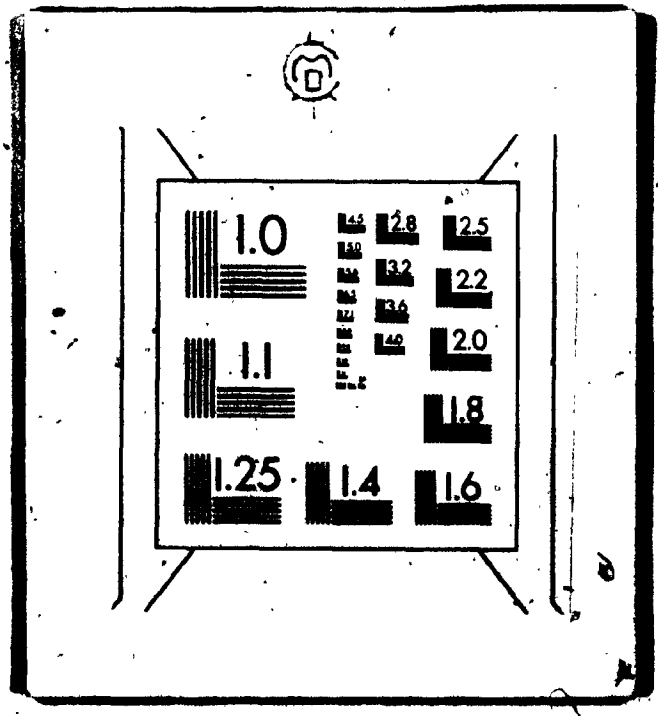
1 1

OF / DE



49498

0-315-01641-8

National Library of Canada | Bibliothèque nationale du Canada

CANADIAN THESES ON MICROFICHE | THÈSES CANADIENNES SUR MICROFICHE

NAME OF AUTHOR/*NOM DE L'AUTEUR* JAN J. GREEN-KRÓTKI

TITLE OF THESIS/*TITRE DE LA THÈSE* MATCHING POLYHEDRA

UNIVERSITY/*UNIVERSITÉ* CARLETON UNIVERSITY

DEGREE FOR WHICH THESIS WAS PRESENTED/ *GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* M.Sc.

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE DEGRÉ* 1980

NAME OF SUPERVISOR/*NOM DU DIRECTEUR DE THÈSE* DR. W. CUNNINGHAM

Permission is hereby granted to the NATIONAL LIBRARY OF CANADA to microfilm this thesis and to lend or sell copies of the film.

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

L'autorisation est, par la présente, accordée à la BIBLIOTHÈQUE NATIONALE DU CANADA de microfilmer cette thèse et de prêter ou de vendre des exemplaires du film.

L'auteur se réserve les autres droits de publication; ni la thèse ni de longs extraits de celle-ci ne doivent être imprimés ou autrement reproduits sans l'autorisation écrite de l'auteur.

DATED/*DATÉ* 21 IV 1980 SIGNED/*SIGNÉ* Jan Green-Krótki

PERMANENT ADDRESS/*RÉSIDENCE FIXE* 72 PATTERSON AVENUE
OTTAWA, ONT.
K1S 1Y3

NL-91 (11-79)

National Library of Canada
Collections Development Branch

Canadian Theses on
Microfiche Service

Bibliothèque nationale du Canada
Direction du développement des collections

Service des thèses canadiennes
sur microfiche

NOTICE

The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us a poor photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30. Please read the authorization forms which accompany this thesis.

AVIS

La qualité de cette microfiche dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de mauvaise qualité.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30. Veuillez prendre connaissance des formules d'autorisation qui accompagnent cette thèse.

THIS DISSERTATION
HAS BEEN MICROFILMED
EXACTLY AS RECEIVED

LA THÈSE A ÉTÉ
MICROFILMÉE TELLE QUE
NOUS L'AVONS REÇUE

Ottawa, Canada
K1A 0N4

NL-339 (Rev. 8/80)

MATCHING POLYHEDRA

by

Jan J. Green-Krótki, B.A.

A thesis submitted to the Faculty of
Graduate Studies in partial fulfilment
of the requirements for the degree of
Master of Science

Department of Mathematics and Statistics

Carleton University

Ottawa, Ontario

April 7, 1980

The undersigned recommend to the Faculty of Graduate Studies acceptance of the thesis "Matching Polyhedra" submitted by Jan J. Green-Krótki, B.A., in partial fulfilment of the requirements for the degree of Master of Science.

W. H. Cunningham

Dr. W. Cunningham
Thesis Supervisor

L. Nel

Dr. L. D. Nel
Chairman, Department of Mathematics and Statistics

Carleton University

Abstract

The matching polyhedron theorem is proved non-algorithmically and it is shown that the general matching problem has a polynomially bounded algorithm.

As applications, the theorem is used to derive the Chinese postman polyhedron and the convex cone of incidence vectors of circuits of a graph.

TABLE OF CONTENTS

	Page
INTRODUCTION	4
CHAPTER 1 - PRELIMINARIES	11
1.1 - General Notation	12
1.2 - Graph Theory	14
1.3 - Convexity	18
1.4 - Linear Programming	20
1.5 - Polyhedral Combinatorics	23
1.6 - Deriving Convex Hulls	27
CHAPTER 2 - MATCHING POLYHEDRA	29
2.1 - The Matching Problem	30
2.2 - The 1-Matching Polyhedron (1-MP)	35
2.3 - The b-Matching Polyhedron (b-MP)	45
2.4 - The Lobe Matching Polyhedron (LMP)	49
2.5 - The Capacitated Matching Polyhedron (CMP)	52
2.6 - The Bidirected Matching Polyhedron (BMP)	56
2.7 - The Matching Polyhedron (MP)	61
2.8 - Bounds for Matching Algorithms	64
CHAPTER 3 - APPLICATIONS	72
3.1 - The Chinese Postman Polyhedron	73
3.2 - Sums of Circuits	77
BIBLIOGRAPHY	84

INTRODUCTION

In its simplest form, in which case it is called a 1-matching, a matching of a graph is a subset of the edges such that each vertex is an end of precisely one of the matching edges. Hence the name: a matching matches the vertices in pairs. Given costs associated with the edges of the graph, the matching problem is to find an optimum (minimum or maximum) cost matching, where the cost of a matching is the sum of the costs of the matching edges. For example, given n processors and n tasks, with costs associated with a particular processor executing a particular task, the problem of assigning exactly one task to each processor, in such a way as to minimize total cost, is a 1-matching problem.

It will be obvious, once we define it, that the matching problem is an integer linear programming (ILP) problem. Now we do not have an efficient algorithm to solve ILP problems in general. One of the reasons we do not have such an algorithm is that we know no "good" criterion for optimality of feasible solutions to a general ILP problem. Such a criterion is necessary for an efficient algorithm since the algorithm must be able to decide, in an efficient manner, when to stop. However, the matching problem does have an efficient algorithm, for which

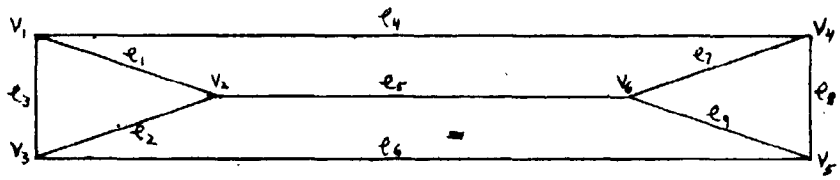
the stopping rule comes from linear programming (LP) duality theory. Every ILP problem with feasible solution set S is equivalent to the LP problem with the same objective function and with the convex hull of S as its feasibility region. This is because every vertex (or extreme point) of the convex hull of S is an element of S, and if an LP problem has an optimal solution, then it has a vertex optimal solution. What is special about the matching problem is that we know an explicit description of the convex hull of its feasible solution set in terms of linear inequalities. This result is a consequence of the so--called blossom algorithm (announced by Edmonds and Johnson [11]) and it is the main goal of this dissertation to prove the result non-algorithmically.

Let us illustrate some of the above ideas on the example of 1-matching. The feasible solution set M of a 1-matching problem on a graph G with vertex-set V and edge-set E is the set of (0,1)-valued vectors $\underline{x}$, indexed by E, satisfying

I.1 $\Sigma(x_e : e \text{ is incident with } v) = 1 \quad \text{for } v \in V.$

Now the feasible solution set of any ILP problem is the set of integer-valued vectors satisfying a finite system L of inequalities. There are ILP problems having the property that the convex hull of the feasible solution set is precisely the set of real-valued vectors satisfying L, in which case an

optimality criterion based on LP duality is immediately at hand. We show this is not the case for the 1-matching problem. Consider the graph



and let M' be the set of incidence vectors of its 1-matchings. For instance, M' contains (0,0,0,1,1,1,0,0,0) and (0,0,1,0,1,0,0,1,0). The claim is that the convex hull of M' is not the set P of nonnegative real-valued vectors satisfying I.1. Let $\underline{x}^*$ be the vertex of P satisfying the equations I.1 and $x^*_{e_4} = x^*_{e_5} = x^*_{e_6} = 0$. Then $x^*_{e_1} + x^*_{e_2} = x^*_{e_1} + x^*_{e_3} = 1$, so $x^*_{e_2} = x^*_{e_3}$. But $x^*_{e_2} + x^*_{e_3} = 1$, so $x^*_{e_1} = x^*_{e_2} = x^*_{e_3} = \frac{1}{2}$ and similarly $x^*_{e_7} = x^*_{e_8} = x^*_{e_9} = \frac{1}{2}$. Therefore $\underline{x}^* \notin M'$ and P is not the convex hull of M'.

It is not hard to see that any 1-matching of the above graph must include at least one of the edges e_4, e_5, e_6. This is because the three (note that three is an odd number) vertices v_1, v_2, v_3 cannot all be incident with matching edges from the set $\{e_1, e_2, e_3\}$. Now if $\underline{x}$ is in the convex hull of M', that is, $\underline{x}$ is a convex combination of incidence vectors of 1-matchings, then of course $x_{e_4} + x_{e_5} + x_{e_6} \geq 1$, an inequality not satisfied by the $\underline{x}^*$ defined above.

The above argument can be generalized (and will be - in section 2.2) to show that any $\underline{x}$ in the convex hull of M must satisfy

I.2 $\Sigma(x_e$: e is incident with precisely one $v \in T) \geq 1$ for subsets T of V such that $|T|$ is odd.

That the convex hull of M is precisely the set of nonnegative real-valued vectors satisfying I.1 and I.2 will be proved in section 2.2.

What does all this have to do with optimality criteria? Given a 1-matching problem with cost vector $\underline{c}$, indexed by the edge-set, we will have shown that this problem is equivalent to the LP problem

I.3 minimize $\underline{c}.\underline{x}$ subject to $\underline{x} \geq \underline{0}$, I.1 and I.2.

We know from LP duality theory that optimal solutions to problem I.3 and its dual are related through complementary slackness. Thus if our algorithm detects primal and dual feasible solutions for which complementary slackness holds, it can stop.

Is such an optimality criterion "good"? After all, the number of inequalities I.2 is exponential in $|V|$, so the number of dual variables is exponential in $|V|$, checking complementary slackness may require verifying an exponential number of inequalities, and an algorithm that requires an amount of work that is exponential in the problem's input size is not considered

to be efficient. However, the number of dual constraints, which is the number of primal variables, is polynomial in the input size, and we know that if an LP problem has an optimal solution, then it has a vertex optimal solution in which the number of variables with value strictly positive does not exceed the number of constraints. Thus if our algorithm maintains vertex primal and dual feasible solutions, it can check complementary slackness in a polynomial amount of work. The algorithm also has a good optimality criterion if it maintains a vertex dual feasible solution and a primal solution that is not necessarily feasible, but such that the two solutions satisfy complementary slackness, for then it can stop as soon as the primal solution becomes feasible; this is the optimality criterion used by the blossom algorithm.

The subject, called polyhedral combinatorics, of exploiting LP duality theory to obtain optimality criteria for ILP problems is the primary motivation of this dissertation, and is dealt with more technically in section 1.5. (The first four sections of chapter 1 consist of standard definitions and theorems without proofs.) The last section of the first chapter is rather important because it contains the basic tools used to prove the polyhedral results that are the content of the subsequent chapters.

In the next chapter, which is the meat of this dissertation, first the matching problem is introduced and then a proof is given of the convex hull result for the special case of 1-matching. The subsequent sections consist of successive generalizations of this special case all the way up to the original problem. At each stage the previous convex hull result is used, together with the convex hull characterizations of section 1.6, to prove the convex hull result for the more general problem. Since these proofs involve polynomially reducing the more general problem to the more special one, and since Cunningham and Marsh (see Marsh [25]) have exhibited a polynomial algorithm for the matching problem of section 2.3, it is also shown, in section 2.8, that the most general matching problem has a polynomial algorithm.

Section 3.1 presents the proof of Edmonds and Johnson [12] of their Chinese postman polyhedron theorem. This proof is based on the main result of chapter 2. A postman tour in a graph is a circuit which traverses every edge at least once, and the Chinese postman problem is to find a minimum cost postman tour, given nonnegative costs associated with each edge. With each postman tour we can associate a nonnegative integer-valued vector, indexed by the edge-set of the graph, which counts the number of times each edge is traversed superfluously (more

than once). The Chinese postman polyhedron is the convex hull of such vectors.

The last section gives a matching-theoretical proof of a theorem, due to Seymour [28], that provides a finite system of linear inequalities determining the set of nonnegative linear combinations of incidence vectors of circuits of a graph.

CHAPTER 1 - PRELIMINARIES

In which definitions and notation are introduced and necessary results from the theories of graphs, convexity and linear programming are stated without proof, in which the motivation behind the main theorem of the dissertation is discussed in the section on polyhedral combinatorics, and in which the last section gives two characterizations of convex hulls that provide the basic framework for subsequent proofs.

Section 1.1 - General Notation

It is assumed that the reader is familiar with the meanings and elementary properties of the following symbols and concepts: $S = \{x: P(x)\}$ = the set of all x satisfying a given condition P; $x \in S$, $x \notin S$; $S \subseteq T$, $T \supseteq S$; $S \cap T$, $S \cup T$, $S - T = \{x \in S: x \notin T\}$; $|S|$ = the cardinality of S; $x \equiv y \pmod z$; $+\infty$, $-\infty$; $|x|$ = the absolute value of x, $\Sigma(x_i: i \in I)$; the empty set $\emptyset$; $\bar{T} = S - T$ when $T \subseteq S$ and S is obvious from context.

$\underline{R}$ and $\underline{Z}$ respectively denote the sets of real and integer numbers. If $\underline{X}$ is either of these symbols, then $\underline{X}_+ = \{x \in \underline{X}: x \geq 0\}$ and, for any set S, $\underline{X}^S$ denotes the set of all functions (or vectors if $|S| < \infty$) defined on S and taking values in $\underline{X}$; similarly for $\{0,1\}^S$.

A vector $\underline{x}$ with index set I is an $|I|$-tuple $\underline{x} = (x_i: i \in I)$; we shall also use the function notation $\underline{x} \in \underline{X}^I$ where $\underline{X}$ is the range of possible values for the coordinates of $\underline{x}$. If $\underline{x}$ and $\underline{y}$ are both indexed by I, then $\underline{x} \leq \underline{y}$ means that $x_i \leq y_i$ for each $i \in I$. The incidence vector of a set $J \subseteq I$ is the vector $\underline{x} \in \{0,1\}^I$ defined by $x_i = 1$ if and only if $i \in J$; when it is obvious what is meant we do not distinguish between a set and its incidence vector. The zero vector is written $\underline{0}$, since the index set will always be obvious from context. For the same reason we do not

distinguish between row and column vectors. For $J \subseteq I$, we use the abbreviation $\underline{x}(J)$ to denote $\Sigma(x_i : i \in J)$. For $\underline{x} \in \underline{X}^I$, $\underline{y} \in \underline{X}^I$ and $s \in \underline{X}$, the dot product is $\underline{x} \cdot \underline{y} = \Sigma(x_i y_i : i \in I) \in \underline{X}$ and the <u>scalar product</u> is $s\underline{x} = (sx_i : i \in I) \in \underline{X}^I$.

A <u>matrix</u> with entries from $\underline{X}$, rows indexed by I and columns indexed by J is denoted by $A = (a_{ij}) \in \underline{X}^{I \times J}$, with the usual matrix addition and multiplication.

The <u>minimum</u> and <u>maximum</u> of a set $S \subseteq \underline{X}$ are respectively written $\min(x : x \in S)$ and $\max(x : x \in S)$, with the convention that $\min(x : x \in \emptyset) = +\infty$ and $\max(x : x \in \emptyset) = 0$.

Section 1.2 - Graph Theory

The single result and most of the definitions of this section can be found in Bondy and Murty [4] and Edmonds and Johnson [11].

A graph $G = (V,E)$ consists of a finite non-empty set V of vertices together with a prescribed set E of unordered pairs of elements of V. Each such pair $e = (u,v) \in E$ is called an edge of G; we say that e joins u and v and is incident with each of its ends u and v. An edge of the form $e = (v,v)$ is called a loop. Given any graph H, the vertex-set and edge-set of H are respectively denoted by $V(H)$ and $E(H)$. The degree of of a vertex is the number of edges incident with it.

A subgraph of G is a graph having all of its vertices and edges in G, and a spanning subgraph of G is a subgraph containing all the vertices of G.

A path from v_0 to v_n in a graph G is an alternating sequence $v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n$ of distinct vertices and edges such that each e_i, for $1 \leq i \leq n$, joins v_{i-1} and v_i. A sequence with the same properties, except that $v_0 = v_n$, is called a circuit, which is an even circuit or an odd circuit according to whether n (the number of edges in the circuit) is even or odd.

A graph G is <u>connected</u> if there exists a path from u to v for every pair of vertices u and v of G. A maximal connected subgraph of G is called a <u>component</u> of G. A component consisting of a single vertex is called an <u>isolated vertex</u>.

The <u>coboundary</u> of $S \subseteq V$, denoted by $\delta_G(S)$, is the set of all edges having exactly one end in S. If $S = \{v\}$ is a singleton, $\delta_G(\{v\})$ is abbreviated to $\delta_G(v)$. The symbol $\gamma_G(S)$ denotes the set of all edges having both ends in S, and $\gamma_G(\{v\})$ is abbreviated to $\gamma_G(v)$ = the set of loops incident with v. The subscript G is suppressed whenever this does not result in ambiguities.

A graph is <u>bipartite</u> if its vertex-set V can be partitioned into two subsets V_1 and V_2 such that $\gamma(V_1)$ and $\gamma(V_2)$ are both empty, or equivalently, $\delta(V_1) = \delta(V_2)$ = the edge-set of the graph. There follows the promised single result.

1.2.1 <u>Theorem</u>: A graph is bipartite if and only if all its circuits are even.

We now specialize and generalize the definition of a graph in various ways. Unless otherwise indicated, the above definitions either extend naturally to, or will not be needed on these different classes of graphs.

A <u>simple graph</u> is a graph in which the vertices of any pair defining an edge are distinct, that is, a simple graph is a graph with no loops.

A directed graph is a graph in which the vertex pairs defining edges are ordered. If $e = (u,v)$ is a (directed) edge of a directed graph, we say that u is the tail of e, v is the head of e, and write $u = t(e)$, $v = h(e)$. Here we partition $\delta(S)$ into $\delta^-(S) = \{e \in E: t(e) \in S, h(e) \notin S\}$, $\delta^+(S) = \{e \in E: h(e) \in S, t(e) \notin S\}$.

A generalized graph is a graph in which some edges, called lobes, may have only one end. A simple generalized graph is a generalized graph with no loops.

A bidirected graph is a generalized graph in which each edge has a direction associated with each vertex with which it is incident. Thus each edge of a bidirected graph has either one or two (not necessarily distinct) ends, and each of these ends is either a tail end or a head end. Here δ and γ are defined as for graphs, $\delta^+(S)$ (respectively $\delta^-(S)$) is the subset of $\delta(S)$ consisting of those edges whose single end in S is a head end (respectively tail end), and $\gamma^+(v)$ (respectively $\gamma^-(v)$) is the set of loops having two head ends (respectively tail ends) at v.

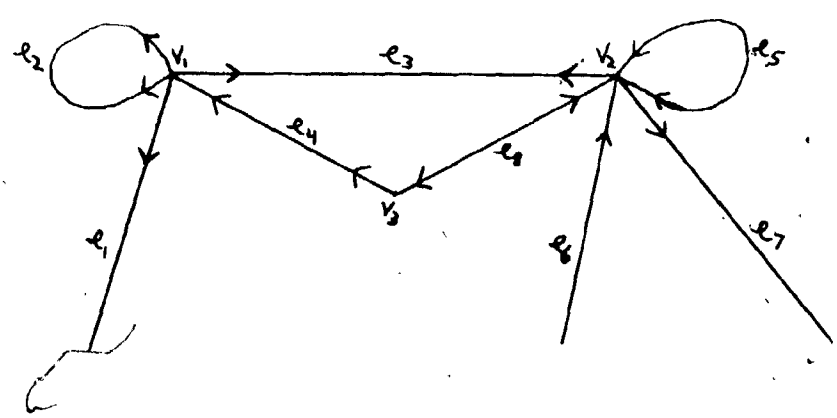
We represent any type of graph by means of its incidence matrix, which we define only for bidirected graphs, the other definitions being obvious specializations. Thus the incidence matrix of a bidirected graph $G = (V,E)$ is $A = (a_{ve}) \in \{-2,-1,0,1,2\}^{V \times E}$ where $a_{ve} = -2,-1,0,1,2$ if respectively, e is a loop with two

tail ends at v, e is not a loop and has one tail end at v, e is not incident with v, e is not a loop and has one head end at v, e is a loop with two head ends at v. The attentive reader will have noticed that this representation does not allow for loops having one tail end and one head end; in our application of bidirected graphs it can be assumed, as will be explained, that such loops do not exist. Clearly $\Sigma(|a_{ve}|: v \in V)$ is equal to 1 or 2 according to whether e is a lobe or not a lobe.

Another, intuitively more useful, representation of graphs is by means of a <u>picture</u> in which vertices, edges and directions are respectively represented by means of points, lines and arrows.

The following incidence matrix and picture represent the same bidirected graph.

$$A = \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} -1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}$$



Section 1.3 - Convexity

The definitions of this section are standard, unless otherwise indicated, and proofs of the results can be found in Grünbaum [18] or Stoer and Witzgall [30].

Given a set $S \subseteq \underline{R}^n$, $\underline{x} \in \underline{R}^n$ is a nonnegative linear combination of elements of S if there exist $\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^k \in S$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \underline{R}_+$ such that

$$\underline{x} = \Sigma(\alpha_i \underline{x}^i : 1 \leq i \leq k).$$

If, in addition, $\Sigma(\alpha_i : 1 \leq i \leq k) = 1$, the vector $\underline{x}$ is said to be a convex combination of elements of S. The set S is said to be convex if it contains all convex combinations of its elements. The convex cone of $S \subseteq \underline{R}^n$, denoted by CC(S), is the set of all nonnegative linear combinations of elements of S, and the convex hull of S, denoted by CH(S), is the set of all convex combinations of elements of S. It is not hard to see that convex cones and convex hulls are convex.

An extreme point of a convex set S is an $\underline{x} \in S$ that cannot be written as a convex combination of more than one element of S, that is, $\underline{x} = \underline{x}$ is the only representation of $\underline{x}$ as a convex combination of elements of S.

We are interested here in a very special kind of convex set, namely one that is the set of feasible solutions to a

finite system of linear inequalities (which may include equations since any equation can be written as two inequalities), which we call a polyhedron. This terminology is not completely standard: some authors, for example Gale [16], use the terms polygon, polyhedron and polytope when $n = 2$, $n = 3$ and $n \geq 3$ respectively.

A vertex of a polyhedron P is an $\underline{x} \in P$ that is the unique solution to a system of equations obtained from the inequalities determining P by setting some of them to equality.

1.3.1 Theorem: If $P \subseteq \underline{R}^n$ is a polyhedron, then $\underline{x} \in \underline{R}^n$ is an extreme point of P if and only if $\underline{x}$ is a vertex of P.

1.3.2 Corollary: Given $S \subseteq \underline{R}^n$, if CH(S) is a polyhedron, then every vertex of CH(S) is an element of S.

1.3.3 Theorem: A bounded convex set is the convex hull of its extreme points.

1.3.4 Corollary: A bounded polyhedron is the convex hull of its vertices.

Section 1.4 - Linear Programming

Some standard references on linear programming are Dantzig [6], Gale [16] and Luenberger [24].

The linear programming problem or LP problem is to optimize a linear objective function $\underline{c}.\underline{x}$, where $\underline{c} \in \underline{R}^n$ is the cost vector and $\underline{x} \in \underline{R}^n$ is the vector of variables, over a polyhedron contained in $\underline{R}^n$, also called the feasibility region.

Since maximizing a linear objective function is equivalent to minimizing its negative, we can write any LP problem as:

1.4.1 General LP: $\min(\underline{c}.\underline{x}: \underline{x} \in P)$ where $P \subseteq \underline{R}^n$ is a polyhedron.

Any $\underline{x} \in P$ is a feasible solution and $\underline{x}^* \in P$ is an optimal solution if $\underline{c}.\underline{x}^* \leq \underline{c}.\underline{x}$ for all $\underline{x} \in P$. The LP is said to be infeasible if $P = \emptyset$ and unbounded if, given any $r \in \underline{R}$, there exists $\underline{x} \in P$ such that $\underline{c}.\underline{x} < r$.

There follow two well-known LP results about polyhedral vertices.

1.4.2 Theorem: If 1.4.1 has an optimal solution, then it has an optimal solution that is a vertex of P.

1.4.3 Theorem: Given any polyhedron $P \subseteq \underline{R}^n$, $\underline{x}^*$ is a vertex of P if and only if there is a vector $\underline{c} \in \underline{R}^n$ such that $\underline{x}^*$ is the unique optimal solution to the LP problem $\min(\underline{c}.\underline{x}: x \in P)$.

The problem of integer linear programming (ILP) is to minimize a linear objective function $\underline{c}.\underline{x}$ subject to the constraints $\underline{x} \in \underline{Z}^n$ and $\underline{x}$ satisfies a finite system of linear inequalities in which all the coefficients and constant terms are integers. It is not hard to see that, by algebraic manipulation of such inequalities, any ILP problem can be written in the form:

1.4.4 General ILP: $\min(\underline{c}.\underline{x}: \underline{0} \leq \underline{x} \leq \underline{u},\ A\underline{x} = \underline{b},\ \underline{x} \in \underline{Z}^n)$ where $\underline{c} \in \underline{R}^n$, $\underline{u} \in (\underline{Z}_+ \cup \{+\infty\})^n$, $A \in \underline{Z}^{m \times n}$, $\underline{b} \in \underline{Z}^m$.

We now summarize linear programming duality theory. Although not needed to prove our subsequent results, it is the motivation for these results, for it explains why it is of interest to know the convex hull, in terms of linear inequalities, of the set of feasible solutions to an ILP.

By algebraic manipulation of inequalities, any LP problem can be written as:

1.4.5 Primal LP: $\min(\underline{c}.\underline{x}: \underline{x} \geq \underline{0},\ A\underline{x} = \underline{b})$ where $\underline{c} \in \underline{R}^n$, $A \in \underline{R}^{m \times n}$, $\underline{b} \in \underline{R}^m$.

The dual to this LP problem is:

1.4.6 Dual LP: $\max(\underline{y}.\underline{b}: \underline{y}A \leq \underline{c})$.

The following theorems relate the objective function values of primal feasible solutions and dual feasible solutions, also giving necessary and sufficient conditions for such solutions to be optimal.

1.4.7 Weak LP Duality Theorem: If $\underline{x}$ and $\underline{y}$ are respectively primal and dual feasible, then $\underline{c}.\underline{x} \geq \underline{y}.\underline{b}$. Furthermore, if one of the two problems is unbounded, then the other is infeasible.

1.4.8 Strong LP Duality Theorem: If the primal LP is feasible and bounded, then there is an optimal primal solution $\underline{x}^*$ and an optimal dual solution $\underline{y}^*$ such that $\underline{c}.\underline{x}^* = \underline{y}^*.\underline{b}$.

1.4.9 Complementary Slackness Theorem: A primal feasible solution $\underline{x}^*$ and a dual feasible solution $\underline{y}^*$ are both optimal if and only if $\Sigma(y_i^* a_{ij} : 1 \leq i \leq m) = c_j$ for each j $(1 \leq j \leq n)$ such that $x_j^* > 0$.

Section 1.5 - Polyhedral Combinatorics

Consider an ILP of the form:

1.5.1 $\min(\underline{c}.\underline{x}: \underline{x} \in S)$ where $S \subseteq \underline{Z}^n$.

Clearly any finite algorithm, in particular any efficient algorithm, for such an ILP problem must have a stopping rule or optimality criterion. This necessary ingredient for a "good" algorithm is often hard to come by for ILP problems, but always exists for LP problems in the form of the duality theorems 1.4.7, 1.4.8 and 1.4.9. Polyhedral combinatorics is the subject of exploiting LP duality theory to find "good" optimality criteria for ILP problems.

This requires finding an explicit description of a finite system of linear inequalities which determine a polyhedron P(S) having the property that 1.5.1 is equivalent to

1.5.2 $\min(\underline{c}.\underline{x}: \underline{x} \in P(S)$, $\underline{x}$ is a vertex of P(S) if the LP $\min(\underline{c}.\underline{x}: \underline{x} \in P(S))$ has an optimal solution).

It should be noted that, in view of theorem 1.4.2, the additional condition that $\underline{x}$ be a vertex of P(S), when an optimal solution exists, does not change the optimal objective value.

There may be more than one such polyhedron for a given S, but this dissertation deals with convex hulls of sets of feasible solutions to matching problems, so we now show that convex hulls have the above property.

1.5.3 Theorem: Given $S \subseteq \underline{Z}^n$, if CH(S) is a polyhedron, then problems 1.5.1 and 1.5.2, with P(S) = CH(S), are equivalent.

Proof: Since $S = \emptyset$ if and only if $CH(S) = \emptyset$, it is immediate that 1.5.1 is infeasible if and only if 1.5.2 is infeasible.

Since $S \subseteq CH(S)$, it is also clear that if 1.5.1 is unbounded, then so is 1.5.2. Now suppose 1.5.2 is unbounded, let $r \in \underline{R}$ be given, and find $\underline{y} = \Sigma(\alpha_i \underline{x}^i : 1 \leq i \leq k)$, a convex combination of elements of S, such that $\underline{c}.\underline{y} < r$. Since $\Sigma(\alpha_i : 1 \leq i \leq k) = 1$, we then have

$$\min(\underline{c}.\underline{x}^i : 1 \leq i \leq k) \leq \Sigma(\alpha_i \underline{c}.\underline{x}^i : 1 \leq i \leq k) = \underline{c}.\underline{y} < r,$$

so 1.5.1 is also unbounded.

If both problems are feasible and bounded, let $\underline{x}^*$ be an optimal solution to 1.5.1 and let $\underline{y}^*$ be a vertex optimal solution to 1.5.2. Since $S \subseteq CH(S)$, it follows that $\underline{c}.\underline{y}^* \leq \underline{c}.\underline{x}^*$. Invoking 1.3.2, which says that every vertex of CH(S) is an element of S, we also have $\underline{y}^* \in S$, so $\underline{c}.\underline{x}^* \leq \underline{c}.\underline{y}^*$. QED

Some ILP problems (notably network flow problems) are equivalent to the corresponding LP problem obtained by dropping the integrality constraint $\underline{x} \in \underline{Z}^n$, so that optimality criteria derived from duality theory are immediately at hand. If an ILP problem does not have this property we say that it is genuine.

There is a remarkably close correlation between those ILP problems for which we know efficient algorithms and those

for which we explicitly know a polyhedron such that 1.5.1 and 1.5.2 are equivalent. "Good" combinatorial algorithms can often be used to prove polyhedral results and, conversely, polyhedral combinatorics often inspires, or at least motivates, efficient solutions to algorithmic problems. Difficulty in obtaining polyhedral results also helps to explain why related algorithmic problems are difficult.

Recently (1980), Grötschel, Lovász and Schrijver [17] exhibited an even closer relationship between efficient (having computation bounds polynomial in the "input size") algorithms for ILP problems and convex hulls of feasibility sets of these problems. Their result is based on the ellipsoid method used by Khachian [21] (see also Gács and Lovász [15]) in his polynomially bounded algorithm for linear programming. Before stating the theorem we make two definitions. Given a convex set $S \subseteq \underline{R}^n$, the <u>separation problem</u> for S is the problem of deciding whether or not a given $\underline{x}$ is in S, and if $\underline{x} \notin S$, finding an inequality satisfied by all the elements of S but not by $\underline{x}$. A <u>convex body</u> is a bounded convex set contained in $\underline{R}^n$ and containing an n-dimensional ball.

1.5.4 <u>Theorem</u>: Let K be a class of convex bodies. There is a polynomial algorithm to solve the separation problem for the members of K if and only if there is a polynomial algorithm to solve the problem of optimizing any linear objective function over any member of K.

To relate the above theorem to polyhedral combinatorics, let K be the class of convex hulls of feasibility sets of instances of an ILP problem. If it so happens that each member of K is a convex body, then the theorem says there is a polynomial algorithm to solve the ILP problem if and only if there is a polynomial algorithm to solve the separation problem for the members of K.

In this respect, it is interesting to note that in 1979 Padberg and Rao [26] efficiently solved the separation problem for convex hulls of b-matchings (see section 2.3 for a definition of b-matchings, which generalize 1-matchings). Although a polynomial algorithm for the 1-matching problem, as an optimization problem, has been available since 1965 (Edmonds [8]), the b-matching problem was not polynomially solved until 1979 (Cunningham and Marsh - see Marsh [25]).

Section 1.6 - Deriving Convex Hulls

In this section we prove two theorems about convex hulls of ILP feasible solution sets that provide the framework for many of our subsequent proofs. These theorems rely on 1.3.4, which says that a bounded polyhedron is the convex hull of its vertices.

1.6.1 Theorem: If $S \subseteq \underline{Z}^n$ is finite, then $P = CH(S)$ provided

(1) P is a bounded polyhedron in $\underline{R}^n$,

(2) $S \subseteq P$,

(3) every vertex of P is an element of S.

Proof: Conditions (1) and (2) guarantee that any convex combination of elements of S is an element of P, so $CH(S) \subseteq P$. For the reverse inclusion, by condition (1) any $\underline{x} \in P$ is a convex combination of vertices of P, and (3) then guarantees that $\underline{x} \in CH(S)$. QED

In our applications of this theorem, (1) will be self--evident, (2) will be readily verifiable, but showing (3) will often be non-trivial. Typically we shall do it by reducing the ILP problem at hand to an ILP problem for which the convex hull result has already been proved. Then condition (3) will have a more manageable form as in the next theorem.

1.6.2 Theorem: Let $S \subseteq \underline{Z}^n$ be finite and let $S' \subseteq \underline{Z}^m$ be such that, for each $\underline{c} \in \underline{R}^n$, there exists $\underline{c}' \in \underline{R}^m$ with $\min(\underline{c}.\underline{x}: \underline{x} \in S)$ equivalent to $\min(\underline{c}'.\underline{y}: \underline{y} \in S')$. If $P' = CH(S')$, then $P = CH(S)$ provided

(1') P is a bounded polyhedron in $\underline{R}^n$,

(2') $S \subseteq P$,

(3') for every vertex $\underline{x}^*$ of P and for every $\underline{c} \in \underline{R}^n$, there exists $\underline{y}^* \in P'$ such that $\underline{c}.\underline{x}^* = \underline{c}'.\underline{y}^*$, where $\underline{c}' \in \underline{R}^m$ is as in the hypotheses.

Proof: In view of the preceding theorem, it suffices to show that, under the given hypotheses, condition (3') implies condition (3). Given a vertex $\underline{x}^*$ of P, invoke theorem 1.4.3 to find $\underline{c} \in \underline{R}^n$ such that $\underline{x}^*$ is the unique optimal solution to $\min(\underline{c}.\underline{x}: \underline{x} \in P)$, and the hypotheses to find $\underline{c}' \in \underline{R}^m$ such that the two ILP problems $\min(\underline{c}.\underline{x}: \underline{x} \in S)$ and $\min(\underline{c}'.\underline{y}: \underline{y} \in S')$ are equivalent. By condition (3'), we also have a $\underline{y}^* \in P'$ such that $\underline{c}.\underline{x}^* = \underline{c}'.\underline{y}^*$.

Since $P' = CH(S') \neq \emptyset$, it follows that $S' \neq \emptyset$ and both problems are feasible. Since S is finite, we also have that both problems are bounded. Therefore they have optimal solutions $\underline{x}' \in S$ and $\underline{y}' \in S'$ such that $\underline{c}.\underline{x}' = \underline{c}'.\underline{y}'$. But $S \subseteq P$, so the optimality of $\underline{x}^*$ yields $\underline{c}.\underline{x}^* \leq \underline{c}.\underline{x}'$. By theorem 1.5.3, $\underline{y}'$ is an optimal solution to $\min(\underline{c}'.\underline{y}: \underline{y} \in P')$, whence $\underline{c}'.\underline{y}' \leq \underline{c}'.\underline{y}^*$. Therefore

$$\underline{c}.\underline{x}^* \leq \underline{c}.\underline{x}' = \underline{c}'.\underline{y}' \leq \underline{c}'.\underline{y}^* = \underline{c}.\underline{x}^*,$$

and in fact $\underline{c}.\underline{x}^* = \underline{c}.\underline{x}'$. By the uniqueness of $\underline{x}^*$, it follows that $\underline{x}^* = \underline{x}' \in S$ and condition (3) of theorem 1.6.1 is proved. QED

CHAPTER 2 - MATCHING POLYHEDRA

In which the matching problem and the main theorem of this dissertation, the matching polyhedron theorem, are stated and explained, in which the strategy for proving this theorem is first outlined and then executed, to wit the theorem is proved for a special case and then successively generalized by reducing a more general matching problem to one for which the theorem has already been proved, and finally in which some comments are made regarding the existence of a polynomial algorithm for the most general matching problem; the reductions we use are taken from the theory of network flows.

Section 2.1 - The Matching Problem

An ILP problem in the general form 1.4.4 is called a matching problem whenever the constraint matrix A is the incidence matrix of a bidirected graph $G = (V,E)$.

2.1.1 Matching Problem: minimize $\underline{c}.\underline{x}$ subject to

2.1.2 $$\underline{x} \in \underline{Z}_+^E$$

2.1.3 $$\underline{x} \leq \underline{u}$$

2.1.4 $$A\underline{x} = \underline{b}$$

where $\underline{c} \in \underline{R}^E$, $\underline{u} \in (\underline{Z}_+ \cup \{+\infty\})^E$, $\underline{b} \in \underline{Z}^V$ and $A = (a_{ve}) \in \{-2,-1,0,1,2\}^{V \times E}$ satisfies $\Sigma(|a_{ve}| : v \in V) \leq 2$ for each $e \in E$.

Since we shall be considering this problem more from a graph-theoretical than from a linear algebra point of view, it is convenient to write 2.1.4 as

2.1.5 $$\underline{x}(\delta^+(v)) - \underline{x}(\delta^-(v)) + 2\underline{x}(\gamma^+(v)) - 2\underline{x}(\gamma^-(v)) = b_v \text{ for } v \in V.$$

Thus a matching (the use of this term will be explained in the next section) of a bidirected graph $G = (V,E)$ is an assignment of nonnegative integers x_e to the edges of G such that no x_e is greater than the capacity u_e of edge e, and the demand b_v at each vertex v is satisfied, in the sense that edge e contributes $-2x_e$, $-x_e$, 0, x_e, $2x_e$ to the demand at v according to whether e has 2 tails, 1 tail, 0 ends, 1 head, 2 heads at v. The terms "capacity" and "demand" come from network flow problems

which are not genuine ILP problems, and which constitute a sub--class of the class of matching problems. The cost of a matching, with respect to a given cost vector $\underline{c} \in \underline{R}^E$, is the sum, over $e \in E$, of costs c_e taken x_e times on each edge e, and the matching problem is to find a minimum cost or optimal matching.

It is now clear why we can assume that a bidirected graph has no directed loops (loops with one tail and one head), which would appear as a column of zeroes in the incidence matrix. Since a directed loop e makes no contribution to the demand at the vertex v with which it is incident, if $c_e < 0$ then the problem is unbounded if and only if it is feasible, while if $c_e \geq 0$ and if the problem has an optimal solution, then it has an optimal solution in which $x_e = 0$.

We now state, in graph-theoretical language, the matching polyhedron theorem announced by Edmonds and Johnson [11].

2.1.6 Theorem: The convex hull of matchings of a bidirected $G = (V,E)$ with capacities $\underline{u} \in (\underline{Z}_+ \cup \{+\infty\})^E$ and demands $\underline{b} \in \underline{Z}^V$, that is, of feasible solutions to the matching problem 2.1.1, is the polyhedron $MP(G,\underline{b},\underline{u})$ of vectors $\underline{x} \in \underline{R}_+^E$ satisfying 2.1.3, 2.1.5 and

2.1.7 $\underline{x}(\delta(T)) - 2\underline{x}(F) \geq 1 - \underline{u}(F)$ for T and F such that $T \subseteq V$, $F \subseteq \delta(T)$, $\underline{b}(T) + \underline{u}(F)$ is an odd integer.

Strategy of the Proof: Note that $MP(G,\underline{b},\underline{u})$ need not be bounded: for example, if a vertex v has a loop e with 2 heads and a

loop f with 2 tails, if $u_e = u_f = +\infty$ and if $\underline{x} \in MP(G,\underline{b},\underline{u})$, then x_e and x_f can be increased without bound, by the same amount, without violating the demand constraint 2.1.5 for v; the inequalities 2.1.7 remain satisfied because $\delta(T)$ does not contain any loops. However, the special cases of the theorem that we consider in sections 2.2 to 2.6 all involve bounded polyhedra, so we may invoke theorems 1.6.1 (in the first proof in section 2.2) and 1.6.2. Conditions (1) and (1') of these theorems will always be obvious.

In order not to have to check conditions (2) and (2') over and over again, we now show that any feasible solution $\underline{x}$ to problem 2.1.1 is an element of $MP(G,\underline{b},\underline{u})$, for which it suffices to check that $\underline{x}$ satisfies the inequalities 2.1.7. From 2.1.5, we have, for each $v \in T$

2.1.8 $$\underline{x}(\delta^+(v)) - \underline{x}(\delta^-(v)) \equiv b_v \pmod 2.$$

Summing the congruences 2.1.8 over $v \in T$ gives

2.1.9 $$\underline{b}(T) \equiv \Sigma(\underline{x}(\delta^+(v)) : v \in T) - \Sigma(\underline{x}(\delta^-(v)) : v \in T) \pmod 2.$$

Now if $e \in \gamma(T)$ and e is not a loop, then x_e appears in the right--hand side of 2.1.9 with a coefficient of -2, 0 or 2 according to whether e has two tail ends, one tail and one head end, or two head ends. Therefore 2.1.9 implies

$$\underline{b}(T) \equiv \underline{x}(\delta^+(T)) - \underline{x}(\delta^-(T)) \pmod 2,$$

whence $\underline{x}(\delta(T)) \equiv \underline{b}(T) \pmod 2$,

$$\underline{x}(\delta(T)) + \underline{u}(F) \equiv \underline{b}(T) + \underline{u}(F) \pmod 2,$$

2.1.10 $$\underline{x}(\delta(T)) + \underline{u}(F) \text{ is an odd integer.}$$

Now $F \subseteq \delta(T)$ and $\underline{x} \geq \underline{0}$, so $\underline{x}(\delta(T)) \geq \underline{x}(F)$, and $\underline{x} \leq \underline{u}$, so $\underline{u}(F) \geq \underline{x}(F)$, from which

2.1.11 $$\underline{x}(\delta(T)) + \underline{u}(F) \geq 2\underline{x}(F).$$

By 2.1.10, the two sides of 2.1.11 have opposite parities, so in fact $\underline{x}(\delta(T)) + \underline{u}(F) \geq 1 + 2\underline{x}(F)$, which is equivalent to 2.1.7.

It should be noted that there is an inequality 2.1.7 for T and F only when $u_e < +\infty$ for each $e \in F$, because of the restriction that $\underline{b}(T) + \underline{u}(F)$ must be an odd integer. In many of the special cases we consider, $u_e = +\infty$ for all edges e, so the inequalities 2.1.7 have the simpler form

$\underline{x}(\delta(T)) \geq 1$ for $T \subseteq V$ such that $\underline{b}(T)$ is an odd integer,

since F is always empty.

To complete the proof of the main theorem 2.1.6, we go through the following steps: in section 2.2 verify condition (3) of theorem 1.6.1 for the special case of 1-matching, that is, we show that every vertex of the 1-matching polyhedron is a 1-matching; in each of sections 2.3 to 2.6, we define a more general matching problem and polyhedron P, and verify condition (3') of theorem 1.6.2, that is, we reduce the more general

problem to an equivalent instance of the matching of the preceding section, for which we will already know the convex hull P' in terms of linear inequalities, and then given a vertex $\underline{x}^*$ of P, we will exhibit $\underline{y}^* \in P'$ such that $\underline{x}^*$ and $\underline{y}^*$ have equal objective values. The non-trivial computation in this respect will be to show that $\underline{y}^*$ satisfies the inequalities corresponding to 2.1.7. In the last step of the proof, in section 2.7, we use a different method since for the first time the polyhedron will be unbounded.

Section 2.2 - The 1-Matching Polyhedron

A 1-matching of a simple graph $G = (V,E)$ is a subset M of E such that each vertex of G is incident with precisely one element of M, that is, the demand is 1 at each vertex. Since the constraint matrix for the 1-matching problem, the problem of finding a minimum cost 1-matching with respect to a given cost vector $\underline{c} \in \underline{R}^E$, is the nonnegative incidence matrix of a simple graph, each edge has an implicit capacity of 1.

2.2.1 1-Matching Problem: minimize $\underline{c}.\underline{x}$ subject to

2.2.2 $$\underline{x} \in \underline{Z}_+^E$$

2.2.3 $$\underline{x}(\delta(v)) = 1 \quad \text{for } v \in V$$

where $\underline{c} \in \underline{R}^E$ and $G = (V,E)$ is a simple graph.

The constraint 2.2.2 could of course be replaced by $\underline{x} \in \{0,1\}^E$. It is now clear how the term matching came to be: a 1-matching matches the vertices of G in pairs (in particular, for G to have a 1-matching, $|V|$ must be even), and matchings of bidirected graphs are generalizations of simple graph 1-matchings.

What we are calling a 1-matching is often called a perfect 1-matching, in order to distinguish it from those subsets M of E having the property that each vertex of G is incident with at most one element of M, which corresponds to the demand constraints $\underline{x}(\delta(v)) \leq 1$ for $v \in V$.

Since all the demand constraints in this chapter are equalities, we dispense with the adjective "perfect". No generality is lost in this respect, since any imperfect matching problem can be transformed into an equivalent perfect matching problem by adjoining zero cost lobes to every vertex having an inequality demand constraint.

Turning now to the convex hull of feasible solutions to problem 2.2.1, a natural first try for the convex hull of feasible solutions to any ILP problem is to simply drop the integrality constraints, which is equivalent to supposing that the problem is not genuine. Given any simple graph $G = (V,E)$, let us therefore investigate what are the vertices of

$$P(G) = \{\underline{x} \in \underline{R}_+^E : \underline{x}(\delta(v)) = 1 \quad \text{for } v \in V\}.$$

Given a vertex $\underline{x}^*$ of $P(G)$, first erase from G all the edges e on which $x_e^* = 0$. Clearly $\underline{x}^*$ is the unique element of $P(G)$ taking the value 0 on all the erased edges (by the definition of a polyhedral vertex).

In the resulting graph, if e is such that $x_e^* = 1$, then certainly e and its two ends form a component.

Consider a component C on which $\underline{0} < \underline{x}^* < \underline{1}$. Every vertex of C has degree at least 2, so C contains at least one circuit. Let

$$\beta = \tfrac{1}{2}\min(x_e^* : e \in E(C))$$

and note that $\beta > 0$.

Suppose that C contains an even circuit

$$v_1, e_1, v_2, e_2, \ldots, v_{2n}, e_{2n}, v_1$$

and define $\underline{x}' \in \underline{R}^E$

$$x'_e = \begin{cases} x^*_e & \text{if } e \neq e_i \text{ for } 1 \leq i \leq 2n, \\ x^*_e + \beta, & \text{if } e = e_i \text{ for } i \text{ odd}, 1 \leq i \leq 2n, \\ x^*_e - \beta, & \text{if } e = e_i \text{ for } i \text{ even}, 1 \leq i \leq 2n. \end{cases}$$

Since $\underline{x}' \geq \underline{0}$ by definition of β,

$$\underline{x}'(\delta(v)) = \underline{x}^*(\delta(v)) = 1 \quad \text{for } v \neq v_i,\ 1 \leq i \leq 2n, \text{ and}$$

$$\underline{x}'(\delta(v_i)) = \underline{x}^*(\delta(v_i)) + \beta - \beta = 1 \quad \text{for } 1 \leq i \leq 2n,$$

we have that $\underline{x}' \in P(G)$, which contradicts the uniqueness of $\underline{x}^*$. Therefore C contains no even circuits.

Now suppose that C contains more than one odd circuit, in particular the two odd circuits

$$v_1, e_1, v_2, e_2, \ldots, v_{2m+1}, e_{2m+1}, v_1$$

$$w_1, f_1, w_2, f_2, \ldots, w_{2m+1}, f_{2m+1}, w_1$$

where (without loss of generality) v_1 and w_1 are joined by the path $\quad v_1, g_1, u_2, g_2, \ldots, u_k, g_k, w_1$.

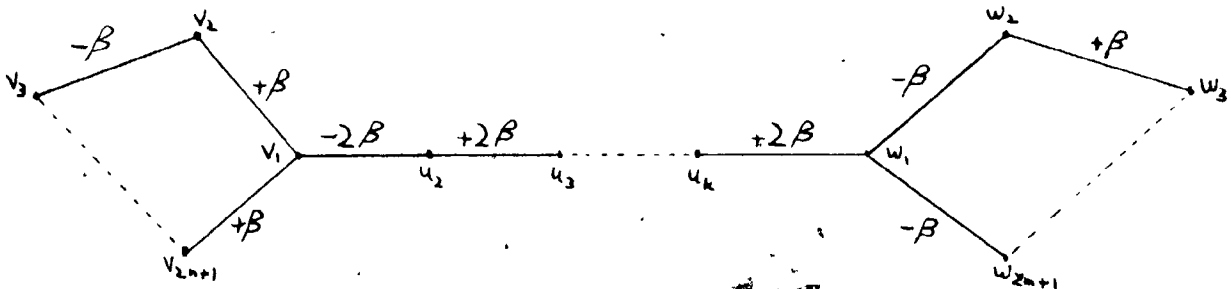
It should be noted that the subsequent argument does not depend on the two odd circuits being disjoint. If they are not disjoint, then of course the path joining them has length zero.

We now define $\underline{x}'$ on E by

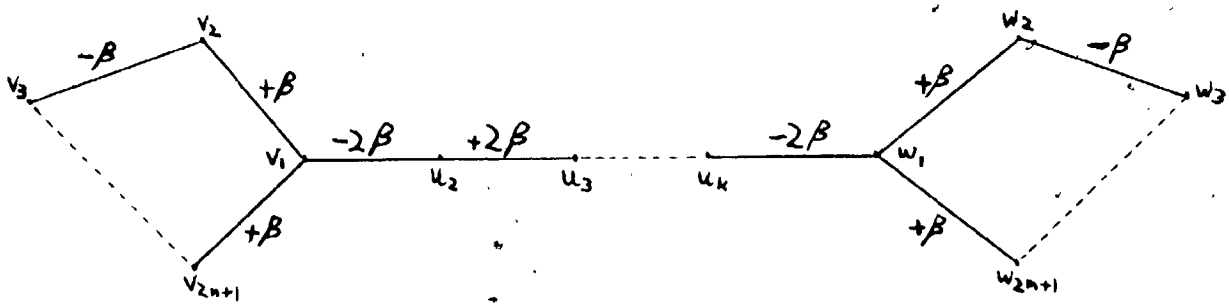
(a) $x'_e = x^*_e$ if e is not an edge of one of the two circuits or of the path,

(b) for the remaining edges of G, depending on whether k is even or odd, we define $\underline{x}'$ according to the schematic diagrams:

k even



k odd



The factor $\frac{1}{2}$ in the definition of β guarantees that $\underline{x}' \geq \underline{0}$. Since the β's cancel at each vertex, we also have that

$$\underline{x}'(\delta(v)) = \underline{x}^*(\delta(v)) = 1 \quad \text{for } v \in V.$$

Therefore $\underline{x}' \in P(G)$ and $\underline{x}' \neq \underline{x}^*$ (since $\beta > 0$), which again contradicts the uniqueness of $\underline{x}^*$, so C contains (and hence is) exactly one odd circuit.

Moreover, the equations

$$\underline{x}^*(\delta(v)) = 1 \quad \text{for } v \in V(C)$$

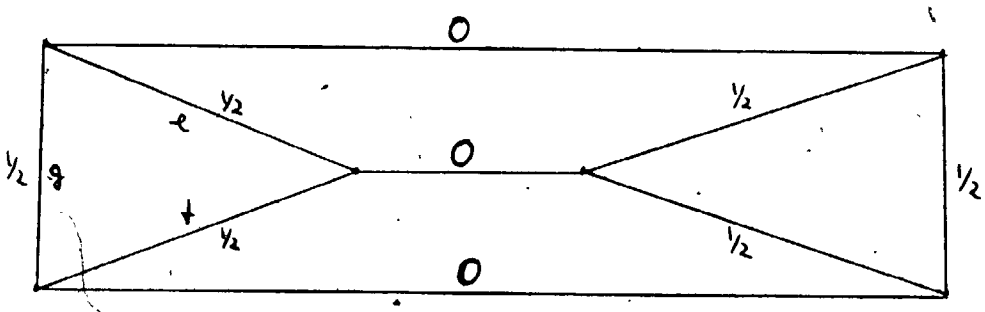
force $x^*_e = \frac{1}{2}$ for each $e \in E(C)$, so we have proved:

2.2.4 <u>Theorem</u>: For any simple graph $G = (V,E)$, the polyhedron

$$P(G) = \{\underline{x} \in \underline{R}^E_+ : \underline{x}(\delta(v)) = 1 \quad \text{for } v \in V\}$$

is the convex hull of vectors $\underline{x}^*$ having the property that the components of the spanning subgraph $(V, \{e \in E : x^*_e > 0\})$ are either single edges (on which $\underline{x}^*$ is 1) or odd circuits (on each edge of which $\underline{x}^*$ is $\frac{1}{2}$).

The following example shows that the second type of component can actually occur, and hence that the 1-matching problem is a genuine ILP problem.



Setting $\underline{x}^*$ equal to 0 on the three horizontal edges forces $\underline{x}^*$ equal to $\frac{1}{2}$ on the other six edges: $x^*_e + x^*_f = x^*_f + x^*_g = 1$ so $x^*_e = x^*_g$; but $x^*_e + x^*_g = 1$ so $x^*_e = x^*_g = \frac{1}{2}$, etcetera.

The next result is an immediate consequence of theorems 2.2.4 and 1.2.1 (bipartite graphs have no odd circuits).

2.2.5 <u>Theorem</u>: If G is bipartite, then P(G) is the convex hull of incidence vectors of 1-matchings of G.

This theorem, which says that a 1-matching problem on a bipartite graph is not a genuine ILP problem, is independently due to Birkhoff [3] and von Neumann [31] in quite different contexts.

What additional linear constraints do we need to obtain the desired convex hull? Let $\underline{x}$ be the incidence vector of a 1-matching and let S be an odd set of vertices (that is, $S \subseteq V$ and $|S|$ is odd). If $e \in \delta(S)$ with $x_e = 1$, then x_e counts one vertex of S, while if $e \in \gamma(S)$ with $x_e = 1$, then x_e counts two vertices of S, so

$$|S| = \underline{x}(\delta(S)) + 2\underline{x}(\gamma(S)).$$

Hence $\quad |S| \geq 2\underline{x}(\gamma(S)) \quad$ (since $\underline{x} \geq \underline{0}$)

and in fact $\quad |S| > 2\underline{x}(\gamma(S)) \quad$ (since $|S|$ is odd).

This gives $\quad \underline{x}(\delta(S)) = |S| - 2\underline{x}(\gamma(S)) > 0$, so

2.2.6 $$\underline{x}(\delta(S)) \geq 1 \quad \text{for odd } S \subseteq V.$$

The necessary (for the convex hull) constraints 2.2.6 are called <u>blossom constraints</u>, because Edmonds' algorithm [8] for the 1-matching problem involves "shrinking" odd circuits (which he called blossoms), such that a 1-matching of the "shrunken" graph uniquely induces a 1-matching of the original graph. The essential point is that if a circuit has 2n + 1 edges

then no 1-matching can use more than n of them, and conversely, if exactly one of the circuit's vertices is met by a matching edge, then the other 2n vertices can always be matched in pairs using n edges of the circuit.

The next theorem, which says that the blossom constraints are also sufficient to determine the convex hull of 1-matchings, has been proved algorithmically by Edmonds [9] and Cunningham and Marsh [5], and non-algorithmically by Balinski [2], Hoffman and Oppenheim [20] and Seymour [29]. The proof given below is based on an idea due to Lovász (oral communication).

2.2.7 Theorem: The convex hull of 1-matchings of a simple graph $G = (V,E)$, that is, of feasible solutions to the 1-matching problem 2.2.1, is the polyhedron 1-MP(G) of vectors $\underline{x} \in \underline{R}_+^E$ satisfying 2.2.3 and

2.2.8 $\underline{x}(\delta(T)) \geq 1$ for $T \subseteq V$ such that $|T|$ is odd.

Proof: If $|V|$ is odd, then 1-MP(G) $= \emptyset$ since $\delta(V) = \emptyset$. This is as it should be, because a simple graph with an odd number of vertices cannot have a 1-matching.

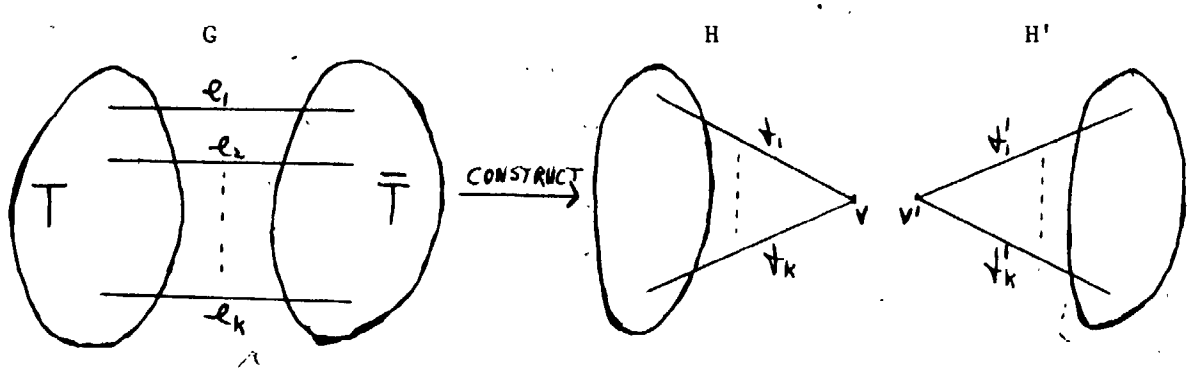
In view of theorem 1.6.1, it suffices to show, which we do by induction on $|V|$, that every vertex of 1-MP(G) is the incidence vector of a 1-matching of G.

The result is obvious for $|V| = 2$, for then 1-MP(G) is empty or a singleton according to whether G has no edges or one edge.

Assume the theorem is true for $|V| \leq 2n$, let $G = (V,E)$ be a simple graph with $|V| = 2n + 2$, and let $\underline{x}^*$ be a vertex of the corresponding 1-MP(G).

Case 1: If $\underline{x}^*(\delta(T)) > 1$ for $T \subseteq V$ such that $|T|$ is odd, $|T| \geq 3$ and $|\bar{T}| \geq 3$, then $\underline{x}^*$ is a vertex of P(G), as defined in theorem 2.2.4, since no blossom constraint is among the equations uniquely determining $\underline{x}^*$. Therefore $\underline{x}^*$ has the property given in this theorem. But if $\underline{x}^*$ is equal to $\frac{1}{2}$ on an odd circuit C, then $\underline{x}^*(\delta(V(C))) = 0$ with $|V(C)|$ odd, contradicting 2.2.8. Hence $\underline{x}^*$ is (0,1)-valued and is the incidence vector of a 1-matching of G.

Case 2: If case 1 does not apply, then there is a subset T of V such that $|T|$ is odd, $|T| \geq 3$, $|\bar{T}| \geq 3$ and $\underline{x}^*(\delta(T)) = 1$. We now use the equality of this (non-trivial) blossom constraint to construct two smaller graphs, H and H', to which the induction hypothesis can be applied, as follows:



Where $\delta_G(T) = \{e_1, e_2, \ldots, e_k\}$, we introduce two new vertices v and v', and form the graphs

$$H = (T \cup \{v\}, \gamma_G(T) \cup \{f_1, f_2, \ldots, f_k\})$$

$$H' = (\bar{T} \cup \{v'\}, \gamma_G(\bar{T}) \cup \{f'_1, f'_2, \ldots, f'_k\})$$

where, for $1 \leq i \leq k$, the ends of f_i (respectively f'_i) in H (respectively H') are v (respectively v') [illegible]e end of e_i in T (respectively $\bar{T}$). Define $\underline{y}$ on $E(H)$ by

$$y_e = \begin{cases} x^*_e & , e \in \gamma_G(T) \\ x^*_{e_i} & , e = f_i \text{ for } 1 \leq i \leq k \end{cases}$$

and $\underline{y}'$ on $E(H')$ similarly.

Clearly $\underline{y} \geq \underline{0}$, for $w \in T$ we have

$$\underline{y}(\delta_H(w)) = \underline{x}^*(\delta_G(w)) = 1, \text{ and}$$

$$\underline{y}(\delta_H(v)) = \Sigma(y_{f_i} : 1 \leq i \leq k) = \Sigma(x^*_{e_i} : 1 \leq i \leq k) = \underline{x}^*(\delta_G(T)) = 1.$$

Furthermore, if $S \subseteq V(H)$ is an odd set of vertices, then $v \notin S$ gives $\underline{y}(\delta_H(S)) = \underline{x}(\delta_G(T)) \geq 1$, and $v \in S$ gives $\underline{y}(\delta_H(S)) = \underline{x}(\delta_G((S \cap T) \cup \bar{T})) \geq 1$ since $|S \cap T| = |S| - 1$ is even, so $|(S \cap T) \cup \bar{T}|$ is odd. This shows that $\underline{y} \in 1\text{-MP}(H)$, and similarly $\underline{y}' \in 1\text{-MP}(H')$.

Since $|V(H)| = |T| + 1 = |V| - |\bar{T}| + 1 \leq |V| - 3 + 1 = 2n$ (and similarly $|V(H')| \leq 2n$), we can apply the induction hypothesis to write $\underline{y}$ and $\underline{y}'$ as

$$\underline{y} = \Sigma(\alpha_i \underline{m}_i : 1 \leq i \leq r)$$

$$\underline{y}' = \Sigma(\alpha'_i \underline{m}'_i : 1 \leq i \leq r')$$

where $\alpha_i > 0$ for $1 \leq i \leq r$, $\alpha'_i > 0$ for $1 \leq i \leq r'$,

$$\Sigma(\alpha_i : 1 \leq i \leq r) = \Sigma(\alpha'_i : 1 \leq i \leq r') = 1,$$

and each $\underline{m}_i$ (respectively $\underline{m}'_i$) is the incidence vector of a 1-matching of H (respectively H').

For each j ($1 \leq j \leq k$) such that $x^*_{e_j} > 0$, the 1-matchings of H and H' satisfying $m_i(f_j) = m'_i(f'_j) = 1$ can be connected in pairs (one from H, one from H') to form 1-matchings of G. Given such a 1-matching $\underline{M}_{is}$ formed from $\underline{m}_i$ and $\underline{m}'_s$, associate with it the coefficient $\beta_{is} = \min(\alpha_i, \alpha'_s)$ and reduce the coefficients α_i and α'_s by the amount β_{is} (so that at least one of them becomes zero). Now find another pair of 1-matchings of H and H' whose coefficients have not been reduced to zero and repeat the process. Since

$$y_{f_j} = y'_{f'_j} = \Sigma(\alpha_i : 1 \leq i \leq r,\ m_i(f_j) = 1) = \Sigma(\alpha'_i : 1 \leq i \leq r',\ m'_i(f'_j) = 1),$$

this procedure finds 1-matchings of G taking the value 1 on e_j and associated coefficients adding up to $y_{f_j} = y'_{f'_j}$. Repeating the procedure for each j, we obtain 1-matchings $\underline{M}_1, \ldots, \underline{M}_t$ of G of which $\underline{x}^*$ is a convex combination.

Since 1-MP(G) is a polyhedron, theorem 1.3.1 tells us that $\underline{x}^*$ is an extreme point of 1-MP(G). By the definition of an extreme point, $t = 1$ and $\underline{x}^* = \underline{M}_1$. <u>QED</u>

Section 2.3 - The b-Matching Polyhedron (b-MP)

The b-matching problem generalizes the 1-matching problem in that vertex demands are allowed to be arbitrary positive integers. The constraint matrix, as the incidence matrix of a simple graph, is nonnegative, so a negative demand at any vertex would result in infeasibility. As in the 1-matching problem, there are implicit capacity constraints and the set of feasible solutions is finite.

2.3.1 <u>b-Matching Problem</u>: minimize $\underline{c}.\underline{x}$ subject to

2.3.2 $$\underline{x} \in \underline{Z}_+^E$$

2.3.3 $$\underline{x}(\delta(v)) = b_v \quad \text{for } v \in V$$

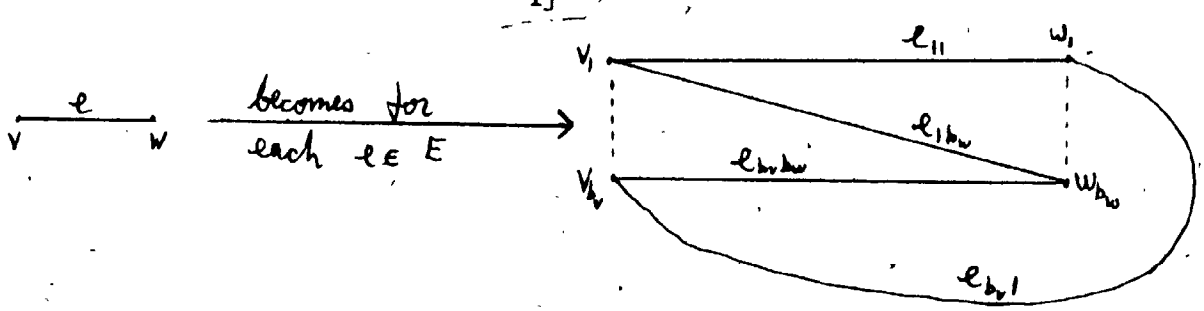
where $\underline{c} \in \underline{R}^E$, $\underline{b} \in \underline{Z}_+^V$ and $G = (V,E)$ is a simple graph.

We first show that any instance of 2.3.1 can be reduced to an equivalent 1-matching problem of the form 2.2.1. Define $G' = (V',E')$ by

$$V' = \{v_1, v_2, \ldots, v_{b_v} : v \in V\}$$

$$E' = \{e_{ij} = (v_i, w_j) : e = (v,w) \in E,\ 1 \le i \le b_v,\ 1 \le j \le b_w\}$$

and define $\underline{c}' \in \underline{R}^{E'}$ by $\quad c'_{e_{ij}} = c_e$ for each $e_{ij} \in E'$.



Thus we have replaced each $v \in V$ by b_v copies of itself, and we have created $b_v b_w$ copies of each $e = (v,w) \in E$, one joining each copy of v to each copy of w.

Given a b-matching $\underline{x}$ of G, we construct a 1-matching $\underline{x}'$ of G' with $\underline{c}.\underline{x} = \underline{c}'.\underline{x}'$ as follows:

Pick $e = (v,w) \in E$ such that $x_e > 0$, and set

$$x'_{e_{ij}} = \begin{cases} 1, & 1 \le i = j \le x_e \\ 0, & i \ne j,\ 1 \le i \le b_v,\ 1 \le j \le b_w. \end{cases}$$

Clearly $c_e x_e = \Sigma(c'_{e_{ij}} x'_{e_{ij}} : 1 \le i \le b_v,\ 1 \le j \le b_w)$. We can now delete from G' all the edges e_{ij}, the vertices $v_1,\ldots,v_{x_e},w_1,\ldots,w_{x_e}$ (since their demands of 1 have been satisfied), and all other edges incident with these vertices (on these edges we set $\underline{x}'$ to zero). The 1-matching problem on this smaller graph is the reduction of the b-matching problem obtained from the original b-matching problem by deleting the edge e and lowering the demands b_v and b_w by the amount x_e. An induction argument yields the desired $\underline{x}'$.

Conversely, suppose $\underline{x}'$ is a 1-matching of G', and define $\underline{x} \in \underline{Z}_+^E$ by

$$x_e = \Sigma(x'_{e_{ij}} : 1 \le i \le b_v,\ 1 \le j \le b_w) \text{ for } e = (v,w) \in E.$$

Now
$$\begin{aligned} \underline{x}(\delta(v)) &= \Sigma(x_e : e = (v,w) \text{ for some } w \in V) \\ &= \Sigma(\Sigma(x'_{e_{ij}} : 1 \le i \le b_v,\ 1 \le j \le b_w) : e = (v,w)) \\ &= \Sigma(\Sigma(x'_{e_{ij}} : e = (v,w),\ 1 \le j \le b_w) : 1 \le i \le b_v) \\ &= \Sigma(\underline{x}'(\delta(v_i)) : 1 \le i \le b_v) = b_v \quad \text{for each } v \in V, \end{aligned}$$

so $\underline{x}$ is a b-matching of G. Since, for each $e = (v,w) \in E$,

$$c_e x_e = \Sigma(c'_{e_{ij}} x'_{e_{ij}} : 1 \leq i \leq b_v,\ 1 \leq j \leq b_w),$$

we also have that $\underline{c}.\underline{x} = \underline{c}'.\underline{x}'$, which completes the proof that the reduced 1-matching problem is equivalent to the original b-matching problem.

Now we use the above reduction to prove the b-matching polyhedron theorem.

2.3.4 <u>Theorem</u>: The convex hull of b-matchings of a simple graph $G = (V,E)$, that is, of feasible solutions to the b-matching problem 2.3.1, is the polyhedron b-MP(G) of vectors $\underline{x} \in \underline{R}_+^E$ satisfying 2.3.3 and

2.3.5 $\underline{x}(\delta(T)) \geq 1$ for $T \subseteq V$ such that $\underline{b}(T)$ is odd.

<u>Proof</u>: As explained in section 2.1, it suffices, given any vertex $\underline{x}^*$ of b-MP(G), to exhibit $\underline{x}' \in$ 1-MP(G') such that $\underline{c}.\underline{x}^* = \underline{c}'.\underline{x}'$, where G' and $\underline{c}'$ are as in the reduction. Define $\underline{x}' \in \underline{R}^{E'}$ by

$$x'_{e_{ij}} = (b_v b_w)^{-1} x^*_e \text{ for } e = (v,w) \in E,\ 1 \leq i \leq b_v,\ 1 \leq j \leq b_w.$$

It is immediate that $\underline{x}' \geq \underline{0}$ and that $\underline{c}.\underline{x}^* = \underline{c}'.\underline{x}'$.

Now, for any $v_i \in V'$, assume without loss of generality that i is the first index of all edges incident with v_i in G', and then $\underline{x}'(\delta_{G'}(v_i)) = \Sigma(\Sigma(x'_{e_{ij}} : 1 \leq j \leq b_w) : e = (v,w) \text{ in } G)$ $= \Sigma(\Sigma((b_v b_w)^{-1} x^*_e : 1 \leq j \leq b_w) : e = (v,w) \text{ in } G) = (b_v)^{-1} \underline{x}^*(\delta_G(v))$ $= (b_v)^{-1} b_v = 1.$

It remains to verify that $\underline{x}'$ satisfies the blossom constraints 2.2.8 for 1-MP(G'). Let T' be any odd subset of V' and let v_i $(1 \leq i \leq b_v)$ be any element of T'. If there exists $j \neq i$ $(1 \leq j \leq b_v)$ such that $v_j \notin T'$, it is not hard to see that $1 = \underline{x}'(\delta_{G'}(v_i)) \leq \underline{x}'(\delta_{G'}(T'))$: for every edge $e_{ik} \in \delta_{G'}(v_i)$ such that $e_{ik} \notin \delta_{G'}(T')$ and the other end of e_{ik} is w_k, we have $w_k \in T'$ and hence $e_{jk} \in \delta_{G'}(T')$ (since $v_j \notin T'$); but $x'_{e_{ik}} = x'_{e_{jk}}$ and the result follows.

Thus we may assume that $\{v_1, v_2, \ldots, v_{b_v}\} \subseteq T'$ whenever $v_i \in T'$, so $\underline{b}(T) = |T'|$ is odd, where $T = \{v \in V: v_1 \in T'\}$. The blossom constraints 2.3.5 of b-MP(G) then guarantee that

$$\underline{x}^*(\delta_G(T)) \geq 1.$$

Now for each $e = (v,w) \in \delta_G(T)$ such that $v \in T$, $w \notin T$, we have $v_i \in T'$, $w_j \notin T'$, and hence $e_{ij} \in \delta_{G'}(T')$, for all i and j in the ranges $1 \leq i \leq b_v$, $1 \leq j \leq b_w$. Therefore

$$\underline{x}'(\delta_{G'}(T')) = \underline{x}^*(\delta_G(T)) \geq 1$$

and $\underline{x}' \in$ 1-MP(G'). <u>QED</u>

Section 2.4 - The Lobe Matching Polyhedron (LMP)

In this section we prove the convex hull theorem for the b-matching problem on simple generalized graphs, that is, simple graphs with lobes (edges having only one end). A b-matching problem on a simple graph $G = (V,E)$ is infeasible whenever $\underline{b}(V)$ is odd, for $\delta(V) = \emptyset$, the blossom constraint $\underline{x}(\delta(V)) \geq 1$ cannot be satisfied, so the convex hull of feasible solutions is empty. On the other hand, if G is allowed to have lobes, then $\delta(V)$ is precisely the set of lobes, so that $\underline{b}(V)$ being even is no longer a necessary condition for feasibility.

2.4.1 Lobe Matching Problem: minimize $\underline{c}.\underline{x}$ subject to

2.4.2 $$\underline{x} \in \underline{Z}_+^E$$

2.4.3 $$\underline{x}(\delta(v)) = b_v \text{ for } v \in V$$

where $\underline{c} \in \underline{R}^E$, $\underline{b} \in \underline{Z}_+^V$ and $G = (V,E)$ is a simple generalized graph.

For the reduction to the problem of section 2.3, let H and H' be two copies of the subgraph of G obtained by deleting all lobes, and define

$$G' = (V(H) \cup V(H'), E(H) \cup E(H') \cup \{(v,v') : v \in V(H) \text{ and } v' \in V(H') \text{ are copies of each other and each is incident with a lobe in } G\}).$$

Note that we may assume no vertex of G is incident with more than one lobe: only the one with least cost need ever be non-zero in an optimal solution.

We define $\underline{b}' \in \underline{Z}_+^{V(G')}$ by letting the demands at $v \in V(H)$ and $v' \in V(H')$ be the same as the demand at the vertex of G of which they are copies. It is convenient to think of G as being the subgraph of G' obtained by deleting V(H') and E(H'). Then $\delta_{G'}(V(H)) = \delta_G(V)$. Defining $\underline{c}' \in \underline{R}^{E(G')}$ to be equal to $\underline{c}$ on E(G) and equal to $\underline{0}$ on E(H') = E(G') - E(G), the equivalence of problem 2.4.1 and the b'-matching problem on G' with cost vector $\underline{c}'$ is immediate.

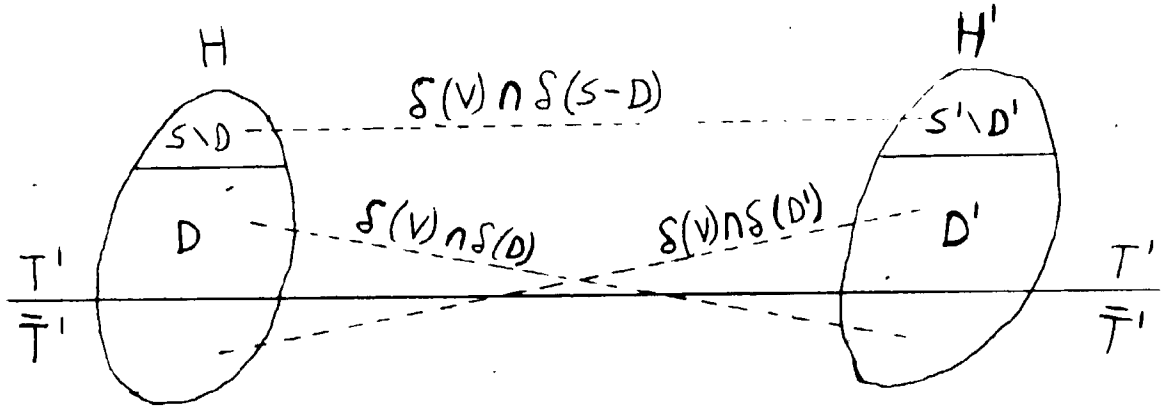
2.4.4 <u>Theorem</u>: The convex hull of b-matchings of a simple generalized graph G = (V,E), that is, of feasible solutions to the lobe matching problem 2.4.1, is the polyhedron LMP(G,$\underline{b}$) of vectors $\underline{x} \in \underline{R}_+^E$ satisfying 2.4.3 and

2.4.5 $\underline{x}(\delta(T)) \geq 1$ for $T \subseteq V$ such that $\underline{b}(T)$ is odd.

<u>Proof</u>: Given $\underline{x}^*$ a vertex of LMP(G,$\underline{b}$), define $\underline{x}' \in \underline{R}_+^{E(G')}$ by $\underline{x}' = \underline{x}^*$ on E(G), and $x'_{e'} = x^*_e$ for $e' \in E(H')$ a copy of e, a non-lobe in G. It is obvious that $\underline{x}'$ satisfies the demands $\underline{b}'$, and that $\underline{c}.\underline{x}^* = \underline{c}'.\underline{x}'$ for any cost vector $\underline{c} \in \underline{R}^E$ and associated (as in the reduction) $\underline{c}' \in \underline{R}^{E(G')}$.

To complete the proof that $\underline{x}' \in \underline{b}'$-MP(G'), we let T' be any subset of V(G') such that $\underline{b}'(T')$ is odd, and show that $\underline{x}'(\delta_{G'}(T')) \geq 1$. Let $S = V(H) \cap T'$, $S' = V(H') \cap T'$, and let D (respectively D') be the subset of S (respectively S') consisting of those vertices whose copies are not in S' (respectively S).

The key to the proof is that every edge in $\delta(V) \cap \delta(D)$ (or $\delta(V) \cap \delta(D')$) is also in $\delta(T')$, as follows immediately from the definition of G'.



Now $\quad \underline{b}'(T') = 2\underline{b}'(S - D) + \underline{b}'(D) + \underline{b}'(D')$

$\quad\quad \equiv \underline{b}'(D) + \underline{b}'(D') \pmod 2$

so $\underline{b}'(D)$ and $\underline{b}'(D')$ have opposite parities. We assume, without loss of generality, that $\underline{b}'(D) = \underline{b}(D)$ is odd, so $\underline{x}^*(\delta_G(D)) \geq 1$ by the assumption that $\underline{x}^* \in LMP(G,\underline{b})$. Now

$$\delta_G(D) = (\delta_G(D) \cap \delta_{G'}(T')) \cup (\delta_G(D) \cap \delta_G(S - D)).$$

But every $\quad e \in (\delta_G(D) \cap \delta_G(S - D))$

has a copy $\quad e' \in (\delta_{G'}(S' - D') \cap \delta_{G'}(\bar{T}' \cap V(H'))) \subseteq \delta_{G'}(T')$

with $\quad x^*_e = x'_e = x'_{e'}$.

Therefore $\quad \underline{x}'(\delta_{G'}(T')) \geq \underline{x}^*(\delta_G(D) \geq 1$. <u>QED</u>

Section 2.5 - The Capacitated Matching Polyhedron (CMP)

The generalization we consider in this section is to a matching problem with finite capacities $\underline{u} \in \underline{Z}_+^E$, and in which the underlying graph is a generalized graph, so that both loops and lobes are allowed. It is worth noting that the constraint $x_e \leq u_e$ is redundant if $u_e \geq \min(b_v : e \in \delta(v))$, for it is this fact we exploit to reduce the capacitated matching problem to problem 2.4.1.

2.5.1 Capacitated Matching Problem: Minimize $\underline{c}.\underline{x}$ subject to

2.5.2 $$\underline{x} \in \underline{Z}_+^E$$

2.5.3 $$\underline{x} \leq \underline{u}$$

2.5.4 $$\underline{x}(\delta(v)) + 2\underline{x}(\gamma(v)) = b_v \quad \text{for } v \in V$$

where $\underline{c} \in \underline{R}^E$, $\underline{u} \in \underline{Z}_+^E$, $\underline{b} \in \underline{Z}_+^V$ and $G = (V,E)$ is a generalized graph.

We assume $\underline{u} > \underline{0}$, for $u_e = 0$ forces $x_e = 0$, and the edge e can be deleted. To reduce the above to the lobe matching problem, we make the following construction:

If e is a loop, then of course v = w both before and after the construction, while if e is a lobe with single end v, then e_2

is a lobe with single end q_e. The result of the construction we denote by $G' = (V',E')$, where

$$V' = V \cup \{p_e, q_e : e \in E\}$$

$$E' = \{e = (v,p_e),\ e_1 = (p_e,q_e),\ e_2 = (q_e,w) : e = (v,w) \in E\}.$$

Note that G' has no loops.

The idea is to set the demand $b'_{p_e} = u_e$ so that the capacity constraint $x_e \leq u_e$ becomes redundant. If we also set $b'_{q_e} = u_e$, then every feasible solution $\underline{x}$ to the new problem satisfies

$$x_e + x_{e_1} = b'_{p_e} = b'_{q_e} = x_{e_1} + x_{e_2}$$

so $x_e = x_{e_2}$, that is, the contribution of edge e to the demand at v is the same as the contribution of e_2 to the demand at w. Thus we define $\underline{b}' \in \underline{Z}_+^{V'}$ by

$$b'_v = \begin{cases} b_v, & v \in V \\ u_e, & v = p_e \text{ or } q_e \text{ for some } e \in E. \end{cases}$$

It is easy to see that the original capacitated matching problem is equivalent to the reduced (uncapacitated) lobe matching problem with cost vector $\underline{c}' \in \underline{R}^{E'}$ given by

$$c'_e = \begin{cases} c_e, & e = (v,p_e) \text{ for some } e = (v,w) \in E \\ 0, & e = e_1 \text{ or } e_2 \text{ for some } e \in E. \end{cases}$$

2.5.5 <u>Theorem</u>: The convex hull of matchings of a generalized graph $G = (V,E)$ with capacities $\underline{u} \in \underline{Z}_+^E$ and demands $\underline{b} \in \underline{Z}_+^V$, that is, of feasible solutions to the capacitated matching problem 2.5.1, is the polyhedron $CMP(G,\underline{b},\underline{u})$ of vectors $\underline{x} \in \underline{R}_+^E$ satisfying 2.5.3, 2.5.4 and

2.5.6 $\underline{x}(\delta(T)) - 2\underline{x}(F) \geq 1 - \underline{u}(F)$ for T and F such that $T \subseteq V$, $F \subseteq \delta(T)$, $\underline{b}(T) + \underline{u}(F)$ is an odd integer.

Proof: It again suffices to show that, given any vertex $\underline{x}^*$ of $CMP(G,\underline{b},\underline{u})$, there exists $\underline{x}' \in LMP(G',\underline{b}')$ such that $\underline{c}.\underline{x}^* = \underline{c}'.\underline{x}'$, where G', $\underline{b}'$ and $\underline{c}'$ are as in the reduction. Define $\underline{x} \in \underline{R}^{E'}$ by

$$x'_e = x'_{e_2} = x^*_e \quad \text{and} \quad x'_{e_1} = u_e - x^*_e \quad \text{for each } e \in E.$$

It is clear from our preceding discussion that $\underline{x}' \geq \underline{0}$, $\underline{x}'(\delta(v)) = b'_v$ for $v \in V'$, and $\underline{c}.\underline{x}^* = \underline{c}'.\underline{x}'$, so it remains to verify that $\underline{x}'$ satisfies the blossom constraints 2.4.5.

Given any $T' \subseteq V'$ such that $\underline{b}'(T')$ is odd, we wish to show that $\underline{x}'(\delta_{G'}(T')) \geq 1$. Consider any edge $e = (v,w) \in E$ with the associated $v,w,p_e,q_e \in V'$ and $e,e_1,e_2 \in E'$. Now if $e,e_1 \in \delta_{G'}(T')$, then

$$\underline{x}'(\delta_{G'}(T')) \geq x'_e + x'_{e_1} = b'_{p_e} = u_e \geq 1$$

and we are done. Similarly $e_1,e_2 \in \delta_{G'}(T')$ also leads to the desired result.

Therefore we may assume that,

whenever	$\delta_{G'}(T') \cap \{e,e_1,e_2\} \neq \emptyset$,
either	(A) $e_1 \in \delta_{G'}(T')$ and $e,e_2 \notin \delta_{G'}(T')$,
or	(B) $e,e_2 \in \delta_{G'}(T')$ and $e_1 \notin \delta_{G'}(T')$.
Now let	$T = V \cap T'$,
	$F' = \{e_1 \in \delta_{G'}(T') : e,e_2 \notin \delta_{G'}(T')\}$
and	$F = \{e \in E : e_1 \in F'\}$.

Case (A) implies exactly one of $v,w \in T'$, so certainly $F \subseteq \delta_G(T)$. But case (B) implies that either $v,w \in T'$ or $v,w \notin T'$, so in fact $F = \delta_G(T)$.

We now show that we can apply the blossom inequality 2.5.6 corresponding to T and F:

case (A) occurs if and only if exactly one of $p_e, q_e \in T'$, so

$$\underline{b}'(T') = \underline{b}(V \cap T') + \Sigma(u_e : p_e \in T') + \Sigma(u_e : q_e \in T')$$

$$\equiv \underline{b}(T) + \Sigma(u_e : \text{exactly one of } p_e, q_e \in T') \pmod 2,$$

$$\underline{b}'(T') \equiv \underline{b}(T) + \underline{u}(F) \pmod 2, \text{ and}$$

$\underline{b}(T) + \underline{u}(F)$ is an odd integer.

This gives

$$\underline{x}^*(\delta_G(T)) - 2\underline{x}^*(F) \geq 1 - \underline{u}(F)$$

whence, using the fact that $x'_{e_1} = u_e - x^*_e$ for every edge of the type e_1, we have

$$\underline{x}'(\delta_{G'}(T')) \geq \underline{x}'(F') = \underline{u}(F) - \underline{x}^*(F) \geq 1 - \underline{x}^*(\delta_G(T)) + \underline{x}^*(F) = 1,$$

since $F = \delta_G(T)$. <u>QED</u>

Section 2.6 - The Bidirected Matching Polyhedron (BMP)

In this second last step of the main theorem's proof, we treat matchings in bidirected graphs with finite capacities on all the edges. The convex hull is bounded and theorem 1.6.2 still applies. In the next section we allow infinite capacities, which, given that coefficients of demand equations can be negative when the underlying graph is bidirected, makes it possible for the convex hull to be unbounded, so that the last step of the proof, in section 2.7, will be somewhat different.

2.6.1 Bidirected Matching Problem: minimize $\underline{c}.\underline{x}$ subject to

2.6.2 $\underline{x} \in \underline{Z}_+^E$

2.6.3 $\underline{x} \leq \underline{u}$

2.6.4 $\underline{x}(\delta^+(v)) + 2\underline{x}(\gamma^+(v)) \cancel{- \underline{x}(\delta^-(v))} - 2\underline{x}(\gamma^-(v)) = b_v$ for $v \in V$

where $\underline{c} \in \underline{R}^E$, $\underline{u} \in \underline{Z}_+^E$, $\underline{b} \in \underline{Z}^V$ and $G = (V,E)$ is a bidirected graph.

The idea for reducing the above problem to the capacitated matching problem is quite simple: if all edges incident with v have heads at v, then $b_v \geq 0$ and the demand constraint

$$\underline{x}(\delta^+(v)) + 2\underline{x}(\gamma^+(v)) = b_v$$

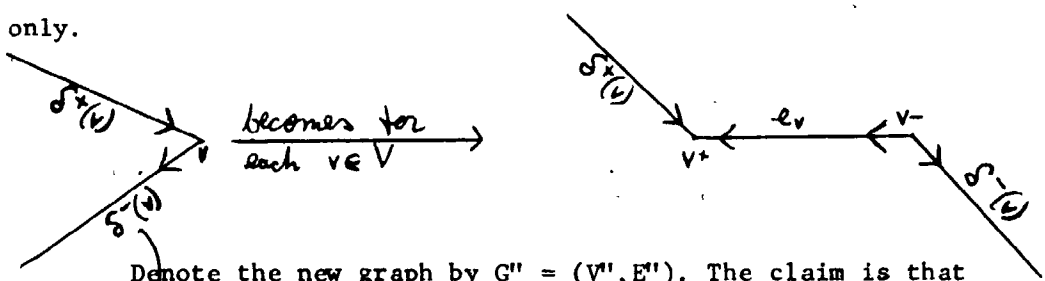
has the same form as a demand constraint in a generalized graph; if all edges incident with v have tails at v, then $b_v \leq 0$ and the demand constraint

$$- \underline{x}(\delta^-(v)) - 2\underline{x}(\gamma^-(v)) = b_v$$

can be multiplied through by -1 to be put into the desired form.

To simplify notation, we assume from now on that G has no loops; it is easy to see that the reduction and the proof of the theorem are not significantly affected by this assumption, mainly because blossom constraints are not dependent on loops.

For each $v \in V$ we create new vertices v+ and v-, joined by a new edge e_v, that are respectively met by heads and tails only.



Denote the new graph by $G'' = (V'', E'')$. The claim is that the two bidirected matching problems on G and G'' will be equivalent if we define

$$c'_{e_v} = 0$$

$$b''_{v+} = b_v + \underline{u}(\delta^-_G(v)) + 1$$

$$b''_{v-} = - \underline{u}(\delta^-_G(v)) - 1$$

$$u'_{e_v} = b''_{v+} + 1$$

leaving $\underline{c}$ and $\underline{u}$ unchanged on the remaining edges.

Given an $\underline{x}$ feasible to 2.6.1, defining

$$x'_{e_v} = b''_{v+} - \underline{x}(\delta^+_G(v))$$

and $\underline{x}' = \underline{x}$ on the remaining edges obviously leaves the objective

value unchanged, and the demand at v+ and the capacity constraint $x'_{e_v} \leq u'_{e_v}$ satisfied. Also,

$$- \underline{x}'(\delta_G^-(v)) - x'_{e_v} = - \underline{x}'(\delta_G^-(v)) + \underline{x}(\delta_G^+(v)) - b_v$$
$$- \underline{u}(\delta_G^-(v)) - 1 = - \underline{u}(\delta_G^-(v)) - 1 = b_{v-}$$

since $\quad b_v = \underline{x}(\delta_G^+(v)) - \underline{x}(\delta_G^-(v)),$

so the demand constraint at v- is satisfied. To check that $x'_{e_v} \geq 0$, we compute

$$x'_{e_v} = b''_{v+} - \underline{x}(\delta_G^+(v)) = b_v + \underline{u}(\delta_G^-(v)) + 1 - \underline{x}(\delta_G^+(v))$$
$$\underline{u}(\delta_G^-(v)) - \underline{x}(\delta_G^-(v)) + 1 \geq 1.$$

The terms +1 and -1 in the definitions of b''_{v+}, b''_{v-} and u'_{e_v} are not necessary for the reduction, but they simplify the proof of the convex hull theorem, precisely because they guarantee $x'_{e_v} \geq 1$ and $x'_{e_v} \leq u'_{e_v} - 1$.

Given an $\underline{x}'$ feasible to the new problem, it is not hard to see that the restriction $\underline{x}$ of $\underline{x}'$ to E is feasible to 2.6.1 and satisfies $\underline{c}.\underline{x} = \underline{c}'.\underline{x}'$, so the two problems are equivalent.

We now transform the new bidirected matching problem on G" into an equivalent capacitated matching problem, of the form 2.5.1, on the graph $G' = (V',E')$, obtained from G" by removing all directions, with demands $\underline{b}' \in \underline{Z}_+^{V'}$ defined by

$$b'_v = |b''_v| \quad \text{for } v \in V'.$$

2.6.5 Theorem: The convex hull of matchings of a bidirected graph $G = (V,E)$ with demands $\underline{b} \in \underline{Z}^V$ and capacities $\underline{u} \in \underline{Z}_+^E$, that is, of feasible solutions to the bidirected matching problem 2.6.1, is the polyhedron $BMP(G,\underline{b},\underline{u})$ of vectors $\underline{x} \in \underline{R}_+^E$ satisfying 2.6.3, 2.6.4 and

2.6.6 $\underline{x}(\delta(T)) - 2\underline{x}(F) \geq 1 - \underline{u}(F)$ for T and F such that $T \subseteq V$, $F \subseteq \delta(T)$, $\underline{b}(T) + \underline{u}(F)$ is an odd integer.

Proof: Given any vertex $\underline{x}^*$ of $BMP(G,\underline{b},\underline{u})$, we define $\underline{x}' \in \underline{R}_+^{E'}$ as in the reduction, namely

$$x'_{e_v} = b'_{v+} - \underline{x}^*(\delta_G^+(v)) \quad \text{for } e_v \in E' - E,$$

and show that $\underline{x}' \in CMP(G',\underline{b}',\underline{u}')$, and $\underline{c}.\underline{x}^* = \underline{c}'.\underline{x}'$ (which follows from the fact that $c'_{e_v} = 0$).

From the reduction, it is immediate that $\underline{0} \leq \underline{x}' \leq \underline{u}'$, and $\underline{x}'(\delta_{G'}(v)) = b'_v$ for $v \in V'$, so it remains to check the blossom constraints 2.5.6 for $T' \subseteq V'$ and $F' \subseteq \delta_{G'}(T')$ such that $\underline{b}'(T') + \underline{u}'(F')$ is an odd integer.

Suppose first that F' contains an edge of the type $e_v \in E' - E$. Then adding the three inequalities

$$- x'_{e_v} \geq 1 - u'_{e_v}$$
$$\underline{x}'(\delta_{G'}(T')) - \underline{x}'(F') \geq 0$$
$$- \underline{x}'(F' - \{e_v\}) \geq - \underline{u}'(F' - \{e_v\})$$

gives the desired result.

Now suppose that $\delta_{G'}(T') - F'$ contains an edge of the type $e_v \in E' - E$. Then we can similarly add

$$x'_{e_v} \geq 1$$
$$\underline{x}'(\delta_{G'}(T') - \{e_v\}) - \underline{x}'(F') \geq 0$$
$$-\underline{x}'(F') \geq -\underline{u}'(F')$$

to obtain the blossom inequality.

It remains to consider the case when

$$\delta_{G'}(T') \cap (E' - E) = \emptyset,$$

which is equivalent to

$$v+ \in T' \text{ if and only if } v- \in T' \quad \text{for all } v \in V.$$

Here we let $T = \{v \in V: v+ \in T'\}$, $F = F'$, and then

$$\underline{b}'(T') + \underline{u}'(F') = \Sigma(b'_{v+} + b'_{v-} : v \in T) + \underline{u}(F)$$
$$= \Sigma(b_v + \underline{u}(\delta^-(v)) + 1 + \underline{u}(\delta^-(v)) + 1 : v \in T) + \underline{u}(F) .$$
$$\equiv \underline{b}(T) + \underline{u}(F) \pmod 2.$$

The assumption that $\underline{x}^* \in BMP(G,\underline{b},\underline{u})$ now gives

$$\underline{x}^*(\delta_G(T)) - 2\underline{x}^*(F) \geq 1 - \underline{u}(F).$$

But $\delta_G(T) = \delta_{G'}(T')$ and $\underline{x}^* = \underline{x}'$ on $\delta_G(T)$, so

$$\underline{x}'(\delta_{G'}(T')) - 2\underline{x}'(F') \geq 1 - \underline{u}'(F'). \quad \underline{\text{QED}}$$

Section 2.7 - The Matching Polyhedron (MP)

We are now ready to deal with the matching problem 2.1.1 in all its glorious generality, and to complete the proof of the main theorem 2.1.6. For the reader's convenience, we first restate the problem and the theorem.

2.7.1 Matching Problem: minimize $\underline{c}.\underline{x}$ subject to

2.7.2 $\underline{x} \in \underline{Z}_+^E$

2.7.3 $\underline{x} \leq \underline{u}$

2.7.4 $\underline{x}(\delta^+(v)) + 2\underline{x}(\gamma^+(v)) - \underline{x}(\delta^-(v)) - 2\underline{x}(\gamma^-(v)) = b_v$ for $v \in V$

where $\underline{c} \in \underline{R}^E$, $\underline{u} \in (\underline{Z}_+ \cup \{+\infty\})^E$, $\underline{b} \in \underline{Z}^V$ and $G = (V,E)$ is a bidirected graph.

2.7.5 Theorem: The convex hull of feasible solutions to the matching problem 2.7.1 is the polyhedron $MP(G,\underline{b},\underline{u})$ of vectors $\underline{x} \in \underline{R}_+^E$ satisfying 2.7.3, 2.7.4 and

2.7.6 $\underline{x}(\delta(T)) - 2\underline{x}(F) \geq 1 - \underline{u}(F)$ for T and F such that $T \subseteq V$, $F \subseteq \delta(T)$, $\underline{b}(T) + \underline{u}(F)$ is an odd integer.

Proof: Let S be the set of feasible solutions to problem 2.7.1. Since S may be infinite we must use a different method from the one we have used so far to prove convex hull theorems. For each positive integer n, define $\underline{u}^n \in \underline{Z}_+^E$ by

$$u_e^n = \begin{cases} u_e, & \text{if } u_e < \infty \\ n, & \text{if } u_e = \infty. \end{cases}$$

Letting S_n be the set of feasible solutions to 2.7.1 when $\underline{u}^n$ is substituted for $\underline{u}$, it is clear that

$$S = \bigcup(S_n : 1 \leq n < \infty)$$

and that, from theorem 2.6.5,

$$CH(S_n) = BMP(G,\underline{b},\underline{u}^n) \quad \text{for } n \geq 1.$$

Since each polyhedron $BMP(G,\underline{b},\underline{u}^n)$ is obviously contained in $CH(S)$, and since each element of $CH(S)$ is an element of $BMP(G,\underline{b},\underline{u}^m)$ for some sufficiently large m, it follows that

$$CH(S) = \bigcup(BMP(G,\underline{b},\underline{u}^n) : 1 \leq n < \infty).$$

Now for any fixed n, the blossom constraints of $MP(G,\underline{b},\underline{u})$ are a subset of the blossom constraints of $BMP(G,\underline{b},\underline{u}^n)$, so

$$CH(S_n) \subseteq MP(G,\underline{b},\underline{u}) \quad \text{for } n \geq 1,$$

and hence $\quad CH(S) \subseteq MP(G,\underline{b},\underline{u})$.

Conversely, given any $\underline{x} \in MP(G,\underline{b},\underline{u})$, we choose m sufficiently large so that $\underline{x} \leq \underline{u}^m$, $2x_e \leq m$ for all e such that $u_e = +\infty$, and m is even, and show that $\underline{x} \in BMP(G,\underline{b},\underline{u}^m)$. This will complete the proof, for it will show that $\underline{x} \in CH(S_m) \subseteq CH(S)$.

Naturally it is only the blossom constraints that need to be verified, so take any $T \subseteq V$, $F \subseteq \delta(T)$ such that $\underline{b}(T) + \underline{u}^m(F)$ is odd. Letting $K = F \cap \{e \in E : u_e = +\infty\}$, and remembering that m is even, it is clear that

$$\begin{aligned} \underline{b}(T) + \underline{u}^m(F) &= \underline{b}(T) + \underline{u}^m(F - K) + m|K| \\ &\equiv \underline{b}(T) + \underline{u}(F - K) \pmod 2. \end{aligned}$$

Since $\underline{x}$ satisfies the blossom constraints 2.7.6, we then have

$$\underline{x}(\delta(T)) - 2\underline{x}(F - K) \geq 1 - \underline{u}(F - K) = 1 - \underline{u}^m(F - K).$$

But m was chosen large enough to ensure that

$$- 2\underline{x}(K) \geq - m|K| = - \underline{u}^m(K).$$

Adding these two inequalities shows that $\underline{x}$ satisfies the blossom constraints 2.6.6 for BMP(G,$\underline{b}$,$\underline{u}^m$). QED

Section 2.8 - Bounds for Matching Algorithms

This section, in which we discuss some known computation bounds for matching algorithms, and some consequences of the reductions in sections 2.3 to 2.6 for an algorithmic solution to the most general matching problem 2.7.1, does not pretend to be rigorous.

A computation bound for an algorithm is an upper bound on the amount of work which the algorithm requires for any input, usually expressed as a function of the input size. If this function is a polynomial, the algorithm is said to be good. We shall not give precise definitions of 'work' and 'input size'. The former is measured in terms of 'elementary operations', which include, for example, additions and comparisons. Thus it is assumed that the amount of work involved in adding or comparing two numbers is independent of the magnitude of the numbers. With respect to input size, we insist that problem instances be input in a 'reasonable' or non-wasteful manner. Otherwise any finite algorithm could be made 'good' by including sufficiently long strings of zeroes in the input. In particular, the input size of a number N in binary notation is on the order of $\log_2|N|$.

The first good matching algorithm is due to Edmonds [8] and solves the 1-matching problem 2.2.1. This is the blossom algorithm we have already mentioned, and its computation bound is on the order of $|V|^2|E|$. The 1-matching problem now has two other good algorithms, one due to Cunningham and Marsh [5] and one to Derigs [7].

Consider now a b-matching problem on $G = (V,E)$. It is not hard to see that its reduction to a 1-matching problem on $G' = (V',E')$, as in section 2.3, can be accomplished in a polynomial amount of work. But after the reduction, $|V'| = \underline{b}(V)$, so applying the blossom algorithm to the reduced problem would give a computation bound on the order of $(\underline{b}(V))^2|E|$, which is not good.

However, Cunningham and Marsh (see Marsh [25]) have found a way to modify the blossom algorithm so that it solves the b-matching problem 2.3.1 with a good computation bound on the order of

$$|V||E|\Sigma(1 + \log(b_v) : v \in V).$$

It is generally supposed that this modification could be extended to yield a good algorithm for the general matching problem 2.7.1, but this has not been done in detail. On the other hand, Edmonds and Johnson [11] have generalized the original blossom algorithm to solve problem 2.7.1. Their

computation bound includes a factor of $\underline{b}(V)$, so the algorithm is strictly speaking not good, although it compares very favourably with bounds for general integer linear programming algorithms.

In the rest of this section we indicate how a good algorithm for problem 2.7.1 could theoretically be devised by reducing it to problem 2.3.1, and then using the modified blossom algorithm of Cunningham and Marsh as a subroutine. The reader can easily convince himself or herself that the reductions of sections 2.4, 2.5 and 2.6 have the following two properties: they can be accomplished in a polynomial amount of work; the input parameters of the reduced problem are polynomials in the input parameters of the more general problem. This shows that the bidirected matching problem 2.6.1 has a good algorithm.

Now the proof of the final polyhedron theorem 2.7.5 does not provide a computation bound for problem 2.7.1, because it does not involve reducing that problem to the bidirected matching problem 2.6.1 with finite capacities. The essential difference between these two problems is that 2.7.1 may be unbounded, whereas 2.6.1 has an optimal solution if and only if it is feasible.

Suppose that we could find a positive integer N such that $x^*_e \leq N$ for all $e \in E$ and for all vertices $\underline{x}^*$ of the matching polyhedron MP. We could then solve the problem with finite capacities $\underline{u}^N \in \underline{Z}^E_+$, as defined in the proof of theorem 2.7.5, to either discover infeasibility, which would imply infeasibility of problem 2.7.1, or an optimal solution $\underline{x}'$ with objective value $\underline{c}.\underline{x}'$. In the latter case, either the given instance of problem 2.7.1 is unbounded, or it is bounded with optimal solution $\underline{x}'$, remembering that if an LP problem has an optimal solution, then it has a vertex optimal solution.

To test for these two possibilities, we solve the problem a second time with finite capacities $\underline{u}^{N+2} \in \underline{Z}^E_+$, obtaining an optimal solution $\underline{x}^2$. The claim is that problem 2.7.1 is bounded if and only if $\underline{c}.\underline{x}' = \underline{c}.\underline{x}^2$. That these objective values are equal if the problem is bounded is immediate from the optimality of $\underline{x}'$.

Conversely, if problem 2.7.1 is unbounded, then it has a feasible solution $\underline{x}^3$ such that $\underline{c}.\underline{x}^3 < \underline{c}.\underline{x}'$. Since the objective function $\underline{c}.\underline{x}$ is linear, it strictly decreases along the straight line segment from $\underline{x}'$ to $\underline{x}^3$. By convexity, this line segment is entirely contained in MP, so any point $\underline{y}$ on it satisfies the the demand constraints and the capacity constraints $\underline{y} \leq \underline{u}$. Certainly we can choose $\underline{y}$ close enough but not equal to $\underline{x}'$ to guarantee $\underline{y} \leq \underline{u}^{N+2}$.

Consider now a blossom constraint of $BMP(G,\underline{b},\underline{u}^{N+2})$ given by $T \subseteq V$, $F \subseteq \delta(T)$, $\underline{b}(T) + \underline{u}^{N+2}(F)$ odd. If $u_e < +\infty$ for all $e \in F$, then this blossom constraint is a constraint of $MP(G,\underline{b},\underline{u})$, and is therefore satisfied by $\underline{y}$. Otherwise note that

$$\underline{b}(T) + \underline{u}^{N+2}(F) \equiv \underline{b}(T) + \underline{u}^{N}(F) \pmod 2,$$

so

$$\underline{x}'(\delta(T)) - 2\underline{x}'(F) \geq 1 - \underline{u}^{N}(F) > 1 - \underline{u}^{N+2}(F).$$

Therefore $\underline{y}$ can be chosen close enough but not equal to $\underline{x}'$ to ensure that

$$\underline{y}(\delta(T)) - 2\underline{y}(F) \geq 1 - \underline{u}^{N+2}(F),$$

and we have exhibited the existence of $\underline{y} \in BMP(G,\underline{b},\underline{u}^{N+2})$ such that $\underline{c}.\underline{y} < \underline{c}.\underline{x}'$, whence it follows that $\underline{c}.\underline{x}^2 < \underline{c}.\underline{x}'$.

We now find such a positive integer N that bounds the coordinates of all the vertices $\underline{x}^*$ of $MP(G,\underline{b},\underline{u})$. The N we find is by no means a tight bound, for our sole purpose is to indicate, using the good computation bound for problem 2.6.1, that problem 2.7.1 also has a good algorithm.

Consider a system of equations, obtained by setting some of the inequalities defining $MP(G,\underline{b},\underline{u})$ to equality, of which $\underline{x}^*$ is the unique solution. Given $e \in E$ such that $u_e = +\infty$, if this system includes an equality blossom constraint

$$\underline{x}^*(\delta(T)) - 2\underline{x}^*(F) = 1 - \underline{u}(F)$$

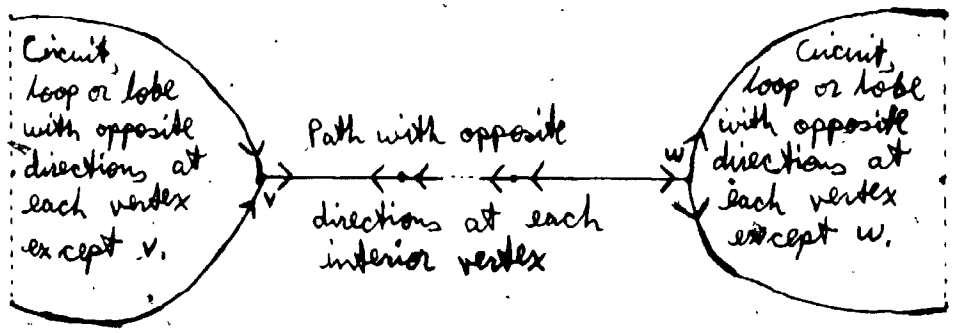
such that $e \in \delta(T)$, then, since $e \notin F$,

$$x^*_e \leq \underline{x}^*(\delta(T)) = 1 - \underline{u}(F) + 2\underline{x}^*(F) \leq 1 + \Sigma(u_e : e \in E,\ u_e < \infty),$$

and we have a bound for x^*_e.

Let $k = 1 + \Sigma(u_e : e \in E, u_e < \infty)$ and consider the spanning subgraph $H = (V, \{e \in E: x^*_e > k\})$. Suppose that every $v \in V$ is either isolated in H, or is a head end of at least one edge of H and a tail end of at least one edge of H. The claim is that we may then increase $\underline{x}^*$ on certain edges of H (provided $E(H) \neq \emptyset$, for otherwise we just take $N = k$), without violating the demand constraints. By the definition of k, capacity constraints and equality blossom constraints will remain satisfied, while inequality blossom constraints will still hold because $\underline{x}^*(F)$ remains unchanged (since $F \cap E(H) = \emptyset$). Thus the uniqueness of $\underline{x}^*$ will be contradicted.

Starting with any edge $e = (v,w)$ of H, pick edges $e_v, e_w \in E(H)$ such that v is a head end (respectively tail end) of e_v if v is a tail end (respectively head end) of e, and similarly for e_w. If e is a loop or lobe, we need pick only one such edge. Do the same at the other ends of e_v and e_w, and continue this process as long as the other ends of the edges picked have not already been used. It is not hard to see that, if the subgraph of H constructed as above does not contain a circuit in which each vertex is a head end of one edge and a tail end of another, and hence in which $\underline{x}^*$ can be increased by the same amount on all the edges without violating demand constraints, then it must be of the form:



Here $\underline{x}^*$ can be increased by 2 on the path and by 1 on a circuit or loop at either end of the path, or by 2 on a lobe at either end of the path, and feasibility is maintained.

Therefore we may assume that H contains a non-isolated vertex v that is not a tail end of any edges (without loss of generality: the argument is much the same if v is not a head end of any edges). We now use the demand constraint at v (considered as a vertex of G) to bound x_e^* for each $e \in \delta^+(v) \cup \gamma^+(v)$. In fact,

$$x_e^* \leq \underline{x}^*(\delta^+(v)) + \underline{x}^*(\gamma^+(v)) = b_v - \underline{x}^*(\gamma^+(v)) + \underline{x}^*(\delta^-(v)) + 2\underline{x}^*(\gamma^-(v))$$
$$\leq \max(|b_v| : v \in V) + 2k|E|.$$

Let k' be the last quantity in the above string of inequalities, and consider the spanning subgraph $H' = (V, \{e \in E: x_e^* > k'\})$. It is clear that H' has at least one more isolated vertex than H did, and, using the same argument all over again, we can bound x_e^* for all the edges e incident in G with some vertex other than v by

$$x_e^* \leq \max(|b_v| : v \in E) + 2k'|E|.$$

Continuing this process at most $|V|$ times gives a bound for x^*_e, for all $e \in E$, on the order of

$$N = |E|^{|V|}(|V|\max(|b_v| : v \in V) + |E|(1 + \Sigma(u_e : e \in E,\ u_e < \infty))).$$

To solve the matching problem 2.7.1, at worst we must solve two bidirected matching problems with finite capacities no greater than order of N. The fact that we solve two problems does not change the order of the computation bound, so we may insert N into the good computation bound for problem 2.6.1 to obtain a computation bound for problem 2.7.1.

The claim is that this computation bound is good. This is because the bound for problem 2.6.1, obtained from the reductions of sections 2.4, 2.5 and 2.6, and from the modified blossom algorithm's bound, includes the parameters u_e only in the argument of a logarithm. Even though N is exponential in $|V|$, logN is only polynomial in $|V|$, $|E|$, $\log|b_v|$ for $v \in V$ and $\log(u_e)$ for $e \in E$ such that $u_e < \infty$.

CHAPTER 3 - APPLICATIONS

In which the Chinese postman polyhedron theorem is shown to be a consequence of the matching polyhedron theorem, and in which a particular system of linear inequalities is proved, again using the matching polyhedron theorem, to determine the convex cone of incidence vectors of circuits of a graph.

Section 3.1 - The Chinese Postman Polyhedron

Most of the material in this section is taken from Edmonds and Johnson [12]. The idea is that the Chinese postman problem, so called by Edmonds [10] because it was first formulated by Mei-Ko Kwan [23], and because it has a natural interpretation in terms of a postman on his rounds, can be solved by first solving a matching problem for which we shall derive a convex hull theorem; it is then necessary to also solve the problem (one of the oldest in graph theory - see Euler [13]) of finding an Euler tour in a graph that is guaranteed to have one, and that is constructed from an optimal solution to the matching problem. We shall not discuss the second problem as it lies outside the scope of this dissertation.

Let $G = (V,E)$ be a connected graph. A <u>postman tour</u> of G is a circuit (in this section circuits are allowed to use the same vertex or edge more than once) of G which contains every $e \in E$ at least once. Note that a connected graph always has postman tours. An <u>Euler tour</u> of G is a circuit of G which contains every $e \in E$ exactly once. It is well known that G has an Euler tour if and only if the degree of every $v \in V$ is even. Given $\underline{c} \in \underline{R}_+^E$, the <u>cost</u> of a tour $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$ is $\Sigma(c_{e_i} : 1 \leq i \leq k)$. The <u>Chinese postman problem</u> is the problem of finding a minimum cost postman tour of G.

Now if G has an Euler tour, then that tour is a postman tour using each $e \in E$ as few times as possible (namely once), so it solves the Chinese postman problem (since $\underline{c} \geq \underline{0}$). On the other hand, given any postman tour of G, let $1 + x_e$ be the number of times edge e is in the tour, and construct $G(\underline{x})$ from G by adjoining to G, for each $e \in E$, x_e additional copies of the edge e. Clearly $G(\underline{x})$ has an Euler tour of the same cost $\underline{c}.\underline{x} + \underline{c}(E)$ as the given postman tour of G.

Therefore the Chinese postman problem can be solved by first obtaining an optimal solution $\underline{x}^*$ to

3.1.1 $\min(\underline{c}.\underline{x}: \underline{x} \in \underline{Z}_+^E,\ (\underline{1} + \underline{x})(\delta(v))$ an even integer for $v \in V)$

and then finding an Euler tour in $G(\underline{x}^*)$ of cost $\underline{c}.\underline{x}^* + \underline{c}(E)$.

We now use the result of the previous chapter to derive the <u>Chinese postman polyhedron</u> CPP(G), which is the convex hull of feasible solutions to 3.1.1. Naturally we must first write 3.1.1 as an equivalent matching problem.

Define $\underline{b} \in \{0,1\}^V$ by

$$b_v = \begin{cases} 0, & \text{if the degree of } v \text{ in } G \text{ is even} \\ 1, & \text{if the degree of } v \text{ in } G \text{ is odd} \end{cases}$$

and observe that $(\underline{1} + \underline{x})(\delta(v))$ is even if and only if

$$\underline{x}(\delta(v)) \equiv |\delta(v)| \pmod 2$$

which is in turn equivalent to

3.1.2 $$\underline{x}(\delta(v)) \equiv b_v \pmod 2.$$

In order to write 3.1.2 as an equation, we use a zero cost loop with two tails at v, that is, we introduce vertex variables $\underline{y} \in \underline{Z}_+^V$, and write the problem 3.1.1 as

3.1.3 $\min(\underline{c}.\underline{x}: \underline{x} \in \underline{Z}_+^E, \underline{y} \in \underline{Z}_+^V, \underline{x}(\delta(v)) - 2y_v = b_v \text{ for } v \in V)$.

By theorem 2.7.5, noting that 3.1.3 has infinite capacities on all edges, the convex hull of its feasible solutions is

$$P(G) = \{(\underline{x},\underline{y}) \in \underline{R}_+^{E \cup V}: \underline{x}(\delta(v)) - 2y_v = b_v \text{ for } v \in V;$$
$$\underline{x}(\delta(T)) \geq 1 \text{ for } T \subseteq V \text{ such that } \underline{b}(T) \text{ is odd}\}.$$

This straightforward application of the matching polyhedron theorem is all we need to prove the result of this section.

3.1.4 <u>Theorem</u>: The Chinese postman polyhedron, that is, the convex hull of feasible solutions to problem 3.1.1, is

$$CPP(G) = \{\underline{x} \in \underline{R}_+^E: \underline{x}(\delta(T)) \geq 1 \text{ for } T \subseteq V \text{ such that } \underline{b}(T) \text{ is odd}\}.$$

<u>Proof</u>: Given $\underline{x} \in CPP(G)$, define $\underline{y} \in \underline{R}^V$ by

$$y_v = \tfrac{1}{2}(\underline{x}(\delta(v)) - b_v) \quad \text{for } v \in V.$$

Note that if $b_v = \underline{b}(\{v\}) = 1$ is odd, then by assumption $\underline{x}(\delta(v)) \geq 1$, so certainly $\underline{y} \geq \underline{0}$ and hence $(\underline{x},\underline{y}) \in P(G)$. Therefore $(\underline{x},\underline{y})$ is a convex combination of feasible solutions to 3.1.3, each of which, when restricted to E, is a feasible solution to 3.1.1, so $\underline{x}$ is a convex combination of feasible solutions to 3.1.1.

Conversely, suppose $\underline{x}$ is a convex combination of feasible solutions $\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^k$ to 3.1.1:

$$\underline{x} = \Sigma(\alpha_i \underline{x}^i : 1 \leq i \leq k).$$

Clearly $\underline{x} \geq \underline{0}$. Moreover, each $\underline{x}^i$ (for $1 \leq i \leq k$) can be extended to a feasible solution $(\underline{x}^i, \underline{y}^i)$ to 3.1.3. Letting

$$\underline{y} = \Sigma(\alpha_i \underline{y}^i : 1 \leq i \leq k),$$

the vector $(\underline{x}, \underline{y})$ is a convex combination of $(\underline{x}^1, \underline{y}^1), \ldots, (\underline{x}^k, \underline{y}^k)$ and hence an element of P(G). This guarantees that

$\underline{x}(\delta(T)) \geq 1$ for $T \subseteq V$ such that $\underline{b}(T)$ is odd,

so $\underline{x} \in$ CPP(G). <u>QED</u>

Section 3.2 - Sums of Circuits

In this section we use the matching polyhedron theorem to prove a theorem, due to Seymour [28], giving the convex cone of incidence vectors of circuits of a graph. It was Edmonds who pointed out that this theorem can be derived from matching theory, which Seymour's original proof does not use.

The reader may reasonably ask why we now turn to a convex cone, which does not have the same connection with a good algorithm as does a convex hull. There is a good reason to suspect that finding the convex hull of circuits of a graph in terms of a finite system of linear inequalities is difficult, if not impossible. This reason is the connection between circuits and Hamiltonian circuits, which are circuits using every vertex exactly once, that is, circuits of length equal to the number of vertices of the graph.

Finding Hamiltonian circuits is one of graph theory's canonically difficult problems, and having the convex hull P of circuits would immediately give the convex hull $P \cap HP$ of Hamiltonian circuits, where HP is the polyhedron of vectors $\underline{x}$ satisfying $\underline{x}(E) = |V|$. If $\underline{x}$ is a convex combination of Hamiltonian circuits, as

$$\underline{x} = \Sigma(\alpha_i \underline{h}^i : 1 \leq i \leq k),$$

then certainly each $\underline{h}^i \in P \cap HP$, so $\underline{x} \in P \cap HP$. On the other hand, if $\underline{x} \in P \cap HP$, then $\underline{x}$ is a convex combination of circuits, namely $\underline{x} = \Sigma(\alpha_i \underline{c}^i : 1 \leq i \leq k)$.

But $\underline{c}^i(E) \leq |V|$ for each i $(1 \leq i \leq k)$, and therefore

$$|V| = \underline{x}(E) = \Sigma(\alpha_i \underline{c}^i(E) : 1 \leq i \leq k) \leq \Sigma(\alpha_i |V| : 1 \leq i \leq k) = |V|$$

since $\Sigma(\alpha_i : 1 \leq i \leq k)$. It follows that $\underline{c}^i(E) = |V|$ for each i, so each $\underline{c}^i$ is actually a Hamiltonian circuit, and $\underline{x}$ is a convex combination of Hamiltonian circuits.

In some sense a convex cone is the next best thing to a convex hull. The algebraic relationship between the two is clear. Geometrically, a convex cone has the property that if $\underline{x} \neq \underline{0}$ is in it, then the half-line, passing through $\underline{x}$ and having $\underline{0}$ as its endpoint, is entirely contained in the convex cone and intersects the corresponding convex hull in at least one point. In other words, the convex cone is the infinite union of all half-lines passing through points of the convex hull and having $\underline{0}$ as their endpoints.

The first step on the road to the promised convex cone theorem is to derive, using the matching polyhedron, the convex hull of SC(G), which stands for sums of circuits of G, and is the set of incidence vectors of unions of disjoint circuits of G = (V,E).

3.2.1 Lemma: $SC(G) = \{\underline{x} \in \{0,1\}^E : \underline{x}(\delta(v)) \in \{0,2\}$ for $v \in V\}$.

Proof: Suppose $\underline{x}$, defined on E, is (0,1)-valued and such that $\underline{x}(\delta(v))$ is equal to 0 or 2 for $v \in V$. Deleting from G all edges e such that $x_e = 0$ gives a graph in which: each component is either an isolated vertex, or a subgraph with each vertex having degree 2. Such a subgraph is simply a circuit, so $\underline{x} \in SC(G)$. The reverse containment is clear. QED

3.2.2 Theorem: The convex hull of SC(G) is

$$SCP(G) = \{\underline{x} \in \underline{R}_+^E : \underline{x} \leq \underline{1};\ \underline{x}(\delta(v)) \leq 2 \text{ for } v \in V;$$
$$\underline{x}(\delta(T)) - 2\underline{x}(F) \geq 1 - |F| \text{ for } T \text{ and } F \text{ such that}$$
$$T \subseteq V,\ F \subseteq \delta(T),\ |F| \text{ is odd}\}.$$

Proof: We use theorem 1.6.2. It is obvious that SCP(G) is a bounded polyhedron. To show that $SC(G) \subseteq SCP(G)$, we need only check the blossom constraints for $\underline{x} \in SC(G)$, which follow from the facts that $\underline{x}(\delta(T))$ is even for $T \subseteq V$, and $\underline{x}(\delta(T)) - 2\underline{x}(F) \geq -|F|$ for $F \subseteq \delta(T)$.

Now let

$$A = \{(\underline{x},\underline{y}) : \underline{x} \in \{0,1\}^E;\ \underline{y} \in \{0,1\}^V;\ \underline{x}(\delta(v)) + 2y_v = 2 \text{ for } v \in V\}.$$

It is routine to check that $\min(\underline{c}.\underline{x} : \underline{x} \in SC(G))$ is equivalent to $\min(\underline{c}.\underline{x} : (\underline{x},\underline{y}) \in A)$ for every $\underline{c} \in \underline{R}^E$. Each variable y_v represents a zero cost loop with two heads at v. From theorem

2.7.5, the convex hull of A is SCP(G) with

$\underline{x}(\delta(v)) \leq 2$ for $v \in V$

replaced by

$\underline{y} \in \underline{R}_+^V$; $\underline{y} \leq \underline{1}$; $\underline{x}(\delta(v)) + 2y_v = 2$ for $v \in V$.

Given $\underline{x} \in$ SCP(G), we define $\underline{y} \in \underline{R}_+^V$ by

$y_v = \frac{1}{2}(2 - \underline{x}(\delta(v)))$ for $v \in V$.

Since blossom constraints are unaffected by loops, it is easy to check that $(\underline{x},\underline{y})$ is in the convex hull of A. QED

We are now ready to state and prove Seymour's convex cone theorem for

$C^*(G) = \{\underline{c}: \underline{c}$ is the incidence vector of a circuit of $G\}$.

3.2.3 Theorem: The convex cone of $C^*(G)$ is

$$CC(C^*(G)) = \{\underline{x} \in \underline{R}_+^E: x_e \leq \underline{x}(\delta(T) - \{e\}) \text{ for } T \text{ and } e \text{ such that } T \subseteq V,\ e \in \delta(T)\}.$$

Proof: Suppose first that $\underline{x}$ is in the convex cone of $C^*(G)$, that is, $\underline{x} = \Sigma(\alpha_c \underline{c}: \underline{c} \in C^*(G))$ for some $\underline{\alpha} \in \underline{R}_+^{C^*(G)}$. Let $\alpha^* = \Sigma(\alpha_c: \underline{c} \in C^*(G))$. If $\alpha^* = 0$, then $\underline{x} = \underline{0}$ and the result follows immediately. Otherwise let

$\underline{x}^* = (\alpha^*)^{-1}\underline{x} = \Sigma((\alpha^*)^{-1}\alpha_c \underline{c}: \underline{c} \in C^*(G))$.

Since $C^*(G) \subseteq SC(G)$ and $\Sigma((\alpha^*)^{-1}\alpha_c: \underline{c} \in C^*(G)) = 1$, theorem 3.2.2 guarantees that $\underline{x}^* \in$ SCP(G). Given any T and e such that $T \subseteq V$ and $e \in \delta(T)$, clearly $|\{e\}|$ is odd, so

$$\underline{x}^*(\delta(T)) - 2x^*_e \geq 1 - |\{e\}| = 0,$$

$$x^*_e \leq \underline{x}^*(\delta(T)) - x^*_e = \underline{x}^*(\delta(T) - \{e\}),$$

$$x_e = \alpha^* x^*_e \leq \alpha^* \underline{x}^*(\delta(T) - \{e\}) = \underline{x}(\delta(T) - \{e\}).$$

Conversely, suppose $\underline{x} \in \underline{R}^E_+$ satisfies Seymour's inequalities. If $\underline{x}(E) = 0$, then $\underline{x} = \underline{0}$ and $\underline{x}$ is trivially in the convex cone of $C^*(G)$. Otherwise let

$$\underline{x}^* = (\underline{x}(E))^{-1}\underline{x}.$$

We show that $\underline{x}^* \in SCP(G)$. Clearly $\underline{x}^* \geq \underline{0}$ since $\underline{x} \geq \underline{0}$. Also, for any $B \subseteq E$, $\underline{x}^*(B) \leq \underline{x}^*(E) = (\underline{x}(E))^{-1}\underline{x}(E) = 1$, so in particular $\underline{x}^* \leq \underline{1}$ and $\underline{x}^*(\delta(v)) \leq 2$ for $v \in V$. To check the blossom constraints, take any T and F such that $T \subseteq V$, $F \subseteq \delta(T)$, $|F|$ is odd, and use the fact that $\underline{x}^*$ satisfies Seymour's inequalities:

$$x^*_e \leq \underline{x}^*(\delta(T) - \{e\}) \text{ for each } e \in F, \text{ so}$$

$$\underline{x}^*(F) \leq \Sigma(\underline{x}^*(\delta(T) - \{e\}) : e \in F) = |F|\underline{x}^*(\delta(T)) - \underline{x}^*(F),$$

$$\underline{x}^*(\delta(T)) - 2\underline{x}^*(F) \geq -(|F| - 1)\underline{x}^*(\delta(T)) \geq 1 - |F|$$

since $\underline{x}^*(\delta(T)) \leq 1$ and $1 - |F| \leq 0$. Therefore $\underline{x}^* \in SCP(G)$, which ensures that $\underline{x}^*$ can be written as a convex combination of elements of $SC(G)$. Since each element of $SC(G)$ is the incidence vector of a union of circuits, it follows that $\underline{x}^*$ can be written as a nonnegative linear combination of incidence vectors of circuits, that is, elements of $C^*(G)$. The same is then true of $\underline{x} = \underline{x}(E)\underline{x}^*$. QED

We close with a generalization of theorem 3.2.3, deriving it from the convex cone result the same way Seymour does [28].

3.2.4 Theorem: Given $\underline{d},\underline{u} \in \underline{R}_+^E$ with $\underline{d} \leq \underline{u}$, $CC(C^*(G)) \cap \{\underline{x} \in \underline{R}_+^E: \underline{d} \leq \underline{x} \leq \underline{u}\} \neq \emptyset$ if and only if

3.2.5 $d_e \leq \underline{u}(\delta(T) - \{e\})$ for T and e such that $T \subseteq V$, $e \in \delta(T)$.

Proof: It suffices to show that if the pair $(\underline{d},\underline{u})$ satisfies 3.2.5, then there exists $\underline{x} \in \underline{R}_+^E$ such that $(\underline{x},\underline{x})$ satisfies 3.2.5, and $\underline{d} \leq \underline{x} \leq \underline{u}$. This we do by induction on

$$N = |\{e \in E: d_e < u_e\}|.$$

If $N = 0$, then of course $\underline{d} = \underline{u}$ and we may take $\underline{x} = \underline{d}$. Otherwise choose $f \in E$ such that $d_f < u_f$.

For any $t \in \underline{R}_+$ such that $d_f \leq t \leq u_f$, define $\underline{d}^t,\underline{u}^t \in \underline{R}_+^E$ by $d_f^t = u_f^t = t$, and $d_e^t = d_e$, $u_e^t = u_e$ for $e \neq f$. Obviously $\underline{d} \leq \underline{d}^t \leq \underline{u}^t \leq \underline{u}$ and $|\{e \in E: d_e^t < u_e^t\}| = N - 1$. To complete the induction we show that t can be chosen such that $(\underline{d}^t,\underline{u}^t)$ satisfies 3.2.5.

If $t = u_f$ works, then we are done. Otherwise there is a $T \subseteq V$ such that $f \in \delta(T)$ and $u_f > \underline{u}(\delta(T) - \{f\})$ so setting

$$t = \min(\underline{u}(\delta(T) - \{f\}): T \subseteq V,\ f \in \delta(T))$$

guarantees that $d_f \leq t < u_f$.

Let $T_1 \subseteq V$ be such that $f \in \delta(T_1)$ and $t = \underline{u}(\delta(T_1) - \{f\})$. To verify the claim that $(\underline{d}^t,\underline{u}^t)$ satisfies 3.2.5, we must

show that

3.2.6 $$d_e = d_e^t \leq \underline{u}^t(\delta(T) - \{e\}) = \underline{u}(\delta(T) - \{e\}) + t - u_f$$

for each $T \subseteq V$ such that $f \in \delta(T)$, and for each $e \in \delta(T) - \{f\}$.

If $e \in \delta(T_1)$, then

$$d_e \leq u_e \leq \underline{u}(\delta(T_1) - \{f\}) = t.$$

Since $u_f \leq \underline{u}(\delta(T) - \{e\})$, 3.2.6 is satisfied.

If $e \notin \delta(T_1)$, then there is a $T_2 \subseteq V$ such that

$$\delta(T_2) \subseteq \delta(T) \cup \delta(T_1),$$

$$e \in \delta(T_2),$$

and $$f \notin \delta(T_2), \text{ whence}$$

$$d_e \leq \underline{u}(\delta(T_2) - \{e\}) \leq \underline{u}(\delta(T) - \{e,f\}) + \underline{u}(\delta(T_1) - \{f\})$$

$$= \underline{u}(\delta(T) - \{e\}) + t - u_f,$$

which is 3.2.6. <u>QED</u>

BIBLIOGRAPHY

[1] A. V. Aho, J. E. Hopcroft and J. D. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, Mass., 1974.

[2] M. L. Balinski, "Establishing the matching polytope", J. Combinatorial Theory B 13 (1972), 1-13.

[3] G. Birkhoff, "Tres observaciones sobre el algebra lineal", Univ. Nac. Tucuman Rev. Ser. A 5 (1946), 147-151.

[4] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan Press, London, 1976.

[5] W. H. Cunningham and A. B. Marsh III, "A primal algorithm for optimum matching", Math. Programming Study 8 (1978), 50-72.

[6] G. B. Dantzig, Linear Programming and Extensions, Princeton Univ. Press, Princeton, N. J., 1963.

[7] U. Derigs, "A shortest augmenting path method for solving minimal perfect matching problems", Report 79-06, Mathematisches Institut der Universität zu Köln, Köln, April 1979.

[8] J. Edmonds, "Paths, trees, and flowers", Can. J. Math. 17 (1965), 449-467.

[9] J. Edmonds, "Maximum matching and a polyhedron with (0.1) vertices", J. Res. Nat. Bur. Stand. 69 B (1965), 125-130.

[10] J. Edmonds, "The Chinese postman problem", Operations Res. 13 Suppl. 1 (1965), 373.

[11] J. Edmonds and E. L. Johnson, "Matching: a well-solved class of integer linear programs", in: R. K. Guy et al. (eds.), Combinatorial Structures and their Applications, Gordon and Breach, New York, 1970 (89-92).

[12] J. Edmonds and E. L. Johnson, "Matching, Euler tours, and the Chinese postman", Math. Programming 5 (1973), 88-124.

[13] L. Euler, "Solutio problematis ad geometriam situs pertinentis", Commentarii Academiae Petropolitanae 8 (1736), 128-140.

[14] L. R. Ford Jr. and D. R. Fulkerson, Flows in Networks, Princeton Univ. Press, Princeton, N. J., 1962.

[15] P. Gács and L. Lovász, "Khachian's algorithm for linear programming", Math. Programming Studies, to appear.

[16] D. Gale, The Theory of Linear Economic Models, McGraw-Hill, New York, 1960.

[17] M. Grötschel, L. Lovász and A. Schrijver, "The ellipsoid method and its consequences in combinatorial optimization", to appear.

[18] B. Grünbaum, Convex Polytopes, John Wiley and Sons, London, 1967.

[19] A. J. Hoffman, "Some recent applications of the theory of linear inequalities to extremal combinatorial analysis", Proc. Symp. Appl. Math. 10 (1960), 113-127.

[20] A. J. Hoffman and R. Oppenheim, "Local unimodularity in the matching polytope", Annals of Discrete Math. 2 (1978), 201-209.

[21] L. G. Khachian, "A polynomial algorithm in linear programming", Doklady Akademii Nauk SSSR 244 (1979), 1093-1096 (English translation: Soviet Math. Dokl. 20 (1979), 191-194).

[22] D. E. Knuth, The Art of Computer Programming, Volume 1, Fundamental Algorithms, Addison-Wesley, Reading, Mass., 1968.

[23] M.-K. Kwan, "Graphic programming using odd or even points", Chinese Math. 1 (1962), 273-277.

[24] D. G. Luenberger, Introduction to Linear and Nonlinear Programming, Addison-Wesley, Reading, Mass., 1973.

[25] A. B. Marsh III, Matching Algorithms, Thesis, Johns Hopkins University, Baltimore, 1979.

[26] M. W. Padberg and M. R. Rao, "Odd minimum cut-sets and b-matchings", Math. of Operations Res., to appear.

[27] W. R. Pulleyblank, Faces of Matching Polyhedra, Thesis, University of Waterloo, Waterloo, 1973.

[28] P. D. Seymour, "Sums of circuits", in: J. A. Bondy and U. S. R. Murty (eds.), Graph Theory and Related Topics, Academic Press, New York, 1979 (341-355).

[29] P. D. Seymour, "On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte", Proc. London Math. Society (3) 38 (1979), 423-460.

[30] J. Stoer and C. Witzgall, Convexity and Optimization in Finite Dimensions I, Springer-Verlag, New York, 1970.

[31] J. von Neumann, "A certain zero-sum two-person game equivalent to the optimal assignment problem", in: H. W. Kuhn and A. W. Tucker (eds.), Contributions to the Theory of Games, Volume 2, Annals of Math. Study 28, Princeton Univ. Press, Princeton, N. J., 1953 (12-15).

END

1	7	1	2	8	1

FIN