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Lemma 1.2.1 Let $0 \leq n \leq m-1$, and let λ and μ be two real numbers. Let A and B be the matrices given by (1.2.4) and (1.2.5), respectively, and let $y(\lambda)$ be the vector given by (1.2.6). Then

$$i) (\lambda - \mu) \sum_{r=0}^n b_r y_r(\lambda) y_r(\mu) = c_n \begin{vmatrix} y_{n+1}(\lambda) & y_{n+1}(\mu) \\ y_n(\lambda) & y_n(\mu) \end{vmatrix}.$$

$$ii) \sum_{r=0}^n b_r (y_r(\lambda))^2 = c_n \begin{vmatrix} y'_{n+1}(\lambda) & y_{n+1}(\lambda) \\ y'_n(\lambda) & y_n(\lambda) \end{vmatrix}.$$

If λ is a complex number, then

$$iii) \sum_{r=0}^n b_r |y_r(\lambda)|^2 = (2i \operatorname{Im}(\lambda))^{-1} c_n \begin{vmatrix} y_{n+1}(\lambda) & y_{n+1}(\bar{\lambda}) \\ y_n(\lambda) & y_n(\bar{\lambda}) \end{vmatrix}.$$

Here the vertical bars denote the determinant of the matrix at hand, and $y'_n(\lambda)$ denotes the derivative of $y_n(\lambda)$.

Proof : See [2, p.98-99, Theorem 4.2.1-4.2.3] ■

Definition 1.2.1 Let A and B be the matrices given by (1.2.4) and (1.2.5), and let $y(\lambda)$ be the vector given by (1.2.6). The normalization constant ρ_r associated with an eigenvalue λ_r of (1.2.3) is defined by

$$\rho_r = \sum_{p=0}^{m-1} b_p |y_p(\lambda_r)|^2. \quad (1.2.7)$$

The set of normalization constants ρ_r associated with the eigenvalues λ_r ($0 \leq r \leq m-1$) is denoted by $\rho(A, B)$.

Lemma 1.2.2 Let A and B be the matrices given by (1.2.4) and (1.2.5), and let $y(\lambda)$ be the vector given by (1.2.6). Let λ_r and ρ_r ($0 \leq r \leq m-1$) be the eigenvalues and the normalization constants of the generalized eigenvalue problem (1.2.3), respectively. Let us assume that the ρ_r are nonzero real numbers, and the λ_r are distinct real numbers. Then

AN INVERSE EIGENVALUE PROBLEM

by

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B.Sc. Karadeniz Technical University

M.Sc. University of New Brunswick

**A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
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AN INVERSE EIGENVALUE PROBLEM

submitted by

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in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

for Dr. Vlastimil Dlab, Chair

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May 1995

To my MOTHER
FATMA BİLGİN

Abstract

In this thesis we introduce the concept of a positive definite sequence (according to Ahiezer and Krein [1]). We apply this theory to solve an inverse generalized eigenvalue problem $Ay = \lambda By$ when B is an indefinite real diagonal $m \times m$ matrix, and A is a symmetric tridiagonal matrix.

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INTRODUCTION

The theory of inverse eigenvalue problems plays a significant role in physics and applied mathematics. This theory is concerned with the construction of a model of a given type; e.g., a mass-spring system, a vibrating string, etc., which has given spectral data. (In inverse problems, data and results are interchanged.) Research on this theory began in the former Soviet Union with the work of M. G. Krein in the 1930's; his primary interest being the inverse problem for the Sturm-Liouville equation.

There are a great many papers in the literature devoted to the theory of inverse eigenvalue problems. The theory is separated into two parts:

- (1) continuous case,
- (2) discrete case.

We shall not be concerned in this thesis with the first case but only the second one. With respect to the first case, we just mention that physical entities form some continuous distribution and lead to the differential equations.

As for the discrete case, there is a discrete sequence of entities, and it is describable by means of a difference or recurrence relation. One can associate different interpretations to the three-term recurrence relation in order to make applications to vibrating strings, mass-spring systems, network theory, atomic physics, probability theory, the area of Markov processes, birth and death processes, and random walks. We illustrate it by means of an example involving a vibrating string:

Consider a weightless string which bears m particles, of masses $b_0, b_1, \dots, b_{m-1}$, with

both ends pinned down. Suppose that the distance between the particles b_r and b_{r+1} is to be $1/c_r$; $r = 0, 1, \dots, m-2$. In addition, the distance between the left hand end and the particle b_0 is $1/c_{-1}$, and the distance between the right hand end and the particle b_{m-1} is $1/c_{m-1}$. The entire string is pulled taut by a unit tension. If u_n is the displacement of the particle b_n at time t then the differential equation of the motion is

$$b_n u_n'' = -c_{n-1}(u_n - u_{n-1}) + c_n(u_{n+1} - u_n),$$

where u_n'' denotes the second derivative of u_n with respect to time t . If we seek a solution of the form $u_n = y_n \cos(\omega t)$, where y_n is the amplitude of oscillation of the particle b_n , then we derive that

$$c_n(y_{n+1} - y_n) - c_{n-1}(y_n - y_{n-1}) + b_n \lambda y_n = 0,$$

where $\omega^2 = \lambda$. Now, we define $y_n(\lambda)$ as the solution of the previous equation with boundary conditions $y_{-1} = y_m = 0$, where we normalize y_0 by setting $c_{-1}y_0 = 1$. This represents a free vibrating string (or hypothetical vibrating string) with both ends pinned down. Atkinson [2] considered the more general boundary condition $y_m(\lambda) + h y_{m-1}(\lambda) = 0$, where h is some fixed real number. We consider the case $h = 0$ in this thesis and leave the interesting case $h \neq 0$ to future study. Using matrix notation it is expressed as:

$$Ay = \lambda By,$$

where

$$A = \begin{bmatrix} c_0 & -c_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -c_0 & c_0 + c_1 & -c_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -c_1 & c_1 + c_2 & -c_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -c_{m-3} & c_{m-3} + c_{m-2} & -c_{m-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -c_{m-2} & c_{m-2} + c_{m-1} \end{bmatrix}$$

is the stiffness matrix,

$$B = \text{diag}(b_0, b_1, \dots, b_{m-1})$$

is the mass matrix and

$$y(\lambda) = (y_0(\lambda), y_1(\lambda), \dots, y_{m-1}(\lambda))^T$$

is the amplitude matrix. Here the matrix B is positive definite, the matrix A is either positive definite or positive semidefinite, and the eigenvalues λ are nonnegative real numbers. For details see [13], [26], [27], [23].

Atkinson [2] solved the inverse generalized eigenvalue problem $Ay = \lambda By$, where B is a positive definite diagonal matrix, and the eigenvalues λ are distinct real numbers. He seeks for a Jacobi matrix A which satisfies the given data. Boucekkine [9] constructed a positive definite Jacobi matrix A for the inverse eigenvalue problem when B is an indefinite matrix in the 2×2 and 3×3 cases. Here B can be interpreted as a matrix of charges (positive or negative) in atomic physics.

In this thesis we generalize Boucekkine's result to any $m \times m$ indefinite diagonal matrix B . We give an explicit formula for a positive definite Jacobi matrix A which solves the inverse generalized eigenvalue problem. Indeed, we find 2^{m-1} different positive definite Jacobi matrices. Among these matrices there is a unique distinguished positive Jacobi matrix A : We use three-term recurrence relations and orthogonal polynomials to construct the matrix A . The positive definiteness of A appears to be difficult to prove using basic matrix theory alone, so we adopted an old technique involving Hankel forms and positive definite sequences in order to arrive at this conclusion.

Chapter 1

AN INVERSE GENERALIZED EIGENVALUE PROBLEM

In Section 1.1 we recall briefly the basic properties of matrix theory. Proofs of these properties can be found in the standard textbooks on matrix theory, for example see [11],[12],[18]. In Section 1.2 we state the basic properties of the Generalized Eigenvalue Problem of Sturm-Liouville type associated with a three-term recurrence relation. We also state an Inverse Generalized Eigenvalue Problem, and give the necessary conditions for the existence of the solution. These properties are explained in [9], see also [2].

1.1 Basic Concepts From Matrix Theory

The vector space of all $m \times m$ real matrices is denoted by $R^{m \times m}$. The **transpose** of a matrix $A = (a_{ij})$ is denoted by $A^T = (a_{ji})$, and the **determinant** of A is denoted by $\det(A)$ or $|A|$.

Definition 1.1.1 Let $A = (a_{ij})$ be an $m \times m$ matrix of real or complex numbers.

i) If $a_{ij} = 0$ for $i \neq j$ then A is called a **diagonal matrix** and denoted by $A = \text{diag}(a_1, a_2, \dots, a_m)$ where $a_{ii} = a_i$ for $i = 1, 2, \dots, m$.

ii) If $a_{ij} = 0$ for $|i - j| > 1$ then A is called a **tridiagonal matrix**.

iii) If $A = A^T$ then A is called a **symmetric matrix**. (or, if $A = \overline{A^T}$, A is **hermitian**, where the bar denotes the complex conjugate matrix of A .)

Definition 1.1.2 Let $A = (a_{ij}) \in R^{m \times m}$ be a symmetric matrix.

- i) If $A = (a_{ij})$ is a tridiagonal matrix then it is called a **Jacobi matrix**.
- ii) If $y^T A y > 0$ for every $0 \neq y \in R^m$ then A is called a **positive definite matrix**.
- iii) If there exist $x, y \in R^m$ such that $(x^T A x)(y^T A y) < 0$ then A is called an **indefinite matrix**.
- iv) If $A^T = A^{-1}$ then A is called an **orthogonal matrix**.

Definition 1.1.3 Let $A = (a_{ij})$ be an $m \times m$ matrix, and let λ be a complex number.

- i) If $Ay = \lambda y$ has a nonzero solution $y \in R^m$ then λ is called an **eigenvalue** of A .
- ii) Any nonzero solution y of $Ay = \lambda y$ corresponding to an eigenvalue λ is called an **eigenvector** of A .
- iii) The set of all eigenvalues of A is called the **spectrum** of A and denoted by $\sigma(A)$.

Lemma 1.1.1 Let A be an $m \times m$ real symmetric matrix. Then,

- i) All the eigenvalues of A are real.
- ii) The eigenvectors associated with distinct eigenvalues of A are orthogonal with respect to usual inner product in R^m .
- iii) A is positive definite if and only if all the eigenvalues of A are positive.
- iv) A is positive definite if and only if $D_k > 0$ ($k = 1, 2, \dots, m$), where D_k is the determinant of the k^{th} leading principal submatrix of A .

Proof : See [6, p.35, Theorem 1; p.36, Theorem 2; p.75, Theorem 1] ■

Definition 1.1.4 Let A and B be two $m \times m$ real matrices, and let λ be a complex number.

- i) If the equation $Ay = \lambda By$ has a nonzero solution $y \in R^m$ then λ is called an **eigenvalue of the generalized eigenvalue problem** for the matrix pair (A, B) .
- ii) The set of all the eigenvalues for the matrix pair (A, B) is called the **spectrum**

of the generalized eigenvalue problem and denoted by $\sigma(A, B)$.

iii) Let λ be an eigenvalue for the matrix pair (A, B) . If a nonzero $y \in R^m$ satisfies the equation $Ay = \lambda By$ then y is called an eigenvector for the matrix pair (A, B) corresponding to the eigenvalue λ .

Note that the generalized eigenvalue problem $Ay = \lambda By$ has m finite eigenvalues if and only if $\text{rank}(B) = m$. If B is rank deficient then $\sigma(A, B)$ may be empty, finite or an infinite set.

An important case for the generalized eigenvalue problem is the case where A and B are symmetric with one of them positive definite.

Lemma 1.1.2 *Let A and B be two $m \times m$ symmetric matrices, and let A be positive definite. Then*

i) *All the eigenvalues of $Ay = \lambda By$ are real.*

ii) *Let λ_1 and λ_2 be in $\sigma(A, B)$, $\lambda_1 \neq \lambda_2$, and let y_1 and y_2 be the corresponding eigenvectors, respectively. Then $(Ay_1, y_2) = 0$, that is, y_1 and y_2 are orthogonal in this sense, where (Ay_1, y_2) denotes $y_2^T Ay_1$, is the usual scalar product in R^m .*

Proof : We leave the proof to the reader. ■

1.2 Three-Term Recurrence Relations

We consider here a generalized eigenvalue problem of Sturm-Liouville type, see [2, Chapter 4] associated with the three-term recurrence relation

$$c_n y_{n+1}(\lambda) = (b_n \lambda - a_n) y_n(\lambda) - c_{n-1} y_{n-1}(\lambda), \quad 0 \leq n \leq m-1, \quad (1.2.1)$$

and the conditions

$$y_{-1}(\lambda) = 0, \quad y_0(\lambda) = 1/c_{-1} \quad \text{and} \quad y_m(\lambda) = 0, \quad (1.2.2)$$

where a_n is an arbitrary finite sequence of real numbers, and b_n and c_n are non-zero real numbers. In Atkinson [2] it is assumed that $b_n > 0$, $c_n > 0$. (Note: The a_n in Atkinson are our " b_n ".)

Since $b_n \neq 0$, it is obvious that the polynomial $y_n(\lambda)$ has degree n for $0 \leq n \leq m$.

A solution of (1.2.1) satisfying the conditions (1.2.2) is a vector

$$y(\lambda) = (y_0(\lambda), y_1(\lambda), \dots, y_{m-1}(\lambda))^T,$$

where $y_{-1}(\lambda) = y_m(\lambda) = 0$. The three-term recurrence relation (1.2.1) is then equivalent to the generalized eigenvalue problem

$$Ay = \lambda By, \quad (1.2.3)$$

where

$$A = \begin{bmatrix} a_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_0 & a_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_1 & a_2 & c_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_{m-3} & a_{m-2} & c_{m-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_{m-2} & a_{m-1} \end{bmatrix}, \quad (1.2.4)$$

$$B = \text{diag}(b_0, b_1, \dots, b_{m-1}) \text{ and} \quad (1.2.5)$$

$$y(\lambda) = (y_0(\lambda), y_1(\lambda), \dots, y_{m-1}(\lambda))^T. \quad (1.2.6)$$

Note that $y_m(\lambda) = 0$ is equivalent to $\det(A - \lambda B) = 0$. Simply expand $\det(A - \lambda B)$ and compare with $y_m(\lambda)$ as given by the recurrence relation (1.2.1). Hence, the roots of $y_m(\lambda) = 0$ are the **eigenvalues** of the generalized eigenvalue problem (1.2.3). We denote the roots of $y_m(\lambda) = 0$ by λ_r ($0 \leq r \leq m-1$). In this thesis, unless otherwise stated, by an **eigenvalue** we mean an eigenvalue of the generalized eigenvalue problem (1.2.3).

Lemma 1.2.1 Let $0 \leq n \leq m-1$, and let λ and μ be two real numbers. Let A and B be the matrices given by (1.2.4) and (1.2.5), respectively, and let $y(\lambda)$ be the vector given by (1.2.6). Then

$$i) (\lambda - \mu) \sum_{r=0}^n b_r y_r(\lambda) y_r(\mu) = c_n \begin{vmatrix} y_{n+1}(\lambda) & y_{n+1}(\mu) \\ y_n(\lambda) & y_n(\mu) \end{vmatrix},$$

$$ii) \sum_{r=0}^n b_r (y_r(\lambda))^2 = c_n \begin{vmatrix} y'_{n+1}(\lambda) & y_{n+1}(\lambda) \\ y'_n(\lambda) & y_n(\lambda) \end{vmatrix}.$$

If λ is a complex number, then

$$iii) \sum_{r=0}^n b_r |y_r(\lambda)|^2 = (2i \operatorname{Im}(\lambda))^{-1} c_n \begin{vmatrix} y_{n+1}(\lambda) & y_{n+1}(\bar{\lambda}) \\ y_n(\lambda) & y_n(\bar{\lambda}) \end{vmatrix}.$$

Here the vertical bars denote the determinant of the matrix at hand, and $y'_n(\lambda)$ denotes the derivative of $y_n(\lambda)$.

Proof : See [2, p.98-99, Theorem 4.2.1-4.2.3] ■

Definition 1.2.1 Let A and B be the matrices given by (1.2.4) and (1.2.5), and let $y(\lambda)$ be the vector given by (1.2.6). The **normalization constant** ρ_r associated with an eigenvalue λ_r of (1.2.3) is defined by

$$\rho_r = \sum_{p=0}^{m-1} b_p |y_p(\lambda_r)|^2. \quad (1.2.7)$$

The set of normalization constants ρ_r associated with the eigenvalues λ_r ($0 \leq r \leq m-1$) is denoted by $\rho(A, B)$.

Lemma 1.2.2 Let A and B be the matrices given by (1.2.4) and (1.2.5), and let $y(\lambda)$ be the vector given by (1.2.6). Let λ_r and ρ_r ($0 \leq r \leq m-1$) be the eigenvalues and the normalization constants of the generalized eigenvalue problem (1.2.3), respectively. Let us assume that the ρ_r are nonzero real numbers, and the λ_r are distinct real numbers. Then

i) If $i \neq j$ then $y(\lambda_i)$ is orthogonal to $y(\lambda_j)$ in the sense that

$$\sum_{p=0}^{m-1} b_p y_p(\lambda_i) y_p(\lambda_j) = \rho_i \delta_{ij}. \quad (1.2.8)$$

ii) If $0 \leq p, q \leq m-1$ then

$$\sum_{r=0}^{m-1} y_p(\lambda_r) y_q(\lambda_r) \rho_r^{-1} = b_p^{-1} \delta_{pq}. \quad (1.2.9)$$

The property ii) is called the dual orthogonality.

Proof : See [2, p.104-106, Theorem 4.4.1-4.4.2] ■

1.3 An Inverse Generalized Eigenvalue Problem

The Problem : Let $B = \text{diag}(b_0, b_1, \dots, b_{m-1})$ be an indefinite real matrix of rank r , and let λ_i and ρ_i be real numbers for $0 \leq i \leq r-1$. We find a **Jacobi** matrix A which satisfies the generalized eigenvalue problem (1.2.3) with

$$\text{i) } \sigma(A, B) = \{\lambda_0, \lambda_1, \dots, \lambda_{r-1}\} \text{ and}$$

$$\text{ii) } \rho(A, B) = \{\rho_0, \rho_1, \dots, \rho_{r-1}\}.$$

This problem is called an **inverse generalized eigenvalue problem**. We also show that under some conditions the matrix A which we constructed is a **positive definite Jacobi matrix**. For background information about inverse eigenvalue problems see [2, Chapter 4.6].

Now let us figure out some necessary conditions regarding the generalized eigenvalues λ_i and the normalization constants ρ_i for the existence of a positive definite solution of this problem.

Theorem 1.3.1 *Let $B = \text{diag}(b_0, b_1, \dots, b_{m-1})$ be an indefinite real matrix of rank r , $2 \leq r \leq m-1$, and let λ_i and ρ_i , $0 \leq i \leq r-1$ be a given set of real numbers. Then, in order that a positive definite Jacobi matrix A solves the generalized eigenvalue problem $Ay = \lambda By$, where the λ_i and ρ_i are the corresponding generalized eigenvalues and normalization constants, it is necessary that*

i) $\lambda_i \rho_i > 0$ for $0 \leq i \leq r-1$, and

ii) If B is nonsingular then the number of negative eigenvalues of $Ay = \lambda By$ is equal to the number of negative eigenvalues of B .

Proof : If B is of rank $r < m$ then, without loss of generality, we may assume that $b_r = b_{r+1} = \dots = b_{m-1} = 0$. It is easy to see that if the rank of B is r then both $\sigma(A, B)$ and $\rho(A, B)$ must have exactly r finite elements, counting the multiplicities. Let $\lambda_i \in \sigma(A, B)$ and let $y^{(i)}$ be the corresponding eigenvector. Then

$$Ay^{(i)} = \lambda_i By^{(i)}.$$

Hence

$$y^{(i)T} Ay^{(i)} = \lambda_i y^{(i)T} By^{(i)} = \lambda_i \rho_i.$$

Since A is positive definite, $\sigma(A, B)$ is a real set by Lemma 1.1.2 and

$$0 < y^{(i)T} Ay^{(i)} = \lambda_i \rho_i.$$

This completes the proof of part i).

Since A is positive definite, A^{-1} is positive definite. Hence, $A^{-1}B$ has the same number of negative eigenvalues as B , see [18, p.465, 7.6.3 Theorem]. Since B is nonsingular, then $\sigma(A, B) = \sigma(B^{-1}A)$. On the other hand, by i) the elements of $\sigma(A, B)$ are nonzero. Thus, we have

$$\lambda \in \sigma(B^{-1}A) \text{ if and only if } 1/\lambda \in \sigma(A^{-1}B).$$

So $B^{-1}A$ has the same number of negative eigenvalues as B , which completes the proof of part ii). ■

Corollary 1.3.1 *Let the notation and hypotheses be as in Theorem 1.3.1. Then there are r finite and nonzero real generalized eigenvalues λ_i such that the corresponding normalization constants ρ_i are nonzero real numbers.*

Theorem 1.3.2 *Let A be an $m \times m$ positive definite Jacobi matrix, and let B be an $m \times m$ nonsingular diagonal matrix. Then all the generalized eigenvalues of*

$$Ay = \lambda By$$

are real and distinct.

Proof : By Section 1.2, we know that the elements of $\sigma(A, B)$ are the roots of $y_m(\lambda) = 0$. So it is enough to show that the roots of $y_m(\lambda) = 0$ are simple.

Suppose λ_i is a multiple root of $y_m(\lambda) = 0$. Then it follows that $y_m(\lambda_i) = y'_m(\lambda_i) = 0$. By Lemma 1.1.2, $\sigma(A, B)$ is a real set and, by Lemma 1.2.1,

$$\rho_i = \sum_{k=0}^{m-1} b_k (y_k(\lambda_i))^2 = c_{m-1} \begin{vmatrix} y'_m(\lambda_i) & y_m(\lambda_i) \\ y'_{m-1}(\lambda_i) & y_{m-1}(\lambda_i) \end{vmatrix} = 0,$$

which is in contradiction with Corollary 1.3.1. Hence all the generalized eigenvalues are distinct. ■

1.4 The Spectral Function

Definition 1.4.1 *The spectral function $\tau(\lambda)$ is defined on the real axis, as a step function having jumps of amount $1/\rho_r$ at the eigenvalues λ_r (points of the spectrum), and being constant in between the eigenvalues, where ρ_r is the normalization constant defined by (1.2.7). More precisely*

$$\tau(\lambda) = \sum_{\lambda_r < \lambda} 1/\rho_r \tag{1.4.1}$$

for all $\lambda \in R$.

The spectral function is also called the **distribution function**. This function unifies the eigenvalues λ_r and the normalization constants ρ_r into a single concept.

Let $B = \text{diag}(b_0, b_1, \dots, b_{m-1})$ be a nonsingular indefinite real matrix, and let λ_i and ρ_i be real numbers for $0 \leq i \leq m-1$. Let $\tau(\lambda)$ be the spectral function given by

(1.4.1). An equivalent reformulation of the problem stated in Section 1.3 is :

Find a positive definite Jacobi matrix A which satisfies the generalized eigenvalue problem (1.2.3) and agrees with the given spectral function.

Note that the spectral function $\tau(\lambda)$ is of bounded variation on R for an indefinite matrix B in comparison with $\tau(\lambda)$ nondecreasing if $B > 0$.

Let us note that, by Theorem 1.3.1, and Theorem 1.3.2 the following conditions must be satisfied :

- i) $\lambda_i \rho_i > 0$; $i = 0, 1, \dots, m-1$.
- ii) $\lambda_i \neq \lambda_j$ if $i \neq j$.
- iii) The number of negative eigenvalues of B is equal to the number of negative generalized eigenvalues of the matrix pair (A, B) .

In this thesis, unless otherwise stated, we will always assume that these three conditions are satisfied. In fact, A is positive definite and B is generally indefinite throughout.

Lemma 1.4.1 *The relation (1.2.9) is equivalent to the Riemann-Stieltjes integral*

$$\int_{-\infty}^{\infty} y_p(\lambda) y_q(\lambda) d\tau(\lambda) = b_p^{-1} \delta_{pq}, \quad 0 \leq p, q \leq m-1. \quad (1.4.2)$$

Proof : See [2, p.106]. ■

Lemma 1.4.2 *The equation (1.4.2) with $p \neq q$ is equivalent to*

$$\int_{-\infty}^{\infty} y_p(\lambda) \lambda^q d\tau(\lambda) = 0, \quad 0 \leq q \leq p-1. \quad (1.4.3)$$

Proof : First assume the equation (1.4.2) holds. Since each $y_n(\lambda)$ has degree n , $\{y_0(\lambda), y_1(\lambda), \dots, y_{m-1}(\lambda)\}$ is a basis for the vector subspace of the polynomials of degree at most $m-1$. Let $0 \leq q \leq p-1$. Then there exist $c_0, c_1, \dots, c_q$ such that $c_q \neq 0$ and

$$\lambda^q = c_0 y_0(\lambda) + c_1 y_1(\lambda) + \dots + c_q y_q(\lambda).$$

Hence

$$\begin{aligned}\int_{-\infty}^{\infty} y_p(\lambda) \lambda^q d\tau(\lambda) &= \int_{-\infty}^{\infty} y_p(\lambda) \left[\sum_{i=0}^q c_i y_i(\lambda) \right] d\tau(\lambda) \\ &= \sum_{i=0}^q c_i \int_{-\infty}^{\infty} y_p(\lambda) y_i(\lambda) d\tau(\lambda) .\end{aligned}$$

Since $0 \leq i \leq q \leq p-1$, $i \neq p$ for each $0 \leq i \leq q$. So, by the equation (1.4.2)

$$\int_{-\infty}^{\infty} y_p(\lambda) y_i(\lambda) d\tau(\lambda) = 0.$$

Then it follows that

$$\int_{-\infty}^{\infty} y_p(\lambda) \lambda^q d\tau(\lambda) = 0 ,$$

which is (1.4.3).

Conversely, assume that the equation (1.4.3) holds. We need to show that for $p \neq q$, $0 \leq p, q \leq m-1$,

$$\int_{-\infty}^{\infty} y_p(\lambda) y_q(\lambda) d\tau(\lambda) = 0.$$

Without loss of generality assume that $p > q$. Let

$$y_q(\lambda) = c_0 + c_1 \lambda + \cdots + c_q \lambda^q .$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} y_p(\lambda) y_q(\lambda) d\tau(\lambda) &= \int_{-\infty}^{\infty} y_p(\lambda) \left(\sum_{i=0}^q c_i \lambda^i \right) d\tau(\lambda) \\ &= \sum_{i=0}^q c_i \int_{-\infty}^{\infty} y_p(\lambda) \lambda^i d\tau(\lambda) ,\end{aligned}$$

where $0 \leq i \leq q < p$. Since $0 \leq i \leq p-1$ then by the equation (1.4.3),

$$\int_{-\infty}^{\infty} y_p(\lambda) \lambda^i d\tau(\lambda) = 0 .$$

Hence for $0 \leq p, q \leq m-1$ and $p \neq q$,

$$\int_{-\infty}^{\infty} y_p(\lambda) y_q(\lambda) d\tau(\lambda) = 0 ,$$

which is (1.4.2). ■

Definition 1.4.2 Let $\tau(\lambda)$ be the spectral function defined by (1.4.1). The scalars

$$\mu_j = \int_{-\infty}^{\infty} \lambda^j d\tau(\lambda), \quad j = 0, 1, \dots \quad (1.4.4)$$

are called the moments of $\tau(\lambda)$.

Note that the equation (1.4.4) is equivalent to

$$\mu_j = \sum_{r=0}^{m-1} \lambda_r^j / \rho_r, \quad j = 0, 1, \dots \quad (1.4.5)$$

and since $\lambda_r \rho_r > 0$, the odd moments μ_{2n+1} are all positive ($n = 0, 1, 2, \dots$).

Definition 1.4.3 Let $\tau(\lambda)$ be the spectral function defined by (1.4.1), and let μ_j be the moments of $\tau(\lambda)$ given by (1.4.5). We define M_n and Δ_n as

$$i) \quad M_n = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{bmatrix} \quad (1.4.6)$$

$$ii) \quad \Delta_n = \det(M_n), \quad \Delta_0 = 1. \quad (1.4.7)$$

We make use of the following remark in Chapter 3 for the construction of the Jacobi matrix A which solves an inverse generalized eigenvalue problem.

Remark 1.4.1 Let the notation be as in the previous definition. Then

$$\begin{aligned} \Delta_{n+1} = & \mu_n \sum_{k=1}^n (-1)^k \mu_{n-1+k} A_{1k} + \mu_{n+1} \sum_{k=1}^n (-1)^{1+k} \mu_{n-1+k} A_{2k} + \cdots \\ & + \mu_{n+r} \sum_{k=1}^n (-1)^{r+k} \mu_{n-1+k} A_{r+1,k} + \cdots \\ & + \mu_{2n-1} \sum_{k=1}^n (-1)^{n-1+k} \mu_{n-1+k} A_{nk} + \mu_{2n} \Delta_n, \end{aligned} \quad (1.4.8)$$

where A_{ij} is the determinant of the matrix obtained from M_n by deleting row i and column j . If $n = 1$ we set $A_{1,1} = 1$.

1.5 3×3 and 4×4 Cases

In this section we recall the results of Boucekkine [9] when $m = 3$, and illustrate how difficult it is to show that A is positive definite for $m = 4$ with his technique.

Theorem 1.5.1 *Let $\tau(\lambda)$ be a spectral function defined by the expression (1.4.1) with $\sigma(A, B) = \{\lambda_0, \lambda_1, \lambda_2\}$ and $\rho(A, B) = \{\rho_0, \rho_1, \rho_2\}$. Let $B = \text{diag}(b_0, b_1, b_2)$ be a nonsingular indefinite real 3×3 matrix. If the conditions*

$$i) b_i \Delta_i \Delta_{i+1} > 0, \quad i = 0, 1, 2$$

$$ii) \lambda_i \rho_i > 0, \quad i = 0, 1, 2$$

iii) *the number of negative eigenvalues of B is equal to the number of negative generalized eigenvalues of the matrix pair (A, B) hold, then*

i) *there exist three real polynomials*

$$y_0(\lambda) = k_0, \quad y_1(\lambda) = k_1(\lambda + \alpha_{1,0}), \quad y_2(\lambda) = k_2(\lambda^2 + \alpha_{2,1}\lambda + \alpha_{2,0})$$

of degrees 0, 1 and 2, respectively, and such that the coefficients of $y_n(\lambda)$ ($n = 0, 1, 2$) are determined up to a change of sign, where

$$k_i^2 = \frac{\Delta_i}{b_i \Delta_{i+1}}, \quad i = 0, 1, 2,$$

$$\alpha_{1,0} = -\frac{\mu_1}{\mu_0}, \quad \alpha_{2,0} = \frac{\mu_2^2 - \mu_1 \mu_3}{\mu_1^2 - \mu_0 \mu_2}, \quad \alpha_{2,1} = \frac{\mu_0 \mu_3 - \mu_1 \mu_2}{\mu_1^2 - \mu_0 \mu_2}.$$

ii) *a positive definite Jacobi matrix A of the form*

$$A = \begin{bmatrix} a_0 & c_0 & 0 \\ c_0 & a_1 & c_1 \\ 0 & c_1 & a_2 \end{bmatrix}, \quad (1.5.1)$$

given by

$$c_i = \frac{b_i k_i}{k_{i+1}}, \quad i = 0, 1$$

$$a_0 = -b_0\alpha_{1,0} , \quad a_1 = -\frac{b_1k_1^2\alpha_{2,0} + a_0k_0^2}{k_1^2\alpha_{1,0}} , \quad a_2 = \frac{b_2k_2^2\lambda_0\lambda_1\lambda_2 - b_1k_1^2\alpha_{1,0}}{k_2^2\alpha_{2,0}}$$

is a solution for the inverse generalized eigenvalue problem $Ay = \lambda By$ which assumes $r(\lambda)$ as its spectral function where $y(\lambda) = (y_0(\lambda), y_1(\lambda), y_2(\lambda))^T$.

Proof: The first part of this theorem is a special case of Theorem 3.1.1. The construction of the Jacobi matrix A is similar to the construction given in Theorem 3.1.2 except that the expression for a_n is more simple in Theorem 3.1.2. Hence we only recall the proof that A is positive definite.

The determinants of the leading principal submatrices of the matrix A are

$$D_1 = \frac{b_0\Delta_0}{\Delta_1}\mu_1 , \quad D_2 = \frac{b_0\Delta_0}{\Delta_1} \cdot \frac{b_1\Delta_1}{\Delta_2}(\mu_1\mu_3 - \mu_2^2) , \quad D_3 = b_0b_1b_2\lambda_0\lambda_1\lambda_2 .$$

By the first and second assumptions, it follows that $D_1 > 0$, and by the third assumption, $D_3 > 0$.

To show that $D_2 > 0$, one needs to show that $\mu_1\mu_3 - \mu_2^2 > 0$. Equation (1.4.5) gives that

$$\begin{aligned} \mu_1\mu_3 - \mu_2^2 = \frac{1}{(\rho_0\rho_1\rho_2)^2} & [\rho_2^2\rho_0\rho_1\lambda_0\lambda_1(\lambda_1 - \lambda_0)^2 + \rho_1^2\rho_0\rho_2\lambda_0\lambda_2(\lambda_0 - \lambda_2)^2 \\ & + \rho_0^2\rho_1\rho_2\lambda_1\lambda_2(\lambda_1 - \lambda_2)^2] , \end{aligned}$$

which is positive by the second assumption. Hence, the Jacobi matrix A is positive definite by Lemma 1.1.1.

Since there are two possible signs for the k_i ($i = 0, 1, 2$) and the sign of c_i depends on the sign of k_i , then there are exactly four positive definite Jacobi matrices A_i ($i = 1, 2, 3, 4$). ■

Using the techniques of Boucekkine [9] to show that A is positive definite when $m \geq 4$ is extremely difficult. We illustrate it for $m = 4$.

Let D_1 , D_2 , D_3 and D_4 be the determinants of the leading principal submatrices of a Jacobi matrix A in Theorem 3.1.2 for $m = 4$. Then we obtain

$$\begin{aligned}
D_1 &= \frac{b_0 \Delta_0}{\Delta_1} \mu_1 , \\
D_2 &= \frac{b_0 \Delta_0}{\Delta_1} \cdot \frac{b_1 \Delta_1}{\Delta_2} (\mu_1 \mu_3 - \mu_2^2) , \\
D_3 &= \frac{b_0 \Delta_0}{\Delta_1} \cdot \frac{b_1 \Delta_1}{\Delta_2} \cdot \frac{b_2 \Delta_2}{\Delta_3} (\mu_1 \mu_3 \mu_5 + \mu_2^2 \mu_5 - \mu_1 \mu_4^2 + 2 \mu_2 \mu_3 \mu_4 - \mu_3^3) , \\
D_4 &= b_0 b_1 b_2 b_3 \lambda_0 \lambda_1 \lambda_2 \lambda_3 .
\end{aligned}$$

It is easy to see that D_1 , D_2 and D_4 are positive. Using equation (1.4.5) and doing a lengthy calculation we obtain

$$\begin{aligned}
D_3 &= \frac{1}{(\rho_0 \rho_1 \rho_2 \rho_3)^2} \left[\rho_0^2 \rho_1 \rho_2 \rho_3 \lambda_1 \lambda_2 \lambda_3 (\lambda_1^2 \lambda_2 - \lambda_1 \lambda_2^2 - \lambda_1^2 \lambda_3 + \lambda_2^2 \lambda_3 + \lambda_1 \lambda_3^2 - \lambda_2 \lambda_3^2)^2 \right. \\
&\quad + \rho_1^2 \rho_0 \rho_2 \rho_3 \lambda_0 \lambda_2 \lambda_3 (\lambda_0^2 \lambda_2 - \lambda_0 \lambda_2^2 - \lambda_0^2 \lambda_3 + \lambda_2^2 \lambda_3 + \lambda_0 \lambda_3^2 - \lambda_2 \lambda_3^2)^2 \\
&\quad + \rho_2^2 \rho_0 \rho_1 \rho_3 \lambda_0 \lambda_1 \lambda_3 (\lambda_0^2 \lambda_1 - \lambda_0 \lambda_1^2 - \lambda_0^2 \lambda_3 + \lambda_1^2 \lambda_3 + \lambda_0 \lambda_3^2 - \lambda_1 \lambda_3^2)^2 \\
&\quad \left. + \rho_3^2 \rho_0 \rho_1 \rho_2 \lambda_0 \lambda_1 \lambda_2 (\lambda_0^2 \lambda_1 - \lambda_0 \lambda_1^2 - \lambda_0^2 \lambda_2 + \lambda_1^2 \lambda_2 + \lambda_0 \lambda_2^2 - \lambda_1 \lambda_2^2)^2 \right] ,
\end{aligned}$$

which is positive.

Since it is virtually impossible when m is large to show that A is positive definite, we need to have a different approach to handle this difficulty. We have found that the use of *positive definite sequences* together with the quadratic form of Hankel works very nicely in this regard.

Chapter 2

POSITIVE DEFINITE SEQUENCES

In Section 2.1 we introduce the concept of a *positive definite sequence* on a real interval J according to Ahiezer and Krein [1], and show how it can be applied to the sequence of the moments of the spectral function $\tau(\lambda)$. We adopt the terminology in [1, Chapter 1] with respect to these "sequences". In Section 2.2 we use the quadratic form of Hankel and these positive definite sequences to prove Theorem 2.2.3 which plays a crucial role in showing that the Jacobi matrix A constructed in Theorem 3.1.2 is positive definite.

2.1 Positive Definite Sequences

In this section we introduce the concept of a *positive definite sequence* on a real interval, J , and prove that the sequence of the moments of the spectral function $\tau(\lambda)$ is a positive definite sequence.

Definition 2.1.1 *Let*

$$(s) = s_0, s_1, \dots, s_n \tag{2.1.1}$$

be a sequence of real numbers, and let $J = (a, b)$ ($-\infty \leq a < b \leq \infty$) be a certain interval in R . If for every polynomial $G_n(u) = A_0 + A_1u + \dots + A_nu^n$ satisfying

$$G_n(u) \neq 0, \quad G_n(u) \geq 0, \quad u \in J,$$

the sequence (s) has the property that

$$A_0 s_0 + A_1 s_1 + \cdots + A_n s_n > 0 ,$$

then the sequence (s) is called a **positive definite sequence on J** .

Here, note that $s_0 > 0$, for any positive definite sequence (s) on J (since we can choose $A_1 = A_2 = \cdots = A_n = 0$ in $G_n(u)$ etc.).

Example 2.1.1 The sequence $\{1, 0, 1\}$ is positive definite on $(-\infty, \infty)$ since $G_2(u) \geq 0$, $u \in R$ is satisfied provided

$$A_1^2 < 4A_0A_2, \quad A_0 > 0 \text{ and } A_2 > 0 .$$

So,

$$A_0 s_0 + A_1 s_1 + A_2 s_2 = A_0 + A_2 > 0 .$$

Theorem 2.1.1 [1, p.2] Let $r_1, r_2, \dots, r_m$ be m positive real numbers, and let $\xi_1 < \xi_2 < \cdots < \xi_m$ be m real numbers. Set

$$s_k = r_1 \xi_1^k + r_2 \xi_2^k + \cdots + r_m \xi_m^k ; \quad k = 0, 1, \dots$$

Then the sequence

$$(s) = s_0, s_1, \dots, s_n$$

is positive definite in every interval (a, b) satisfying $-\infty \leq a < \xi_1$ and $\xi_m < b \leq \infty$ for $n \leq 2m - 1$.

Proof : Let $G_n(u) = A_0 + A_1 u + \cdots + A_n u^n$ be a polynomial satisfying $G_n(u) \not\equiv 0$ and $G_n(u) \geq 0$ in the interval (a, b) . Then

$$\begin{aligned} & A_0 s_0 + A_1 s_1 + \cdots + A_n s_n \\ &= A_0(r_1 + r_2 + \cdots + r_m) + A_1(r_1 \xi_1 + r_2 \xi_2 + \cdots + r_m \xi_m) + \cdots \\ &+ A_n(r_1 \xi_1^n + r_2 \xi_2^n + \cdots + r_m \xi_m^n) \\ &= r_1 G_n(\xi_1) + r_2 G_n(\xi_2) + \cdots + r_m G_n(\xi_m) \geq 0. \end{aligned}$$

Now,

$$r_1 G_n(\xi_1) + r_2 G_n(\xi_2) + \cdots + r_m G_n(\xi_m) = 0$$

if and only if

$$G_n(\xi_1) = G_n(\xi_2) = \cdots = G_n(\xi_m) = 0$$

since $G_n(\xi_i) \geq 0$ over (a, b) by hypothesis (i.e., $\xi_1, \xi_2, \dots, \xi_m$ are real roots of $G_n(u)$). Since they must occur with even multiplicity (as $G_n(u) \geq 0, u \in J$, by hypothesis), it follows that $\deg G_n(u) \geq 2m$.

So, if $n \leq 2m - 1$, not all $G_n(\xi_i) = 0$. Hence

$$r_1 G_n(\xi_1) + r_2 G_n(\xi_2) + \cdots + r_m G_n(\xi_m) > 0,$$

i.e., the sequence $(s) = s_0, s_1, \dots, s_n$ is positive definite for $n \leq 2m - 1$. ■

In the following theorem we show that a sequence of moments of a spectral function is a positive definite sequence on $(-\infty, \infty)$. We use this theorem to prove Theorem 2.2.3 which enables us to prove that the Jacobi matrix A in Theorem 3.2.2 is positive definite.

Theorem 2.1.2 *Let λ_i and ρ_i ($0 \leq i \leq m-1$) be the eigenvalues and the normalization constants of the generalized eigenvalue problem (1.2.3). If $\lambda_i \rho_i > 0$ for $0 \leq i \leq m-1$ then the sequence of moments given by (1.4.5), viz.,*

$$\mu_1, \mu_2, \dots, \mu_{2m}$$

is a positive definite sequence in the interval $(-\infty, \infty)$.

Proof : Define the real numbers

$$r_k = \frac{\lambda_{k-1}}{\rho_{k-1}}, \quad 1 \leq k \leq m$$

$$\xi_k = \lambda_{k-1}, \quad 1 \leq k \leq m$$

and set

$$s_k = r_1 \xi_1^k + r_2 \xi_2^k + \cdots + r_m \xi_m^k, \quad (k = 0, 1, \dots).$$

Then $r_k > 0$ for $1 \leq k \leq m$. Hence, by Theorem 2.1.1, the sequence $(s) = s_0, s_1, \dots, s_{2m-1}$ is positive definite on $(-\infty, \infty)$. By equation (1.4.5), $s_k = \mu_{k+1}$ ($k = 0, 1, \dots$), and so

$$\mu_1, \mu_2, \dots, \mu_{2m}$$

is a positive definite sequence in the interval $(-\infty, \infty)$. ■

2.2 The Quadratic Form of Hankel

In this section we make use of the quadratic form of Hankel to obtain some important properties of the moments of the spectral function $\tau(\lambda)$.

Definition 2.2.1 *A real quadratic form in $x_1, x_2, \dots, x_n$ is an expression that can be written as*

$$(x_1, x_2, \dots, x_n)^T A (x_1, x_2, \dots, x_n),$$

where A is a symmetric matrix.

Note that the matrix $A = (a_{ij})$ is positive definite if and only if the corresponding quadratic form

$$(x_1, x_2, \dots, x_n)^T A (x_1, x_2, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$$

is positive definite (in the sense of quadratic forms [6, chapter 4]).

Theorem 2.2.1 *The sequence (2.1.1) is positive definite in the interval $(-\infty, \infty)$ if and only if the quadratic form of Hankel, viz.,*

$$\sum_{i=0}^m \sum_{j=0}^m s_{i+j} x_i x_j$$

is positive, where $m = [n/2]$, (denotes the largest integer not exceeding $n/2$).

Proof : See [1, p.3; Theorem 1]. ■

Theorem 2.2.2 *Let λ_i and ρ_i ($0 \leq i \leq m-1$) be the eigenvalues and the normalization constants of the generalized eigenvalue problem (1.2.3). Let $\lambda_i \rho_i > 0$ ($0 \leq i \leq m-1$) and let*

$$\mu_1, \mu_2, \dots, \mu_{2m-3}$$

be a sequence of moments given by (1.4.5). Then the quadratic form

$$\sum_{i=0}^{m-2} \sum_{j=0}^{m-2} \mu_{i+j+1} x_i x_j$$

is positive definite.

Proof : By Theorem 2.1.2, the sequence

$$\mu_1, \mu_2, \dots, \mu_{2m-3}$$

is positive definite (since it is a "subsequence" of a positive definite sequence on $(-\infty, \infty)$). By Theorem 2.2.1, the quadratic form

$$\sum_{i=0}^{m-2} \sum_{j=0}^{m-2} \mu_{i+j+1} x_i x_j$$

is positive (since $[n/2] = [(2m-3)/2] = m-2$). ■

Recall that, by Lemma 1.1.1, the quadratic form

$$\sum_{i=0}^m \sum_{j=0}^m s_{i+j} x_i x_j$$

is positive if and only if the determinants of the principal submatrices are positive, i.e

$$\begin{vmatrix} s_0 & s_1 & \cdots & s_k \\ s_1 & s_2 & \cdots & s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_k & s_{k+1} & \cdots & s_{2k} \end{vmatrix} > 0$$

for $k = 0, 1, \dots, m$.

Finally we end this chapter with the following important theorem, which plays a crucial role to prove that the Jacobi matrix A in Theorem 3.2.2 is positive definite.

Theorem 2.2.3 Let λ_i and ρ_i ($0 \leq i \leq m-1$) be the eigenvalues and the normalization constants of the generalized eigenvalue problem (1.2.3). Let $\lambda_i \rho_i > 0$ ($0 \leq i \leq m-1$) and let

$$\mu_1, \mu_2, \dots, \mu_{2m-3}$$

be the sequence of moments given by (1.4.5). Then

$$\begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \end{vmatrix} > 0$$

for $n = 1, 2, \dots, m-1$.

Proof : By Theorem 2.2.1, the quadratic form

$$\sum_{i=0}^{m-2} \sum_{j=0}^{m-2} \mu_{i+j+1} x_i x_j$$

is positive definite. Then, the determinants of the principal submatrices are positive, i.e.,

$$\begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \end{vmatrix} > 0$$

for $n = 1, 2, \dots, m-1$. ■

Chapter 3

A SOLUTION FOR AN INVERSE GENERALIZED EIGENVALUE PROBLEM

In this chapter, unless otherwise stated, B is assumed to be a fixed nonsingular indefinite diagonal real $m \times m$ matrix. We seek an $m \times m$ positive definite Jacobi matrix A which solves the inverse generalized eigenvalue problem $Ay = \lambda By$.

3.1 Construction of a Solution

In this section we construct a Jacobi matrix A which solves the inverse generalized eigenvalue problem.

Theorem 3.1.1 *Let $\tau(\lambda)$ be the spectral function defined by (1.4.1), and let $B = \text{diag}(b_0, b_1, \dots, b_{m-1})$ be a nonsingular indefinite real $m \times m$ matrix. If the condition*

$$b_n \Delta_n \Delta_{n+1} > 0 \text{ for } 0 \leq n \leq m-1 \quad (3.1.1)$$

holds then there exist m real polynomials $y_n(\lambda)$ of degree n ($0 \leq n \leq m-1$) which satisfy the dual orthogonality property (1.4.2), and the polynomials $y_n(\lambda)$ are determined up to a change of sign.

Proof: We seek polynomials of the form

$$y_n(\lambda) = k_n(\lambda^n + \sum_{r=0}^{n-1} \alpha_{nr} \lambda^r), \quad 0 \leq n \leq m-1, \quad (3.1.2)$$

where $y_0(\lambda) = k_0$ and $k_n \neq 0$. It follows from the equations (1.4.2) and (1.4.4) that

$$\int_{-\infty}^{\infty} (y_0(\lambda))^2 d\tau(\lambda) = b_0^{-1} \text{ or } k_0^2 \mu_0 = b_0^{-1}.$$

Hence

$$k_0^2 = 1/\mu_0 b_0 = \Delta_0/b_0 \Delta_1, \quad (3.1.3)$$

which is positive by hypothesis. It follows from (3.1.2) that

$$(y_n(\lambda))^2 = k_n^2 [\lambda^{2n} + \sum_{r=0}^{n-1} \alpha_{nr} \lambda^{n+r}] + k_n y_n(\lambda) \sum_{r=0}^{n-1} \alpha_{nr} \lambda^r, \quad 1 \leq n \leq m-1. \quad (3.1.4)$$

It follows from (1.4.2), (1.4.3) and (3.1.4) that

$$b_n^{-1} = k_n^2 (\mu_{2n} + \sum_{r=0}^{n-1} \alpha_{nr} \mu_{n+r}). \quad (3.1.5)$$

This gives the k_n in terms of the α_{nr} and other given spectral data. Now we determine α_{nr} . By equation (1.4.3) we know that

$$\int_{-\infty}^{\infty} y_n(\lambda) \lambda^r d\tau(\lambda) = 0, \quad (3.1.6)$$

where $1 \leq n \leq m-1$ and $0 \leq r \leq n-1$. Substituting $y_n(\lambda)$ given by (3.1.2) into the equation (3.1.6) and using the equation (1.4.4), we obtain

$$\mu_{n+r} + \sum_{i=0}^{n-1} \alpha_{ni} \mu_{r+i} = 0, \quad (3.1.7)$$

where $1 \leq n \leq m-1$ and $0 \leq r \leq n-1$. Using matrix notation this means

$$M_n(\alpha_{n0}, \alpha_{n1}, \dots, \alpha_{n,n-1})^T = (-\mu_n, -\mu_{n+1}, \dots, -\mu_{2n-1})^T. \quad (3.1.8)$$

It follows from this equation that

$$\alpha_{nr} = (\sum_{k=1}^n (-1)^{k+r} \mu_{n-1+k} A_{r+1,k}) / \Delta_n, \quad (3.1.9)$$

where $1 \leq n \leq m-1$ and $0 \leq r \leq n-1$. Substituting the equation (1.4.8) into the equation (3.1.9), we obtain

$$\Delta_{n+1} = \Delta_n (\mu_{2n} + \sum_{r=0}^{n-1} \alpha_{nr} \mu_{n+r}). \quad (3.1.10)$$

Using equation (3.1.5) in equation (3.1.10) we obtain

$$k_n^2 = \Delta_n / \Delta_{n+1} b_n \quad (3.1.11)$$

which is positive by the hypothesis for $1 \leq n \leq m-1$. Hence by (3.1.3) and (3.1.11)

$$k_n^2 = \Delta_n / \Delta_{n+1} b_n, \quad 0 \leq n \leq m-1. \quad (3.1.12)$$

This completes the proof. ■

Theorem 3.1.2 *Let $\tau(\lambda)$ be a spectral function defined by the expression (1.4.1), and let $B = \text{diag}(b_0, b_1, \dots, b_{m-1})$ be a nonsingular indefinite real $m \times m$ matrix. If the condition*

$$b_n \Delta_n \Delta_{n+1} > 0, \quad 0 \leq n \leq m-1 \quad (3.1.13)$$

holds then the Jacobi matrix A of the form

$$A = \begin{bmatrix} a_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_0 & a_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_1 & a_2 & c_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_{m-3} & a_{m-2} & c_{m-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_{m-2} & a_{m-1} \end{bmatrix} \quad (3.1.14)$$

given by

$$a_0 = -b_0 \alpha_{1,0} \quad (3.1.15)$$

$$a_n = b_n (\alpha_{n,n-1} - \alpha_{n+1,n}), \quad 1 \leq n \leq m-2 \quad (3.1.16)$$

$$a_{m-1} = b_{m-1} (\alpha_{m-1,m-2} + \sigma_{m-1}) \quad (3.1.17)$$

$$\alpha_{nr} = \left(\sum_{k=1}^n (-1)^{k+r} \mu_{n-1+k} A_{r+1,k} \right) / \Delta_n, \quad 1 \leq n \leq m-1 \text{ and } 0 \leq r \leq n-1$$

$$c_n = b_n k_n / k_{n+1}, \quad 0 \leq n \leq m-2 \quad (3.1.18)$$

$$k_n^2 = \Delta_n / \Delta_{n+1} b_n, \quad 0 \leq n \leq m-1$$

is a solution for an inverse generalized eigenvalue problem $Ay = \lambda By$ which assumes $\tau(\lambda)$ as its spectral function, where

$$y(\lambda) = (y_0(\lambda), y_1(\lambda), \dots, y_{m-1}(\lambda))^T, \quad (3.1.19)$$

$$y_n(\lambda) = k_n(\lambda^n + \sum_{r=0}^{n-1} \alpha_{nr} \lambda^r), \quad 0 \leq n \leq m-1, \quad (3.1.20)$$

and A_{ij} is the determinant of the matrix obtained from M_n by deleting row i and column j , where $i, j = 1, 2, \dots, m-1$. If $n = 1$ we set $A_{1,1} = 1$.

Proof: By the recurrence relation (1.2.1), we have

$$c_n y_{n+1}(\lambda) + c_{n-1} y_{n-1}(\lambda) + a_n y_n(\lambda) = \lambda b_n y_n(\lambda), \quad 0 \leq n \leq m-1. \quad (3.1.21)$$

By substituting (3.1.2) for $n = 0, 1$ into the equation (3.1.21) we obtain

$$c_0 k_1 (\lambda + \alpha_{1,0}) = (b_0 \lambda - a_0) k_0.$$

Equating the coefficients of λ and the constant terms in the above equation gives

$$c_0 = b_0 k_0 / k_1 \text{ and } a_0 = -b_0 \alpha_{1,0},$$

respectively. Now we substitute equation (3.1.2) into equation (3.1.21), and equate the coefficients of λ^{n+1} and λ^n . Then we obtain

$$c_n k_{n+1} = b_n k_n \text{ and}$$

$$c_n k_{n+1} \alpha_{n+1,n} + a_n k_n = b_n k_n \alpha_{n,n-1},$$

respectively, where $1 \leq n \leq m-2$. In this pair of equations by substituting the first equation into the second equation we have, since $k_n \neq 0$ for any n ,

$$a_n = b_n (\alpha_{n,n-1} - \alpha_{n+1,n}), \quad 1 \leq n \leq m-2.$$

Thus we obtain

$$c_n = b_n k_n / k_{n+1},$$

$$a_n = b_n (\alpha_{n,n-1} - \alpha_{n+1,n})$$

where $0 \leq n \leq m-2$ and $\alpha_{0,-1} = 0$.

Finally we need to determine a_{m-1} in order to determine the matrix A completely. To do this we have to determine $y_m(\lambda)$ and c_{m-1} so that $y_m(\lambda)$ has $\lambda_0, \lambda_1, \dots, \lambda_{m-1}$ as its zeros and

$$c_{m-1}y_m(\lambda) = (b_{m-1}\lambda - a_{m-1})y_{m-1} - c_{m-2}y_{m-2}(\lambda). \quad (3.1.22)$$

Hence we define $y_m(\lambda)$ by (see Theorem 1.3.2)

$$\begin{aligned} y_m(\lambda) &= (\lambda - \lambda_0)(\lambda - \lambda_1) \cdots (\lambda - \lambda_{m-1}) \\ &= \lambda^m - \sigma_{m-1}\lambda^{m-1} + \cdots + (-1)^m \sigma_0, \end{aligned} \quad (3.1.23)$$

where $\sigma_{m-1} = \lambda_0 + \lambda_1 + \cdots + \lambda_{m-1}$ and $\sigma_0 = \lambda_0\lambda_1 \cdots \lambda_{m-1}$, so that $y_m(\lambda)$ will have the correct zeros. Now substituting $n = m-1, m-2$ into equation (3.1.2), (3.1.23) into (3.1.22), and equating the coefficients of λ^m and λ^{m-1} , we obtain

$$c_{m-1} = b_{m-1}k_{m-1} \quad (3.1.24)$$

$$-\sigma_{m-1}c_{m-1} + a_{m-1}k_{m-1} = b_{m-1}k_{m-1}\alpha_{m-1,m-2}, \quad (3.1.25)$$

respectively. Then by using the equation (3.1.24) in the last equation, we obtain

$$a_{m-1} = b_{m-1}(\alpha_{m-1,m-2} + \sigma_{m-1}).$$

This completes the proof. ■

Note that the matrix A found in Theorem 3.1.2 is not unique. The following theorem gives the number of such distinct matrices.

Theorem 3.1.3 *Let $\tau(\lambda)$ be the spectral function defined by the expression (1.4.1), and let $B = \text{diag}(b_0, b_1, \dots, b_{m-1})$ be a nonsingular indefinite real $m \times m$ matrix. Assume that $b_n \Delta_n \Delta_{n+1} > 0$ holds for $0 \leq n \leq m-1$. Then there are 2^{m-1} distinct Jacobi matrices A of the form (3.1.14) that solve the inverse generalized eigenvalue problem $Ay = \lambda By$ which assumes $\tau(\lambda)$ as its spectral function.*

Proof: By equation (3.1.12), we see that there are two possible signs for k_n . Since the diagonal entries a_n of A do not depend on k_n then the number of the distinct matrices A depends only on the c_n (see (3.1.16) and (3.1.18)). Hence we obtain

$$\binom{m-1}{0} + \binom{m-1}{1} + \cdots + \binom{m-1}{m-1} = 2^{m-1}$$

distinct matrices A depending upon the choice of the c_n . ■

In Theorem 3.1.3 we determined the number of distinct Jacobi matrices A which solves the inverse generalized eigenvalue problem. Now in the next theorem we classify those Jacobi matrices in terms of an Abelian group G of order 2^{m-1} with exponent 2.

Theorem 3.1.4 *Let the notation be as in Theorem 3.1.2, and let A be the Jacobi matrix with the corresponding eigenvector*

$$y(\lambda) = (y_0(\lambda), y_1(\lambda), \dots, y_{m-1}(\lambda))^T$$

for which all the k_n are positive. Let G be the Abelian group of order 2^{m-1} given by

$$G = \{ \text{diag}(1, d_1, d_2, \dots, d_{m-1}) \mid d_i = \pm 1 \} .$$

Then for every matrix $D \in G$, DAD also solves the inverse generalized eigenvalue problem; in other words, all the solutions of the inverse generalized eigenvalue problem are classified as the set of conjugates of A with respect to G .

Proof: Let $D \in G$. Set $z(\lambda) = Dy(\lambda)$. Then, using $D^2 = I$, $BD = DB$ and $D^T = D$ we obtain

$$DADz(\lambda) = DADDy(\lambda) = DAy(\lambda) = D\lambda By(\lambda) = \lambda BDy(\lambda) = \lambda Bz(\lambda),$$

and

$$\rho_i = y(\lambda_i)^T By(\lambda_i) = (Dz(\lambda_i))^T BDz(\lambda_i) = z(\lambda_i)^T Bz(\lambda_i) .$$

Hence DAD is a solution with the eigenvector $z(\lambda) = Dy(\lambda)$. ■

Note that the group G in the above theorem is isomorphic to the group

$$Z_2 \oplus Z_2 \oplus \cdots \oplus Z_2, \text{ (m-1 summands) ,}$$

where Z_2 denotes the group of order 2. See [19, p. 63].

Atkinson solved the inverse generalized eigenvalue problem $Ay = \lambda By$ where $B = \text{diag}(b_0, b_1, \dots, b_{m-1})$ is a positive definite $m \times m$ matrix. (see [2, Chapter 4]). In that case the normalization constants ρ_r ($0 \leq r \leq m-1$) are positive (see (1.2.7)). Hence spectral function $\tau(\lambda)$ (see (1.4.1)) is a nondecreasing step function with m points of increase, and so by equation (1.4.2)

$$\int_{-\infty}^{\infty} (y_n(\lambda))^2 d\tau(\lambda) = b_n^{-1} > 0, \quad 0 \leq n \leq m-1,$$

and by equation (3.1.11)

$$b_n k_n^2 = \frac{\Delta_n}{\Delta_{n+1}}, \quad 0 \leq n \leq m-1.$$

Since $\Delta_0 = 1$, it follows that $\Delta_n > 0$ for $n \geq 1$. So the conditions

$$b_n \Delta_n \Delta_{n+1} > 0, \quad 0 \leq n \leq m-1$$

are automatically satisfied. In this case Atkinson has shown that there exist unique polynomials $y_0(\lambda), y_1(\lambda), \dots, y_{m-1}(\lambda)$ satisfying equation (1.4.2), and such that $y_n(\lambda)$ has degree n , with the requirement that the coefficient k_n of λ^n in the polynomial $y_n(\lambda)$ ($0 \leq n \leq m-1$) be positive.

It follows that since

$$c_n = b_n k_n / k_{n+1} \quad (0 \leq n \leq m-2), \quad c_{m-1} = b_{m-1} k_{m-1},$$

all the c_n are positive. So he ends up with a *unique* solution matrix A which is not positive definite unless all the elements of $\sigma(A, B)$ are positive. Indeed he could take k_n of any sign, and he could have obtained 2^{m-1} distinct solutions for $m \times m$ case, just as we did.

In our case, even if all the k_n are positive we still have some negative c_n since the matrix B is indefinite. But we can choose the sign of k_n in a way that we can have all the c_n positive (see (3.1.18)). (Recall that in the solution of Atkinson the c_n are necessarily positive.) Since the matrix A is positive definite all the diagonal entries a_n ($0 \leq n \leq m-1$) are positive. Thus we have a unique Jacobi matrix A such that all the entries of A are positive.

We present an example of Atkinson type in Section 3.4.

3.2 A Positive Definite Solution

In this section we show that the matrix A found in Theorem 3.1.2 is, in fact, **positive definite** given some additional assumptions on $\sigma(A, B)$ and $\sigma(B)$.

Theorem 3.2.1 *Let A be the matrix obtained in Theorem 3.1.2. Then the determinants D_n of the leading principal submatrices of A are*

$$D_n = b_0 b_1 \cdots b_{n-1} (-1)^n \alpha_{n,0} \quad \text{for } 1 \leq n \leq m-1. \quad (3.2.1)$$

Proof: Let A be the matrix found in Theorem 3.1.2. In equation (1.2.1), by replacing n by $n-1$, we obtain

$$c_{n-1} y_n(\lambda) = (b_{n-1} \lambda - a_{n-1}) y_{n-1}(\lambda) - c_{n-2} y_{n-2}(\lambda). \quad (3.2.2)$$

Substituting equation (3.1.2) into the previous equation, and equating the constant terms give

$$c_{n-1} k_n \alpha_{n,0} = -a_{n-1} k_{n-1} \alpha_{n-1,0} - c_{n-2} k_{n-2} \alpha_{n-2,0}.$$

Substituting equation (3.1.16) with n replaced by $n-1$ into the previous equation gives

$$c_{n-1} k_n \alpha_{n,0} = -b_{n-1} (\alpha_{n-1,n-2} - \alpha_{n,n-1}) k_{n-1} \alpha_{n-1,0} - c_{n-2} k_{n-2} \alpha_{n-2,0}$$

Replacing n by $n-1$ in equation (3.1.18) and substituting it into the previous equation we obtain

$$\alpha_{n,0} = -\alpha_{n-1,0}(\alpha_{n-1,n-2} - \alpha_{n,n-1}) - \frac{c_{n-2}k_{n-2}}{c_{n-1}k_n}\alpha_{n-2,0}.$$

Again using the equations (3.1.18) and (3.1.12) for appropriate indices, we find

$$\alpha_{n,0} = -\alpha_{n-1,0}(\alpha_{n-1,n-2} - \alpha_{n,n-1}) - \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}\alpha_{n-2,0}. \quad (3.2.3)$$

Now we prove by induction that

$$D_n = b_0b_1 \cdots b_{n-1}(-1)^n\alpha_{n,0}. \quad (3.2.4)$$

Clearly $D_1 = a_0$ and by equation (3.1.15), $a_0 = -b_0\alpha_{1,0}$, and so $D_1 = (-1)b_0\alpha_{1,0}$. Using equation (3.1.18) with n replaced by $n-2$ and equation (3.1.12) with n replaced by $n-2$ and $n-1$, we see that

$$c_{n-2}^2 = \frac{b_{n-2}b_{n-1}\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}. \quad (3.2.5)$$

By the induction hypothesis we also assume that the equation (3.2.4) is true for n replaced by $n-1$. Then we have

$$\begin{aligned} D_n &= \begin{vmatrix} a_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_0 & a_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_1 & a_2 & c_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_{n-3} & a_{n-2} & c_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_{n-2} & a_{n-1} \end{vmatrix} \quad (3.2.6) \\ &= a_{n-1}D_{n-1} - c_{n-2}^2D_{n-2} \\ &= b_{n-1}(\alpha_{n-1,n-2} - \alpha_{n,n-1})b_0b_1 \cdots b_{n-2}(-1)^{n-1}\alpha_{n-1,0} \\ &\quad - \frac{b_{n-2}b_{n-1}\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}b_0b_1 \cdots b_{n-3}(-1)^{n-2}\alpha_{n-2,0} \\ &= b_0b_1 \cdots b_{n-1}(-1)^{n-1}[\alpha_{n-1,0}(\alpha_{n-1,n-2} - \alpha_{n,n-1}) + \alpha_{n-2,0}\frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}] \\ &= b_0b_1 \cdots b_{n-1}(-1)^n\alpha_{n,0} \quad (\text{by (3.2.3)}). \end{aligned}$$

Hence the proof is complete by induction. ■

Theorem 3.2.2 *Let A be the matrix obtained in Theorem 3.1.2. Then A is positive definite if and only if*

i) $\lambda_i \rho_i > 0$, $0 \leq i \leq m-1$ and

ii) *the number of negative elements of $\sigma(A, B)$ is equal to the number of negative elements of $\sigma(B)$.*

Proof: Let A be the matrix found in Theorem 3.1.2. Suppose i) and ii) are satisfied.

We will show that A is positive definite. By Theorem 3.2.1, we have

$$D_n = b_0 b_1 \cdots b_{n-1} (-1)^n \alpha_{n,0}.$$

This can be rewritten as

$$D_n = \frac{b_0 \Delta_0}{\Delta_1} \frac{b_1 \Delta_1}{\Delta_2} \cdots \frac{b_{n-1} \Delta_{n-1}}{\Delta_n} (-1)^n \alpha_{n,0} \Delta_n. \quad (3.2.7)$$

On the other hand, using equation (3.1.8) we obtain

$$(-1)^n \alpha_{n,0} \Delta_n = \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \end{vmatrix}. \quad (3.2.8)$$

Hence, $D_n > 0$ if and only if (because of (3.1.1))

$$\begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \end{vmatrix} > 0,$$

which is true by Theorem 2.2.3.

Finally, it is easy to see that (comparing (3.2.6) and (3.1.14))

$$D_m = |A| = b_0 b_1 \cdots b_{m-1} \lambda_0 \lambda_1 \cdots \lambda_{m-1}. \quad (3.2.9)$$

Since the number of negative eigenvalues of $\sigma(A, B)$ is equal to the number of negative eigenvalues of $\sigma(B)$ we obtain that $D_m > 0$. Hence by Lemma 1.1.1, A is positive definite.

The converse of the theorem follows from Theorem 1.3.1. ■

3.3 A Positive Definite Solution when B is Singular

In this section we solve the inverse generalized eigenvalue problem $Ay = \lambda By$, when B is singular. For the sake of the simplicity of the calculations we present the solution when $B = \text{diag}(b_0, b_1, \dots, b_{m-2}, b_{m-1})$ with $b_0, b_1, \dots, b_{m-2}$ are nonzero and $b_{m-1} = 0$. All the other cases can be treated similarly.

Theorem 3.3.1 *Let $\tau(\lambda)$ be the spectral function defined by the expression (1.4.1), and let $B = \text{diag}(b_0, b_1, \dots, b_{m-1})$ be an indefinite real $m \times m$ matrix of rank $m - 1$ with $b_{m-1} = 0$. If the condition*

$$b_n \Delta_n \Delta_{n+1} > 0, \quad 0 \leq n \leq m - 2 \quad (3.3.1)$$

holds then the Jacobi matrix A of the form

$$A = \begin{bmatrix} a_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_0 & a_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_1 & a_2 & c_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_{m-3} & a_{m-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{m-1} \end{bmatrix} \quad (3.3.2)$$

given by

$$a_0 = -b_0 \alpha_{1,0} \quad (3.3.3)$$

$$a_n = b_n (\alpha_{n,n-1} - \alpha_{n+1,n}), \quad 1 \leq n \leq m-2 \quad (3.3.4)$$

$$a_{m-2} = b_{m-2}(\alpha_{m-2,m-3} + \sigma_{m-2}) \quad (3.3.5)$$

$$\alpha_{ns} = \left(\sum_{k=1}^n (-1)^{k+s} \mu_{n-1+k} A_{s+1,k} \right) / \Delta_n, \quad 1 \leq n \leq m-2 \text{ and } 0 \leq s \leq n-1$$

$$c_n = b_n k_n / k_{n+1}, \quad 0 \leq n \leq m-3 \quad (3.3.6)$$

$$k_n^2 = \Delta_n / \Delta_{n+1} b_n, \quad 0 \leq n \leq m-2$$

is a solution for an inverse generalized eigenvalue problem $Ay = \lambda By$ which assumes $\tau(\lambda)$ as its spectral function, where

$$y(\lambda) = (y_0(\lambda), y_1(\lambda), \dots, y_{m-2}(\lambda), 0)^T \quad (3.3.7)$$

and

$$y_n(\lambda) = k_n(\lambda^n + \sum_{s=0}^{n-1} \alpha_{ns} \lambda^s), \quad 0 \leq n \leq m-2. \quad (3.3.8)$$

The matrix A is positive definite if and only if

i) $\lambda_i \rho_i > 0$, $0 \leq i \leq m-2$ and

ii) the number of negative elements of $\sigma(A, B)$ is equal to the number of negative elements of $\sigma(B)$.

Proof: Let $B_1 = \text{diag}(b_0, b_1, \dots, b_{m-2})$. Since B_1 is nonsingular we can apply Theorem 3.1.2 to obtain a Jacobi matrix A_1 with the eigenvector $z(\lambda) = (z_0(\lambda), z_1(\lambda), \dots, z_{m-2}(\lambda))$. Then the matrix

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & a_{m-1} \end{bmatrix} \quad (3.3.9)$$

with the eigenvector $y(\lambda) = (z(\lambda), 0)$ is a solution of the inverse generalized eigenvalue problem, where a_{m-1} is any real number. By Theorem 3.2.2, the matrix A_1 is positive definite. Hence, the matrix A is positive definite if $a_{m-1} > 0$. ■

We illustrate Theorem 3.3.1 with the Example 3.4.4.

3.4 Numerical Examples

In this section we give two numerical examples to illustrate Theorem 3.1.2 and Theorem 3.1.3 for the cases $m = 3$ and 4, and one example to discuss the uniqueness of the matrix of Atkinson.

Example 3.4.1 *Let $m = 3$ in Theorem 3.1.2 and let $B = \text{diag}(1, -1, 1)$. Let*

$$\sigma(A, B) = \{\lambda_0 = -1, \lambda_1 = 2, \lambda_2 = 3\}$$

and let the corresponding normalization constants be

$$\rho_0 = -1, \rho_1 = 1, \rho_2 = 1.$$

Then:

$$b_0 \Delta_0 \Delta_1 = 1,$$

$$b_1 \Delta_1 \Delta_2 = 24,$$

$$b_2 \Delta_2 \Delta_3 = 3456.$$

So the condition (3.1.13) holds. By Theorem 3.1.2,

$$k_0^2 = 1, k_1^2 = 1/24, k_2^2 = 1/6.$$

and

$$a_0 = 6, a_1 = 9/2, a_2 = 5/2,$$

$$c_0 = k_0/k_1, c_1 = -k_1/k_2.$$

We use Theorem 3.1.4 to determine all the Jacobi matrices which solve the inverse generalized eigenvalue problem. First we find the Jacobi matrix A for which

$$k_0, k_1 \text{ and } k_2 > 0.$$

By Theorem 3.1.2, the corresponding Jacobi matrix is

$$A = \begin{bmatrix} 6 & 2\sqrt{6} & 0 \\ 2\sqrt{6} & 9/2 & -1/2 \\ 0 & -1/2 & 5/2 \end{bmatrix} .$$

The corresponding eigenvector is given by

$$y(\lambda) = (1, (\lambda - 6)/2\sqrt{6}, (2\lambda^2 - 3\lambda - 6)/2\sqrt{6})^T .$$

Substituting the generalized eigenvalues

$$\lambda_0 = -1 , \lambda_1 = 2 , \lambda_2 = 3$$

into the previous equation we obtain the following eigenvectors:

$$(1, -7/2\sqrt{6}, -1/2\sqrt{6})^T ,$$

$$(1, -2/\sqrt{6}, -2/\sqrt{6})^T ,$$

$$(1, -3/2\sqrt{6}, 3/2\sqrt{6})^T ,$$

for the matrix A . It is easy to see that

$$D_1 = 6 , D_2 = 3 , D_3 = 6 .$$

Hence by Lemma 1.1.1, the matrix A is positive definite. Note that the equations

$$\rho_i = \sum_{p=0}^2 b_p (y_p(\lambda_i))^2 ; i = 0, 1, 2$$

are also satisfied.

By Theorem 3.1.4, all the solutions are given by DAD with the eigenvector $z(\lambda) = Dy(\lambda)$, where $D \in G$ and

$$G = \{diag(1, 1, 1), diag(1, 1, -1), diag(1, -1, 1), diag(1, -1, -1)\} .$$

Example 3.4.2 Let $m = 4$ in Theorem 3.1.2 and let $B = \text{diag}(1, -1, 2, -2)$. Let

$$\sigma(A, B) = \{\lambda_0 = 2, \lambda_1 = -2, \lambda_2 = 5, \lambda_3 = -1\}$$

and let the corresponding normalization constants be

$$\rho_0 = 1, \rho_1 = -1, \rho_2 = 2, \rho_3 = -3.$$

Then:

$$b_0 \Delta_0 \Delta_1 = 1/6,$$

$$b_1 \Delta_1 \Delta_2 = 67/9,$$

$$b_2 \Delta_2 \Delta_3 = 328032,$$

$$b_3 \Delta_3 \Delta_4 = 2798240256.$$

So the condition (3.1.13) holds. By Theorem 3.1.2,

$$k_0^2 = 6, k_1^2 = 1/268, k_2^2 = 67/5427, k_3^2 = 17/3528.$$

and

$$a_0 = 41, a_1 = 2642/67, a_2 = 26542/3417, a_3 = 148/51,$$

$$c_0 = k_0/k_1, c_1 = -k_1/k_2, c_2 = 2k_2/k_3, c_3 = -2k_3.$$

We use Theorem 3.1.4 to determine all the Jacobi matrices which solve the inverse generalized eigenvalue problem. First we find the Jacobi matrix A for which

$$k_0, k_1, k_2 \text{ and } k_3 > 0.$$

By Theorem 3.1.2, the corresponding Jacobi matrix is

$$A = \begin{bmatrix} 41 & \sqrt{1608} & 0 & 0 \\ \sqrt{1608} & 2642/67 & -9\sqrt{34}/67 & 0 \\ 0 & -9\sqrt{34}/67 & 26542/3417 & 14\sqrt{67}/51 \\ 0 & 0 & 14\sqrt{67}/51 & 148/51 \end{bmatrix}.$$

The corresponding eigenvector is given by

$$y(\lambda) = (\sqrt{6}, (1/2\sqrt{67})(\lambda - 41), (1/18\sqrt{2278})(67\lambda^2 - 105\lambda - 586), \\ (1/6\sqrt{714})(51\lambda^3 - 278\lambda^2 - 120\lambda + 1091))^T.$$

Substituting the generalized eigenvalues

$$\lambda_0 = 2, \lambda_1 = -2, \lambda_2 = 5, \lambda_3 = -1$$

into the previous equation we obtain the following eigenvectors:

$$y(2) = (\sqrt{6}, \frac{-39}{2\sqrt{67}}, \frac{-88}{3\sqrt{2278}}, \frac{49}{2\sqrt{714}})^T,$$

$$y(-2) = (\sqrt{6}, \frac{-43}{2\sqrt{67}}, \frac{-6}{\sqrt{2278}}, \frac{-63}{2\sqrt{714}})^T,$$

$$y(5) = (\sqrt{6}, \frac{-18}{\sqrt{67}}, \frac{94}{3\sqrt{2278}}, \frac{-14}{\sqrt{714}})^T,$$

$$y(-1) = (\sqrt{6}, \frac{-21}{\sqrt{67}}, \frac{-23}{\sqrt{2278}}, \frac{147}{\sqrt{714}})^T$$

for the matrix A . It can be seen that

$$D_1 = 41, D_2 = 586/67, D_3 = 32182/51, D_4 = 80.$$

Thus by Lemma 1.1.1, the matrix A is positive definite. Note that the equations

$$\rho_r = \sum_{p=0}^3 b_p(y_p(\lambda_r))^2; \quad r = 0, 1, 2, 3$$

are also satisfied. By Theorem 3.1.4, all the solutions are given by DAD with the eigenvector $z(\lambda) = Dy(\lambda)$, where $D \in G$ and

$$G = \{\text{diag}(1, d_1, d_2, d_3) \mid d_1, d_2, d_3 = \mp 1\}.$$

Example 3.4.3 *In this example we discuss the uniqueness of the solution of Atkinson's matrix in the 2×2 case. Let $B = \text{diag}(1, 2)$, $\sigma(A, B) = \{-2, 3\}$, $\rho(A, B) = \{1, 1/2\}$. Then,*

$$\Delta_1 = 3, \Delta_2 = 50, k_0^2 = 1/3, k_1^2 = 3/100, \alpha_{1,0} = -4/3$$

We have two possibilities:

- i) $k_0 > 0$ and $k_1 > 0$,
- ii) $k_0 > 0$ and $k_1 < 0$.

The first case is the case that Atkinson used. Let us consider each possibility in turn:

$$\text{i) } k_0 = 1/\sqrt{3}, k_1 = \sqrt{3}/10.$$

In this case the corresponding Jacobi matrix is

$$A_1 = \begin{bmatrix} 4/3 & 10/3 \\ 10/3 & -2/3 \end{bmatrix}.$$

The corresponding eigenvector is given by

$$y(\lambda) = \left(\frac{1}{\sqrt{3}}, \frac{3\lambda - 4}{10\sqrt{3}} \right)^T.$$

Substituting the generalized eigenvalues $\lambda_0 = -2$ and $\lambda_1 = 3$ in to the previous equation we obtain the following eigenvectors, respectively:

$$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)^T, \left(\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}} \right)^T.$$

$$\text{ii) } k_0 = 1/\sqrt{3}, k_1 = -\sqrt{3}/10.$$

In this case the corresponding Jacobi matrix is

$$A_2 = \begin{bmatrix} 4/3 & -10/3 \\ -10/3 & -2/3 \end{bmatrix}.$$

The corresponding eigenvector is given by

$$y(\lambda) = \left(\frac{1}{\sqrt{3}}, -\frac{3\lambda - 4}{10\sqrt{3}} \right)^T.$$

Substituting the generalized eigenvalues $\lambda_0 = -2$ and $\lambda_1 = 3$ in to the previous equation we obtain the following eigenvectors:

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T, \left(\frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right)^T,$$

respectively. Note that

$$\det A_1 = \det A_2 = -12 < 0,$$

so A_1, A_2 are not positive definite.

Example 3.4.4 Let $m = 3$ in Theorem 3.4.1 and let $B = \text{diag}(1, -1, 0)$. Let

$$\sigma(A, B) = \{\lambda_0 = 2, \lambda_1 = -1\}$$

and let the corresponding normalization constants be

$$\rho_0 = 1/2, \rho_1 = -2.$$

Recall that since B is of rank 2, both $\sigma(A, B)$ and $\rho(A, B)$ have exactly 2 elements.

Set $B_1 = \text{diag}(1, -1)$. Then B_1 is an indefinite nonsingular 2×2 matrix. Hence we can apply Theorem 3.1.2 to obtain a positive definite Jacobi matrix A_1 with the eigenvector

$$z(\lambda) = (z_0(\lambda), z_1(\lambda))$$

such that $A_1 z(\lambda) = \lambda B_1 z(\lambda)$. After a calculation we find that

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & a \end{bmatrix}$$

with the eigenvector

$$y(\lambda) = (z(\lambda), 0) = (2/\sqrt{6}, (\lambda - 3)/\sqrt{6}, 0)$$

is a solution for the inverse generalized eigenvalue problem $Ay = \lambda By$, where $a > 0$.

Conclusion

In this thesis the theory of positive definite sequences was introduced (according to Ahiezer and Krein [1]). This theory was applied in order to solve the inverse generalized eigenvalue problem $Ay = \lambda By$ when B is an indefinite real diagonal $m \times m$ matrix, and A is positive definite. Future work will focus on solving the inverse generalized eigenvalue problem when B is a symmetric $m \times m$ matrix.

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