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A GENERALIZATION OF THE LOGISTIC EQUATION

by

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A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
Master of Science

Carleton University

Ottawa, Ontario

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The undersigned recommend to
the Faculty of Graduate Studies and Research
acceptance of the thesis,

A GENERALIZATION OF THE LOGISTIC EQUATION

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Valérie Poulin, B.Sc.

in partial fulfillment of the requirements for
the degree of Master of Science

A handwritten signature in black ink, appearing to read 'A. B. Mingarelli', written over a horizontal line.

Dr. Angelo B. Mingarelli, Thesis Supervisor

A handwritten signature in black ink, appearing to read 'Kenneth S. Williams', written over a horizontal line.

Dr. Kenneth S. Williams, Director, School of Mathematics and Statistics

Carleton University

November, 1999

To my grandmother
Lauria.

Abstract

Two families of non-conjugate functions which generalize the well-known logistic map $x_{n+1} = \lambda x_n(1 - x_n)$ for $\lambda = 4$ are defined, and some of their basic properties studied. It is shown that the Lyapunov exponent of all the maps contained in these families can be expressed by a simple formula and that, despite the dissimilarity of their polynomial expressions, they all possess the same invariant density. The Lyapunov exponents are calculated by constructing the conjugate maps of the functions and are shown to be positive which is a strong indication that all the functions are chaotic. The chaoticity properties are further studied using several other methods. Moreover, the two families of polynomial maps are briefly looked at from an algebraic point of view.

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Contents

1	Introduction	1
2	General Definitions	3
2.1	Orbits	3
2.2	Chaotic Behavior	6
2.3	Probabilistic Approach	10
3	Logistic Family	14
3.1	Definition of the Logistic Map	14
3.2	Bifurcation Diagram	15
3.3	Lyapunov Exponents	17
3.4	$\lambda = 4$. Exact Results	19
4	Generalization of the Logistic Family	22
4.1	Definition of two Sets of Functions: $\mathcal{S}$ and $\mathcal{C}$	22
4.2	Particularities of $\mathcal{S}$ and $\mathcal{C}$	28
4.3	Conjugate Maps of functions in $\mathcal{S}$ and $\mathcal{C}$	34
4.4	Lyapunov Exponents of $SN(x)$ and $CN(x)$	39
5	Conclusion	47

Chapter 1

Introduction

The *logistic map* was originally of interest to the ecologists who wanted to model the population of species. To simplify the problem, they tried to model the world in terms of discrete time intervals. One of the simplest models proposed (Maynard Smith 1968) was the following:

$$x_{n+1} = \lambda x_n(1 - x_n)$$

now known as the *logistic equation*. In this model x_0 is the initial population. x_n is the population n years (or any time intervals) later and the λ -value represents the reproduction rate of the studied population. The large fascination for this function is explained by the effect of the λ -value on the model itself: for various values of λ , the model behaves completely differently. At $\lambda = 4$, for instance, one can get a formula to express the population at any time in terms of n : $x_n = f(n)$.

Nowadays, the logistic equation is probably the most popular example found in the Dynamical Systems and Chaos literature. The reason for it, is that it is a very simple function which arises naturally in many contexts and it has a wide variety of types of dynamics.

In this thesis, we generalize the logistic function to two sets of functions $\mathcal{S}$ and $\mathcal{C}$ that exhibit very similar behavior. We begin in Chapter 2 by giving the necessary terminology and by introducing various concepts that are studied in the following chapters: sensitive dependence on initial conditions, Lyapunov exponent, bifurcation diagram, invariant measure, etc.

Chapter 3 is devoted to the study of the logistic function. We study its chaotic behavior by analyzing, through different approaches, several key properties of chaos. Every result presented in this chapter can be found in basic literature [5, 12, 14], its principal role is to motivate the observations made on functions of $\mathcal{S}$ and $\mathcal{C}$ in the next chapter.

Finally, Chapter 4 deals with the generalization itself. It contains the definitions of the sets $\mathcal{S}$ and $\mathcal{C}$ and it underlines the similarities between the study of the logistic function and the functions in $\mathcal{S}$ and in $\mathcal{C}$. Moreover, we briefly describe an algebraic property of these two sets and we make some connections between elements of $\mathcal{S}$ and elements of $\mathcal{C}$.

Chapter 2

General Definitions

In this chapter, we introduce the basic definitions and results relevant to the study of the logistic function. These definitions can be found in standard textbooks on “Chaos theory” or “Discrete Dynamical Systems”, [5, 12, 14].

2.1 Orbits

In this section, we consider a map $f : I \rightarrow I$ where $I \subset \mathbb{R}$. We say that we *iterate* f when we evaluate it over and over using the output of the previous application as the input for the next: $f(\dots(f(x))\dots)$, or if f is composed with itself a number of times.

Definition 1 *A dynamical system consists of a set of possible states, together with a rule that determines the present state in terms of past states. We denote the **initial state** by x_0 . The state changes with time under the **rule** $x_{n+1} = f(x_n)$. Here, the variable n stands for time, and x_n designates the state at time n .*

Definition 2 *Given $x_0 \in \mathbb{R}$, we define the **orbit of x_0 under f** to be the sequence of iterations of f , $\{x_0, f(x_0), f^2(x_0), \dots\}$. We use the following notation: $x_n = f^n(x_0)$.*

Definition 3 We call p a **periodic point of period k** if $f^k(p) = p$, and if k is the smallest such positive integer. The orbit with initial point p (which consists of k points) is called a **periodic orbit of period k** . When $k = 1$ i.e., $f(p) = p$, we say that p is a **fixed point** of the map f . The orbit $\{x_0, x_1, \dots\}$ is **eventually periodic** if there exists an integer N for which $\{x_N, x_{N+1}, \dots\}$ is a periodic orbit. Finally, the orbit $\{x_0, x_1, \dots\}$ is said to be **asymptotically periodic** if there exists a periodic orbit $\{y_0, y_1, \dots\}$ such that $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$.

There are mainly two different types of fixed points: *attracting* and *repelling* fixed points, also called *sinks* and *sources*.

Definition 4 Let p be a fixed point. We say that p is a **sink** (attracting fixed point) if there is an $\varepsilon > 0$ such that for all x in the ε -neighborhood $N_\varepsilon(p)$, $\lim_{k \rightarrow \infty} f^k(x) = p$. We call p a **source** (repelling fixed point) if there is an ε -neighborhood $N_\varepsilon(p)$ such that each x in $N_\varepsilon(p)$, except for p itself, eventually maps outside of $N_\varepsilon(p)$.

Note that the above definitions can also be used to characterize periodic points (*attracting* and *repelling* periodic points) since a k -periodic point of f is a fixed point of f^k . An attracting periodic point of f is a sink of f^k and a repelling periodic point of f is a source of f^k .

Theorem 2.1.1 Let f be a continuous function for which derivatives of all orders exist (i.e., f is a **smooth** function), and assume that p is a fixed point of f .

1. If $|f'(p)| < 1$, then p is a sink.
2. If $|f'(p)| > 1$, then p is a source.

Proof: See [14, p.10, Theorem 1.5]. □

Example 1 Let

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1/2 \\ 2(1-x), & \text{if } 1/2 \leq x \leq 1 \end{cases}.$$

This map is usually called the **tent map**. In Figure 1, we can see that f has only two fixed points: 0 and $2/3$. Those fixed points are sources since $|f'(x)| = 2 > 1$ for all x .

Figure 1 Fixed points of the tent map $f(x)$: $f(x) = x$ at 0 and $2/3$.

The points 0 and $2/3$ are shown in Figure 2 since they satisfy the equation $f^2(x) = x$. The other two points $2/5$ and $4/5$ are thus period-two points.

Figure 2 Fixed points and period-two points of the tent map:

$$f^2(x) = x \text{ at } 0, 2/5, 2/3 \text{ and } 4/5.$$

Notice that every periodic point of f is a repelling periodic point since $|(f^k)'(x)| = 2^k > 1$ for all positive integers k and for all points x for which the derivative of f^k exists.

Lemma 2.1.1 *If $|(f^k)'(x)| > 1$ for all positive integers k and for all points x for which the derivative of f^k exists then f does not have any asymptotically periodic orbit other than eventually periodic orbits.*

Proof: Let the orbit $\{x_0, x_1, \dots\}$ be an asymptotically periodic orbit of f , i.e., there exists a periodic orbit of period k , $\{y_0, y_1, \dots\}$ such that $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$. Note that $\{y_0, y_k, y_{2k}, \dots\}$ is a constant sequence and $\lim_{n \rightarrow \infty} x_{nk} = y_0$ which implies that for all $\varepsilon > 0$, there exists an integer N such that $|(f^k)^n(x_0) - y_0| < \varepsilon$ for all $n > N$. But y_0 is a repelling fixed point of f^k since $|(f^k)'(y_0)| > 1$, i.e., there exists $\varepsilon > 0$ such that each $x \in N_\varepsilon(y_0)$ except for y_0 itself eventually maps outside of $N_\varepsilon(y_0)$ when we iterate f^k . This implies that $(f^k)^n(x_0) = y_0$ for all $n > N$ and so $\{x_0, x_1, x_2, \dots\}$ is eventually periodic. $\square$

2.2 Chaotic Behavior

Maps of the real line are said to have *chaotic behavior* when they have *chaotic orbits*. As for a source, an orbit that is chaotic is one that exhibits an unstable behavior but that is not itself fixed or periodic. Chaoticity is generally a very complicated property to verify. Because of similarities between the orbits of a map and those of a map's conjugate, conjugacies are useful in characterizing the behavior of some functions under iteration.

Definition 5 *Let $f : I \rightarrow I$ and $g : J \rightarrow J$ be two maps. We say that f and g are **conjugate** if there is a homeomorphism $H : I \rightarrow J$ (bijective and continuous map), such that $H \circ f = g \circ H$. The function H is called a **conjugacy**.*

An interesting fact is that conjugacy is an equivalence relation on the set of functions of the form $f : I \rightarrow I$. It is easy to show that when x_0 is a periodic point of f of period k , then $H(x_0)$ is a periodic point of g of period k .

Lemma 2.2.1 *Let f, g be two conjugate maps with $f \circ H = H \circ g$. Let $\{x_0, x_1, x_2, \dots\}$ be an orbit of g . Then $\{H(x_0), H(x_1), H(x_2), \dots\}$ is an orbit of f .*

Proof: Consider $y_0 = H(x_0)$ and $\{y_0, y_1, y_2, \dots\}$ the orbit of y_0 under f . By induction, we have that

$$y_n = f(H(x_{n-1})) = H(g(x_{n-1})) = H(x_n). \quad \square$$

Example 2 *Consider the tent map*

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1/2 \\ 2(1-x), & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

and the logistic map $g(x) = 4x(1-x)$ (this function is studied in section 3.4). The function $H(x) = \sin^2(\frac{\pi x}{2})$ is a conjugacy since $(g \circ H)(x) = \sin^2(\pi x) = (H \circ f)(x)$. From Example 1, we know that 0 and 2/3 are fixed points of f and 2/5 and 4/5 are period two points of f : therefore $H(0) = 0$ and $H(2/3) = \sin^2(\frac{\pi}{3}) = (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$ are both fixed points of g and $H(2/5) = \sin^2(\frac{\pi}{5})$ and $H(4/5) = \sin^2(\frac{2\pi}{5})$ are both period two points of g .

In order to define the concept of “chaotic orbits”, we introduce a number, the *Lyapunov exponent* (or *Lyapunov number*) and the notion of *sensitive dependence*. Sensitive dependence on initial conditions refers loosely to the property that pairs of points, initially close together, will eventually move apart as we iterate f .

Definition 6 *Let f be a map on $\mathbb{R}$. A point x_0 has **sensitive dependence on initial conditions** if there exists $d > 0$ such that any neighborhood N of x_0 contains a point x such that $|f^k(x) - f^k(x_0)| \geq d$ for some nonnegative integer k .*

Figure 3 Time Evolution of $x_0 = 0.17$ and $y_0 = 0.17 + 10^{-3}$ under $g(x) = 4x(1 - x)$.

To illustrate this property, we depict in Figure 3 the time evolution of two nearby data under the map $g(x) = 4x(1 - x)$. The difference between the two initial points is 10^{-3} : it could account for the various sources of imprecision or error. We observe that after a certain time the two curves deviate: so, the point $x_0 = 0.17$ has sensitive dependence on initial conditions.

In order to quantify such a behavior we introduce the Lyapunov number which gives an average rate of separation of points x very close to some specific point x_0 .

Definition 7 Let f be a smooth map of the real line $\mathbb{R}$. The **Lyapunov number** $L(x_0)$ of the orbit $\{x_0, x_1, \dots\}$ is defined as

$$L(x_0) = \lim_{n \rightarrow \infty} (|f'(x_0)| \dots |f'(x_n)|)^{1/n}.$$

if this limit exists. The **Lyapunov exponent** $h(x_0)$ is defined as

$$h(x_0) = \lim_{n \rightarrow \infty} \frac{1}{n} (\ln |f'(x_0)| + \dots + \ln |f'(x_n)|)$$

if this limit exists.

Note that the Lyapunov exponent is an average along an orbit (a time average) of the logarithm of the derivative. We will see in the next section that this average is equal (under certain restrictions) to the space average (the integral) of the logarithm of the derivative for almost every point.

Example 3 Consider the tent map $f : I \rightarrow I$ from Example 2:

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1/2 \\ 2(1-x), & \text{if } 1/2 \leq x \leq 1 \end{cases}.$$

Note that $|f'(x)| = 2$ for all $x \in I$ except at $x = 0, 1/2, 1$ where the derivative does not exist as a two-sided derivative. Then, for any $x_0 \in I$, such that $x_i \neq 0, 1/2, 1$ for all $i = 0, 1, 2, \dots$ we have

$$\begin{aligned} h(x_0) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \ln |f'(x_i)| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \ln(2) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln(2^n) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} n \ln(2) \\ &= \ln(2). \end{aligned}$$

Definition 8 Let f be a map of the real line $\mathbb{R}$, and let $\{x_0, x_1, \dots\}$ be a bounded orbit of f . The orbit is **chaotic** if

1. $\{x_0, x_1, \dots\}$ is not asymptotically periodic, and
2. The Lyapunov exponent $h(x_0)$ is greater than zero, $h(x_0) > 0$.

The following result is remarkable for the simplicity of its statement and the facility it procures to verify if a map has a chaotic behavior or not.

Theorem 2.2.1 Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous and has an orbit of period three. Then f has chaotic orbits.

Proof: See [8, p.300-301]. □

Remark: There exists an order $\prec$ defined on the natural numbers $\mathbb{N}$ called the *Sharkovskii's ordering*:

$$\begin{aligned} 3 \prec 5 \prec 7 \prec 9 \prec 11 \prec \dots &\prec 2 * 3 \prec 2 * 5 \prec 2 * 7 \prec \dots \\ \prec 2^2 * 3 \prec 2^2 * 5 \prec 2^2 * 7 \prec \dots &\prec 2^3 * 3 \prec 2^3 * 5 \prec \dots \\ \prec 2^4 * 3 \prec 2^4 * 5 \prec 2^4 * 7 \prec \dots &\prec 2^4 \prec 2^3 \prec 2^2 \prec 2 \prec 1. \end{aligned}$$

This ordering on $\mathbb{N}$, where $*$ denotes ordinary multiplication, allows the following result: Assume f is a continuous map on an interval and has a period- p orbit. If $p \prec q$, then f has a period- q orbit. As a corollary, we have that if f has a period-3 orbit, then it has a period- n orbit for all integers n (since 3 is the smallest integer with respect to Sharkovskii's ordering). For a proof of this result, see [12, section 3.1].

2.3 Probabilistic Approach

In the previous section, we have introduced the concept of Lyapunov exponent as a particular property of the orbit of a point x_0 under a function f . The following result tells us that the Lyapunov exponent characterizes the function f itself, the concept becomes therefore more useful since it gives information on the map independently from the orbits.

In order to present this result, we briefly introduce the concepts of *ergodicity* and *invariant measure*. A full description and proofs of the results can be found in [9, 11]. The idea behind a probabilistic description of a dynamical system, $f : I \rightarrow I$, is to consider a small region of points in I (instead of a single point) that we iterate under f . The central quantity to evaluate is the probability density $\rho_n(x)$ to be in state x at time n . Recall that a *probability density* is a *normalized Lebesgue integrable function*, $\int |\rho_n(x)| dx = 1$. The probability density $\rho_n(x)$ evolves in time and, according to the so-called Frobenius-Perron

equation [9], we get:

$$\rho_{n+1}(x) = \int_I \delta(x - f(t)) \rho_n(t) dt$$

where $\delta(x - f(t))$ denotes the Dirac measure at $x = f(t)$.

We call the *invariant probability density* the expression $\rho(x)$ that satisfies

$$\rho(x) = \int_I \delta(x - f(t)) \rho(t) dt \quad (2.1)$$

see [11]. This density is particularly interesting since it induces a measure $\mu(x)$ that is preserved by f (by “preserved” we mean that $\mu(C') = \mu(C)$ where $C' = \{x : f(x) \in C\}$). This measure is known as the *invariant measure* and is expressed as follows:

$$\mu(C) = \int_C \rho(x) dx$$

for C a Borel set.

A dynamical system is called *ergodic* if every invariant set C , ($f(C) = C$), is either of measure 0 or 1. In other words, if we cannot decompose the domain of f into invariant measurable subsets other than the two previous ones.

Proposition 2.3.1 *The tent map $f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1/2 \\ 2(1-x), & \text{if } 1/2 \leq x \leq 1 \end{cases}$ is ergodic.*

Proof: This is shown in Chapter 4. Theorem 4.4.1 □

The following theorem establishes a relationship between ergodicity and $\rho(x)$.

Theorem 2.3.1 *If the dynamical system $f(x)$ is ergodic, then there is exactly one invariant probability density $\rho(x)$ which is Lebesgue integrable. Furthermore, if there is a unique invariant probability density $\rho(x)$ and $\rho(x) > 0$ almost everywhere, then f is ergodic.*

Proof: See [9]. □

Example 4 Consider the tent map $f(x)$, (see Ex 1). Then (2.1) becomes

$$\begin{aligned}\rho(x) &= \int_0^{1/2} \delta(x - 2x_0)\rho(x_0)dx_0 + \int_{1/2}^1 \delta(x + 2x_0 - 2)\rho(x_0)dx_0 \\ &= 1/2[\rho(x/2) + \rho(1 - x/2)]\end{aligned}$$

which admits the solutions $\rho(x) = K$ for any constant K . We must normalize to get the invariant probability density $\rho(x) \equiv 1$, it is unique according to Proposition 2.3.1 and Theorem 2.3.1. The invariant measure of a subset C for the tent map is thus the length of the corresponding interval (Lebesgue measure). One can derive from this result the invariant measure of the logistic map. Recall that the tent map generates the logistic map through the transformation $y = \frac{2}{\pi} \arcsin(\sqrt{x})$. (Example 2). Indeed, using the conservation probability, [13], we write

$$\underbrace{\rho(x)}_{\text{logistic}} = \underbrace{\rho(y)}_{\text{tent}} \left| \frac{dy}{dx} \right|$$

but with $\rho(y) = 1$.

$$\rho(x) = \left| \frac{d}{dx} \left(\frac{2}{\pi} \arcsin(\sqrt{x}) \right) \right|$$

and finally,

$$\rho(x) = \frac{1}{\pi \sqrt{x(1-x)}}.$$

This invariant probability density is uniquely determined and $\rho(x) > 0$ for all $x \in [0, 1]$. Hence, by Theorem 2.3.1 we have that the logistic equation is ergodic. Moreover, $\rho(x)$ induces the following invariant measure for C a Borel set:

$$\mu(C) = \frac{1}{\pi} \int_C \frac{1}{\sqrt{x(1-x)}} dx.$$

Remark : By “almost everywhere”, we mean that the condition is fulfilled for all points except for a subset of measure zero, the measure to be used being the invariant measure.

Now, let us make some connection between the Lyapunov exponent and the invariant measure for an ergodic dynamical system. The following result is very useful since it allows one to compute the Lyapunov exponent of a map without actually computing any orbit.

Theorem 2.3.2 *If $f : I \rightarrow I$ has an invariant measure μ , and if f is ergodic with respect to μ , then the Lyapunov exponent $h(x)$ is constant almost everywhere and*

$$h_f(x) = \int_I \rho(x) \ln |f'(x)| dx.$$

Proof: This can be proved with Birkhoff's Ergodic Theorem. [3]. □

The point to understand from this discussion is that if the map preserves a reasonable measure then the Lyapunov exponent is constant almost everywhere. The value $\int \rho(x) \ln |f'(x)| dx$ is the space average of the function $\ln |f'(x)|$; the value $h(p)$ is the time average of the function $\ln |f'(x)|$ along the orbit of p . Thus, for any integrable function, the time average along almost all orbits is equal to the space average of the function. This indicates that if $f(x)$ is the rule of an ergodic dynamical system, the orbit of a point p must be dense for almost all points.

Chapter 3

Logistic Family

We will define the logistic family and study some of its properties. Every concept and result presented in this chapter are already known and are part of all basic textbooks [5, 12, 14]. They concern the logistic map and they are to be generalized over two sets of polynomials in the next chapter. We will see that a function as simple as the logistic map can give rise to a very rich mathematical theory.

3.1 Definition of the Logistic Map

The logistic family is defined as follows:

$$f_\lambda(x) = \lambda x(1 - x)$$

where $0 < \lambda \leq 4$ and $f_\lambda : [0, 1] \rightarrow [0, \lambda/4]$.

As can be noticed, 0 is a fixed point of f_λ for all values of λ and since $|f'_\lambda(0)| = |\lambda|$ it is stable for $0 < \lambda < 1$. In fact, every orbit of f_λ is attracted to the origin for $0 < \lambda < 1$, Figure 4 (a).

Figure 4 (a) $\lambda = 1/2$ (b) $\lambda = 2$ (c) $\lambda = 3.5$

The properties of f_λ change dramatically for various values of λ . Consider $1 < \lambda < 3$. In this case, see Figure 4(b), the map has a sink at $x = \frac{(\lambda-1)}{\lambda}$ since $|f'_\lambda(\frac{(\lambda-1)}{\lambda})| = |1-\lambda| < 1$. For $\lambda > 3$, as in Figure 4(c), the fixed point $x = \frac{(\lambda-1)}{\lambda}$ is unstable since $|f'_\lambda(\frac{(\lambda-1)}{\lambda})| = |1-\lambda| > 1$. The case $\lambda = 4$ can be solved in exact fashion analytically: we will see it later in this chapter.

3.2 Bifurcation Diagram

The *bifurcation diagram* is a computer image that captures the dynamics of F_λ , a family of functions, for many λ -values. In that diagram, we plot the parameter λ on the horizontal axis versus the *asymptotic orbit* of x_0 under F_λ on the vertical axis where x_0 can be any point of the domain. By asymptotic orbit we mean that we plot the iterations of x_0 but we do not plot the first few iterations (usually 100) of x_0 . That way, we allow the orbit to “settle down” and to reach its eventual behavior (k -periodic orbit, chaotic orbit,...). The dynamics of the logistic family are very complex as we can see in Figure 5.

Figure 5 Bifurcation Diagram of the Logistic Function

$$F_{\lambda}(x) = \lambda x(1 - x).$$

All the stable orbits are shown in this picture. As explained in Section 3.1, 0 and $\frac{\lambda-1}{\lambda}$ are both fixed points for all values of λ . However, the diagram does not show the $\frac{\lambda-1}{\lambda}$ part for $0 < \lambda < 1$ and the 0 part when $1 < \lambda < 3$ because they are “unstable” fixed points for these specific values of λ . For $3 < \lambda < 1 + \sqrt{6}$, we see a period-two sink that also becomes unstable when λ grows above $1 + \sqrt{6} \approx 3.45$.

Figure 6 $F_{\lambda}(x) = \lambda x(1 - x)$ for $3.82 < \lambda < 3.85$.

The bifurcations give rise to period 2, 4, 8, 16, ... (phenomenon called *period-doubling*). If we zoom into the region $3.825 < \lambda < 3.85$ as in Figure 6, we see that a stable period three orbit appears and then the period-doubling starts again to give orbits of period 6, 12, 24, This structure has the property of *self-similarity*, i.e., some regions once enlarged offer a resemblance to the entire graphic.

3.3 Lyapunov Exponents

The bifurcation diagram contains a large quantity of information for an entire family of functions. It is as well possible to capture in a single image the Lyapunov exponents of a family of functions. With the use of Theorem 2.3.2, we can compute explicitly the value of the Lyapunov exponent of f_λ for $\lambda = 4$. From Example 4, recall that the invariant probability density of f_4 is

$$\rho(x) = \frac{1}{\pi \sqrt{x(1-x)}}$$

hence,

$$\begin{aligned} h(f_4) &= \frac{1}{\pi} \int_0^1 \frac{\ln |4 - 8x|}{\sqrt{x(1-x)}} dx \\ &= \ln(2) \end{aligned}$$

For other λ -values, we need to find the invariant probability density which is not obvious. Therefore, in order to plot it, we approximate the Lyapunov exponent. To do so, we compute the Lyapunov exponent of a function f_λ starting with a random value $x \in [0, 1]$ for as many values of λ as possible. Then we plot this value $h(f_\lambda)$ versus λ . The starting value x does not influence the result obtained (Theorem 2.3.2). In Figure 7, $h(f_\lambda)$ is computed as follows:

$$h(f_\lambda) = \frac{1}{1000} \sum_{n=1}^{1000} \ln |f'_\lambda(x_n)|$$

where $\{x_0, x_1, x_2, \dots\}$ is the orbit of x_0 under f_λ .

It is very interesting to examine the Lyapunov exponents of a family of functions versus the bifurcation diagram since they simultaneously express the behavior of the dynamical system. One can notice from Figure 7 that the Lyapunov exponent rises and hits 0 at period doublings: at $\lambda = 3$ and $\lambda \approx 3.45$ for instance. It becomes positive in the chaotic region (e.g., we have $h(f_4) = \ln(2) > 0$) and it drops below 0 when there is a periodic attractor.

(a) Bifurcation Diagram of the Logistic Function

Figure 7 (b) Lyapunov exponent of the Logistic Function versus the parameter λ :
the Horizontal Direction is the Parameter λ between 2 and 4;
the Vertical Direction is the Lyapunov exponent of f_λ .

3.4 $\lambda = 4$, Exact Results

We can express explicitly all orbits of the logistic equation in terms of simple functions when $\lambda = 4$, i.e., for $f_4(x) = 4x(1 - x)$. Since $x_n \in [0, 1]$, let $x_n = \sin^2(\pi\theta_n)$. Then

$$\begin{aligned}\sin^2(\pi\theta_{n+1}) &= x_{n+1} \\ &= f_4(x_n) \\ &= 4\sin^2(\pi\theta_n)(1 - \sin^2(\pi\theta_n)) \\ &= 4\sin^2(\pi\theta_n)\cos^2(\pi\theta_n) \\ &= \sin^2(2\pi\theta_n)\end{aligned}$$

That is equivalent to $\theta_{n+1} = 2\theta_n \bmod 1$ which has the explicit solution

$$\theta_n = 2^n\theta_0 \bmod 1 \quad (3.1)$$

where $\theta_0 = \frac{\arcsin \sqrt{x_0}}{\pi}$. It also gives us a formula for the orbit of a point x_0 under f_4 ; namely,

$$x_n = \sin^2(2^n\pi\theta_0) \quad \text{where } \theta_0 = \frac{\arcsin \sqrt{x_0}}{\pi}.$$

If we express θ_0 as a binary number

$$\theta_0 = 0.b_0b_1b_2\dots = \sum_{i=0}^{\infty} \frac{b_i}{2^{i+1}}, \quad b_i \in \{0, 1\},$$

then applying (3.1) to θ_0 is equivalent to applying a left shift to $0.b_0b_1b_2\dots$ and dropping the integer part: we call this function, S . In other words,

$$\begin{aligned}\theta_n &= 2^n\theta_0 \bmod 1 \\ &= 0.b_nb_{n+1}b_{n+2}\dots \\ &= S^n(0.b_0b_1b_2\dots), \text{ where } S^n = S \circ S^{n-1} \text{ for } n > 1 \text{ and } S^1 = S.\end{aligned}$$

With such an observation, it is fairly easy to find points of period k for any given integer k . A rational number $\theta_0 \in \mathbb{Q}$ of period k generates an orbit of period k , i.e., $x_0 = \sin^2(\pi\theta_0)$

is a periodic point of period k under f_4 . In fact, since any rational number is eventually periodic, we have that for all $\theta_0 \in \mathbb{Q}$, $\{x_0 = \sin^2(\pi\theta_0), x_1 = f_4(x_0), x_2 = f_4^2(x_0), \dots\}$ is an eventually periodic orbit. The converse is also true.

Example 5 Consider $f_4(x) = 4x(1 - x)$ and $x_0 = \sin^2(\frac{6\pi}{7})$. We have that $\theta_0 = \frac{6}{7} = 0.110110110\dots$ in binary expansion. The initial condition θ_0 is a rational number, in fact, it is a period three number. Hence, we expect a period three orbit.

$$\begin{aligned} x_1 &= f_4(x_0) = \sin^2\left(\frac{12\pi}{7}\right) = \sin^2\left(\frac{5\pi}{7}\right) \\ x_2 &= f_4(x_1) = \sin^2\left(\frac{10\pi}{7}\right) = \sin^2\left(\frac{3\pi}{7}\right) \\ x_3 &= f_4(x_2) = \sin^2\left(\frac{6\pi}{7}\right) = x_0 \end{aligned}$$

Although it is not obvious from the value of $x_0 \approx 0.1882551$, $\{x_0, x_1, x_2, \dots\}$ is a period three orbit. By Theorem 2.2.1 we conclude that f_4 has chaotic orbits.

Theorem 3.4.1 The logistic map $f_4(x) = 4x(1 - x)$ has chaotic orbits.

Proof: By Theorem 2.2.1 and Example 5. □

With the use of the binary expansion many concepts become more intuitive: the periodicity of an orbit or the sensitive dependence on initial conditions, for instance. We can show the latter as follows.

Consider $\theta_0 = 0.b_0b_1b_2\dots$, $x_0 = \sin^2(\pi\theta_0)$ and $d < 1/2$ a positive real number. We want to show that for all $\varepsilon > 0$, there exists y_0 such that $|x_0 - y_0| < \varepsilon$ and $|x_k - y_k| = |f_4^k(x_0) - f_4^k(y_0)| \geq d$ for some nonnegative integer k .

Given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|\theta_0 - \psi_0| < \delta \Rightarrow |\sin^2(\pi\theta_0) - \sin^2(\pi\psi_0)| < \varepsilon$$

by continuity of the function $\sin^2(\pi x)$. Let k be the smallest integer such that $\delta > \frac{1}{2^k}$. Now, consider $\theta_k = S^k(0.b_0b_1b_2\dots) = 0.b_kb_{k+1}b_{k+2}\dots$ and

$$\Delta = \begin{cases} 1/2 - \theta_k, & \text{if } |\sin^2(\pi\theta_k)| \leq 1/2 \\ -\theta_k, & \text{if } |\sin^2(\pi\theta_k)| > 1/2 \end{cases}.$$

Let $\psi_0 = \theta_0 + \frac{\Delta}{2^k}$ and $y_0 = \sin^2(\pi\psi_0)$. Note that $\psi_k = \theta_k + \Delta$. We have that $|x_0 - y_0| < \varepsilon$ since $|\theta_0 - \psi_0| = |\frac{\Delta}{2^k}| \leq \frac{1}{2^k} < \delta$. Furthermore,

$$\begin{aligned} |x_k - y_k| &= |\sin^2(\pi\theta_k) - \sin^2(\pi\psi_k)| \\ &= \begin{cases} |\sin^2(\pi\theta_k) - \sin^2(\pi/2)|, & \text{if } |\sin^2(\pi\theta_k)| \leq 1/2 \\ |\sin^2(\pi\theta_k)|, & \text{if } |\sin^2(\pi\theta_k)| > 1/2 \end{cases} \\ &\geq 1/2 > d. \end{aligned}$$

Chapter 4

Generalization of the Logistic Family

In this chapter we define two sets of functions $\mathcal{S}$ and $\mathcal{C}$. We show that the dynamical behavior of the maps in $\mathcal{S}$ and in $\mathcal{C}$ is similar to the logistic map's behavior: In fact, the logistic map is a member of the set $\mathcal{S}$. Furthermore, as for the logistic map when $\lambda = 4$, these polynomials have orbits that can be solved exactly, thus yielding analytic closed formulas. Finally, and surprisingly, the invariant measure is the same for all the functions in $\mathcal{S}$ and in $\mathcal{C}$.

4.1 Definition of two Sets of Functions: $\mathcal{S}$ and $\mathcal{C}$

As we saw in Section 3.4, the logistic equation $f_4(x) = 4x(1 - x)$ can be solved explicitly with the use of the change variable $x = \sin^2(\pi\theta)$. This comes from the fact that

$$\begin{aligned}\sin(2\theta) &= 2 \sin \theta \sqrt{1 - \sin^2 \theta} \\ \Rightarrow \sin^2(2\theta) &= 4 \sin^2 \theta (1 - \sin^2 \theta).\end{aligned}$$

The same idea can be used to construct another family obtained by expressing $\cos^2(2\theta)$ in terms of $\cos^2 \theta$: $\cos^2(2\theta) = 4(\cos^2 \theta)^2 - 4(\cos^2 \theta) + 1$ which induces the map $g(x) = 4x^2 - 4x + 1$ and even the family $g_\lambda = \lambda(x^2 - x + 1/4)$, for $\lambda \in (0, 4]$. More generally, it

is possible to build two large sets of functions: one which expresses $\sin^2(N\theta)$ in terms of $\sin^2\theta$ and the other one, $\cos^2(N\theta)$ in terms of $\cos^2\theta$. The first set, the *Sine Functions Set* is defined as follows:

$$\mathcal{S} = \{SN(x) : SN(x) = s_N^2(\sqrt{x}) \text{ and } N = 1, 2, 3, \dots\} \quad (4.1)$$

where

$$\begin{aligned} s_1(x) &= x \\ s_2(x) &= 2x\sqrt{1-x^2} \\ &\dots \\ s_N(x) &= 2\sqrt{1-x^2}s_{N-1}(x) - s_{N-2}(x), \quad N \geq 3. \end{aligned} \quad (4.2)$$

In a similar manner we can build the *Cosine Functions Set*:

$$\mathcal{C} = \{CN(x) : CN(x) = c_N^2(\sqrt{x}) \text{ and } N = 1, 2, 3, \dots\}$$

where

$$\begin{aligned} c_1(x) &= x \\ c_2(x) &= 2x^2 - 1 \\ &\dots \\ c_N(x) &= 2xc_{N-1}(x) - c_{N-2}(x), \quad N \geq 3. \end{aligned} \quad (4.3)$$

Remark: The equations (4.2) and (4.3) come from the fact that

$$\begin{aligned} \sin(n\theta) &= 2\cos\theta\sin((n-1)\theta) - \sin((n-2)\theta) \\ \cos(n\theta) &= 2\cos\theta\cos((n-1)\theta) - \cos((n-2)\theta). \end{aligned}$$

This will be used later in this chapter. Note that

$$S2(x) = s_2^2(\sqrt{x}) = (2\sqrt{x}\sqrt{1-x})^2 = 4x(1-x)$$

is the logistic map. Moreover, the polynomials $c_N(x)$, with the standardization $c_N(1) = 1$, are in fact the Tschebysheff Polynomials of the first kind restricted to the interval $[0, 1]$ instead of $[-1, 1]$.

Lemma 4.1.1 *For $N = 1, 2, \dots$, we have that*

$$s_N(x) = \begin{cases} xp_N(x), & \text{if } N \text{ is odd} \\ x\sqrt{1-x^2}p_N(x), & \text{if } N \text{ is even} \end{cases}$$

where $p_N(x)$ are polynomials defined as

$$p_N(x) = \begin{cases} 2(1-x^2)p_{N-1}(x) - p_{N-2}(x), & \text{if } N \text{ is odd} \\ 2p_{N-1}(x) - p_{N-2}(x), & \text{if } N \text{ is even} \end{cases}$$

with $p_1(x) = 1$ and $p_2(x) = 2$.

Proof: We show this result by induction on N . If $N = 1$, $s_1(x) = x = xp_1(x)$ where $p_1(x) = 1$. If $N = 2$, we have $s_2(x) = 2x\sqrt{1-x^2} = x\sqrt{1-x^2}p_2(x)$ where $p_2(x) = 2$. Consider the result true for $N-1$ and $N-2$. Then, if N is odd we have

$$\begin{aligned} s_N(x) &\stackrel{(4.2)}{=} 2\sqrt{1-x^2}s_{N-1}(x) - s_{N-2}(x) \\ &= 2\sqrt{1-x^2}(x\sqrt{1-x^2}p_{N-1}(x)) - xp_{N-2}(x) \\ &= x(2(1-x^2)p_{N-1}(x) - p_{N-2}(x)) \\ &= xp_N(x) \end{aligned}$$

where $p_N(x) = 2(1-x^2)p_{N-1}(x) - p_{N-2}(x)$. On the other hand, if N is even, we have

$$\begin{aligned} s_N(x) &\stackrel{(4.2)}{=} 2\sqrt{1-x^2}s_{N-1}(x) - s_{N-2}(x) \\ &= 2\sqrt{1-x^2}xp_{N-1}(x) - x\sqrt{1-x^2}p_{N-2}(x) \\ &= x\sqrt{1-x^2}(2p_{N-1}(x) - p_{N-2}(x)) \\ &= x\sqrt{1-x^2}p_N(x) \end{aligned}$$

where $p_N(x) = 2p_{N-1}(x) - p_{N-2}(x)$. $\square$

Because of the square roots appearing in the definition of functions of $\mathcal{S}$ and of $\mathcal{C}$, what follows is an important and non-trivial result.

Proposition 4.1.1 *$SN(x)$ and $CN(x)$ are polynomials of degree N for all integers N .*

Proof: The proof for $CN(x)$ is omitted since it is very similar to the proof for $SN(x)$. Using the same notation as in the previous lemma, and letting $\Phi_N(x) = p_N(\sqrt{x})$ we can write

$$SN(x) = s_N^2(\sqrt{x}) = \begin{cases} x\Phi_N^2(x), & \text{if } N \text{ is odd} \\ x(1-x)\Phi_N^2(x), & \text{if } N \text{ is even} \end{cases} \quad (4.4)$$

where

$$\Phi_N(x) = \begin{cases} 2(1-x)\Phi_{N-1}(x) - \Phi_{N-2}(x), & \text{if } N \text{ is odd} \\ 2\Phi_{N-1}(x) - \Phi_{N-2}(x), & \text{if } N \text{ is even} \end{cases} \quad (4.5)$$

We can find the values of $\Phi_1(x)$ and $\Phi_2(x)$ explicitly:

$$\begin{aligned} x = S1(x) = x\Phi_1^2(x) &\Rightarrow \Phi_1(x) = 1 \\ 4x(1-x) = S2(x) = x(1-x)\Phi_2^2(x) &\Rightarrow \Phi_2(x) = 2 \end{aligned}$$

Since $\Phi_1(x)$ and $\Phi_2(x)$ are both polynomials, by (4.5) we conclude that $\Phi_N(x)$ are all polynomials and so are $SN(x)$, by (4.4). The degree of $SN(x)$ is N since the degree of $\Phi_N(x)$ is

$$\deg(\Phi_N) = \begin{cases} \frac{N-1}{2}, & \text{if } N \text{ is odd} \\ \frac{N-2}{2}, & \text{if } N \text{ is even} \end{cases}$$

which is easy to show by induction. $\square$

Remark: The construction of the polynomials to prove the previous result for $CN(x)$ is the following:

$$CN(x) = \begin{cases} x\Psi_N^2(x), & \text{if } N \text{ is odd} \\ \Psi_N^2(x), & \text{if } N \text{ is even} \end{cases} \quad (4.6)$$

$$\Psi_N(x) = \begin{cases} 2\Psi_{N-1}(x) - \Psi_{N-2}(x), & \text{if } N \text{ is odd} \\ 2x\Psi_{N-1}(x) - \Psi_{N-2}(x), & \text{if } N \text{ is even} \end{cases} \quad (4.7)$$

with $\Psi_1(x) = 1$ and $\Psi_2(x) = 2x - 1$.

Note that the absolute value of the leading coefficient, i.e., the coefficient of the highest degree term is 4^{N-1} for $SN(x)$ as well as for $CN(x)$. Hence, from each function $SN(x)$ and $CN(x)$ we define a family of functions: for $0 < \lambda \leq 4^{N-1}$ we have

$$\begin{aligned} SN_\lambda : [0, 1] &\rightarrow [0, 1] \\ x &\mapsto \lambda \frac{SN(x)}{4^{N-1}}. \end{aligned}$$

and

$$\begin{aligned} CN_\lambda : [0, 1] &\rightarrow [0, 1] \\ x &\mapsto \lambda \frac{CN(x)}{4^{N-1}}. \end{aligned}$$

The first few members of the sets $\mathcal{S}$ and $\mathcal{C}$ are plotted in Figure 9 and are described in the Table 1 below. Simply looking at the polynomials in the Table, one can claim that $SN(x) = CN(x)$ when N is odd and, otherwise, $SN(x)$ and $CN(x)$ add up to the constant 1. This result, and many other observations on the behavior of these polynomials are proved in the next sections.

$SN(x)$	$CN(x)$
$S1(x) = x$	$C1(x) = x$
$S2(x) = -4x^2 + 4x$	$C2(x) = 4x^2 - 4x + 1$
$S3(x) = 16x^3 - 24x^2 + 9x$	$C3(x) = 16x^3 - 24x^2 + 9x$
$S4(x) = -64x^4 + 128x^3 - 80x^2 + 16x$	$C4(x) = 64x^4 - 128x^3 + 80x^2 - 16x + 1$
$S5(x) = 256x^5 - 640x^4 + 560x^3$ $-200x^2 + 25x$	$C5(x) = 256x^5 - 640x^4 + 560x^3$ $-200x^2 + 25x$
$S6(x) = -1024x^6 + 3072x^5 - 3456x^4$ $+1792x^3 - 420x^2 + 36x$	$C6(x) = 1024x^6 - 3072x^5 + 3456x^4$ $-1792x^3 + 420x^2 - 36x + 1$
$S7(x) = 4096x^7 - 14336x^6 + 19712x^5$ $-13440x^4 + 4703x^3 - 784x^2 + 49x$	$C7(x) = 4096x^7 - 14336x^6 + 19712x^5$ $-13440x^4 + 4703x^3 - 784x^2 + 49x$
...	...

Table 1 Few members of the sets $\mathcal{S}$ and $\mathcal{C}$.

4.2 Particularities of $\mathcal{S}$ and $\mathcal{C}$

The functions in $\mathcal{S}$ and in $\mathcal{C}$ are constructed so that they all have *exact* solutions. It is, in fact, the principal idea behind this generalization. However, surprisingly, the sets $\mathcal{S}$ and $\mathcal{C}$ exhibit interesting algebraic properties such as the closure under composition. We prove this in the present section, but first, we express $SN(x)$ and $CN(x)$ in terms of trigonometric functions in order to simplify the proofs.

Lemma 4.2.1 *Consider $CN(x)$ and $SN(x)$ as defined in the previous section. We have that*

$$\begin{aligned} SN(\sin^2 \theta) &= \sin^2(N\theta) \\ CN(\cos^2 \theta) &= \cos^2(N\theta) \end{aligned} \tag{4.8}$$

for $N = 1, 2, 3, \dots$

Proof: Notice, by Trigonometry,

$$\begin{aligned} \sin(n\theta) &= 2 \cos \theta \sin((n-1)\theta) - \sin((n-2)\theta) \\ \cos(n\theta) &= 2 \cos \theta \cos((n-1)\theta) - \cos((n-2)\theta) \end{aligned} \tag{4.9}$$

for all integers $n \geq 2$. The result is obvious for $N = 1$ and 2 . Suppose (4.8) true for $N-1$ and $N-2$, that is to say $s_{N-1}^2(\sin \theta) \stackrel{(4.1)}{=} S[N-1](\sin^2 \theta) \stackrel{(4.8)}{=} \sin^2((N-1)\theta)$, and the same equalities hold for $N-2$, so

$$\begin{aligned} s_{N-1}(\sin \theta) &= \sin((N-1)\theta) \quad \text{and} \\ s_{N-2}(\sin \theta) &= \sin((N-2)\theta). \end{aligned} \tag{4.10}$$

Then, by (4.1), for $\theta \in [0, \pi/2]$,

$$\begin{aligned}
SN(\sin^2 \theta) &= s_N^2(\sin \theta) \\
&\stackrel{(4.2)}{=} \left(2\sqrt{1 - \sin^2 \theta} s_{N-1}(\sin \theta) - s_{N-2}(\sin \theta)\right)^2 \\
&\stackrel{(4.10)}{=} \left(2 \cos \theta \sin((N-1)\theta) - \sin((N-2)\theta)\right)^2 \\
&\stackrel{(4.9)}{=} \sin^2(N\theta) \quad \square
\end{aligned}$$

Lemma 4.2.2 *Consider $CN(x)$ and $SN(x)$ as previously defined, we have that*

$$\begin{aligned}
SN(x) &= \sin^2(N \arcsin \sqrt{x}) \quad \text{and} \\
CN(x) &= \cos^2(N \arccos \sqrt{x})
\end{aligned} \tag{4.11}$$

for all $x \in [0, 1]$.

Proof: Let $x \in [0, 1]$. There exist $\theta \in [0, \pi/2]$ such that $x = \sin^2 \theta$. Thus, $\theta = \arcsin \sqrt{x}$. Now, from Lemma 4.2.1 we have that

$$\begin{aligned}
SN(x) &= SN(\sin^2 \theta) \\
&= \sin^2(N\theta) \\
&= \sin^2(N \arcsin \sqrt{x}) \quad \square
\end{aligned}$$

The next result enables us to obtain a direct and simple relation between elements of $\mathcal{S}$ and elements of $\mathcal{C}$.

Proposition 4.2.1 *Consider $CN(x) \in \mathcal{C}$ and $SN(x) \in \mathcal{S}$. We have that*

$$SN(x) = \begin{cases} CN(x), & \text{if } N \text{ is odd} \\ 1 - CN(x), & \text{if } N \text{ is even} \end{cases}$$

for all $x \in [0, 1]$.

Proof: Recalling from Trigonometry that $\arcsin \theta = \pi/2 - \arccos \theta$ and that $\sin(\pm\pi/2 + \theta) = \pm \cos \theta$, we can write that

$$\begin{aligned}
SN(x) &\stackrel{(4.11)}{=} \sin^2(N \arcsin \sqrt{x}) \\
&= \sin^2(N\pi/2 - N \arccos \sqrt{x}) \\
&= \begin{cases} \cos^2(-N \arccos \sqrt{x}), & \text{if } N \text{ is odd} \\ \sin^2(-N \arccos \sqrt{x}), & \text{if } N \text{ is even} \end{cases} \\
&= \begin{cases} \cos^2(N \arccos \sqrt{x}), & \text{if } N \text{ is odd} \\ 1 - \cos^2(N \arccos \sqrt{x}), & \text{if } N \text{ is even} \end{cases} \\
&\stackrel{(4.11)}{=} \begin{cases} CN(x), & \text{if } N \text{ is odd} \\ 1 - CN(x), & \text{if } N \text{ is even.} \end{cases} \quad \square
\end{aligned}$$

We can deduce more from Lemma 4.2.2 which allows us to classify the sets $\mathcal{S}$ and $\mathcal{C}$. In fact, we have that $\mathcal{S}$ and $\mathcal{C}$ form *abelian monoids* with respect to the *composition* of functions, $\circ$. Recall that a *monoid* is a non-empty set M together with a binary operation $*$, i.e., $x * y \in M$ for all $x, y \in M$. There is an element $e \in M$, called the identity element, for which $x * e = e * x = x$ for all $x \in M$. Finally, the operation $*$ is associative. Moreover, if the operation is commutative, we say that the monoid is *abelian*. In $\mathcal{S}$ and in $\mathcal{C}$, the identity elements are $S1(x) = x$ and $C1(x) = x$ respectively. The composition of function is clearly associative. The last conditions left to check are that $\mathcal{S}$ and $\mathcal{C}$ are closed under composition and that this composition is commutative (which is generally not the case with composition of functions).

Theorem 4.2.1 Consider $N_1, N_2 \in \mathbb{N}$ any positive integers. We have that

$$\begin{aligned} SN_1 \circ SN_2 &= SN_2 \circ SN_1 = S[N_1 N_2], \text{ and} \\ CN_1 \circ CN_2 &= CN_2 \circ CN_1 = C[N_1 N_2]. \end{aligned} \quad (4.12)$$

Proof: For N_1 and N_2 given, take $SN_1(x)$ and $SN_2(x)$ as defined previously. Then, for any $x \in [0, 1]$, we have

$$\begin{aligned} SN_1(SN_2(x)) &\stackrel{(4.11)}{=} \sin^2 \left(N_1 \arcsin(\sqrt{SN_2(x)}) \right) \\ &\stackrel{(4.11)}{=} \sin^2 \left(N_1 \arcsin \left(\sin(N_2 \arcsin \sqrt{x}) \right) \right) \\ &= \sin^2(N_1 N_2 \arcsin \sqrt{x}) \\ &\stackrel{(4.11)}{=} S[N_1 N_2](x). \end{aligned}$$

Obviously, $S[N_1 N_2](x) = S[N_2 N_1](x)$, so the composition is commutative. $\square$

Definition 9 Consider $SN(x) \in \mathcal{S}$ and $CN(x) \in \mathcal{C}$. We say that $SN(x)$ ($CN(x)$ respectively) is a **prime element of $\mathcal{S}$** (of $\mathcal{C}$ respectively) if N is a prime number.

Corollary 4.2.1 Any polynomial $SN(x) \in \mathcal{S}$ and $CN(x) \in \mathcal{C}$ can uniquely be expressed as a composition of prime elements of $\mathcal{S}$ and of $\mathcal{C}$ respectively. In this sense, $\{SN(x) : N \text{ prime}\}$ and $\{CN(x) : N \text{ prime}\}$ are called **generators of $\mathcal{S}$** and of $\mathcal{C}$ respectively.

Proof: Consider $N \in \mathbb{N}$. From the Fundamental Theorem of Arithmetic, we can uniquely find prime numbers $P_1, \dots, P_s$ such that $N = P_1^{\alpha_1} \dots P_s^{\alpha_s}$, for $0 < \alpha_i \in \mathbb{N}$. Hence, from the previous Theorem, we clearly have that

$$SN = (SP_1)^{\alpha_1} \circ (SP_2)^{\alpha_2} \circ \dots \circ (SP_s)^{\alpha_s}. \quad \square$$

Another useful observation that follows from the Theorem 4.2.1 is that if one is interested in studying the k -periodic points of a certain polynomial $SN(x)$ (or $CN(x)$), then

one only has to look at the fixed points of the function $S[N^k](x)$ since this last one is, according to (4.12), $SN^k(x)$, (see Example 1).

Remark: It is important to be very careful with the notation in order to avoid confusion: $SN^i(x)$ could be interpreted as $S[N^i](x)$ as well as $(SN)^i(x)$. For the rest of this paper, the notation $SN^i(x)$ holds for $(SN)^i(x)$.

We now construct closed formulas to express $x_n = SN^n(x_0)$, an orbit under $SN(x)$ and $y_n = CN^n(y_0)$, an orbit under $CN(x)$. Notice that formulas for orbits under prime elements of $\mathcal{S}$ and of $\mathcal{C}$ would be enough to deduce formulas under any element of $\mathcal{S}$ and of $\mathcal{C}$ because of Corollary 4.2.1. However, we do not proceed this way since it is simpler to treat the general case (any integer $N \geq 0$).

Theorem 4.2.2 *Let $\{x_0, x_1, x_2, \dots\}$ be the orbit of x_0 under $SN(x)$ and $\{y_0, y_1, y_2, \dots\}$ be the orbit of y_0 under $CN(x)$. If we write $x_n = \sin^2(\pi\theta_n)$ and $y_n = \cos^2(\pi\psi_n)$ we get that*

1. $\theta_n = N^n\theta_0 \bmod 1$. where $\theta_0 = \frac{\arcsin(\sqrt{x_0})}{\pi}$ and
2. $\psi_n = N^n\psi_0 \bmod 1$. where $\psi_0 = \frac{\arccos(\sqrt{x_0})}{\pi}$

Proof: The case $N = 1$ is obvious and the case $N = 2$ is proved in Section 3.4. Now, to prove the general case, $N > 2$, we use the Lemma 4.2.1:

$$\begin{aligned}
 \sin^2(\pi\theta_{n+1}) &= x_{n+1} \\
 &= SN(x_n) \\
 &= SN(\sin^2(\pi\theta_n)) \\
 &= \sin^2(N\pi\theta_n)
 \end{aligned}$$

This implies that $\theta_{n+1} = N\theta_n \bmod 1$ for all $n \in \mathbb{N}$ and all $N = 1, 2, 3, \dots$. This equation has the solution

$$\theta_n = N^n \theta_0 \bmod 1 \quad (4.13)$$

where $\theta_0 = \frac{\arcsin(\sqrt{x_0})}{\pi}$. $\square$

For $\{x_0, x_1, x_2, \dots\}$ an orbit of $SN(x)$ and $\{y_0, y_1, y_2, \dots\}$ an orbit of $CN(x)$, the last theorem enables us to write

$$\begin{aligned} x_n &= \sin^2(N^n \pi \theta_0) & \text{where } \theta_0 &= \frac{\arcsin \sqrt{x_0}}{\pi} \\ y_n &= \cos^2(N^n \pi \psi_0) & \text{where } \psi_0 &= \frac{\arccos \sqrt{y_0}}{\pi} \end{aligned} \quad (4.14)$$

for all $n \in \mathbb{N}$.

In the case $N > 1$, if we express θ_0 in base N ,

$$\theta_0 = \sum_{i=0}^{\infty} \frac{b_i}{N^{i+1}} \quad \text{where } b_i \in \{0, 1, \dots, N-1\}$$

then (4.13) is equivalent to applying S (defined in Section 3.4 : i.e., shift and drop the integer part). It implies that if $\theta_0 = 0.b_0b_1b_2\dots$ (in base N) then $x_n = \sin^2(\pi\theta_n)$ where $\theta_n = 0.b_nb_{n+1}b_{n+2}\dots$. As for the logistic map, this notation allows us to understand the nature of sensitive dependence on initial conditions. If we choose two numbers $\theta_0 = 0.b_0b_1b_2\dots$ and $\overline{\theta_0} = 0.\overline{b_0}\overline{b_1}\overline{b_2}\dots$ very close to one another but different, there must exist a $k \in \mathbb{N}$ such that $b_k \neq \overline{b_k}$. If the smallest such k is arbitrarily large then the difference between θ_0 and $\overline{\theta_0}$ is negligible but it becomes more and more important as we iterate θ_0 and $\overline{\theta_0}$ since b_k and $\overline{b_k}$ are shifted to the left at each iteration.

4.3 Conjugate Maps of functions in $\mathcal{S}$ and $\mathcal{C}$

As we explained earlier in Section 2.2, a conjugate map contains a great deal of information. To be precise, it can be used to transform a dynamical system whose behavior is unknown or unproved into another one whose dynamics is known. In the previous chapter, many results concerning the logistic map were easy to prove because of its conjugacy with a simpler map: the tent map. The same idea is used for $SN(x)$ and $CN(x)$. This is the major motivation of this Section.

Let $TN(x)$ be a piecewise linear function defined on subintervals $[k/N, (k+1)/N]$ of $[0, 1]$ by setting

$$TN(x) = \begin{cases} Nx - k, & \text{for } k \text{ even} \\ -Nx + k + 1, & \text{for } k \text{ odd} \end{cases}$$

with $k = 0, 1, \dots, N-1$, (see Figure 9). Note that $T2(x)$ is the tent map.

Theorem 4.3.1 *Consider $SN(x)$ and $CN(x)$ as defined in Section 4.1 and $TN(x)$ defined previously. We have that $SN(x)$ and $CN(x)$ are both conjugate to $TN(x)$, i.e., there exist $H_s(x)$ and $H_c(x)$ such that*

1. $SN \circ H_s = H_s \circ TN$ for $N = 1, 2, 3, \dots$, and
2. $CN \circ H_c = H_c \circ TN$ for $N = 1, 2, 3, \dots$

where $H_s(x) = \sin^2(\frac{\pi x}{2})$ and $H_c(x) = \cos^2(\frac{\pi x}{2})$.

Proof: First, note from Figure 8 that $H_s(x) = \sin^2(\frac{\pi x}{2})$ and $H_c(x) = \cos^2(\frac{\pi x}{2})$ are both continuous and bijective maps on the interval $[0, 1]$.

Figure 8 Conjugacies between $SN(x)$ and $TN(x)$
and between $CN(x)$ and $TN(x)$ respectively.

We have that $SN(H_s(x)) = SN(\sin^2(\frac{\pi x}{2})) = \sin^2(\frac{N\pi x}{2})$. On the other hand,

$$\begin{aligned}
 H_s(TN(x)) &= \begin{cases} \sin^2(N\pi x/2 - k\pi/2), & \text{for } k \text{ even} \\ \sin^2(-N\pi x/2 + (k+1)\pi/2), & \text{for } k \text{ odd} \end{cases} \\
 &= \left(\pm \sin\left(\frac{N\pi x}{2}\right) \right)^2 \\
 &= \sin^2\left(\frac{N\pi x}{2}\right)
 \end{aligned}$$

The result for $CN(x)$ can be proved similarly. □

Corollary 4.3.1 *For all integers $N \geq 1$, we have that $SN(x)$ is conjugate to $CN(x)$ with conjugacy map $H(x) = 1 - x$.*

Proof: This is a consequence from the fact that conjugacy is an equivalence relation but we show it explicitly. Note that $H(x) = 1 - x = H^{-1}(x) = \sin^2(\arccos \sqrt{x}) =$

$\cos^2(\arcsin \sqrt{x})$. Therefore, we have that

$$\begin{aligned}
H^{-1}(SN(H(x))) &= H^{-1}(SN(\sin^2(\arccos \sqrt{x}))) \\
&= H^{-1}(\sin^2(N \arccos \sqrt{x})) \\
&= 1 - \sin^2(N \arccos \sqrt{x}) \\
&= \cos^2(N \arccos \sqrt{x}) \\
&= CN(x)
\end{aligned}$$

by Lemma 4.2.1 and Lemma 4.2.2. $\square$

In Figure 9. below, we depict the graphs of CN , TN and SN for $2 \leq N \leq 7$. In each case note that $SN(x)$ is a “smooth” version of $TN(x)$.

$C_2(x)$

$T_2(x)$

$S_2(x)$

$C_3(x)$

$T_3(x)$

$S_3(x)$

$C_4(x)$

$T_4(x)$

$S_4(x)$

Figure 9 CN 's, TN 's and SN 's for $N = 2, \dots, 7$.

4.4 Lyapunov Exponents of $SN(x)$ and $CN(x)$

In this section we calculate the Lyapunov exponents of the functions $SN(x)$ and $CN(x)$. We compare the plots of the Lyapunov exponents with the bifurcation diagrams of the families of functions $SN_\lambda(x)$ and $CN_\lambda(x)$. However, we must previously find the *invariant measure* of those functions.

Theorem 4.4.1 *Consider $f : [0, 1] \rightarrow [0, 1]$. Let (I_j) be a finite family of disjoint open intervals in $[0, 1]$ such that*

1. $\mathcal{M}([0, 1] \setminus \cup_j I_j) = 0$. ($\mathcal{M}$ denotes Lebesgue measure).
2. For every j , there exists a set K of indices such that $f(I_j) = \cup_{k \in K} I_k \pmod{0}$. (i.e., $\mathcal{M}((f(I_j) \setminus \cup_{k \in K} I_k) \cup (\cup_{k \in K} I_k \setminus f(I_j))) = 0$).
3. For every $x \in \cup_j I_j$, the derivative of f exists and satisfies $|f'(x)| \geq \alpha$ for some fixed $\alpha > 0$.
4. There exists $\beta > 1$ and $n_0 > 0$ such that if $x_m \in \cup_j I_j$, ($x_m = f^m(x_0)$), for all $0 \leq m \leq n_0 - 1$ we have $|(f^{n_0})'(x_0)| \geq \beta$.
5. There exists $m > 0$ such that $\mathcal{M}(f^{-m}(I_j) \cap I_i) \neq 0$ for every i, j .
6. There exist $C > 0$ and $0 < \gamma < 1$ such that

$$\left| \frac{f'(x)}{f'(y)} - 1 \right| \leq C|x - y|^\gamma.$$

Then there exists a unique f -invariant probability measure μ which is absolutely continuous with respect to Lebesgue measure. This also implies that f is ergodic (according to Theorem 2.3.1).

Proof: See [10]. $\square$

To find the invariant measure of $SN(x)$ and of $CN(x)$, we proceed as we did in Example 4 for the logistic equation, i.e., we first show that the maps $TN(x)$ are ergodic.

Corollary 4.4.1 *The tent maps $TN(x)$ are ergodic, for $N \geq 2$.*

Proof: Consider $TN(x)$ with $N \geq 2$. We have that $TN(x)$ satisfies the conditions 1 to 6 of the previous Theorem. Many conditions are trivial, we show conditions 4 and 5 only.

For $TN(x)$, $I_j = (\frac{j}{N}, \frac{j+1}{N})$, $j = 0, 1, \dots, N-1$. Hence, $\cup_j I_j = [0, 1] \setminus \{j/N : j = 0, 1, \dots, N\}$. Take $\beta \in (1, 2)$. We have that $|f'(x)| = N$, for all $x \in \cup_j I_j$ so

$$|(TN^{n_0})'(x_0)| = \prod_{i=0}^{n_0-1} |TN'(x_i)| = N^{n_0} \geq 2^{n_0} > \beta.$$

To prove condition 5, take $m = 1$. We have that

$$\mathcal{M}(TN^{-1}(I_j) \cap I_i) = \frac{1}{N^2}.$$

for every N $\square$

.

Now, let us find the invariant probability density of the function $TN(x)$:

If N is even,

$$\begin{aligned} \rho(x) &= \sum_{k=0}^{N/2} \left(\int_{2k/N}^{(2k+1)/N} \delta(x - Nx_0 + k) \rho(x_0) dx_0 + \int_{(2k+1)/N}^{(2k+2)/N} \delta(x + Nx_0 - k - 1) \rho(x_0) dx_0 \right) \\ &= \sum_{k=0}^{N/2} \left(\frac{1}{N} \left[\rho\left(\frac{x+k}{N}\right) + \rho\left(\frac{k+1-x}{N}\right) \right] \right), \end{aligned}$$

while, if N is odd,

$$\rho(x) = \sum_{k=0}^{(N-1)/2} \left(\frac{1}{N} \left[\rho\left(\frac{x+k}{N}\right) + \rho\left(\frac{k+1-x}{N}\right) \right] \right) + \frac{1}{N} \rho\left(\frac{x-N+1}{N}\right),$$

for which $\rho(x) = 1$ is the unique normalized solution, since $TN(x)$ is ergodic (see Theorem 2.3.1). The invariant measure μ_T of $TN(x)$ is thus the Lebesgue measure. Now, because $H_s(x) = \sin^2(\frac{\pi x}{2})$ is the conjugacy map of $TN(x)$ and $SN(x)$, we have that (see Section 2.3):

$$\underbrace{\rho(x)}_{SN(x)} = \underbrace{\rho(y)}_{TN(y)} \left| \frac{dy}{dx} \right|$$

but with $\rho(y) = 1$,

$$\rho(x) = \left| \frac{d}{dx} \left(\frac{2}{\pi} \arcsin(\sqrt{x}) \right) \right|$$

and finally,

$$\rho(x) = \frac{1}{\pi \sqrt{x(1-x)}}.$$

which induces the invariant measure

$$\mu_S(C) = \frac{1}{\pi} \int_C \frac{1}{\sqrt{x(1-x)}} dx$$

where C is a Borel set, see [13, 2.1]. Since $H_c(x) = \cos^2(\frac{\pi x}{2})$ is the conjugacy of $TN(x)$ and $CN(x)$ we have that

$$\begin{aligned} \underbrace{\rho(x)}_{CN(x)} &= \underbrace{\rho(y)}_{TN(y)} \left| \frac{dy}{dx} \right| \\ \rho(x) &= \left| \frac{d}{dx} \left(\frac{2}{\pi} \arccos(\sqrt{x}) \right) \right| \\ \rho(x) &= \frac{1}{\pi \sqrt{x(1-x)}}, \end{aligned}$$

which induces the invariant measure

$$\mu_C(C) = \frac{1}{\pi} \int_C \frac{1}{\sqrt{x(1-x)}} dx$$

where C is a Borel set. Hence all the functions from both sets $\mathcal{S}$ and $\mathcal{C}$ have the *same* invariant measure that we denote $\mu = \mu_S = \mu_C$.

The result we want to show at this point, concerns the Lyapunov exponent of $SN(x)$ and of $CN(x)$ which is calculated to be $\ln(N)$. To do so, we need to prove few lemmas, but first, we compute the Lyapunov exponent of $TN(x)$.

Let us define $\mathcal{D}_N = \{k/N : k = 0, 1, \dots, N\}$. This is the set of points for which $TN(x)$ is not differentiable and for which $SN'(x)$ and $CN'(x)$ are zero. Note that the image of a point $x \in \mathcal{D}_N$ under $SN(x)$ or $CN(x)$ is either 0 or 1.

Lemma 4.4.1 *For given N , let $\{x_0, x_1, x_2, \dots\}$ be an orbit such that $x_i \notin \mathcal{D}_N$ for $i = 0, 1, 2, \dots$. Then the Lyapunov exponent of x_0 is $\ln(N)$ under $TN(x)$.*

Proof: We have that $|TN'(x)| = |\pm N| = N$ for all $x \in [0, 1] \setminus \mathcal{D}_N$, i.e., for all x for which the derivative of $TN(x)$ is defined. Consider x_0 such that the orbit $\{x_0, x_1, x_2, \dots\}$ under $TN(x)$ is such that $x_i \notin \mathcal{D}_N$ for $i = 0, 1, 2, \dots$. Then

$$\begin{aligned} h(x_0) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \ln |TN'(x_i)| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \ln(N) \\ &= \ln(N). \quad \square \end{aligned}$$

In order to generalize this last Lemma to the functions of $\mathcal{S}$ and of $\mathcal{C}$, we need to define two more sets of points. Let $\mathcal{D}_\mathcal{S} = \{x \in [0, 1] : \frac{\arcsin \sqrt{x}}{\pi} \in \mathbb{Q}\}$ and $\mathcal{D}_\mathcal{C} = \{x \in [0, 1] : \frac{\arccos \sqrt{x}}{\pi} \in \mathbb{Q}\}$ be these two sets of points. The rest of the argument is done for $SN(x)$ only since the same reasoning enables us to prove the result for $CN(x)$.

Lemma 4.4.2 *The set $\mathcal{D}_\mathcal{S}$ has Lebesgue measure zero.*

Proof: The function $g(x) = \frac{\arcsin \sqrt{x}}{\pi}$ being bijective on the interval $[0, 1]$, we have that the Lebesgue measure of $\mathcal{D}_\mathcal{S}$ is equal to the Lebesgue measure of the set $g(\mathcal{D}_\mathcal{S}) = \{g(x) : x \in \mathcal{D}_\mathcal{S}\}$. But, by definition of $\mathcal{D}_\mathcal{S}$, we have that $g(\mathcal{D}_\mathcal{S}) \subset \mathbb{Q} \cap [0, 1]$ which have Lebesgue measure zero. $\square$

Lemma 4.4.3 *Let $x_0 \in [0, 1]$, $x_i = SN^i(x_0)$ and $N > 1$. If $x_0 \notin \mathcal{D}_S$ then the orbit of x_0 under $SN(x)$ is not asymptotically periodic and $x_i \neq 0$ or 1 for all $i = 0, 1, 2, \dots$*

Proof: From Theorem 4.2.2, recall that if we write $x_i = \sin^2(\pi\theta_i)$, then $\theta_i = N^i\theta_0 \bmod 1$ where $\theta_0 = \frac{\arcsin \sqrt{x}}{\pi}$. By continuity of the function $\frac{\arcsin \sqrt{x}}{\pi}$, we have that the orbit $\{x_i\}$ is asymptotically periodic if and only if the orbit $\{\theta_i\}$ is. Note that $\{\theta_i\}$ is the orbit of θ_0 under the function $f(x) = Nx \bmod 1$. Hence, according to Lemma 2.1.1, since $|(f^k)'(x)| = N^k > 1$, when the derivative exists, we have that if $\{\theta_i\}$ is asymptotically periodic then it must be eventually periodic. But, clearly, $\{\theta_i\}$ is eventually periodic if and only if $\theta_0 \in \mathbb{Q}$, so if $x_0 \in \mathcal{D}_S$.

Furthermore, note that $x_i = 0$ or 1 if and only if $\theta_i = 0, 1/2$ or 1 . So suppose $x_{i_0} = 0$ or 1 , we have $\theta_{i_0} = 0, 1/2$ or 1 . But, this implies that the orbit $\{\theta_i\}$ is eventually periodic, so $\theta_0 \in \mathbb{Q}$ and $x_0 \in \mathcal{D}_S$. $\square$

Remark: The definition of the Lyapunov exponent of a function $f(x)$ at x_0 is

$$\begin{aligned} h(x_0) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \ln |f'(x_i)| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \prod_{i=0}^n f'(x_i) \right| \end{aligned}$$

where $x_i = f^i(x_0)$. Using the Chain Rule that tells us that $(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$ we can write that

$$h(x_0) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln |(f^n)'(x_0)|.$$

Theorem 4.4.2 *The Lyapunov exponent of $SN(x)$ and $CN(x)$ is $\ln(N)$ for almost all $x \in [0, 1]$.*

Proof: As usual, we restrict our proof to $SN(x)$, the proof for $CN(x)$ being similar. We prove that the result is true for all $x \notin \mathcal{D}_S$, we already proved that the set $\mathcal{D}_S$ has measure zero. Take $x_0 \in [0, 1] \setminus \mathcal{D}_S$. Recall that $SN(x) = H_s \circ TN \circ H_s^{-1}(x)$ and

$SN^i(x) = H_s \circ TN^i \circ H_s^{-1}(x)$ where $H_s(x) = \sin^2(\frac{\pi x}{2})$. Note that $|H_s(x)| \leq 1$ for $x \in [0, 1]$ (see Figure 8). We use the notation $y_i = H_s^{-1}(x_i)$ where $x_i = SN^i(x_0)$. We have that $y_0 \notin \mathbb{Q}$ since $x_0 \notin \mathcal{D}_S$ and so, $TN'(y_i) = N$ for all $i = 0, 1, 2, \dots$. Hence,

$$\begin{aligned}
h(x_0) &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln |(SN^n)'(x_0)| \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \ln |(H_s \circ TN^n \circ H_s^{-1})'(x_0)| \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \ln |H_s'(y_n)(TN^n)'(y_0)(H_s^{-1})'(x_0)| \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} (\ln |H_s'(y_n)| + \ln |(TN^n)'(y_0)| + \ln |(H_s^{-1})'(x_0)|) \\
&\leq \lim_{n \rightarrow \infty} \frac{1}{n} (\ln(1) + n \ln(N) + \ln |(H_s^{-1})'(x_0)|) \\
&= \ln(N)
\end{aligned}$$

On the other hand, since $x_0 \notin \mathcal{D}_S$ and according to Lemma 4.4.3, we can choose a sequence of integers n_j going to infinity such that $x_{n_j} \in [\delta, 1 - \delta]$. Also $|H_s'(x)| > 0$ in the open interval $(0, 1)$, so for all $\delta > 0$ there is a bound $K_\delta > 0$ such that $K_\delta < |H_s'(x)|$ for $x \in [\delta, 1 - \delta]$. Note that if $x_i \in [\delta, 1 - \delta]$, then $y_i = H_s^{-1}(x_i) \in [\delta, 1 - \delta]$ since $H_s^{-1}(x) > x$ when $x \in (0, 1/2)$ and $H_s^{-1}(x) < x$ when $x \in (1/2, 1)$. Then,

$$\begin{aligned}
h(x_0) &= \lim_{n_j \rightarrow \infty} \frac{1}{n_j} \ln |(SN^{n_j})'(x_0)| \\
&= \lim_{n_j \rightarrow \infty} \frac{1}{n_j} (\ln |H_s'(y_{n_j})| + \ln |(TN^{n_j})'(y_0)| + \ln |(H_s^{-1})'(x_0)|) \\
&\geq \lim_{n_j \rightarrow \infty} \frac{1}{n_j} (\ln(K_\delta) + n_j \ln(N) + \ln |(H_s^{-1})'(x_0)|) \\
&= \ln(N)
\end{aligned}$$

Hence, $h(x) = \ln(N)$ for all $x \notin \mathcal{D}_S$. $\square$

Remark: The last theorem is an important fact for our study. Indeed, it confirms that the functions $SN(x)$ and $CN(x)$ are *not all* conjugate to each other. Furthermore,

considering the function

$$h : \mathcal{S} \text{ (or } \mathcal{C}) \rightarrow \mathbb{R}$$

that sends $SN(x)$ (or $CN(x)$) to its Lyapunov exponent, and using the Theorem 4.2.1, we have that

$$\begin{aligned} h(SN_1 \circ SN_2) &= h(S[N_1N_2]) \\ &= \ln(N_1N_2) \\ &= \ln(N_1) + \ln(N_2) \\ &= h(SN_1) + h(SN_2). \end{aligned}$$

That is to say, h is an *homomorphism of monoids* from the monoid $\mathcal{S}$ (or $\mathcal{C}$), with respect to composition, to the monoid $\mathbb{R}$, with respect to addition. In fact, it is an injective homomorphism which enables us to say that there is an isomorphism of abelian monoids from $\mathcal{S}$ or $\mathcal{C}$ to the image of h in $\mathbb{R}$, which is $\{\ln(N) : N = 1, 2, 3, \dots\}$.

Figure 10 Bifurcation Diagrams vs Lyapunov Exponents

Chapter 5

Conclusion

We have found two abelian monoids $\mathcal{S}$ and $\mathcal{C}$ of non-conjugate polynomial maps that exhibit chaotic behavior : they have chaotic orbits (except for $S1(x)$ and $C1(x)$). Indeed, a chaotic orbit is one that is not asymptotically periodic and for which the Lyapunov exponent is greater than zero [14]. The Lyapunov exponent of $SN(x)$ and $CN(x)$ is $\ln(N)$ (Theorem 4.4.2). Moreover, choosing $x_0 = \sin^2(\pi\theta_0)$ (or $y_0 = \cos^2(\pi\psi_0)$) with $\theta_0 \notin \mathbb{Q}$ (or $\psi_0 \notin \mathbb{Q}$) assures that the orbit of x_0 (or y_0) is not asymptotically periodic (Lemma 4.4.3).

Another way to convince ourselves that the polynomials of $\mathcal{S}$ and $\mathcal{C}$ have chaotic orbits, is to find a number x_0 of period 3 for each polynomial in the sets (Theorem 2.2.1). For a fixed $1 < N \in \mathbb{N}$, consider the number

$$x_0 = \sin^2(\pi\theta_0)$$

where

$$\begin{aligned}\theta_0 &= \frac{1}{N^3 - 1} \\ &= 0.001001\dots \text{ (in base } N\text{)}.\end{aligned}$$

Hence, the orbit of x_0 under $SN(x)$ is $x_n = \sin^2(N^n \pi \theta_0)$, that is

$$\begin{aligned} x_0 &= \sin^2(\pi * 0.001001...) \\ x_1 &= \sin^2(\pi * 0.010010...) \\ x_2 &= \sin^2(\pi * 0.100100...) \\ x_3 &= \sin^2(\pi * 0.001001...) \dots \end{aligned}$$

The point x_0 generates the desired period 3 orbit. We thus showed that the chaoticity properties, originally found in the logistic map, can be extended to infinite families of polynomial maps, namely $\mathcal{S}$ and $\mathcal{C}$, which include the logistic map.

Furthermore, through this thesis, we have restricted our study to functions of the form $SN(x) = \sin^2(N \arcsin \sqrt{x})$ and $CN(x) = \cos^2(N \arccos \sqrt{x})$ for N a positive integer. One could be interested in generalizing this to any function of the type $S\alpha(x) = \sin^2(\alpha \arcsin \sqrt{x})$ and $C\alpha(x) = \cos^2(\alpha \arccos \sqrt{x})$ for $\alpha \in \mathbb{Q}$ or even $\alpha \in \mathbb{R}$. However, as for the functions $S1(x)$ and $C1(x)$, the functions $S\alpha(x)$ or $C\alpha(x)$ don't exhibit any interesting properties for $0 < \alpha \leq 1$.

For instance, the function $S\frac{1}{2}(x) = \sin^2(1/2 \arcsin \sqrt{x})$ is, in fact, conjugate to the function $g(x) = x/2$, which has no fixed point (other than zero) and no periodic point: all the orbits under $S\frac{1}{2}(x)$ are attracted to the origin. Hence, $S\frac{1}{2}(x)$ doesn't have any chaotic orbits.

To generalize the results of this thesis to functions of the form $S\alpha(x)$ and $C\alpha(x)$, one could try to answer the following questions:

1. Is $\ln(\alpha)$ the Lyapunov exponent of $S\alpha(x)$ and $C\alpha(x)$ for every $\alpha \in \mathbb{R}$?
2. Do $S\alpha(x)$ and $C\alpha(x)$ admit chaotic behavior if and only if $|\alpha| > 1$?

It is not possible to answer this last question using exactly the same approach as for $SN(x)$ and $CN(x)$ for $N > 0$, an integer (see Theorem 4.2.2). Indeed, to prove the

chaotic behavior of the functions $SN(x)$ and $CN(x)$ we used the expression of a point in “base N ” which makes sense if N is an integer greater than 1 only.

The families $\{S\alpha(x) : \alpha \in \mathbb{R}\}$ and $\{C\alpha(x) : \alpha \in \mathbb{R}\}$ might well generalized the idea of the present thesis.

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