathcal{A}$ be an (m, n) -circular arrangement with at least three components. The *standard sub-arrangements* of $\mathcal{A}$ are the arrangements defined as follows: for each $i \in \{1, \dots, n\}$ such that C_i is in component H and C_{i-1} is in component H' with $H \neq H'$, let $C_{i'}$ be the cycle such that $[C_i, C_{i'}]_{\mathcal{C}} \subseteq H$ and $C_{i'+1} \notin H$. Let C_j be the cycle such that $[C_i, C_{j-1}]_{\mathcal{C}} \subseteq H \cup H'$ and $C_j \notin H \cup H'$. Now define $\mathcal{A}_i = (C_{i-1}, C_j, [C_i, C_{i'}]_{\mathcal{C}}, \mathcal{P}, \psi)$.

Theorem 5.7 *Let $\mathcal{A}$ be a non-separating (m, n) -circular arrangement with at least three components, and let $\mathcal{A}_i$ be the standard sub-arrangements.*

1. *If there do not exist i, j such that (S_i, T_j, S_j, T_i) is a subsequence of $\mathcal{C}$, then the standard sub-arrangements partition $\mathcal{A}$ and $cr(\mathcal{A}) \geq (m-2)n$;*
2. *If there exists i, j such that (S_i, T_j, S_j, T_i) is a subsequence of $\mathcal{C}$, then $\mathcal{A}$ has exactly three components H_1, H_2, H_3 such that:*

- $S_i, S_j \in H_1$; $C_i \in H_2$; and $C_j \in H_3$;
- $T_j = C_i$ and $T_i = C_j$;
- $\langle S_i, S_j \rangle_{\mathcal{C}} \subseteq (H_1 \cup H_2)$;
- $\langle S_j, S_i \rangle_{\mathcal{C}} \subseteq (H_1 \cup H_3)$;

In particular, if $\mathcal{A}$ has at least four components, then $cr(\mathcal{A}) \geq (m-2)n$.

Proof: Every $C_i \in \mathcal{C}$ is in some standard sub-arrangement $\mathcal{A}_i$ since either C_i and C_{i-1} are in different components (so $C_i \in \mathcal{C}_i$) or they are in the same component (so C_i is in the same arrangement as C_{i-1}). Furthermore, for every $\mathcal{A}_i$, by definition S_i , T_i , and C_i are in different components, and are therefore disjoint.

Suppose (1) holds, i.e., there are no i, j such that (S_i, T_j, S_j, T_i) is a subsequence of $\mathcal{C}$. If $\mathcal{C}_i \cap \langle S_j, T_j \rangle_{\mathcal{C}} \neq \emptyset$, then, since $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ and S_j immediately precedes C_j in $\mathcal{C}$, $C_i \in \langle S_j, T_j \rangle_{\mathcal{C}}$ and in particular $S_i \neq S_j$ and $S_i \neq T_j$. Symmetrically, if $\mathcal{C}_j \cap \langle S_i, T_i \rangle_{\mathcal{C}} \neq \emptyset$, then $S_j \neq S_i$ and $S_j \neq T_i$. Since (S_i, S_j, T_i) and (S_j, S_i, T_j) are subsequences of $\mathcal{C}$, it follows that (S_i, T_j, S_j, T_i)

is a subsequence of $\mathcal{C}$, a contradiction. Therefore, $\mathcal{A}$ with the standard subarrangements is a partitioned circular arrangement by Theorem 4.1.

Now suppose that (S_i, T_j, S_j, T_i) occurs. Then $\mathcal{C} \subseteq \langle S_i, T_i \rangle_{\mathcal{C}} \cup \langle S_j, T_j \rangle_{\mathcal{C}}$. Let H_1, H_2, H_3 be distinct components such that $S_i \in H_1$, $C_i \subseteq H_2$, and $T_i \in H_3$. By definition of T_i , $\langle S_i, T_i \rangle_{\mathcal{C}} \subseteq H_1 \cup H_2$. In particular, $S_j \in H_1 \cup H_2$ and $T_j \in H_1 \cup H_2$. Since S_j and T_j are in different components, one is in H_1 and the other is in H_2 .

Let $H' \in \{H_1, H_2\}$ be the component containing S_j and $H'' \in \{H_1, H_2\}$ be the component containing T_j . Since C_j is in a different component than S_j and T_j , $C_j \cap (H_1 \cup H_2) = \emptyset$. Since $C_{j-1} = S_j$, either $C_j \in \langle C_i, T_i \rangle_{\mathcal{C}}$ or $C_j = T_i$. In the first case, this contradicts $C_j \cap (H_1 \cup H_2) = \emptyset$. Therefore, $C_j = T_i$ and, symmetrically, $C_i = T_j$. Consequently, $H'' = H_2$, $H' = H_1$, and $C_j \subseteq H_3$. Thus, $\langle S_j, T_j \rangle_{\mathcal{C}} \subseteq H_1 \cup H_3$, so $\mathcal{C} \subseteq \langle S_i, T_i \rangle_{\mathcal{C}} \cup \langle S_j, T_j \rangle_{\mathcal{C}} \subseteq H_1 \cup H_2 \cup H_3$.

If $\mathcal{A}$ has at least four components, then there is a cycle $C \notin (H_1 \cup H_2 \cup H_3)$, contradicting the earlier conclusion that $\mathcal{C} \subseteq \langle S_i, T_i \rangle_{\mathcal{C}} \cup \langle S_j, T_j \rangle_{\mathcal{C}}$. Therefore $\mathcal{A}$ has three components. **Q.E.D.**

5.5 2-Restricted Drawings

In [4], the main results of which were recounted in Chapter 2, a *restricted* drawing of $C_m \times C_n$ is a drawing in which either each m -cycle intersects at most one other m -cycle, or each n -cycle intersects at most one other n -cycle. This is a 2-restricted drawing, using the current notation. Chapter 2 recounts the attempt in [4] to prove the following theorem (previously denoted as Theorem 2.3), which will be proven here:

Theorem 5.8 *Let m, n be such that $n \geq m \geq 3$. Then every drawing of $C_m \times C_n$ such that either the n m -cycles or the m n -cycles are 2-restricted has at least $(m - 2)n$ crossings.*

Theorem 5.8 follows from the following fact (previously denoted as Lemma 2.4).

Theorem 5.9 *Let $\mathcal{A}$ be a non-separating, 2-restricted (m, n) -circular arrangement with $n \geq 5$. Then $cr(\mathcal{A}) \geq (m - 2)n$.*

Proof of Theorem 5.8: This will be shown by induction.

Note that if $m \leq 5$, then the result is known by [8].

For $n \geq m$, assume that $cr_{br}(C_{m-1} \times C_n) \geq (m - 3)n$. If an n -cycle C separates two other n -cycles C', C'' , then there are at least n crossings of C . Let $\mathcal{D}'$ be the drawing of $\mathcal{D}$ removing C . This is now a blue-restricted drawing of $C_{m-1} \times C_n$, and by induction there are at least $(m - 3)n$ crossings of $\mathcal{D}'$. Thus, adding in the crossings of C , there are at least $(m - 3)n + n = (m - 2)n$ crossings in $\mathcal{D}$. If no n -cycle separates two others, then Theorem 5.9 applies and $cr(\mathcal{D}) \geq (n - 2)m \geq (m - 2)n$ as required.

If $m = n$ and $\mathcal{D}$ is a red-restricted drawing of $C_m \times C_n$, then this is also a blue-restricted drawing of $C_n \times C_m$, and the previous result applies. Now assume $n > m$ and $cr_{rr}(C_m \times C_{n-1}) \geq (m - 2)(n - 1)$. If some m -cycle C separates two other m -cycles C', C'' , then there are at least m crossings of C . If $\mathcal{D}'$ is the drawing obtained by removing C from $\mathcal{D}$, then $\mathcal{D}'$ is a red-restricted drawing of $C_m \times C_{n-1}$ and thus $cr(\mathcal{D}') \geq (m - 2)(n - 1)$. Adding in

the crossings of C gives $(m-2)(n-1) + m > (m-2)n$ crossings. If there is no separating cycle, then $\mathcal{D}$ is a red-restricted drawing of $C_m \times C_n$, so apply Theorem 5.9 to get $cr(\mathcal{D}) \geq (m-2)n$. **Q.E.D.**

Proof of Theorem 5.9: If $n \geq 7$, then there are at least four components, and the result is a consequence of Theorem 5.7. (More generally, if $\mathcal{A}$ is a non-separating k -restricted (m, n) -circular arrangement with $n \geq 3k + 1$, then $cr(\mathcal{A}) \geq (m-2)n$.)

If $n \in \{5, 6\}$, it only remains to demonstrate a set of sub-arrangements that satisfy the conditions of a partitioned circular arrangement. Label the cycles as $\mathcal{C} = (C_1, C_2, \dots, C_n)$. If the subsequence (S_i, T_j, S_j, T_i) does not arise from the standard sub-arrangements, then by Theorem 5.7, $\mathcal{A}$ is partitioned. So assume that (S_i, T_j, S_j, T_i) arises from the standard sub-arrangements. Then by Theorem 5.7, there are exactly 3 components each containing two cycles of $\mathcal{C}$ and $S_i, S_j \in H_1$, $C_i \in H_2$ and $C_j \in H_3$. Without loss of generality, we may assume that $S_i = C_1$.

For $n = 6$, by Theorem 5.7, we can conclude that $S_j = C_4$, $C_2, C_3 \in H_2$, and $C_5, C_6 \in H_3$. With this configuration, we can make a new set of sub-arrangements:

- $\mathcal{A}_1 = (C_6, C_2, (C_1));$
- $\mathcal{A}_2 = (C_1, C_5, (C_2, C_3));$
- $\mathcal{A}_4 = (C_3, C_5, (C_4));$
- $\mathcal{A}_5 = (C_3, C_1, (C_5, C_6)).$

In order to check that these arrangements partition $\mathcal{A}$, it suffices to verify that, for $i \neq j$, $\langle S_i, T_i \rangle_{\mathcal{C}} \cup \langle S_j, T_j \rangle_{\mathcal{C}} \neq \mathcal{C}$ and that if $C \in \langle S_i, T_i \rangle_{\mathcal{C}} \setminus \mathcal{C}_i$, then, for $C \in \mathcal{C}_j$, $\mathcal{C}_i \cap \langle S_j, T_j \rangle_{\mathcal{C}} = \emptyset$. These verifications have been left to the reader.

For $n = 5$, there are two cases. First, $H_1 = \{C_1, C_3\}$, $H_2 = \{C_2\}$ and $H_3 = \{C_4, C_5\}$. Secondly, $H_1 = \{C_1, C_4\}$, $H_2 = \{C_2, C_3\}$ and $H_3 = \{C_5\}$. Note, however, that these two cases are identical by interchanging H_2 and H_3 and rotating the indices. Therefore, only the first case has to be shown. Consider the following set of sub-arrangements:

- $\mathcal{A}_1 = (C_5, C_2, (C_1));$
- $\mathcal{A}_2 = (C_1, C_4, (C_2));$
- $\mathcal{A}_3 = (C_2, C_4, (C_3));$
- $\mathcal{A}_4 = (C_2, C_1, (C_4, C_5)).$

It is now easy to verify that $\mathcal{A}$ with these sub-arrangements forms a partitioned circular arrangement. **Q.E.D.**

The following corollary and proof are taken from [4]:

Corollary 5.10 $cr(C_6 \times C_n) \geq 4n$ for $n \geq 6$.

Proof: This will be shown by induction. It is known that $cr(C_6 \times C_6) = 24$ [1].

Now let $n > m$ and suppose that $cr(C_6 \times C_{n-1}) = 4(n-1)$. Let $\mathcal{D}$ be a drawing of $C_6 \times C_n$ with $n > 6$. Hence, $\mathcal{D}$ has n 6-cycles. If any one of the n 6-cycles in $\mathcal{D}$, say R , has at least four crossings, then we let $\mathcal{D}'$ be the

drawing of $C_6 \times C_{n-1}$ obtained by removing the edges of the cycle R from $\mathcal{D}$. Hence, $cr(\mathcal{D}) \geq cr(D') + 4 = 4(n-1) + 4 = 4n$.

If every 6-cycle in D has fewer than four crossings, then each 6-cycle intersects at most one other 6-cycle and in at most two points. Hence, the family of n 6-cycles is restricted and, by Theorem 5.8, $cr(\mathcal{D}) \geq 4n$. **Q.E.D.**

We know that any $(5, 5)$ -mesh has at least 15 crossings [10] and any $(6, 6)$ -mesh has at least 24 crossings [1]. Therefore, the same induction as in Corollary 5.10 yields:

Corollary 5.11 *Let $\mathcal{A}$ be an (m, n) -circular arrangement with $m \leq 6$ and $n \geq m$. Then $cr(\mathcal{A}) \geq (m-2)n$.*

5.6 3-Restricted Drawings

The next result is needed to show $cr(C_7 \times C_n) \geq 5n$.

Theorem 5.12 *Let $\mathcal{A}$ be a non-separating, 3-restricted (m, n) -circular arrangement with $n \geq 7$. Then $\mathcal{A}$ is partitioned.*

Proof: If $n \geq 10$, then $\mathcal{A}$ has at least four components and we are done by Theorem 5.7. For $7 \leq n \leq 9$, we may assume $\mathcal{A}$ has exactly three components $\{H_1, H_2, H_3\}$. Furthermore, we can assume S_i, T_j, S_j, T_i is a subsequence of $\mathcal{C}$ for some $i \neq j$ in the standard sub-arrangements. Therefore, $S_i, S_j \in H_1$, $T_j = C_i \in \mathcal{C}_i \subseteq H_2$ and $T_i = C_j \in \mathcal{C}_j \subseteq H_3$.

If $n = 9$, each of the three components is made up of three cycles. Therefore, $\langle S_i, S_j \rangle_{\mathcal{C}}$ has at least three cycles, and $\langle S_i, S_j \rangle_{\mathcal{C}}$ has at least three cycles.

Since $S_i = C_1$, it follows that $S_j \in \{C_5, C_6\}$. Also note that $S_i = C_1$ and $S_j = C_5$ gives the same configuration as $C_i = C_1$ and $C_j = C_6$ by interchanging the i and j and renumbering the cycles. Therefore, there is only one possibility to be considered.

Since $S_i = C_1$ and $S_j = C_5$, then $C_2, C_3, C_4 \in H_2$, $C_1, C_5 \in H_1$, and $C_6 \in H_3$. Since H_1 has three cycles, one of $\{C_7, C_8, C_9\}$ is in H_1 . The list below gives all the three possible configurations and, in each case, a set of arrangements. It is routine to show that in each case the set of arrangements defines a partitioned circular arrangement:

- $C_1, C_5, C_9 \in H_1, C_2, C_3, C_4 \in H_2, C_6, C_7, C_8 \in H_3$.

- $\mathcal{A}_1 = (C_1, C_6, (C_2, C_3, C_4))$

- $\mathcal{A}_2 = (C_8, C_2, (C_9, C_1))$

- $\mathcal{A}_3 = (C_4, C_6, (C_5))$

- $\mathcal{A}_4 = (C_4, C_9, (C_6, C_7, C_8))$

- $C_1, C_5, C_8 \in H_1, C_2, C_3, C_4 \in H_2, C_6, C_7, C_9 \in H_3$.

- $\mathcal{A}_1 = (C_9, C_2, (C_1))$

- $\mathcal{A}_2 = (C_1, C_6, (C_2, C_3, C_4))$

- $\mathcal{A}_3 = (C_4, C_6, (C_5))$

- $\mathcal{A}_4 = (C_4, C_8, (C_6, C_7))$

- $\mathcal{A}_5 = (C_4, C_9, (C_8))$

- $\mathcal{A}_6 = (C_8, C_2, (C_9))$

- $C_1, C_5, C_7 \in H_1, C_2, C_3, C_4 \in H_2, C_6, C_8, C_9 \in H_3$.

- $\mathcal{A}_1 = (C_9, C_2, (C_1))$
- $\mathcal{A}_2 = (C_1, C_6, (C_2, C_3, C_4))$
- $\mathcal{A}_3 = (C_4, C_6, (C_5))$
- $\mathcal{A}_4 = (C_4, C_7, (C_6))$
- $\mathcal{A}_5 = (C_4, C_8, (C_7))$
- $\mathcal{A}_6 = (C_7, C_2, (C_8, C_9))$

If $n = 8$, then two of the components have three cycles, and the third has two cycles. Similar to above, we get that $S_i = C_1$ and $S_j \in \{C_4, C_5, C_6\}$. Again, note that $(S_i, S_j) = (C_1, C_4)$ is identical to $(S_i, S_j) = (C_1, C_6)$ by interchanging the indices. Therefore, there are two possibilities to check.

If $S_j = C_4$, then $\{C_2, C_3\} = H_2$ and H_1 and H_3 have three cycles. Since it is known that $C_1, C_4 \in H_1$ and $C_5 \in H_3$, then one of $\{C_5, C_6, C_7, C_8\}$ is the third member of H_1 . The three possibilities with the partitioning arrangements are given below.

- $C_1, C_4, C_8 \in H_1, C_2, C_3 \in H_2, C_5, C_6, C_7 \in H_3$.

- $\mathcal{A}_1 = (C_1, C_5, (C_2, C_3))$
- $\mathcal{A}_2 = (C_7, C_2, (C_8, C_1))$
- $\mathcal{A}_3 = (C_3, C_5, (C_4))$
- $\mathcal{A}_4 = (C_3, C_8, (C_5, C_6, C_7))$

- $C_1, C_4, C_7 \in H_1, C_2, C_3 \in H_2, C_5, C_6, C_8 \in H_3$.

- $\mathcal{A}_1 = (C_8, C_2, (C_1))$
- $\mathcal{A}_2 = (C_1, C_5, (C_2, C_3))$
- $\mathcal{A}_3 = (C_3, C_5, (C_4))$
- $\mathcal{A}_4 = (C_3, C_7, (C_5, C_6))$
- $\mathcal{A}_5 = (C_3, C_8, (C_7))$
- $\mathcal{A}_6 = (C_7, C_2, (C_8))$

- $C_1, C_4, C_6 \in H_1, C_2, C_3 \in H_2, C_5, C_7, C_8 \in H_3.$

- $\mathcal{A}_1 = (C_8, C_2, (C_1))$
- $\mathcal{A}_2 = (C_1, C_5, (C_2, C_3))$
- $\mathcal{A}_3 = (C_3, C_5, (C_4))$
- $\mathcal{A}_4 = (C_3, C_6, (C_5))$
- $\mathcal{A}_5 = (C_3, C_7, (C_6))$
- $\mathcal{A}_6 = (C_6, C_2, (C_7, C_8))$

If $S_j = C_5$, then there are three possibilities. First, there are two cycles in H_1 . These have already been identified as S_i and S_j , so we have:

- $C_1, C_5 \in H_1, C_2, C_3, C_4 \in H_2, C_6, C_7, C_8 \in H_3.$

- $\mathcal{A}_1 = (C_8, C_2, (C_1))$
- $\mathcal{A}_2 = (C_1, C_5, (C_2, C_3, C_4))$
- $\mathcal{A}_3 = (C_4, C_6, (C_5))$
- $\mathcal{A}_4 = (C_5, C_1, (C_6, C_7, C_8))$

If H_2 has two cycles, then one of $\{C_3, C_4\}$ is in H_1 . This gives two possibilities:

- $C_1, C_3, C_5 \in H_1, C_2, C_4 \in H_2, C_6, C_7, C_8 \in H_3$.

- $\mathcal{A}_1 = (C_8, C_2, (C_1))$

- $\mathcal{A}_2 = (C_8, C_3, (C_2))$

- $\mathcal{A}_1 = (C_2, C_6, (C_3))$

- $\mathcal{A}_2 = (C_3, C_6, (C_4))$

- $\mathcal{A}_3 = (C_4, C_6, (C_5))$

- $\mathcal{A}_4 = (C_4, C_1, (C_6, C_7, C_8))$

- $C_1, C_4, C_5 \in H_1, C_2, C_3 \in H_2, C_6, C_7, C_8 \in H_3$.

- $\mathcal{A}_1 = (C_8, C_2, (C_1))$

- $\mathcal{A}_2 = (C_1, C_6, (C_2, C_3))$

- $\mathcal{A}_3 = (C_3, C_6, (C_4, C_5))$

- $\mathcal{A}_4 = (C_3, C_1, (C_6, C_7, C_8))$

If H_3 has two cycles, this is identical to the case where H_2 has two cycles, again by interchanging the indices i and j .

If $n = 7$, then we get that $S_i = C_1$ and $S_j \in \{C_3, C_4, C_5, C_6\}$. Again, note that $(S_i, S_j) = (C_1, C_3)$ is identical to $(S_i, S_j) = (C_1, C_6)$ and that $(S_i, S_j) = (C_1, C_4)$ is identical to $(S_i, S_j) = (C_1, C_5)$ by interchanging the indices. Therefore, there are two possibilities to check.

If $S_j = C_3$, then $\{C_2\} = H_2$ and H_1 and H_3 have three cycles. Since it is known that $C_1, C_3 \in H_1$ and $C_4 \in H_3$, then one of $\{C_5, C_6, C_7\}$ is the third member of H_1 . The three possibilities with the arrangements to partition the circular arrangement are given below.

- $C_1, C_3, C_7 \in H_1, C_2 \in H_2, C_4, C_5, C_6 \in H_3$.

- $\mathcal{A}_1 = (C_1, C_4, (C_2))$

- $\mathcal{A}_2 = (C_6, C_2, (C_7, C_1))$

- $\mathcal{A}_3 = (C_2, C_4, (C_3))$

- $\mathcal{A}_4 = (C_2, C_7, (C_4, C_5, C_6))$

- $C_1, C_3, C_6 \in H_1, C_2 \in H_2, C_4, C_5, C_7 \in H_3$.

- $\mathcal{A}_1 = (C_7, C_2, (C_1))$

- $\mathcal{A}_2 = (C_1, C_4, (C_2))$

- $\mathcal{A}_3 = (C_2, C_4, (C_3))$

- $\mathcal{A}_4 = (C_2, C_6, (C_4, C_5))$

- $\mathcal{A}_5 = (C_2, C_7, (C_6))$

- $\mathcal{A}_6 = (C_6, C_2, (C_7))$

- $C_1, C_3, C_5 \in H_1, C_2 \in H_2, C_4, C_6, C_7 \in H_3$.

- $\mathcal{A}_1 = (C_7, C_2, (C_1))$

- $\mathcal{A}_2 = (C_1, C_4, (C_2))$

- $\mathcal{A}_3 = (C_2, C_4, (C_3))$

- $\mathcal{A}_4 = (C_2, C_5, (C_4))$
- $\mathcal{A}_5 = (C_2, C_6, (C_5))$
- $\mathcal{A}_6 = (C_5, C_2, (C_6, C_7))$

If $S_j = C_4$, then there are three possibilities. First, there are two cycles in H_1 . These have already been identified as S_i and S_j , so we have:

- $C_1, C_4 \in H_1, C_2, C_3 \in H_2, C_5, C_6, C_7 \in H_3$.
 - $\mathcal{A}_1 = (C_7, C_2, (C_1))$
 - $\mathcal{A}_2 = (C_1, C_4, (C_2, C_3))$
 - $\mathcal{A}_3 = (C_3, C_6, (C_4))$
 - $\mathcal{A}_4 = (C_4, C_1, (C_5, C_6, C_7))$

If H_1 has three cycles, then one of $\{C_3, C_6, C_7\}$ is in H_1 . This gives three possibilities:

- $C_1, C_3, C_4 \in H_1, C_2 \in H_2, C_5, C_6, C_7 \in H_3$.
 - $\mathcal{A}_1 = (C_7, C_2, (C_1))$
 - $\mathcal{A}_2 = (C_7, C_3, (C_2))$
 - $\mathcal{A}_3 = (C_2, C_5, (C_3, C_4))$
 - $\mathcal{A}_4 = (C_4, C_1, (C_5, C_6, C_7))$
- $C_1, C_4, C_6 \in H_1, C_2, C_3 \in H_2, C_5, C_7 \in H_3$.
 - $\mathcal{A}_1 = (C_7, C_2, (C_1))$

$$- \mathcal{A}_2 = (C_1, C_5, (C_2, C_3))$$

$$- \mathcal{A}_3 = (C_3, C_5, (C_4))$$

$$- \mathcal{A}_4 = (C_3, C_6, (C_5))$$

$$- \mathcal{A}_5 = (C_5, C_1, (C_6))$$

$$- \mathcal{A}_6 = (C_6, C_2, (C_7))$$

$$\bullet C_1, C_4, C_7 \in H_1, C_2, C_3 \in H_2, C_5, C_6 \in H_3.$$

$$- \mathcal{A}_1 = (C_7, C_2, (C_1))$$

$$- \mathcal{A}_2 = (C_1, C_5, (C_2, C_3))$$

$$- \mathcal{A}_3 = (C_3, C_5, (C_4))$$

$$- \mathcal{A}_4 = (C_3, C_7, (C_5, C_6))$$

$$- \mathcal{A}_5 = (C_6, C_2, (C_7))$$

Thus, in all cases, $\mathcal{A}$ is a partitioned circular arrangement, hence $cr(\mathcal{A}) \geq (m - 2)n$. **Q.E.D.**

5.7 Crossing Number of $C_7 \times C_n$

This section uses the previous work to establish that the crossing number of $C_7 \times C_n = 5n$. This is a new result. An *optimal drawing* $\mathcal{D}$ of $\mathcal{G} = C_m \times C_n$ is a drawing in which $cr(\mathcal{D}) = cr(\mathcal{G})$.

Lemma 5.13 *Let $\mathcal{D}$ be a drawing of $C_7 \times C_n$. If $\mathcal{D}$ has a component with at least four cycles, then either some 7-cycle has at least 5 crossings or $\mathcal{D}$ is not optimal.*

Proof: We assume that no 7-cycle has more than four crossings. In particular, no 7-cycle is separating. Let H be a component with four or more cycles and let $C_i, C_j \in H$. We claim that $|C_i \cap C_j| \leq 2$.

To see this, note that $|H| \geq 3$ implies one of C_i and C_j (say C_i) intersects another cycle in H and so if $|C_i \cap C_j| \geq 4$, then C_i would have at least five crossings.

Now suppose $C_i, C_j \in H$ intersect. Let S_i be the segment of C_i in I_{C_j} and let S_j be the segment of C_j in I_{C_i} . If there is no vertex of C_i on S_j and no vertex of C_j on S_i , then create the cycles C'_i and C'_j by deleting the segment S_i (resp S_j) and replacing it with the segment S_j (resp S_i). Then the drawing $\mathcal{D}'$ formed by removing C_i and C_j from $\mathcal{D}$ and replacing them with C'_i and C'_j has two fewer crossings. Therefore, $\mathcal{D}$ is not optimal.

First, assume that one cycle is intersected by three other cycles. This gives 6 crossings for that cycle, so we may assume that each cycle intersects at most two others. Without loss of generality, assume that C_1 intersects C_2 and C_3 . Then there are four crossings of C_1 . If there are any vertices of C_2 or C_3 in the interior of C_1 , then the path associated with this vertex must cross C_1 for a total of five crossings. Therefore, there are no vertices of C_2 or C_3 in the interior of C_1 . If $C_2 \cap C_3 \neq \emptyset$, then both C_2 and C_3 have four crossings, so C_4 must be disjoint, or else one of C_1, C_2, C_3 has more than five crossings. So C_2 and C_3 are disjoint.

If C_2 intersects C_4 , then C_2 has four crossings - two with C_1 and two with C_4 . Therefore, there are no vertices of C_2, C_3 , or C_4 in the interior of C_1 , as above, and likewise there are no vertices of C_1, C_3 , or C_4 in the interior

of C_2 . Therefore, $\mathcal{D}$ is not optimal as we can eliminate the intersections in $C_1 \cap C_2$. **Q.E.D.**

Theorem 5.14 $cr(C_7 \times C_n) \geq 5n$ if $n \geq 7$.

Proof: It is known that $cr(C_7 \times C_7) = 35$ [2]. Let $\mathcal{D}$ be an optimal drawing of $C_7 \times C_n$ for some $n > 7$. Assume that $cr(C_7 \times C_{n-1}) \geq 5(n-1)$.

If some 7-cycle C has five intersections, form $\mathcal{D}'$ by removing C from $\mathcal{D}$ to yield a drawing of $C_7 \times C_{n-1}$. From the induction hypothesis, $cr(\mathcal{D}') \geq 5(n-1)$, so $cr(\mathcal{D}) \geq 5(n-1) + 5 = 5n$ as required.

If some 7-cycle C separates two other 7-cycles, then there are at least 7 crossings of this cycle, and the previous result applies.

By Lemma 5.13, the n 7-cycles form at least three components. By Theorems 5.7 and 5.12, $\mathcal{D}$ is a drawing of a partitioned, non-separating $(7, n)$ -circular arrangement. By Theorem 4.1, $cr(\mathcal{D}) \geq 5n$. **Q.E.D.**

We know that any $(7, 7)$ -mesh has at least 35 crossings [2]. This provides the base of the induction for the following generalization of Theorem 5.14:

Theorem 5.15 *Let $\mathcal{A}$ be a $(7, n)$ -circular arrangement with $n \geq 7$. Then $cr(\mathcal{A}) \geq 5n$.*

Chapter 6

Future Work on Arrangements

6.1 Introduction

This work on arrangements has pushed the knowledge of crossing numbers further than it was before, but it has the potential to lead to even more impressive results, and maybe to provide the final proof of the HKS-conjecture.

6.2 Symmetry of Force

Up to now, the definitions of force have been symmetric. That is, the m n -cycles $C_1, \dots, C_m$ have two orderings in $C_m \times C_n$. They can be viewed in the order $(C_1, \dots, C_m)$, or in reverse order $(C_m, \dots, C_1)$. The crossings associated with the force of a cycle C_i have been independent of the ordering of the cycles. This thesis deviates from this viewpoint. The crossings in the force of a cycle are now dependent upon the direction in which the cycles are traversed. This is an important shift that was essential to producing the results on arrangements, and can perhaps be extended and further exploited with evolving definitions of the force of a cycle.

6.3 Disjoint S and T

Most of the theory of Chapter 3 requires S and T to be disjoint. This is a requirement that can probably be eliminated. In fact, using the Fundamental Theorem of Arrangements, it can be shown that the crossing number for any $(m, 1)$ -arrangement is at least $m - 2$, as required. However, for arrangements with two or more cycles, there is a potential for double-counting the crossings in $S \cap T$ using these techniques. Nevertheless, we conjecture that this can be dealt with, giving the result for any (m, n) -arrangement.

6.4 Exterior Faces

The main limitation to the theory of arrangements involves the notion of an exterior face. Let $\mathcal{A} = (S, T, \mathcal{C}, \mathcal{P}, \psi)$ be an arrangement. The definition of an arrangement says that S and T are in the same face of the arrangement, and this face is well-defined. This is an important condition. The whole notion of counting crossing numbers of arrangements deals with paths starting from the “outside”, meeting the boundary and potentially entering “inside”, then returning to the “outside”. Without this notion, the theory falls apart. The Fundamental Theorem of Arrangements requires this fact, as do the subsequent results.

6.5 Disjointness of S and $\mathcal{C}$

Throughout most of this paper, it has been assumed that S is disjoint from $\mathcal{C}$. This is not strictly required. Examining the proof of Lemma 3.7 shows

that the only requirement is that for each $C_i \in \mathcal{C}$ and $P \in \mathcal{P}$, there is a $j < i$ such that $\psi_{\mathcal{P}}(C_i, P) \in E_{C_j}$ and $C_j \neq T$. A disjoint S satisfies this condition; however, it seems it is not strictly required. This fact may be useful in attempting to expand the theory of arrangements.

This also suggests that the notion of an exterior face can be relaxed. As noted above, the true requirement is that there is a $j < i$ such that $\psi_{\mathcal{P}}(C_i, P) \in E_{C_j}$ and $C_j \neq T$. The use of $E_{\mathcal{C}}$ is convenient in places, but is not strictly required.

6.6 Generalized Arrangements

A *generalized (m, n) -arrangement* is a triple $\mathcal{A} = (\mathcal{C}, \mathcal{P}, \psi)$ in which $\mathcal{C} = (C_0, C_1, \dots, C_{n+1})$ is a sequence of cycles, $\mathcal{P} = \{P_1, P_2, \dots, P_m\}$ is a set of paths, each starting on C_0 and ending on C_{n+1} , and ψ assigns to each path-cycle pair (P, C) a vertex, so that the vertices on P occur in the order $C_0, C_1, \dots, C_{n+1}$. Furthermore, for each $i = 1, 2, \dots, n$, there exist $b(i), a(i)$ such that $0 \leq b(i) < i < a(i) \leq n+1$ such that $C_{b(i)}, C_i$, and $C_{a(i)}$ are pairwise disjoint, and $C_{b(i)}$ and $C_{a(i)}$ are both in the same component of $\mathbf{R}^2 \setminus C_i$. We conjecture (generalizing Theorem 3.12):

Conjecture 6.1 *If $\mathcal{A}$ is a generalized (m, n) -arrangement, then $cr(\mathcal{A}) \geq (m - 2)n$.*

The techniques used in the proof of Theorem 3.12 seem to be inadequate for proving Conjecture 6.1. There is a difficulty in ensuring the C_i satisfy the disjointness condition that either $C_i \cap \langle C_{b(j)}, C_{a(j)} \rangle_{\mathcal{C}} = \emptyset$ or $C_j \cap \langle C_{b(i)}, C_{a(i)} \rangle_{\mathcal{C}} = \emptyset$.

6.7 Non-partitioned Circular Arrangements

This thesis has dealt with non-separating, partitioned circular arrangements. The difference between a circular arrangement and a partitioned circular arrangement is that the “S” and “T” cycles in a circular arrangement are defined for individual cycles, whereas in a partitioned circular arrangement, they are defined for groups of cycles, with non-overlapping conditions. Proving results for the non-partitioned circular arrangements is the next natural step in generalizing the notion of an arrangement. This leads to the following conjecture:

Conjecture 6.2: *A non-separating, 2-disjoint (m, n) -circular arrangement has at least $(m - 2)n$ crossings.*

In attempting a proof of this conjecture, most of the proofs for an arrangement in Chapters 3 and 4 can be followed very closely. However, some problems do arise. First, in defining the force for C_j , if C_i is a cycle between S_j and C_j , there is no guarantee that C_i is disjoint from T_j . This is important in applying the present theory. Secondly, when the paths are partitioned into disjoint sets, it is much more difficult to show that the crossings counted among $rest(i, j)$ are disjoint, as in theorem 4.2. It is unclear at this point whether this requires a redefinition of the force of a cycle, or if in fact the crossings are actually disjoint. However, the problem with intersecting cycles remains.

This conjecture has the following consequences:

Consequence 6.3: *If the HKS-conjecture is true for $C_m \times C_m$, then it is*

true for $C_m \times C_n$.

Consequence 6.4: For each m , there is an $N = N(m)$ such that, if $n \geq N$, then $cr(C_m \times C_n) = (m - 2)n$.

Proof of Consequence 6.3: Suppose $n > m$ and, inductively, $cr(C_m \times C_{n-1}) = (m - 2)(n - 1)$. If some m -cycle has at least $m - 2$ crossings, then deleting its edges leaves a drawing of $C_m \times C_{n-1}$, which inductively has at least $(m - 2)(n - 1)$ crossings, so the original drawing has at least $(m - 2)(n - 1) + (m - 2) = (m - 2)n$ crossings.

If, on the other hand, the drawing of $C_m \times C_n$ is such that no cycle has at least $m - 2$ crossings, then:

- there is no m -cycle that separates two other m -cycles; and
- each m -cycle intersects at most $(m - 3)/2$ other m -cycles, which implies that the drawing is 2-disjoint, with the m -cycles as $\mathcal{C}$.

By Conjecture 6.2, the drawing has at least $(m - 2)n$ crossings, as required.

Q.E.D.

Proof of Consequence 6.4: For each $n \geq m + 2$, either some m -cycle has at least m crossings or the drawing is separation-free and 2-disjoint. Now suppose $n \geq m^2 + 2m$.

If no m -cycle has at least m crossings, then, by Conjecture 6.2, the drawing has at least $(m - 2)n$ crossings and we are done. More generally, suppose $i_1, i_2, \dots, i_k$ are indices such that, for $j = 1, 2, \dots, k$, the m -cycle C_{i_j} has at least m crossings in the drawing obtained by deleting the edges of

$C_{i_1}, C_{i_2}, \dots, C_{i_{j-1}}$. Then the original drawing obviously has at least km crossings, so if $k \geq (m-2)n/m$, then the original drawing has at least $(m-2)n$ crossings.

Therefore, we can suppose $k < (m-2)n/m$ and, after deleting the edges of $C_{i_1}, C_{i_2}, \dots, C_{i_k}$, no remaining m -cycle has at least m crossings. Since $n - k \geq n - ((m-2)n/m) = n(2/m) \geq 2m + 2 \geq m + 2$, the resulting drawing is separation-free and 2-disjoint. By Conjecture 6.2, this drawing has at least $(m-2)(n-k)$ crossings, so the original drawing has at least $(m-2)(n-k) + km \geq (m-2)n$ crossings, as required. **Q.E.D.**

Thus, we see that Conjecture 6.2 would do all the inductive parts of the proof of the HKS-conjecture (Consequence 6.3). Moreover, it would imply the HKS-conjecture is true for all but finitely many n , for each m (Consequence 6.4). A major advance toward Conjecture 6.2 would occur if Conjecture 6.1 were proved.

We thank Gelasio Salazar for pointing out that 6.4 is a consequence of 6.2.

6.8 $C_8 \times C_8$ And Beyond

It seems very difficult to prove that every $(8,8)$ -mesh has at least 48 crossings. It is more likely that someone will show that every non-separating $(8,8)$ -circular arrangement has at least 48 crossings, although how to do this is not evident at this time. An $(8,8)$ -circular arrangement with fewer than 48 crossings probably has at most two components. In this work, such arrangements have not been considered.

The induction of $C_8 \times C_n$ has not been considered in detail, although perhaps 4-restricted arrangements would be helpful there. But very soon, k -restricted arrangements are not going to be useful, since even for $C_9 \times C_n$, we need 7 crossings for the induction and it is quite conceivable that there is only one component and yet each cycle has at most 6 crossings. There is no evident method other than Conjecture 6.2 for making significant progress here.

Arrangements as developed in this thesis do not present an obvious approach for the specific case of $C_m \times C_m$. Perhaps if Conjecture 6.2 were proven, this approach could also be used in these cases.

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