Fourier-Bessel Analysis of Cyllindrically Symmetric
Dielectric Structures, Slot Channel Refractive Index
Sensors

by

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Abstract

Maxwell's wave equations can be solved using different techniques in order to extract optical properties of a variety of dielectric structures. For structures that contain an extended axis, for instance cylindrical symmetry, it is shown that an expansion of the fields and inverse of relative permittivity using a set of basis functions of Fourier-Bessel terms, provide access to an eigenvalue formulation from which the eigen-states can be computed. For cylindrically symmetric structures the computational technique provides a significantly reduced matrix order to be populated when compared to the plane wave method applied to these structures. The steps used to convert Maxwell's equation into an eigenvalue formulation are discussed and the technique is tested on various dielectric structures. Several novel refractive index sensors based on slot channel configuration are presented and the sensitivity of each structure is discussed. Using the Fourier-Bessel technique the eigenvalues are calculated and the slot channel field profiles are plotted based on the computed eigenvectors.

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List of Symbols and abbreviations

Most symbols define their normal meaning in the equations.

FFB	Fourier-Fourier-Bessel
PWEM	Plane wave Expansion Method
FDTD	Finite Difference Time Domain
FEM	Finite Element Method
TE	Transverse electric
TM	Transverse magnetic
Ω	Omega; Inverse of Relative Permittivity
ε	Permittivity
μ	Permeability of the material
κ	Expansion Coefficient with subscripts further specifying field or inverse of relative permittivity
G_n	Reciprocal Lattice Vectors

Chapter 1. Introduction

Maxwell's wave equations and the constitutive parameters defining a medium have been employed to determine the waveguide, resonator properties and performance of numerous optical devices [1]. The advent of photonic crystals (1987) and at the same time quasi-crystals provided researchers and device designers with a new set of optical structures which possess interesting optical properties [2]. The introduction of dielectric structures with higher order rotational symmetries, than those dictated by crystallography, introduced computational difficulties for applying traditional numerical techniques such as the plane wave expansion method (PWEM). In order to better match the numerical technique to the rotational symmetry present in the dielectric structure, the Fourier-Bessel basis expansion was proposed as a refit to the procedural process of the PWEM [3]. The technique has been successfully applied to a number of 2-D and 3-D dielectric configurations [4]. Some of the recent publications show results for structures resembling bottle resonators [5], cylindrical space slot channel waveguides [6], perfect periodic quasi-crystals [7], and even the simple Bragg resonator and fiber [4]. Recently Spherical space localized modes solver employing the same approach was introduced by other members of professor Gauthier's group using Bessel-Legendre-Fourier technique [8].

By using a Fourier-Bessel set of basis functions to express the field vector components of the wave equations then are transformed into a system-matrix that can be solved as an eigenvalue problem. Matching the basis functions to the rotational symmetry of the dielectric significantly reduces the order of the matrix to be diagonalized which results in faster computation process and also is possible to achieve with fair amount of memory.

The mathematical steps required to obtain the system-matrix is to first expand the double curl of the $\vec{E}$ or $\vec{H}$ wave equation in cylindrical coordinates followed by taking all required derivatives of the field components. The proposed solver makes research in the cylindrically symmetric dielectric structures efficient as other computational techniques such as plane wave (PWEM), finite element method (FEM), and finite-difference time domain (FDTD) exceed computer resources due to the large grid size required (FDTD, FEM) and large number of plane waves (PWEM) needed for well converged numerical results.

Chapter 2 will detail the concept of defining the two wave equations starting from Maxwell's equations. Maxwell's wave equations for the electric and magnetic field vectors are expressed with the assumption of homogeneous dielectric material with no current or free charges, followed by transverse electric (TE) and transverse magnetic (TM) modes characteristics. The plane wave expansion method is explained since the Fourier-Fourier-Bessel (FFB) technique proposed in this thesis follows the same procedure. Although PWEM is considered to be an efficient technique for the analysis of photonic crystals, it requires significant computational resources to achieve converged results, especially for the dielectric structures considered in this work. The cylindrical symmetry of the dielectric (and fields) proposed in this thesis implies that Fourier-Bessel basis functions are best suited for establishing the system-matrix to reduce the order of the matrix.

Chapter 3 details the matrix development in FFB technique process related to casting Maxwell's equations into an eigenvalue problem for cylindrically symmetric dielectric structures. The choice of the basis space functional form is justified through the

simplifications it introduces in the matrix element generation expressions. All necessary expressions related to populating the system-matrix are provided in the appendices.

Chapter 4 will detail the use of the technique in computing the steady state of several cylindrically symmetric dielectric structures. The technique's application is highlighted by examining the 1-D and 2-D Bragg structures then by combination of a Bragg fiber into which a cylindrical (orbital) slot channel waveguide is defined on the outer perimeter of the support Bragg structure. Different types of modes are identified and the convergence test performed for two localized modes of interest for the Bragg structure.

Chapter 5 presents the application of Fourier-Bessel technique by examining several refractive index sensors based on the slot channel waveguide configuration. These index sensors consist of having four high index (Silicon) channels, two stacked on top of two with the filling medium in between to be sensed. The field profiles are plotted for desired wavelength for both inner and outer channels with respect to the index change. The sensitivity of the slot channel sensors is high and comparable to the published research paper.

Chapter 6 will close this thesis by providing the conclusion on Fourier-Bessel approach advantages and challenges. Summary is included along with a discussion on the possible future works.

Novel configurations of the slot channel index sensors are proposed which show a sensitive behavior. The results from this thesis have lead to 1 journal publication and 2 conference proceedings along with 1 international conference presentation. FFB technique was developed by my thesis supervisor Dr. Robert Gauthier. Testing, design and examination of dielectric structures performed by author of this thesis.

Chapter 2. Cartesian Numerical Solver

2.1 Introduction

Maxwell's equations for electromagnetism describe how electric field, magnetic field, and even light behave. Maxwell's equations can be written, depending on whether there is vacuum, a charge present, matter present, and in differential or integral form. In differential form they are:

$$\vec{\nabla} \cdot \vec{D} = \rho \quad (2.1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (2.2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (2.3)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \frac{\partial \vec{D}}{\partial t} \quad (2.4)$$

where $\vec{E}$ is the electric field vector, $\vec{D}$ is the electric flux displacement, $\vec{B}$ is the magnetic field, $\vec{J}$ is current density, ρ is free charge and μ_0 is the permeability of free space. The fields are functions of position and time, the mediums are functions of position only. Within the contest of this thesis the mediums are assumed to have a real, frequency independent relative dielectric constant ϵ_r and non-magnetic with free space permeability [9]. Charge and current density are also set to zero. The resulting simplified Maxwell's equations are:

$$\vec{\nabla} \cdot \vec{H}(\vec{r}, t) = 0 \quad (2.5)$$

$$\vec{\nabla} \cdot [\epsilon(\vec{r}) \vec{E}(\vec{r}, t)] = 0 \quad (2.6)$$

$$\vec{\nabla} \times \vec{E}(\vec{r}, t) + \mu_o \frac{\partial \vec{H}(\vec{r}, t)}{\partial t} = 0 \quad (2.7)$$

$$\vec{\nabla} \times \vec{H}(\vec{r}, t) - \varepsilon_o \varepsilon_r(\vec{r}) \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} = 0 \quad (2.8)$$

where $\vec{H}$ is the magnetic field strength vector.

The time varying nature of the electric and magnetic fields is treated by expressing them as spatial mode profiles multiplied by a time varying exponential with angular frequency ω .

$$\vec{H}(\vec{r}, t) = \vec{H}(\vec{r}) e^{j\omega t} \quad (2.9)$$

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}) e^{j\omega t} \quad (2.10)$$

Introducing these fields into the curl equations provides the following relationship between $\vec{E}$ and $\vec{H}$.

$$\vec{\nabla} \times \vec{H}(\vec{r}) = j\omega \varepsilon_r(\vec{r}) \varepsilon_o \vec{E}(\vec{r}) \quad (2.11)$$

$$\vec{\nabla} \times \vec{E}(\vec{r}) = -j\omega \mu_o \vec{H}(\vec{r}) \quad (2.12)$$

These two curl equations can be combined to provide wave equations that have been called the master equations by photonic crystal researchers [9].

$$\vec{\nabla} \times \left(\frac{1}{\varepsilon_r(\vec{r})} \vec{\nabla} \times \vec{H}(\vec{r}) \right) = \left(\frac{\omega}{c} \right)^2 \vec{H}(\vec{r}) \quad (2.13)$$

$$\frac{1}{\varepsilon_r(\vec{r})} \vec{\nabla} \times (\vec{\nabla} \times \vec{E}(\vec{r})) = \left(\frac{\omega}{c} \right)^2 \vec{E}(\vec{r}) \quad (2.14)$$

The speed of light in free space is $c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$. Through the use of the master equation

the field, either H or E , can be completely determined for a given relative dielectric distribution $\varepsilon_r(\vec{r})$.

There are a number of methods for numerically solving the differential equation (2.13) and (2.14). The most commonly used in the analysis of PC and PQC is the plane wave expansion method (PWEM) [9, 10]. Techniques based on solving the curl equations are also possible such as the finite-difference time-domain (FDTD). PWEM is used for its efficiency with periodic structures in the frequency domain and FDTD for its flexibility in the time domain. The theory of PWEM technique will be reviewed as it follows exactly the same procedure as the proposed Fourier-Fourier-Bessel (FFB) method in this thesis. In the following section the case of solving these equations in 3-D is examined.

2.2 Cartesian 3-D Wave Equations

Expanding the curl equations and the fields in 3-D provides relationships between field components and exploring the basic steps required in their solutions. In this section the $(\vec{r}, t)$ dependence for the fields is hidden. The curl expressions can be expanded in the Cartesian coordinate system (x, y, z) , the electric field as $\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$ and the magnetic field as $\vec{H} = H_x \hat{x} + H_y \hat{y} + H_z \hat{z}$ where $(\hat{x}, \hat{y}, \hat{z})$ are the unite vectors. The curl of $\vec{E}$ is :

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{x} - \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{z} \quad (2.15)$$

By substituting (2.15) into (2.7) gives a set of scalar equations relating the components of the electric and magnetic field:

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) = -\mu_0 \frac{\partial H_x}{\partial t} \quad (2.16)$$

$$\left(-\frac{\partial E_z}{\partial x} + \frac{\partial E_x}{\partial z} \right) = -\mu_0 \frac{\partial H_y}{\partial t} \quad (2.17)$$

$$\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = -\mu_0 \frac{\partial H_z}{\partial t} \quad (2.18)$$

The curl expression can also be applied to the magnetic field in equation (2.8).

$$\vec{\nabla} \times \vec{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \hat{x} - \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) \hat{y} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \hat{z} \quad (2.19)$$

By substituting (2.19) into equation (2.8) results in the following three expressions

relating the field components:

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) = \varepsilon(\vec{r}) \frac{\partial E_x}{\partial t} \quad (2.20)$$

$$\left(-\frac{\partial H_z}{\partial x} + \frac{\partial H_x}{\partial z} \right) = \varepsilon(\vec{r}) \frac{\partial E_y}{\partial t} \quad (2.21)$$

$$\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) = \varepsilon(\vec{r}) \frac{\partial E_z}{\partial t} \quad (2.22)$$

In general, for 3-D analysis the 6 field components are coupled. However, for two dimensional analysis in (x, y) and uniform infinite in z -direction, the field components separate into two sets, one for the TM polarization with (H_x, H_y, E_z) and one for the TE polarization with (E_x, E_y, H_z) . The next section will examine the special case of a 2-D structure with aforementioned two sets of polarization equations.

2.3 TM Polarization

Consider the slab waveguide shown in figure 2.1, in which the direction of propagation is along z -axis. The TM polarization (H_x, H_y, E_z) is characterized with the electric field parallel to axis of propagation E_z and with $E_x = 0, E_y = 0$. For a uniform-infinite medium of in the z -direction, derivatives of any field component with respect to z can be set to zero, $\frac{\partial}{\partial z} \rightarrow 0$. Based on this, equations (2.16), (2.17) and (2.22) can be isolated from the original 6 equations:

$$\left(\frac{\partial E_z}{\partial y} \right) = -\mu_0 \frac{\partial H_x}{\partial t} \quad (2.23)$$

$$\left(-\frac{\partial E_z}{\partial x} \right) = -\mu_0 \frac{\partial H_y}{\partial t} \quad (2.24)$$

$$\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) = \varepsilon(\vec{r}) \frac{\partial E_z}{\partial t} \quad (2.25)$$

Taking the time derivatives of equation (2.19) gives:

$$\left(\frac{\partial}{\partial t} \frac{\partial H_y}{\partial x} - \frac{\partial}{\partial t} \frac{\partial H_x}{\partial y} \right) = \varepsilon(\vec{r}) \frac{\partial^2 E_z}{\partial t^2} \quad (2.26)$$

Taking the y -derivative of equation (2.17)

$$\left(\frac{\partial^2 E_z}{\partial y^2} \right) = -\mu_0 \frac{\partial}{\partial y} \frac{\partial H_x}{\partial t} \quad (2.27)$$

Taking the x -derivative of equation (2.18)

$$\left(-\frac{\partial^2 E_z}{\partial x^2} \right) = -\mu_0 \frac{\partial}{\partial x} \frac{\partial H_y}{\partial t} \quad (2.28)$$

By combining equations (2.26), (2.27) and (2.28), and recognizing the temporal and spatial derivative order may be interchanged gives the wave equation for the E_z field component:

$$\frac{1}{\epsilon_r(\vec{r})} \left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} \right) = \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2} \quad (2.29)$$

Equation (2.29) involves only one component and is simpler to solve than the general vector wave equation. Once the E_z component is solved using equation (2.29), the field components H_x, H_y can be obtained using equation (2.23) and (2.24).

Figure 2.1 Slab waveguide infinite in z -direction

2.4 TE Polarization

Consider the slab waveguide once again. The TE polarization is characterized with the magnetic field parallel to the z -direction, and with magnetic field components H_z and with $H_x = 0, H_y = 0$. For a uniform-infinite medium in the z -direction, derivatives of

any field component with respect to z can be set to zero, $\frac{\partial}{\partial z} \rightarrow 0$. Equations (2.18), (2.20) and (2.21) can be isolated from the original 6 equations and can be rewritten as:

$$\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = -\mu_o \frac{\partial H_z}{\partial t} \quad (2.30)$$

$$\left(\frac{\partial H_z}{\partial y} \right) = \varepsilon(\vec{r}) \frac{\partial E_x}{\partial t} \quad (2.31)$$

$$\left(-\frac{\partial H_z}{\partial x} \right) = \varepsilon(\vec{r}) \frac{\partial E_y}{\partial t} \quad (2.32)$$

Equation (2.31) and (2.32) can be written as:

$$\frac{1}{\varepsilon(\vec{r})} \left(\frac{\partial H_z}{\partial y} \right) = \frac{\partial E_x}{\partial t} \quad (2.33)$$

$$\frac{1}{\varepsilon(\vec{r})} \left(-\frac{\partial H_z}{\partial x} \right) = \frac{\partial E_y}{\partial t} \quad (2.34)$$

Taking the time derivatives of equation (2.30) gives,

$$\left(\frac{\partial}{\partial t} \frac{\partial E_y}{\partial x} - \frac{\partial}{\partial t} \frac{\partial E_x}{\partial y} \right) = -\mu_o \frac{\partial^2 H_z}{\partial t^2} \quad (2.35)$$

Taking the y -derivative of equation (2.33) gives

$$\frac{\partial}{\partial y} \left(\frac{1}{\varepsilon(\vec{r})} \left(\frac{\partial H_z}{\partial y} \right) \right) = \frac{\partial}{\partial y} \frac{\partial E_x}{\partial t} \quad (2.36)$$

Taking the x -derivative of equation (2.34)

$$\frac{\partial}{\partial x} \left(\frac{1}{\varepsilon(\vec{r})} \left(-\frac{\partial H_z}{\partial x} \right) \right) = \frac{\partial}{\partial x} \frac{\partial E_y}{\partial t} \quad (2.37)$$

By combining equations (2.35), (2.36) and (2.37), and recognizing the temporal and spatial derivative order may be interchanged gives the wave equation for the H_z field component.

$$\frac{\partial}{\partial x} \left(\frac{1}{\epsilon_r(\vec{r})} \left(\frac{\partial H_z}{\partial x} \right) \right) + \frac{\partial}{\partial y} \left(\frac{1}{\epsilon_r(\vec{r})} \left(\frac{\partial H_z}{\partial y} \right) \right) = \frac{1}{c^2} \frac{\partial^2 H_z}{\partial t^2} \quad (2.38)$$

Once the H_z component is found, the components E_x and E_y can be solved using equations (2.31) and (2.32).

In 2-D z -infinite cylindrical coordinate (r, ϕ, z) , the TE polarization is still characterized with H_z only and the TM polarization is characterized with E_z only. This is utilized when examining FFB steady states which are uniform-infinite in z .

2.5 Plane Wave Expansion Method 2-D (PWEM)

In science and engineering a basis set is described as a set of functions that can be used to describe any vector within a given vector space. In their most fundamental sense a basis set is linked to a coordinate system with the coefficients of the basis function forming the projection on the coordinate axis. As such $\hat{x}$, $\hat{y}$ and $\hat{z}$ form the basis of the 3-D Cartesian vector space. Basis sets are routinely used to express complicated quantities in terms of the mathematically simpler basis functions. The theory addressing the behavior of waves in periodic media is known as Bloch theory [11].

The plane wave expansion method [10] is a basis expansion technique through which both the field and dielectric layout are expressed as series of plane waves, $e^{-j\vec{k} \cdot \vec{r}}$, where $\vec{k}$ is the wave vector with a magnitude of $\frac{2\pi}{\lambda}$, with λ defined as medium dependent wavelength. In addition to working with plane waves it is necessary to understand that the expansions are Fourier transforms, changing the spatial space of the photonic crystal lattice to a spatial frequency domain known as k -space or momentum space. If a 3-D

lattice has lattice vectors $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ then there is a reciprocal lattice in k-space such that $\vec{a}_i \cdot \vec{b}_j = 2\pi\delta_{ij}$ where δ_{ij} is the Kroeneker delta. The expansions for any component of $\vec{E}$ or $\vec{H}$ fields, and inverse of relative permittivity can be written as:

$$(E_i, H_i) = \sum_{G_m} (\kappa_{G_m}^{E_i}, \kappa_{G_m}^{H_i}) e^{j\vec{G}_m \cdot \vec{r}} e^{j\vec{k} \cdot \vec{r}} \quad (2.39)$$

$$\frac{1}{\epsilon_r(\vec{r})} = \sum_{G_n} \kappa_{G_n}^{\epsilon} e^{j\vec{G}_n \cdot \vec{r}} \quad (2.40)$$

where $(\kappa_{G_m}^{E_i}, \kappa_{G_m}^{H_i})$ and $\kappa_{G_n}^{\epsilon}$ are the expansion coefficients over all possible reciprocal lattice vectors G_m and G_n which are expressed as $G_m = m_1 b_1 + m_2 b_2 + m_3 b_3$ where (m_1, m_2, m_3) are integers. The method works by relying on the periodicity of the relative dielectric material, which in turn imposes a periodicity on the modes which are supported. The expansions are performed over the smallest unit of periodicity in the reciprocal lattice, known as the Brillouin zone [12]. Due to the use of Bloch theory and the Fourier expansions, the mathematical assumption is that the unit cell or Brillouin zone are infinitely repeating, reconstruction of the dielectric using the coefficients, $\kappa_{G_n}^{\epsilon}$ would describe an infinite structure. Structures that lack translational symmetry, either due to lattice defects or that they are aperiodic structures, require a supercell of sufficient size that its repetition accurately represents the structure. When modeling resonator modes this requirement is satisfied by ensuring that the field is zero at the boundary of the supercell.

The solution of the master equation (2.13) continues by selecting a k vector along the boundary of the Brillouin zone and using the expansions in one of the master equations to

obtain the eigenvalue expression. The solution of this expression provides a vertical slice of the band structure. Repetition for k vectors along the border of the Brillouin zone provides the band structure. If a mode profile is required, the eigenvector that corresponds to the specified eigenvalue provides the coefficients ($\kappa_{G_m}^{Field}$) and can be used in (2.39) to reconstruct the mode profile.

By defining the series for the material (dielectric structure) and field component into the master equation and performing all the derivatives then collecting the equations for the expansion coefficients in matrix form of, for H_z :

$$\begin{bmatrix} H_{xx} & H_{xy} & H_{xz} \\ H_{yx} & H_{yy} & H_{yz} \\ H_{zx} & H_{zy} & H_{zz} \end{bmatrix} \begin{bmatrix} \kappa_{m_1}^{H_z} \\ \kappa_{m_2}^{H_z} \\ \kappa_{m_3}^{H_z} \end{bmatrix} = \left(\frac{\omega}{c} \right)^2 \begin{bmatrix} \kappa_{m_1}^{H_z} \\ \kappa_{m_2}^{H_z} \\ \kappa_{m_3}^{H_z} \end{bmatrix} \quad (2.41)$$

The series representation in Maxwell's equations allows the equation to be rearranged as an eigenvalue expression where the expansion coefficients of H_z (or E_z) are the eigenvectors and $\left(\frac{\omega}{c} \right)^2$ are the eigenvalues. Considering that the frequency might be complex, square root of the eigenvalue gives real and imaginary contributions, $\omega = \omega_{real} + j\omega_{imag}$. Then the frequencies can be obtained from the eigenvalues determined by diagonalizing the large matrix in (2.41). Now the band structure of TM polarization (in this case) on a unite cell, figure 2.2, can be generated. The plane wave method for band structure calculations requires that the unit cell to be repeated over all space. When defect states are built into the structure supercell method is used to find the localized steady state. Defect introduces a flat line to the band diagram. When the defect is

confined to a single region the structure may localize or trap light in the vicinity of the defect.

Figure 2.2 Band structure of a photonic crystal square lattice with band gap region in grey.

Inset: $\epsilon_{black} = 8.9$ and $\epsilon_{white} = 1$ with $r=0.2a$

The plane wave expansion method is considered to be an efficient and optimized technique for the analysis of photonic crystals. This method is based on Fourier expansion of the unit cell, and the solutions are based on Bloch wave which are modulated by the periodicity of the structures. Therefore, the accuracy of the results depends on the number of the plane waves used. However, when the structure does not have translational symmetry, supercells are required. The difficulties encountered is that the number of plane waves required to obtain a sampling density of plane waves within the supercell that matches the density that achieved converged results for a single cell,

can exceed computational resources. PWEM can't be used for the structures being examined in this thesis since they would require very large 3-D supercells. Therefore the Fourier-Fourier-Bessel (FFB) technique was developed, where matching the basis functions to the rotational symmetry reduces the order of the matrix that can be solved on a desktop P.C. FFB follows the same developmental procedure as PWEM and is detailed in the next chapter.

Chapter 3. Fourier-Bessel Mode Solver

3.1 Introduction

As discussed at the end of chapter two, resonator (steady) state calculation for dielectric structures using PWEM will require very large number of plane waves, therefore it is not a practical approach. The FFB computational technique provides a significantly reduced matrix order to be populated which results in a system of equations that can be solved on a desktop P.C. The Fourier and Bessel functions are commonly used when in polar or cylindrical coordinates. These functions have been used in the study of Bessel beams [13], the Cann-Hilliard equation in mathematical physics [14], or even the Schrodinger equation [15]. In photonics, the most common application is its use in modelling fields within proximity of circular scatterers such as in the multiple scattering matrix method [16] or when studying photonic crystal fibers [17]. The proposed application of the Fourier-Bessel function is to express the material and field components in series form allowing Maxwell's equations to be arranged as an eigenvalue expression that can be solved numerically. The solutions provide the frequencies and spatial profiles for the steady state modes that are supported. The development of the FFB technique follows next, for any 3-D structure of cylindrical symmetry.

3.2 3-D FFB (Fourier-Fourier-Bessel)

The purpose of this section is to present key steps in the development of the system-matrix and expressions in cylindrical coordinates for the Fourier-Fourier-Bessel (FFB) equivalent of the plane wave method. Bessel functions are used in the radial direction. Fourier basis functions are used for the angular direction (φ) and the out of plane

direction (z). Figure 3.1 shows a dielectric structure within a cylindrical boundary. In the $(x, y) \Rightarrow (r, \varphi)$ the dielectric structure has a rotational symmetry with respect to the center of the polar coordinate. R is the radial limit in the (r, φ) plane. In the z -direction the domain has height T and is centered on the polar coordinate system. The optical analysis is based on Maxwell's vector wave equations written for the electric field $\vec{E}$ (2.14) and magnetic field strength $\vec{H}$ (2.13). The medium is selected to be non-magnetic and as such the permeability is that of free space μ_o . The development will commence by working towards the eigenvalue expression for $\vec{H}$. Afterwards the expression for the electric field $\vec{E}$ can be obtained, as the $\vec{E}$ equation can be extracted from the $\vec{H}$ equation which is shown in appendix A. Theoretical development of FFB follows the steps of PWEM detailed in the previous chapter.

Figure 3.1 Dielectric structure for analysis consisting of a cylindrical dielectric membrane.

Computation domain radius R and height T .

3.3 Theoretical derivation for the magnetic field

The starting point for the $\vec{H}$ field analysis is the wave equation written in (2.13):

$$\vec{\nabla} \times \frac{1}{\epsilon_r(\vec{r})} \vec{\nabla} \times \vec{H} = \left(\frac{\omega}{c}\right)^2 \vec{H} \quad (2.13)$$

In this expression it is assumed that the medium is non-magnetic, uncharged, no currents are present, a time dependence of the form $e^{j\omega t}$, where ω is frequency and c is the free space speed of light. Since the analysis is performed in 3-D, the field components for the magnetic field are in general coupled. The analysis proceeds by writing out the curls in cylindrical coordinates, details are contained in appendix B. The inverse of the relative permittivity (Ω) is a function of position and has in general non-zero derivatives.

$$\Omega = \frac{1}{\epsilon_r(\vec{r})} \quad (3.1)$$

In the FFB formulation, the basis functions are of the form:

$$J_o\left(\rho_p \frac{r}{R}\right) e^{jq\varphi} e^{jG_n z} \quad (3.2)$$

where ρ_p is the p^{th} zero of the lowest order Bessel function, R is the in-plane radius of the computation domain and G_n is reciprocal lattice used in the z -axis direction. In FFB technique the lowest order of Bessel function of the first kind $J_o(r)$ and it's first and second order derivatives are used, which are presented in appendix C, equation (C.3) and appendix A, equation (A.15) respectively. The basis functions are indexed using (pqn) with p linked to the radial expansion of the Bessel functions, q linked to the angular Fourier expansion and n is linked to the axial direction z . The expansion for Ω in this basis space is:

$$\Omega = \sum_{\Omega} \kappa_{\Omega} J_o\left(\rho_{p\Omega} \frac{r}{R}\right) e^{jq_{\Omega}\varphi} e^{jG_{n\Omega} z} = \sum_{\Omega} \kappa_{\Omega} FFB_0(\Omega) \quad (3.3)$$

The compressed notation is introduced using:

$$FFB_0(\Omega) = J_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) e^{jq_\Omega \varphi} e^{jG_{n_\Omega} z} \quad (3.4)$$

As it is shown in appendix B and C, only the left most curl acts on Ω . As a result, Ω is subjected to first derivatives only with respect to (r, φ, z) . These derivatives are determined in Appendix C and transcribed here:

$$\frac{\partial \Omega}{\partial r} = \sum_{\Omega} \kappa_{\Omega} \left(-\frac{\rho_{p_\Omega}}{R} \right) FFB_1(\Omega) \quad (3.5)$$

$$\frac{\partial \Omega}{\partial \varphi} = \sum_{\Omega} \kappa_{\Omega} (jq_{\Omega}) FFB_0(\Omega) \quad (3.6)$$

$$\frac{\partial \Omega}{\partial z} = \sum_{\Omega} \kappa_{\Omega} (jG_{n_\Omega}) FFB_0(\Omega) \quad (3.7)$$

with the simplification:

$$FFB_1(\Omega) = J_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) e^{jq_\Omega \varphi} e^{jG_{n_\Omega} z} \quad (3.8)$$

3.4 Determination of the relative permittivity's expansion coefficients

Noting that Ω is known, the expansion for the inverse of relative permittivity, in this basis space has been specified in equation (3.3). The expansion coefficients can be obtained by making use of the orthogonally relation of the basis functions. This is accomplished by multiplying both sides of (3.3) by one of the basis functions using primes on the indices, $(p'_\Omega, q'_\Omega, n'_\Omega)$, and then integrating over the volume of the computation cylinder domain with base of radius R and height T .

After multiplication and integration applied, equation (3.3) becomes:

$$\begin{aligned}
& \iiint_{0,0,-\frac{T}{2}}^{R,2\pi,\frac{T}{2}} \Omega J_o\left(\rho_{p_\Omega'} \frac{r}{R}\right) e^{-jq_\Omega' \varphi} e^{-jG_{n_\Omega'} z} r dr d\varphi dz \\
&= \iiint_{0,0,-\frac{T}{2}}^{R,2\pi,\frac{T}{2}} \sum_{\Omega} \kappa_{\Omega} J_o\left(\rho_{p_\Omega} \frac{r}{R}\right) e^{jq_\Omega \varphi} e^{jG_{n_\Omega} z} J_o\left(\rho_{p_\Omega'} \frac{r}{R}\right) e^{-jq_\Omega' \varphi} e^{-jG_{n_\Omega'} z} r dr d\varphi dz
\end{aligned} \tag{3.9}$$

The orthogonally of the Bessel are specified for a normalized range $0 \leq r \leq 1$ as:

$$\int_0^1 r J_o(\rho_{p_\Omega} r) J_o(\rho_{p_\Omega'} r) dr = \frac{\delta_{p_\Omega, p_\Omega'}}{2} [J_1(\rho_{p_\Omega})]^2 \tag{3.10}$$

and as a result will be non-zero only when $p_\Omega = p_\Omega'$ as dictated by the delta function.

Examination of the limits of the radial integration requires that the radial range R be rescaled to 1. This is obtained by performing the change of variable $r = R\zeta$ and $dr = Rd\zeta$. Expression (3.9) becomes:

$$\begin{aligned}
& R^2 \iiint_{0,0,-\frac{T}{2}}^{1,2\pi,\frac{T}{2}} \Omega J_o(\rho_{p_\Omega'} \zeta) e^{-jq_\Omega' \varphi} e^{-jG_{n_\Omega'} z} \zeta d\zeta d\varphi dz \\
&= R^2 \iiint_{0,0,-\frac{T}{2}}^{1,2\pi,\frac{T}{2}} \sum_{\Omega} \kappa_{\Omega} J_o(\rho_{p_\Omega} \zeta) e^{-jq_\Omega \varphi} e^{jG_{n_\Omega} z} J_o(\rho_{p_\Omega'} \zeta) e^{-jq_\Omega' \varphi} e^{-jG_{n_\Omega'} z} \zeta d\zeta d\varphi dz
\end{aligned} \tag{3.11}$$

Simplifying and using the Bessel orthogonality gives:

$$\begin{aligned}
& R^2 \iiint_{0,0,-\frac{T}{2}}^{1,2\pi,\frac{T}{2}} \Omega J_o(\rho_{p_\Omega'} \zeta) e^{-jq_\Omega' \varphi} e^{-jG_{n_\Omega'} z} \zeta d\zeta d\varphi dz \\
&= \frac{[J_1(\rho_{p_\Omega'})]^2}{2} R^2 \int_{0,-\frac{T}{2}}^{\frac{T}{2}} \sum_{q_\Omega n_\Omega} \kappa_{p_\Omega', q_\Omega n_\Omega}^\Omega e^{-jq_\Omega \varphi} e^{-jG_{n_\Omega} z} e^{jq_\Omega' \varphi} e^{jG_{n_\Omega'} z} d\varphi dz
\end{aligned} \tag{3.12}$$

The orthogonally condition for the φ based exponentials is:

$$\int_0^{2\pi} e^{jq_\Omega\varphi} e^{-jq_\Omega'\varphi} d\varphi = 2\pi\delta_{q_\Omega q_\Omega'} \quad (3.13)$$

and as a result will be non-zero only when $q_\Omega = q_\Omega'$ as dictated by the delta function.

Simplifying and using the φ coordinate orthogonality gives:

$$\begin{aligned} & R^2 \iiint_{0,0,-\frac{T}{2}}^{1,2\pi,\frac{T}{2}} \Omega J_o(\rho_{p_\Omega'}\zeta) e^{jq_\Omega'\varphi} e^{jG_{n_\Omega}'z} \zeta d\zeta d\varphi dz \\ &= 2\pi \frac{[J_1(\rho_{p_\Omega'})]^2}{2} R^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n_\Omega} \kappa_{p_\Omega'q_\Omega'n_\Omega}^\Omega e^{jG_{n_\Omega}z} e^{-jG_{n_\Omega}'z} dz \end{aligned} \quad (3.14)$$

The application of the orthogonality along the z direction requires that the dielectric structure be rendered periodic in z . It is thus assumed that the structure repeats from $-\infty$ to $+\infty$ with a lattice constant equal to the thickness T . Note that T is not necessarily the thickness of the dielectric structure and in most cases is sufficient such that a super cell approach commonly used for the plane wave analysis is employed. With this, G_{n_Ω} is equivalent to a reciprocal lattice “vector” when written as $G_{n_\Omega}\hat{z}$ and expressed as:

$$G_{n_\Omega} = \frac{2\pi}{T} n_\Omega \quad (3.15)$$

The orthogonality integral in the z -direction is:

$$\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{jG_{n_\Omega}z} e^{-jG_{n_\Omega}'z} dz = T\delta_{n_\Omega n_\Omega'} \quad (3.16)$$

The application of the 3-D cylindrical space orthogonality integration gives:

$$R^2 \iiint_{0,0,-\frac{T}{2}}^{1,2\pi,\frac{T}{2}} \Omega J_o(\rho_{p_\Omega'}\zeta) e^{-jq_\Omega'\varphi} e^{-jG_{n_\Omega}'z} \zeta d\zeta d\varphi dz = 2\pi T R^2 \frac{[J_1(\rho_{p_\Omega'})]^2}{2} \kappa_{p_\Omega'q_\Omega'n_\Omega'}^\Omega \quad (3.17)$$

From which the expansion coefficient can be extracted and written as:

$$\kappa_{p_{\Omega}', q_{\Omega}', n_{\Omega}'}^{\Omega} = \frac{R^2}{\pi T R^2 [J_1(\rho_{p_{\Omega}'})]^2} \iiint_{0,0,-\frac{T}{2}}^{1,2\pi,\frac{T}{2}} \Omega J_o(\rho_{p_{\Omega}'} \zeta) e^{-jq_{\Omega}' \varphi} e^{-jG_{n_{\Omega}'} z} \zeta d\zeta d\varphi dz \quad (3.18)$$

To solve the integral (3.18) using a computer program, there are 3 nested "for loops" needed to compute the integral. Summation is required every time the loop index changes (loop index here is the steps along the integral (for example the steps of $\int_0^{2\pi}$ can be written as $= 0, 0.1, 0.2, \dots$ until 2π). The integral is solved repeatedly for all coefficient indices $(p_{\Omega}', q_{\Omega}', n_{\Omega}')$. These values are saved for future use in the matrix building expressions.

The magnetic field is a vector with three components

$$\vec{H} = H_r \hat{r} + H_{\varphi} \hat{\varphi} + H_z \hat{z} \quad (3.19)$$

Each of the components can be expanded using the FFB basis functions as:

$$H_r = \sum_r \kappa_r FFB_0(r) \quad (3.20)$$

$$H_{\varphi} = \sum_{\varphi} \kappa_{\varphi} FFB_0(\varphi) \quad (3.21)$$

$$H_z = \sum_z \kappa_z FFB_0(z) \quad (3.22)$$

Examination of the curl of the curl of the magnetic field indicates that the derivatives show up as either single or second order. For the H_r component derivatives are required with respect to the following, $(zz, \varphi\varphi, rz, r\varphi, \varphi, z)$. These are developed in Appendix A and transcribed here:

$$\frac{\partial H_r}{\partial \varphi} = \sum_r \kappa_r (jq_r) FFB_0(r) \quad (3.23)$$

$$\frac{\partial^2 H_r}{\partial \varphi^2} = \sum_r \kappa_r (-q_r^2) FFB_0(r) \quad (3.24)$$

$$\frac{\partial H_r}{\partial z} = \sum_r \kappa_r (jG_{n_r}) FFB_0(r) \quad (3.25)$$

$$\frac{\partial^2 H_r}{\partial z^2} = \sum_r \kappa_r (-G_{n_r}^2) FFB_0(r) \quad (3.26)$$

$$\frac{\partial^2 H_r}{\partial r \partial z} = \sum_r \kappa_r \left(-\frac{\rho_{p_r}}{R} \right) (jG_{n_r}) FFB_1(r) \quad (3.27)$$

$$\frac{\partial^2 H_r}{\partial r \partial \varphi} = \sum_r \kappa_r \left(-\frac{\rho_{p_r}}{R} \right) (jq_r) FFB_1(r) \quad (3.28)$$

For the H_φ component derivatives are required with respect to the following,

$(zz, rr, \varphi z, r\varphi, r, \varphi, z)$. These are developed in Appendix A and transcribed here:

$$\frac{\partial H_\varphi}{\partial r} = \sum_\varphi \kappa_\varphi \left(-\frac{\rho_{p_\varphi}}{R} \right) FFB_1(\varphi) \quad (3.29)$$

$$\frac{\partial H_\varphi}{\partial \varphi} = \sum_\varphi \kappa_\varphi (jq_\varphi) FFB_0(\varphi) \quad (3.30)$$

$$\frac{\partial H_\varphi}{\partial z} = \sum_\varphi \kappa_\varphi (jG_{n_\varphi}) FFB_0(\varphi) \quad (3.31)$$

$$\frac{\partial^2 H_\varphi}{\partial r^2} = - \sum_\varphi \kappa_\varphi \left(\frac{\rho_{p_\varphi}}{R} \right)^2 FFB_0(\varphi) + \sum_\varphi \kappa_\varphi \left(\frac{\rho_{p_\varphi}}{rR} \right) FFB_1(\varphi) \quad (3.32)$$

$$\frac{\partial^2 H_\varphi}{\partial z^2} = \sum_\varphi \kappa_\varphi (-G_{n_\varphi}^2) FFB_0(\varphi) \quad (3.33)$$

$$\frac{\partial^2 H_\varphi}{\partial \varphi \partial r} = \sum_\varphi \kappa_\varphi (jq_\varphi) \left(-\frac{\rho_{p_\varphi}}{R} \right) FFB_1(\varphi) \quad (3.34)$$

$$\frac{\partial^2 H_\varphi}{\partial z \partial \varphi} = \sum_{\varphi} \kappa_{\varphi} \left(-G_{n_{\varphi}} q_{\varphi} \right) FFB_0(\varphi) \quad (3.35)$$

For the H_z component derivatives are required with respect to the following, $(rr, \varphi\varphi, \varphi z, r\varphi, r, \varphi)$. These are developed in Appendix A and transcribed here:

$$\frac{\partial H_z}{\partial r} = \sum_z \kappa_z \left(-\frac{\rho_{p_z}}{R} \right) FFB_1(z) \quad (3.36)$$

$$\frac{\partial H_z}{\partial \varphi} = \sum_z \kappa_z (j q_z) FFB_0(z) \quad (3.37)$$

$$\frac{\partial^2 H_z}{\partial r \partial \varphi} = \sum_z \kappa_z \left(-\frac{\rho_{p_z}}{R} \right) (j q_z) FFB_1(z) \quad (3.38)$$

$$\frac{\partial^2 H_z}{\partial z \partial \varphi} = \sum_z \kappa_z (-G_{n_z} q_z) FFB_0(z) \quad (3.39)$$

$$\frac{\partial^2 H_z}{\partial \varphi^2} = \sum_z \kappa_z (-q_z^2) FFB_0(z) \quad (3.40)$$

$$\frac{\partial^2 H_z}{\partial r^2} = - \sum_z \kappa_z \left(\frac{\rho_{p_z}}{R} \right)^2 FFB_0(z) + \sum_z \kappa_z \left(\frac{\rho_{p_z}}{rR} \right) FFB_1(z) \quad (3.41)$$

$$\frac{\partial^2 H_z}{\partial r \partial z} = \sum_z \kappa_z \left(-\frac{\rho_{p_z}}{R} \right) (j G_{n_z}) FFB_1(z) \quad (3.42)$$

3.5 Expression for the radial component of the master equation

The key expression in the radial component is obtained by taking the right hand side of equation (B.16) in appendix B and setting it equal to $\left(\frac{\omega}{c}\right)^2 H_r$ giving the following:

$$\begin{aligned}
& \frac{H_\varphi}{r^2} \frac{\partial \Omega}{\partial \varphi} + \frac{1}{r} \frac{\partial H_\varphi}{\partial r} \frac{\partial \Omega}{\partial \varphi} - \frac{1}{r^2} \frac{\partial H_r}{\partial \varphi} \frac{\partial \Omega}{\partial \varphi} + \frac{\Omega}{r^2} \frac{\partial H_\varphi}{\partial \varphi} + \frac{\Omega}{r} \frac{\partial^2 H_\varphi}{\partial \varphi \partial r} - \frac{\Omega}{r^2} \frac{\partial^2 H_r}{\partial \varphi^2} - \frac{\partial H_r}{\partial z} \frac{\partial \Omega}{\partial z} + \frac{\partial H_z}{\partial r} \frac{\partial \Omega}{\partial z} \\
& - \Omega \frac{\partial^2 H_r}{\partial z^2} + \Omega \frac{\partial^2 H_z}{\partial z \partial r} = \left(\frac{\omega}{c} \right)^2 H_r
\end{aligned} \tag{3.43}$$

Each of the derivatives in (3.43) has been worked out for the FFB expansion of the inverse of relative permittivity and field components. What is required is to collect the derivatives, multiply them by the appropriate factor and then simplify as much as possible. The best way to proceed is to segment equation (3.43) into terms. Starting on the left side is term 1, and proceeds to term 10 just before the equal sign. Each of the terms has the inverse of relative permittivity multiplied by a field component. This will result in the product of the two summations involved. The term by term development follows in the appendix D. It should be noted that the product of two sums can be written as a single sum extending of the combined indices.

3.5.1 Collection of terms and formation of the H_r equation

Table 3-1 shows the collection of terms based on the order of the Bessel function in both the field and inverse of relative permittivity, generated for (3.43).

FFB _o (Ω)	FFB ₁ (Ω)	Ω/H _r
Term(3+6+7+9)	---	FFB _o (r)
---	---	FFB ₁ (r)
Term(1+4)	---	FFB _o (φ)
Term(2+5)	---	FFB ₁ (φ)
---	---	FFB _o (z)
Term(8+10)	---	FFB ₁ (z)

Table 3-1 Collection of terms in Ω and H_r

Using the simplified notation the radial equation is:

$$\begin{aligned}
& \sum_{r\Omega} \kappa_r \left(\kappa_\Omega(Z) \frac{q_r q_\Omega + q_r^2}{r^2} + \kappa_\Omega(\varphi) G_{n_r} G_{n_\Omega} + \kappa_\Omega(\varphi) G_{n_r}^2 \right) FFB_o(r) FFB_o(\Omega) \\
& + \sum_{\varphi\Omega} \kappa_\varphi \kappa_\Omega(Z) \left(\frac{j q_\Omega + j q_\varphi}{r^2} \right) FFB_o(\varphi) FFB_o(\Omega) \\
& + \sum_{\varphi\Omega} \kappa_\varphi \kappa_\Omega(Z) \left(-j \frac{\rho_{p\varphi}}{rR} \right) (q_\Omega + q_\varphi) FFB_1(\varphi) FFB_o(\Omega) \\
& + \sum_{z\Omega} \kappa_z \kappa_\Omega \kappa_\Omega(\varphi) \left(-j \frac{\rho_{pz}}{R} \right) (G_{n_\Omega} + G_{n_z}) FFB_1(z) FFB_o(\Omega) = \left(\frac{\omega}{c} \right)^2 H_r
\end{aligned} \tag{3.44}$$

3.6 Expression for the angular component of the master equation

The key expression in the angular component is obtained by taking the right hand side of equation (B.22) in appendix B and setting it equal to $\left(\frac{\omega}{c} \right)^2 H_\varphi$, giving the following:

$$\begin{aligned}
& \frac{1}{r} \frac{\partial H_z}{\partial \varphi} \frac{\partial \Omega}{\partial z} - \frac{\partial H_\varphi}{\partial z} \frac{\partial \Omega}{\partial z} + \frac{\Omega}{r} \frac{\partial^2 H_z}{\partial z \partial \varphi} - \Omega \frac{\partial^2 H_\varphi}{\partial z^2} - \frac{H_\varphi}{r} \frac{\partial \Omega}{\partial r} - \frac{\partial H_\varphi}{\partial r} \frac{\partial \Omega}{\partial r} + \frac{1}{r} \frac{\partial H_r}{\partial \varphi} \frac{\partial \Omega}{\partial r} + \frac{\Omega}{r^2} H_\varphi \\
& - \frac{\Omega}{r} \frac{\partial H_\varphi}{\partial r} - \Omega \frac{\partial^2 H_\varphi}{\partial r^2} - \frac{\Omega}{r^2} \frac{\partial H_r}{\partial \varphi} + \frac{\Omega}{r} \frac{\partial^2 H_r}{\partial r \partial \varphi} = \left(\frac{\omega}{c} \right)^2 H_\varphi
\end{aligned} \tag{3.45}$$

Each of the derivatives in (3.45) has been worked out for the FFB expansion of the inverse of relative permittivity and field components. What is required is to collect the derivatives, multiply them by the appropriate factor and then simplify as much as possible. The best way to proceed is to segment equation (3.45) into terms. Starting on the left side is term 1, and proceeds to term 12 just before the equal sign. The term by term development follows.

3.6.1 Collection of terms and formation of the H_φ equation

Table 3-2 shows the collection of terms based on the order of the Bessel function in both the field and inverse of relative permittivity, generated for (3.45).

$FFB_o(\Omega)$	$FFB_1(\Omega)$	Ω/H_φ
Term(11)	Term(7)	$FFB_o(r)$
Term(12)	---	$FFB_1(r)$
Term(2+4+8+10_1)	Term(5)	$FFB_o(\varphi)$
Term(9+10_2)	Term(6)	$FFB_1(\varphi)$
Term(1+3)	---	$FFB_o(z)$
---	---	$FFB_1(z)$

Table 3-2Collection of terms in Ω and H_φ

Using the simplified notation the collected equation is:

$$\begin{aligned}
& \sum_{r\Omega} \kappa_r \kappa_\Omega(Z) \left(-\frac{j q_r}{r^2} \right) FFB_o(r) FFB_o(\Omega) + \sum_{r\Omega} \kappa_r \kappa_\Omega(Z) \left(-\frac{j q_r \rho_{p_r}}{r R} \right) FFB_1(r) FFB_o(\Omega) \\
& + \sum_{\varphi\Omega} \kappa_\varphi \left(\kappa_\Omega(r) G_{n_\varphi} G_{n_\Omega} + \kappa_\Omega(r) G_{n_\varphi}^2 + \kappa_\Omega(Z) \frac{1}{r^2} \right. \\
& \left. + \kappa_\Omega(Z) \frac{\rho_{p_\varphi}^2}{R^2} \right) FFB_o(\varphi) FFB_o(\Omega) \\
& + \sum_{z\Omega} \kappa_z \kappa_\Omega(Z) \left(-\frac{q_z G_{n_\Omega}}{r} - \frac{q_z G_{n_z}}{r} \right) FFB_o(z) FFB_o(\Omega) \\
& + \sum_{r\Omega} \kappa_r \kappa_\Omega(Z) \left(-\frac{j q_r \rho_{p_\Omega}}{r R} \right) FFB_o(r) FFB_1(\Omega) \\
& + \sum_{\varphi\Omega} \kappa_\varphi \kappa_\Omega(Z) \left(\frac{\rho_{p_\Omega}}{r R} \right) FFB_o(\varphi) FFB_1(\Omega) \\
& + \sum_{\varphi\Omega} \kappa_\varphi \kappa_\Omega(Z) \left(-\frac{\rho_{p_\varphi} \rho_{p_\Omega}}{R^2} \right) FFB_1(\varphi) FFB_1(\Omega) = \left(\frac{\omega}{c} \right)^2 H_\varphi
\end{aligned} \tag{3.46}$$

3.7 Expression for the axial component of the master equation

The key expression in the axial component is obtained by taking the right hand side of equation (B.28) in appendix B and setting it equal to $\left(\frac{\omega}{c} \right)^2 H_z$, giving the following:

$$\begin{aligned}
& \frac{\partial H_r}{\partial z} \frac{\partial \Omega}{\partial r} - \frac{\partial H_z}{\partial r} \frac{\partial \Omega}{\partial r} + \frac{\Omega}{r} \frac{\partial H_r}{\partial z} - \frac{\Omega}{r} \frac{\partial H_z}{\partial r} + \Omega \frac{\partial^2 H_r}{\partial r \partial z} - \Omega \frac{\partial^2 H_z}{\partial r^2} - \frac{1}{r^2} \frac{\partial H_z}{\partial \varphi} \frac{\partial \Omega}{\partial \varphi} + \frac{1}{r} \frac{\partial H_\varphi}{\partial z} \frac{\partial \Omega}{\partial \varphi} \\
& - \frac{\Omega}{r^2} \frac{\partial^2 H_z}{\partial \varphi^2} + \frac{\Omega}{r} \frac{\partial^2 H_\varphi}{\partial \varphi \partial z} = \left(\frac{\omega}{c} \right)^2 H_z
\end{aligned} \tag{3.47}$$

Each of the derivatives in (3.47) has been worked out for the FFB expansion of the inverse of relative permittivity and field components. What is required is to collect the derivatives, multiply by them by the appropriate factor and then simplify as much as

possible. The best way to proceed is to segment equation (3.47) into terms. Starting on the left side is term 1, and proceeds to term 10 just before the equal sign. The term by term development follows.

3.7.1 Collection of terms and formation of the H_z equation

Table 3-3 shows the collection of terms based on the order of the Bessel function in both the field and inverse of relative permittivity, generated for (3.47).

$FFB_o(\Omega)$	$FFB_1(\Omega)$	Ω/H_z
Term(3)	Term(1)	$FFB_o(r)$
Term(5)	---	$FFB_1(r)$
Term(8+10)	---	$FFB_o(\varphi)$
---	---	$FFB_1(\varphi)$
Term(6_1+7+9)	---	$FFB_o(z)$
Term(4+6_2)	Term(2)	$FFB_1(z)$

Table 3-3Collection of terms in Ω and H_z

Using the simplified notation the collected equation is:

$$\begin{aligned}
& \sum_{r\Omega} \kappa_r \kappa_\Omega(\varphi) \left(\frac{jG_{n_r}}{r} \right) FFB_o(r) FFB_o(\Omega) + \sum_{r\Omega} \kappa_r \kappa_\Omega(\varphi) \left(-\frac{jG_{n_r} \rho_{p_r}}{R} \right) FFB_1(r) FFB_o(\Omega) \\
& + \sum_{\varphi\Omega} \kappa_\varphi \kappa_\Omega(r) \left(-\frac{G_{n_\varphi} q_\Omega}{r} - \frac{G_{n_\varphi} q_\varphi}{r} \right) FFB_o(\varphi) FFB_o(\Omega) \\
& + \sum_{z\Omega} \kappa_z \left(\kappa_\Omega(\varphi) \left(\frac{\rho_{p_z}}{R} \right)^2 + \kappa_\Omega(r) \frac{q_z q_\Omega}{r^2} + \kappa_\Omega(r) \frac{q_z^2}{r^2} \right) FFB_o(z) FFB_o(\Omega) \\
& + \sum_{r\Omega} \kappa_r \kappa_\Omega(\varphi) \left(-\frac{jG_{n_r} \rho_{p_\Omega}}{R} \right) FFB_o(r) FFB_1(\Omega) \\
& + \sum_{z\Omega} \kappa_z \kappa_\Omega(\varphi) \left(-\frac{\rho_{p_z} \rho_{p_\Omega}}{R^2} \right) FFB_1(z) FFB_1(\Omega) = \left(\frac{\omega}{c} \right)^2 H_z
\end{aligned} \tag{3.48}$$

3.8 The formation of the complete system-matrix into an eigen-value problem

The system-matrix results from the coupling of the three field components into three separate and unique equations. The matrix will have the following overall form:

$$\begin{bmatrix} R_r & \Phi_r & Z_r \\ R_\varphi & \Phi_\varphi & Z_\varphi \\ R_z & \Phi_z & Z_z \end{bmatrix} \begin{bmatrix} R \\ \Phi \\ Z \end{bmatrix} = \left(\frac{\omega}{c} \right)^2 I \begin{bmatrix} R \\ \Phi \\ Z \end{bmatrix} \tag{3.49}$$

Column vector $\begin{bmatrix} R \\ \Phi \\ Z \end{bmatrix}$ is formed by the expansion coefficients of the each field component.

The R represents all of the expansion coefficients related to H_r . The Φ represents all the expansion coefficients related to H_φ . The Z represents all the expansion coefficients related to H_z . Each of the field components is expanded using three indices (p, q, n) with p linked to the radial expansion and the Bessel functions, q linked to the angular Fourier expansion and n is linked to the axial direction z . The ordering of the expansion coefficients based on the indices is crucial to establishing the matrix and needs to be

respected in all sub-blocks of the matrix. Since the indices are (p, q, n) then they are written by first specifying p then q then n . Consider the following series as the guide:

$$(p, q, n) \Rightarrow \sum_p \sum_q \sum_n \kappa_{p,q,n} \quad (3.50)$$

Identity matrix I has order equal to 3 times the number of basis functions used to expand the field. Equation (3.49) has the standard form of an eigenvalue equation in which the eigenvalues are the set of $\left(\frac{\omega}{c}\right)^2$ determined by diagonalizing the large matrix. This system-matrix contains both real and complex elements. The eigenvalues can be written in complex form with both real and imaginary parts:

$$\left(\frac{\omega}{c}\right)^2 = W_{real} + jW_{imag} \quad (3.51)$$

Considering that the frequency might be complex and written as $\omega = \omega_{real} + j\omega_{imag}$ with the real part indicating oscillations and the complex part indicating attenuation or gain, then the frequencies can be obtained from the eigenvalues through the following:

$$\omega_{real}^2 - \omega_{imag}^2 = c^2 W_{real} \quad (3.52)$$

$$2\omega_{real}\omega_{imag} = c^2 W_{imag} \quad (3.53)$$

These can be inverted to give the field's complex frequency.

$$\omega_{imag} = \frac{c}{\sqrt{2}} \sqrt{-W_{real} \pm \sqrt{W_{real}^2 + W_{imag}^2}} \quad (3.54)$$

$$\omega_{real} = \frac{c}{\sqrt{2}} \sqrt{W_{real} \pm \sqrt{W_{real}^2 + W_{imag}^2}} \quad (3.55)$$

Sub matrix blocks R_i , Φ_i and Z_i are square matrices that contains an equal number of rows and columns with the number equal to the number of expansion coefficients used in the corresponding field component (radial, angular, axial). The subscript indicated the field component equation that the sub block matrix corresponds to. The actual matrix elements in the sub block are formed by the numerical value of the term associated with the particular field coefficient. The order of the elements in the matrix must have the numbers that follow the order in the column (expansion coefficient) vector.

3.8.1 Expressions for the sub block matrices

In order to compact the following expressions, resulting integrals are expressed as S, T, U, V which are defined in Appendix E along with the conditions that are applied here.

Following equations are related to the expansion coefficient κ_r

R_r matrix elements are generated from the following for H :

$$\frac{2\pi T}{(\pi R^2 T)[J_1(\rho_{p_r^*})]^2} \left\{ \sum_{\Omega} \kappa_{\Omega} \left([q_r q_{\Omega} + q_r^2] S_{-1}(p_r, p_{\Omega}, p_{r^*}) \right. \right. \\ \left. \left. + [G_{n_r} G_{n_{\Omega}} + G_{n_r}^2] R^2 S_1(p_r, p_{\Omega}, p_{r^*}) \right) \delta_{q_r+q_{\Omega}, q_{r^*}} \right\}_{n_r+n_{\Omega}, n_{r^*}} \quad (3.56)$$

for E :

$$\frac{2\pi T}{(\pi R^2 T)[J_1(\rho_{p_r^*})]^2} \left\{ \sum_{\Omega} \kappa_{\Omega} \left([q_r^2] S_{-1}(p_r, p_{\Omega}, p_{r^*}) + [G_{n_r}^2] R^2 S_1(p_r, p_{\Omega}, p_{r^*}) \right) \delta_{q_r+q_{\Omega}, q_{r^*}} \right\}_{n_r+n_{\Omega}, n_{r^*}} \quad (3.57)$$

R_φ matrix elements are generated from the following for H :

$$\frac{2\pi T}{[\pi R^2 T][J_1(\rho_{p_\varphi^*})]^2} \sum_{\Omega} \kappa_{\Omega} [-jq_r S_{-1}(p_r, p_{\Omega}, p_{\varphi^*}) - jq_r \rho_{p_r} T_0(p_r, p_{\Omega}, p_{\varphi^*}) - jq_r \rho_{p_{\Omega}} U_0(p_r, p_{\Omega}, p_{\varphi^*})] \delta_{q_r+q_{\Omega}, q_{\varphi^*}} \delta_{n_r+n_{\Omega}, n_{\varphi^*}} \quad (3.58)$$

for E :

$$\frac{2\pi T}{[\pi R^2 T][J_1(\rho_{p_\varphi^*})]^2} \sum_{\Omega} \kappa_{\Omega} [-jq_r S_{-1}(p_r, p_{\Omega}, p_{\varphi^*}) - jq_r \rho_{p_r} T_0(p_r, p_{\Omega}, p_{\varphi^*})] \delta_{q_r+q_{\Omega}, q_{\varphi^*}} \delta_{n_r+n_{\Omega}, n_{\varphi^*}} \quad (3.59)$$

R_z matrix elements are generated from the following for H :

$$\frac{2\pi T}{[\pi R^2 T][J_1(\rho_{p_z^*})]^2} \sum_{\Omega} \kappa_{\Omega} \delta_{q_r+q_{\Omega}, q_{z^*}} R^2 \left[\left(\frac{jG_{n_r}}{R} \right) S_o(p_r, p_{\Omega}, p_{z^*}) + \left(-\frac{jG_{n_r} \rho_{p_r}}{R} \right) T_1(p_r, p_{\Omega}, p_{z^*}) + \left(-\frac{jG_{n_r} \rho_{p_{\Omega}}}{R} \right) U_1(p_r, p_{\Omega}, p_{z^*}) \right] \quad (3.60)$$

for E :

$$\frac{2\pi T}{[\pi R^2 T][J_1(\rho_{p_z^*})]^2} \sum_{\Omega} \kappa_{\Omega} \delta_{q_r+q_{\Omega}, q_{z^*}} R^2 \left[\left(\frac{jG_{n_r}}{R} \right) S_o(p_r, p_{\Omega}, p_{z^*}) + \left(-\frac{jG_{n_r} \rho_{p_r}}{R} \right) T_1(p_r, p_{\Omega}, p_{z^*}) \right] \quad (3.61)$$

Following equations are related to the expansion coefficient κ_φ .

Φ_r matrix elements are generated from the following for H :

$$\frac{2\pi T}{(\pi R^2 T) [J_1(\rho_{p_r^*})]^2} \left\{ \sum_{\Omega} \kappa_{\Omega} j(q_{\Omega} + q_{\varphi}) \left[S_{-1}(p_{\varphi}, p_{\Omega}, p_{r^*}) - (\rho_{p_{\varphi}}) T_o(p_{\varphi}, p_{\Omega}, p_{r^*}) \right] \delta_{q_{\varphi}+q_{\Omega}, q_{r^*}} \right\} \delta_{n_{\varphi}+n_{\Omega}, n_{r^*}} \quad (3.62)$$

for E :

$$\frac{2\pi T}{(\pi R^2 T) [J_1(\rho_{p_r^*})]^2} \left\{ \sum_{\Omega} \kappa_{\Omega} j(q_{\varphi}) \left[S_{-1}(p_{\varphi}, p_{\Omega}, p_{r^*}) - (\rho_{p_{\varphi}}) T_o(p_{\varphi}, p_{\Omega}, p_{r^*}) \right] \delta_{q_{\varphi}+q_{\Omega}, q_{r^*}} \right\} \delta_{n_{\varphi}+n_{\Omega}, n_{r^*}} \quad (3.63)$$

Φ_{φ} matrix elements are generated from the following for H :

$$\begin{aligned} \frac{2\pi T}{[\pi R^2 T] [J_1(\rho_{p_{\varphi^*}})]^2} \sum_{\Omega} \kappa_{\Omega} \delta_{q_{\varphi}+q_{\Omega}, q_{\varphi^*}} R^2 \left[\left(\left\{ G_{n_{\varphi}} G_{n_{\Omega}} + G_{n_{\varphi}}^2 + \frac{\rho_{p_{\varphi}}^2}{R^2} \right\} S_1(p_{\varphi}, p_{\Omega}, p_{\varphi^*}) \right. \right. \\ \left. \left. + \frac{1}{R^2} S_{-1}(p_{\varphi}, p_{\Omega}, p_{\varphi^*}) \right) + \left(\frac{\rho_{p_{\Omega}}}{R^2} \right) U_0(p_{\varphi}, p_{\Omega}, p_{\varphi^*}) \right. \\ \left. \left. + \left(-\frac{\rho_{p_{\varphi}} \rho_{p_{\Omega}}}{R^2} \right) V_1(p_{\varphi}, p_{\Omega}, p_{\varphi^*}) \right] \right] \end{aligned} \quad (3.64)$$

for E :

$$\begin{aligned} \frac{2\pi T}{[\pi R^2 T][J_1(\rho_{p_{\varphi^*}})]^2} \sum_{\Omega} \kappa_{\Omega} \delta_{q_{\varphi}+q_{\Omega}, q_{\varphi^*}} R^2 \left[\left\{ G_{n_{\varphi}}^2 + \frac{\rho_{p_{\varphi}}^2}{R^2} \right\} S_1(p_{\varphi}, p_{\Omega}, p_{\varphi^*}) \right. \\ \left. + \frac{1}{R^2} S_{-1}(p_{\varphi}, p_{\Omega}, p_{\varphi^*}) \right] \end{aligned} \quad (3.65)$$

Φ_z matrix elements are generated from the following for H :

$$\frac{2\pi T}{[\pi R^2 T][J_1(\rho_{p_{z^*}})]^2} \sum_{\Omega} \kappa_{\Omega} \delta_{q_{\varphi}+q_{\Omega}, q_{z^*}} R \left(-G_{n_{\varphi}} q_{\Omega} - G_{n_{\varphi}} q_{\varphi} \right) S_0(p_{\varphi}, p_{\Omega}, p_{z^*}) \quad (3.66)$$

for E :

$$\frac{2\pi T}{[\pi R^2 T][J_1(\rho_{p_{z^*}})]^2} \sum_{\Omega} \kappa_{\Omega} \delta_{q_{\varphi}+q_{\Omega}, q_{z^*}} R \left(-G_{n_{\varphi}} q_{\varphi} \right) S_0(p_{\varphi}, p_{\Omega}, p_{z^*}) \quad (3.67)$$

Following equations are related to the expansion coefficient κ_z .

Z_r matrix elements are generated from the following for H :

$$\frac{2\pi T}{(\pi R^2 T)[J_1(\rho_{p_r^*})]^2} \left\{ \sum_{\Omega} \kappa_{\Omega} (-j\rho_{p_z} R) (G_{n_{\Omega}} + G_{n_z}) T_1(p_z, p_{\Omega}, p_{r^*}) \delta_{q_z+q_{\Omega}, q_{r^*}} \right\} \quad (3.68)$$

for E :

$$\frac{2\pi T}{(\pi R^2 T)[J_1(\rho_{p_r^*})]^2} \left\{ \sum_{\Omega} \kappa_{\Omega} (-j\rho_{p_z} R) (G_{n_z}) T_1(p_z, p_{\Omega}, p_{r^*}) \delta_{q_z+q_{\Omega}, q_{r^*}} \right\} \quad (3.69)$$

Z_φ matrix elements are generated from the following for H :

$$\frac{2\pi T}{[\pi R^2 T][J_1(\rho_{p_{\varphi^*}})]^2} \sum_{\Omega} \kappa_{\Omega} \delta_{q_z+q_{\Omega}, q_{\varphi^*}} R(-q_z G_{n_{\Omega}} - q_z G_{n_z}) S_0(p_z, p_{\Omega}, p_{\varphi^*})$$

(3.70)

for E :

$$\frac{2\pi T}{[\pi R^2 T][J_1(\rho_{p_{\varphi^*}})]^2} \sum_{\Omega} \kappa_{\Omega} \delta_{q_z+q_{\Omega}, q_{\varphi^*}} R(-q_z G_{n_z}) S_0(p_z, p_{\Omega}, p_{\varphi^*})$$

(3.71)

Z_z matrix elements are generated from the following for H :

$$\begin{aligned} \frac{2\pi T}{[\pi R^2 T][J_1(\rho_{p_{z^*}})]^2} \sum_{\Omega} \kappa_{\Omega} \delta_{q_z+q_{\Omega}, q_{z^*}} R^2 \left[\left(\frac{\rho_{p_z}}{R} \right)^2 S_1(p_z, p_{\Omega}, p_{z^*}) \right. \\ \left. + \left(\frac{q_z q_{\Omega} + q_z^2}{R^2} \right) S_{-1}(p_z, p_{\Omega}, p_{z^*}) + \left(-\frac{\rho_{p_z} \rho_{p_{\Omega}}}{R^2} \right) V_1(p_z, p_{\Omega}, p_{z^*}) \right] \end{aligned}$$

(3.72)

for E :

$$\frac{2\pi T}{[\pi R^2 T][J_1(\rho_{p_{z^*}})]^2} \sum_{\Omega} \kappa_{\Omega} \delta_{q_z+q_{\Omega}, q_{z^*}} R^2 \left[\left(\frac{\rho_{p_z}}{R} \right)^2 S_1(p_z, p_{\Omega}, p_{z^*}) + \left(\frac{q_z^2}{R^2} \right) S_{-1}(p_z, p_{\Omega}, p_{z^*}) \right]$$

(3.73)

3.9 Solving Procedure and Computer Program

In this section is outlined the steps required in the computation process. First the dielectric is defined and discretized in the polar cylindrical grid. The expansion coefficients of the dielectric series are determined as mentioned above. A sufficient number of basis functions are used to ensure proper series representation. The matrix-system (3.49) is populated by first selecting the number of basis functions for the field expansions, this defines the order of matrix, the individual elements of the matrix are obtained using matrix generating expressions (3.56) to (3.73). The eigenvalues and eigenvectors are determined using Matlab© eig() function. One eigenvalue is obtained for each combination of the basis indices used in the series expansions. The localized states present may be hidden amongst thousands of eigenvalues. A few simple search algorithms are available to facilitate the localized state identification. The simplest and most time consuming is to plot each eigenvector and examine the individual profiles for the wavelength range of interest.

3.10 Summery

For structures that contain an extended axis which serve for the reference for cylindrical symmetry, it is shown that an expansion of the fields and inverse of the relative dielectric profile using a simplified and complete set of basis functions of Fourier-Bessel terms provide access to an eigenvalue formulation from which the eigen-states can be computed. The basis functions used in the Fourier-Fourier-Bessel technique are the product of lowest order Bessel function (J_0) in the radial r and Fourier functions ($e^{jq\phi}$) in the angular direction ϕ , and ($e^{jG_n z}$) in the axial direction z . The steps used to convert

Maxwell's equation into an eigenvalue formulation were presented. In this chapter, theory covers the $\vec{H}$ field equation, which is more complicated due to the curl of the inverse of relative permittivity. $\vec{E}$ field equation is simpler and is a subset of $\vec{H}$ field, as the inverse of relative permittivity is external to the double curl operator. The components of $\vec{E}$ field can be found in section 3.8.1 as well as appendix B, equations (B.17), (B.23), and (B.29). The results from FFB technique strongly agree with other numerical methods such as PWEM and FDTD. Due to the 3-D nature of the structures being examined in this thesis PWEM and FDTD require several days of computation provided that the computer resources are available.

The next chapter will discuss the application of the technique. For cylindrically symmetric structures, the FFB computational technique provides a significantly reduced matrix order to be populated.

Chapter 4. Results and Application

4.1 Introduction

The steps of obtaining the eigenvalue formulation from which the eigen-states can be computed, was detailed in the previous chapter. It is to be mentioned that in FFB technique the basis functions impose that all field components go to zero at the radial limit thus the infinite dielectric presence is minimized for localized states that are well confined to the central region of the radial domain. The application of the FFB technique requires that the structure under analysis be well removed from the radial limit such that localized states of the structure have negligible fields near $r = R$ where r is the radius of the dielectric structure under consideration, and R is the radius of the computation domain. As discussed at the end of chapter two, a similar approach is imposed in applications of PWEM for steady state computation using a super-cell with the defect at the center [18].

In this chapter several structures will be examined using FFB technique to compute the steady states. Convergence test is performed by examining the Bragg structure using several Bessel numbers. Convergence and accuracy of the FFB results can be tested by examining the shift in the eigenvalues and field profiles when the radial index number is increased and by comparing the results returned from FFB to those returned from PWEM for dielectric structures for which both techniques are applicable, such as, square and hexagonal array photonic crystals.

4.2 Dielectric structures and FFB computation

The dielectric structures considered here are shown in figure 4.1. The dielectric regions in white are silicon with a relative dielectric constant of 12.1104, the black regions

correspond to a glass medium with relative dielectric constant of 2.3716. The top-left shows the Bragg structure in the (r, φ) plane; top-right shows the (r, z) plane cross-section of the concentric cylinder resonator; bottom-left and right show the (r, z) cross-section for the uniform silicon and alternating medium slot channel configuration on Bragg fiber supports. The large outer black regions in the concentric cylinders image and lower two profiles is air with relative dielectric constant of 1.000. The geometrical parameters for each structure are indicated in the numerical computation sections that follow. These particular structures show no dielectric variations with respect to the angular coordinate φ . As a result, the series expansion of the inverse relative permittivity using the basis functions of (3.2) has non-zero expansion coefficients for $q = 0$ only. This uniform rotational symmetry present transforms the computation process to one that depends on only two indices (p, n) , effectively converting a 3-D structure into a 2-D numerical computation problem. Such a step greatly reduces the order of the matrix required to represent the structure and return well converged eigenvalues. When the matrix is populated using the expressions in chapter 3, independent sub-matrices can be extracted for each mode type and solved individually (monopole $q_i = 0$, dipole $q_i = \pm 1$, quadruple $q_i = \pm 2, \dots$). Mode tuning further reduces the order of the matrix (3.49) and permits the process to focus on a particular mode type.

4.3 Bragg structure

The Bragg ring structure in figure 4.1 top-left is composed of 0.105 μm wide silicon and 0.250 μm wide glass. These parameters are taken from quarter-wave Bragg reflection theory [19,20]. The central disk is silicon and has a radius of 0.725 μm . The rings

extend out radially to $5\text{ }\mu\text{m}$ which is the radius of the computation domain. This dielectric structure is uniform in the axial z -direction. As a result the non-zero matrix elements are characterized with $n = 0$ in addition to $q = 0$. The localized states of infinite extent Bragg rings can be examined using the equivalent of a 1-D numerical computation technique. The infinite axial extent permits the decoupling of the TE and TM polarizations.

Figure 4.1 Top-left; Bragg rings. Top-right; Concentric cylinders. Bottom-left; Uniform slot channel on Bragg fiber. Bottom-right; Alternating medium slot channel on Bragg fiber. White is silicon with $\epsilon_{r,Si} = 12.1104$ and black is a glass medium with $\epsilon_{r,Gl} = 2.3716$. Top right and lower images, large external black regions are air.

Results are presented in figure 4.2 for all selected eigenvalues with those for the TM polarization with field components (H_x, H_y, E_z) circled. Field profiles are produced for

the E_z component. The choice of field component and polarization state is selected based on the geometry and orientation of the slot channel structures, that are examined later.

The eigen-wavelengths were computed using 100 Bessel terms ($p = 1 \rightarrow 100$) for the monopole states with values in the 1.00 to 3.00 μm range plotted in figure 4.2. The step in the eigenvalues with respect to wavelength corresponds to the in-plane bandgap range of the Bragg structure. The centrally localized monopoles are indicated and an intensity profile of the E_z field component is shown.

Figure 4.2 Eigen-wavelengths between 1 and 3 μm returned for the monopole states of the Bragg structure. Circled values have the eigenvector dominated by the E_z field component. Localized monopole state of wavelength 2.111 μm and 1.556 μm present in the bandgap shown with E_z field intensity plotted in the insert.

Amongst tens of monopole states of the Bragg structure two were chosen to indicate different types of non-localized modes. Figure 4.3 shows edge state at $\lambda = 1.274 \mu\text{m}$ and non-localized mode at $\lambda = 2.635 \mu\text{m}$ with E_z field component.

Figure 4.3 E_z field component, Left: Non-localized mode of the Bragg structure in (x,y) plane, left: $\lambda = 2.635\mu\text{m}$ and right: edge state at $\lambda = 1.274\mu\text{m}$

Figure 4.4 shows the convergence test for the Bragg structure. The Bessel number was changed from 10 to 200 in increments of 10. Converged final value at Bessel number of 200 is wavelength $2.093 \mu\text{m}$. At Bessel number 100, which is the number of basis function used to find the steady states in infinite Bragg, results are well converged for $\lambda = 2.111 \mu\text{m}$ and the other at $\lambda = 1.556 \mu\text{m}$.

Figure 4.4 Convergence test of the localized monopole states of wavelength 2.111 μm (top) and 1.556 μm (bottom) of the Bragg structure, with 5% error bars.

The effect of restricting the height of the Bragg structure is considered next. The computation domain height T , was set to 7 μm and the Bragg region, centered on the coordinate plane had a thickness of 5 μm . The introduction of dielectric variations in the axial direction converts the computations to 2-D requiring basis functions with indices (p, n) . The monopole states were computed with 70 Bessel terms and 81 axial terms.

Plots of the E_z field component intensity for the localized state (wavelength = 2.177 μm) in the (r, z) and (x, y) planes are shown in figure 4.5. The width of the central zone of the Bragg structure was modified to 0.625 μm in order to obtain a strongly localized state.

Figure 4.5 Intensity plot for the E_z field profile shown in the (r, z) plane, left, and (x, y) plane, right, for the localized state of the membrane Bragg structure. Membrane thickness is $5\text{ }\mu\text{m}$, state wavelength is $2.177\text{ }\mu\text{m}$.

4.4 Concentric cylinders

An alternate approach to restricting the out-of-plane extent of the Bragg structure is to introduce a sequence of Bragg planes oriented along the z axis. In effect the radial Bragg and the axial Bragg regions can be combined producing a series of concentric alternating index dielectric cylinders. Such a structure is the cylindrical space equivalent of alternating index dielectric spheres [8]. Concentric cylinders have come to attention as distributed Bragg reflector (DBR) microwave resonators [21] as well. The radial Bragg parameters were unchanged from those given above. The axial Bragg was composed of $0.222\text{ }\mu\text{m}$ thick silicon and $0.504\text{ }\mu\text{m}$ thick glass. The central cylinder was selected to have a height of $10\text{ }\mu\text{m}$. Due to the lack of dielectric variations in the φ direction the computation of the eigenstates is a 2-D numerical equivalent. The eigen-wavelengths were computed for the monopole states using 75 Bessel terms and 81 axial terms. A strongly localized monopole is observed at a wavelength of $1.785\text{ }\mu\text{m}$. The E_z field component of this mode is plotted in the (r, z) and (x, y) planes in figure 4.6.

Figure 4.6 Intensity plot for the E_z field component of the concentric cylinder centrally localized state at wavelength $1.785 \mu\text{m}$.

4.5 Uniform slot channel on Bragg fiber

An interesting configuration for optical field confinement is the slot channel waveguide [6]. Two thin and closely spaced high relative dielectric constant ridges can enhance the channel amplitude for the electric field component perpendicular to the channel walls. Such a structure is usually configured as a linear channel resting on a substrate support [22]. Recently it has been theorized and experimentally demonstrated that the slot channel waveguide properties are present in cylindrical configurations [6]. The structures shown in the bottom of figure 4.1 have the high dielectric ridges running the perimeter of a Bragg fiber support. In practice they could be fabricated through selective etching into the Bragg fiber or by rolling up a released silicon-oxide membrane [23]. The structure on the bottom left of figure 4.1 has silicon ridges (width $0.3 \mu\text{m}$, height $0.25 \mu\text{m}$) separated by 200 nm . These parameters are considered with reference to a published paper of our group [6]. The Bragg fiber support is chosen to have the dielectric properties of the infinity extended Bragg ring configuration shown in figure 4.1 top-left examined above.

The Bragg support has an outer radius of 4.63 μm and at this radius the circumference along the slot channel is 29.09 μm . Given that the medium in the slot channel is air, the anticipated lowest azimuthal mode order for channel confined states should lie close to 25 for a wavelength in the 1.55 μm range. This is an estimation and calculated from angular momentum considerations:

$$m\hbar = r \frac{2\pi}{\lambda} n_{eff} \hbar \quad (4.1)$$

where m is mode order, $\hbar$ is reduced Planck constant, r is the radius and n_{eff} is the effective index of the mode, assumed to be 2.

The eigen-wavelength space was computed for several azimuthal mode orders using 60 Bessel and 61 axial terms forming the basis functions. The computation domain is still 2-D for this structure as the dielectric presents no variations with the φ coordinate. The E_z field component profiles in the channel region were plotted. Two strongly slot localized state were observed for azimuthal mode order 25, one at a wavelength of 2.036 μm and the other at 2.026 μm . The intensity of the E_z field component is plotted in the (x, y) plane, right, and (r, z) plane, left, of figure 4.7 for the 2.036 μm state. The top-down arrow indicates the outer edge of the Bragg support. The two small squares indicate the size of the silicon ridges residing on the Bragg support.

Figure 4.7 Intensity plot for the E_z field component shown in the (r, z) , left, and (x, y) plane, right, for an azimuthal order 25 slot channel localized state. State wavelength is $2.036 \mu\text{m}$.

4.6 Alternating medium slot channel on Bragg fiber

The alternating layer slot channel waveguide is composed of silicon and glass layers and would result when the Bragg region of the support fiber is etched into. In the slot ridge region the silicon layers have a width of $0.105 \mu\text{m}$ and the glass medium has a width of $0.25 \mu\text{m}$. The ridges are made of 2 alternating layer pairs. The eigen-wavelength space was computed for several azimuthal mode orders using 60 Bessel and 61 axial terms forming the basis functions. The intensity profiles for the E_z field component with azimuthal order 25 and at a wavelength $2.007 \mu\text{m}$ is shown plotted in figure 4.8. Forming the channels directly by etching into the side wall of the Bragg region should provide a structure capable of supporting slot channel modes.

Figure 4.8 Plot for the E_z field component shown in the (r, z) intensity, left, and amplitude (x, y) plane, right, for an azimuthal order 25 slot channel localized state. State wavelength is $2.007 \mu\text{m}$.

4.7 Summery

In this chapter it is shown that the FFB technique is suitable for obtaining the steady states in 1-D infinite Bragg, 2-D by restricting the height of the infinite Bragg and slot channel configurations. The convergence issue that might come to mind was identified. The test that was performed for infinite Bragg structure, shows that the results are well converged within the number of radial basis number chosen. In the next chapter FFB technique is applied to novel slot channel configurations that are suitable for sensor consideration.

Chapter 5. Refractive Index Sensors Based on Slot Channel

Waveguides

5.1 Introduction

Researchers have been attracted towards optical sensors because of their remarkable characteristics such as immunity to interference, remote sensing, and also the availability of well-implemented technologies from telecommunication industries. The healthcare development has come into a conjugation with photonics based sensors [24]. An interesting feature of optical sensing is that a wide range of physical properties of bio-related materials in nature can be detected and examined using optical frequencies [25]. When light propagates through a waveguide, although a major amount of optical mode power travels in the core, there is still a portion of light (field) in the cladding or substrate regions. When two high relative permittivity (ϵ_r) waveguides are very close the evanescent light field can overlap constructively to produce a high field in the low interstitial index region. An alternate guidance technique, known as the slot channel waveguide, exploits the continuity of the normal component (as a result of boundary condition) of the electric flux density to provide a large electric field in a low relative dielectric medium sandwiched between thin silicon strips. The outstanding configuration of slot channel waveguide based sensors allow light to propagate and strongly confined inside a nano-scale region of low refractive index, making it possible to efficiently sense the medium in between two high ϵ_r channels. Novel structures such as photonic biosensor based on vertically stacked ring resonators [26], double-slot ring waveguides optical temperature sensors [27], LC filling slot waveguide temperature sensor [28], and

on-chip chemical sensing devices based on mid-infrared spectrometer using opto-nanofluidic slot channel [29], have been studied by researchers.

In this chapter, several novel configurations of the slot channel waveguide based sensors in cylindrical domain are simulated and examined using the FFB mode solving technique. The confined slot modes are plotted and investigated with respect to the index of refraction change in the slot region. The optical properties of the slot channel waveguide in cylindrical space can best be examined through the E_z field component, as discussed in chapter 4 for the geometry of the rings and coordinate system. In the cylindrical geometry this is the field component that is discontinuous by the ratio of the relative dielectric constants when passing the air-silicon interfaces in the z -direction. All the proposed dielectric structures in this chapter have rotational symmetry about the z -axis and therefore the only value for the q_Ω index (φ direction) that generates non-zero dielectric expansion coefficients is zero ($q_\Omega = 0$). The dielectric expansion using the FFB basis functions compresses to a 2-D expansion requiring only summations on the p_Ω (r direction) and n_Ω (z direction) indices, as discussed in chapter 3.

5.2 Basic structure of Silicon channels in the Air

Figure 5.1 3-D view of the stacked silicon ring slot channel waveguide configuration in air. Design parameters can be found in text.

The structure considered consists of 4 high relative permittivity rings of silicon placed in a square array as shown in figure 5.1. Each ring has a radial width of $W = 0.3 \mu\text{m}$ and height of $H = 0.25 \mu\text{m}$. The spacing between ring stacks is $P = 0.2 \mu\text{m}$. The inner stacked ring pair have a radius $R_1 = 1.7 \mu\text{m}$ (axis to inner edge of ring) while the outer stacked rings have a radius $R_2 = 2.29 \mu\text{m}$. This gives an edge to edge radial ring spacing of $S = 0.29 \mu\text{m}$. For computational purposes related to the general response of the structure to resonant field propagation is performed with the ring configuration placed in air. The structure parameters provided are those that were explored in the sensor configuration detailed in the next section. For exploratory purposes, the width, radial and axial spacing, inner and outer radius were varied. Given this configuration and the components of the electric field, one anticipates that slot channel waveguide

configurations may be present in the axial direction through the boundary condition of the axial field component, or, between inner and outer rings through the boundary conditions of the radial component.

5.3 Ring structure resonator states

An estimation of the expected mode order for the outer silicon channel can be obtained using the equation (4.1). As discussed in chapter 4, based on the ring structure shown in figure 5.1 the circumference of the outer silicon region is calculated as $2\pi r = 15.33 \mu m$, where r is; $1.7 \mu m$ (radius of glass core) + $0.3 \mu m$ (width of the inner silicon channel) + $0.29 \mu m$ (radial distance between two silicon channels) + $0.15 \mu m$ (half of the outer silicon channel). In order to get an estimate of the azimuthal mode order, the circumference is divided by the wavelength of interest ($1.55 \mu m$), then multiplied by n_{eff} of 2, as an assumption, gives 19.8. This means to fill the circumference of outer channel, 20 of $\lambda = 1.55 \mu m$ are needed which implies 40 peaks in intensity meaning mode order of 20. Several azimuthal mode order possibilities were also examined and it is found that the mode 20 produced strongly well confined states and suitable sensing performances.

Exploring results for several pairs of radial and axial indices limits (p, n) , such as (45, 83), (55, 103), and (65, 123), shows that the last pair gives well converged results. The system-matrix for azimuthal mode order 20 was constructed using 123 axial exponentials $-61 \leq n_{\Omega,i} \leq +61$ and 65 Bessels $1 \leq p_{\Omega,i} \leq 65$ and generating symmetry reduced matrix order of 23595. Not all the states needed to be examined but as mentioned before

in this section it is expected that the outer slot channel modes should be within a wavelength range that corresponds to the physical dimensions of the structure.

Figure 5.2 shows plots of non-slot channel states for E_r ($\lambda = 1.296 \mu\text{m}$) and E_z ($\lambda = 1.203$ and $1.698 \mu\text{m}$) field component intensity in the (r, z) plane. The E_r field component is confined in the inner silicon channels at $\lambda=1.296 \mu\text{m}$ as shown in figure 5.2. This field component does not fulfill the boundary condition for slot channel states discussed in the beginning of this chapter, therefore it is not confined in the slot channel. In order to get the radial slot channel the parameters of the structure needed to be changed. Slot channel states are examined in the next section.

Figure 5.2 Eigen-wavelength between 1.2 and 1.7 μm for the four channels in the air showing different types of modes that are not slot channel, in (r, z) plane.

One common application of the slot channel waveguide configuration is to sense minute changes in the ambient index of refraction [30]. Figure 5.3 shows a plot of the E_z

field component of the eigenvector expansion coefficients, plotted for increasing ambient relative permittivity and index of refraction. The relative permittivity change is from $\epsilon_r = 1$ for air up to $\epsilon_r = 2.4$ with increments of 0.05. It is observed that the eigenvector is well converged with the basis numbers selected. The high field and its confinement to both inner and outer channels is plotted for different ϵ_r (and n) values.

Figure 5.3 E_z Field profiles plotted in (r, z) plane for channels in the air structure, λ changes with the index change for both inner channel (bottom) and outer channel (top), with 1% error bars.

As it can be seen the slot channel state of both inside and outside channels happens at the same λ for index 1 (air). This particular hybrid resonator state is still under investigation. For $\epsilon_r=1.05$ the outer channel state jumps to higher λ and inner channel state drops to a lower λ and the outer and inner states decoupled. Eliminating first three

points of ϵ_r , the absolute value difference of the inside and outside state's wavelength values is plotted in figure 5.4. A linear relationship is obtained.

Figure 5.4 Sensitivity of channels in the air structure. Absolute value difference of inner and outer states with respect to the index change, with 5% error bars.

The change of $|\lambda_{inner} - \lambda_{outer}|$ with respect to ϵ_r (or n), is such that it can be considered a more sensitive sensor compare to published research papers [25,31] . The resulting RI sensitivity is calculated to be $\Delta\lambda/\Delta n = 269 \frac{nm}{RIU}$ where $\Delta\lambda$ is the resonance wavelength difference shift and Δn is the index variation. The fact that the difference of both inner and outer resonances are taken into account has lead to a more sensitive device compare to [25,31].

5.4 Sensing configurations

In order to bring this novel dielectric ring configuration to the real world applications, some sort of support must be added to the ring structure. Three different support configurations are examined in the following.

5.4.1 Suspended Slot Channel

Figure 5.5 shows the support glass added to the channels in the air. The inner Silicon channel is not touching the core glass support, allowing the medium change to be sensed effectively by the inner channel as well as outer channel.

Figure 5.5 Suspended dual ring stack sensor configuration. Silicon rings supported by glass material, fluid/gas infiltration into channel regions. 2-D in (r, z) plane.

Observing figure 5.6 indicates that the behavior of suspended slot channel is pretty much the same as the one that is seen in the previous section, channels in the air. The

reason could come from the fact that both structures are almost the same and there is no glass infiltration in between inside/outside silicon waveguides. In both structures the inner and outer states are exposed to the medium resulting in the sharp change of λ due to the relative permittivity (index of refraction) change.

Figure 5.6 shows a plot of the E_z field component for increasing ambient relative permittivity and index of refraction. This structure shows pretty much same behavior as previous one. The slot channel state of both inside and outside channels happens at the same λ for index 1 (air) and separates thereafter. The inner state varies from $1.497\ \mu\text{m}$ to $1.32\ \mu\text{m}$, and the outer state varies from $1.497\ \mu\text{m}$ to $1.54\ \mu\text{m}$ which is within the same range as the previous structure, 4 channels all in the ambient medium.

Figure 5.6 E_z Field profiles plotted in (r, z) plane for suspended structure. λ changes with the index change for both inner channel (bottom) and outer channel (top), with 1% error bars.

As it comes to the sensitivity discussion, it is expected that adding the glass as support, has minor effects on the sensitivity of the structure since both inner and outer states are well exposed to the medium change. Figure 5.7 shows the sensitivity of the suspended slot channel structure, which is slightly more sensitive compare to previous structure. Eliminating the first two points of ϵ_r , the resulting RI sensitivity was measured to be $277 \frac{nm}{RIU}$. This type of refractive index sensor is highly sensitive for such a small range of ϵ_r .

Figure 5.7 Sensitivity of suspended structure. Absolute value difference of inner and outer states with respect to index change, with 5% error bars.

5.4.2 Buried Slot Channel Structure

In this structure the inside silicon waveguide is surrounded by glass, figure 5.8. Isolating the inner channel was with the hope of having it as a reference that does not see the ambient relative permittivity (ambient index of refraction) change. Referring to the computation results in figure 5.9, it is seen that the outer states follow the same path as previous structures with a dip in the beginning and gradually rise thereafter. Outer channel states are nearly within the same range of λ as previous structures, the small difference is due to the glass support extended to the inside edge of outer channel. The

inner states don't show a dramatic change as the channel is buried inside the glass. In this case λ_{inside} varies from $1.347 \mu\text{m}$ for $\epsilon_r = 1$ to $1.323 \mu\text{m}$ for $\epsilon_r = 2.4$.

Figure 5.8 Buried dual ring stack sensor configuration. Inner Silicon ring buried in glass material, outer silicon ring supported by glass, fluid/gas infiltration into outer channel region. 2-D in (r, z) plane.

Figure 5.9 E_z Field profiles plotted in (r, z) plane for buried structure, λ changes with the index change for both inner channel (bottom) and outer channel (top), with 1% error bars.

As mentioned before the initial design of the structure intended for a flat line of inner states, bottom line of figure 5.9, to be considered as a reference. Although this line is not perfectly flat but there is a very small shift of wavelength compare to the index variation.

Sensitivity of this structure is expected to be less compare to both previous structures, since the inner channel is surrounded by glass and isolated from medium change. By eliminating first four points of ϵ_r , figure 5.10 shows the sensitivity response of the buried slot channel structure. This sensor is less sensitive in first curved part ($\epsilon_r = 1.2$ to 1.8) with $132 \frac{nm}{RIU}$ and more sensitive in the flat range ($\epsilon_r = 1.85$ to 2.4) with $342 \frac{nm}{RIU}$. In general this structure is less sensitive in the curved part and more sensitive in the flat

region, compare to the previous structures. Referring to figure 5.9, in the flat range ($\epsilon_r = 1.85$ to 2.4) the inner state is slightly decaying, which results in a higher difference between inner and outer states. While in other sensor configurations, mentioned before, the inner state is almost flat at this range and doesn't change much.

Figure 5.10 Sensitivity of buried structure. Absolute value difference of inner and outer states with respect to the index change, with 5% error bars.

5.4.3 Regular Slot Channel with glass infiltration

Regular slot channel configuration links back to the one examined in chapter 4. The glass support between inner and outer silicon channels added. As shown in figure 5.11.

Figure 5.11 Regular slot channel dual ring stack sensor configuration. Silicon rings supported by glass material (horizontally), fluid/gas infiltration into channel regions between top and bottom silicon waveguides. 2-D in (r, z) plane

This structure shows pretty much the same behavior as the buried slot channel structure. The reason could be the glass infiltration between inner silicon channel and outer silicon channel. The structure parameters are chosen such that the outer channel state is confined approximately at $\lambda = 1.55 \mu\text{m}$.

The outer slot channel states show the same behavior as the previous structures, figure 5.12. Outer states vary within the range of λ similar to the structures mentioned

before. The inner state change is similar to the buried slot channel structure, in the way that the λ variation range is small.

Figure 5.12 E_z Field profiles plotted in (r, z) plane for regular slot channel structure, λ changes with the index change for both inner channel (bottom) and outer channel (top)

Eliminating the first three points of ϵ_r , the sensitivity graph, figure 5.13, can be divided into two sections. The first curved part ($\epsilon_r = 1.15$ to 1.8) with $92 \frac{nm}{RIU}$ and second curved part ($\epsilon_r = 1.85$ to 2.4) with $241 \frac{nm}{RIU}$. Overall this sensor configuration is the least sensitive compare to the previous ones and comparable to published sensors [25, 31].

Figure 5.13 Sensitivity of regular slot channel structure. Absolute value difference of inner and outer states with respect to the index change, with 5% error bars.

5.5 Summery

In this chapter the FFB computation technique was used to obtain the steady states of several novel refractive index sensors based on the slot channel configuration. First structure, 4 channels in the air, was for exploratory proposes then the support glass was added such a way that three different structures proposed and examined. The 4 channels in the air and the suspended structures show pretty much same behavior and sensitivity to the ambient medium change. These two structures are more sensitive compare to those typically published [25,31], which is due to the consideration of both inner and outer ring slot channel states. The idea of designing the buried slot channel sensor was to have the inner state as reference since it is not exposed to the index of refraction change, but it still changes slightly, which makes this type of sensor supper sensitive in the flat region. The outer slot channel state shows the same behavior for all four structures and varies almost in the same range for all of them, since it is well exposed to the ambient medium change.

Chapter 6. Conclusion

6.1 Summary of work

This work has presented an outline of the numerical steps based on expanding the inverse of relative permittivity and fields of the vector wave equation that convert Maxwell's wave equation in cylindrical space to an eigenvalue problem that returns information on the wavelengths and field profiles of the localized states. Through the use of Fourier-Fourier-Bessel expansions it has been shown that in cylindrical coordinate, structures with rotational symmetry can be examined to find the localized steady states. The master equation is also expressed in cylindrical coordinates to illustrate how it could be simplified through the use of rotational order within the localized modes mentioned. It is indicated that the FFB technique can be tuned for a particular mode type and that the order of the system-matrix can be significantly reduced through the rotational symmetry of the dielectric structure. It is discussed that PWEM will require very large number of plane waves to obtain the steady states of proposed dielectric structures in chapter 5 of this thesis, therefore it is not a practical approach. Numerical computations are presented for the states supported by various 1-D, 2-D and 3-D Bragg configurations. A slot channel design supported on a Bragg fiber is shown to provide access to slot channel confined modes that orbit the axis of the Bragg structure. The analysis is further extended by considering four different slot channel waveguide structures, which are considered as refractive index sensors. The structure parameters are chosen such that the outer channel state is confined approximately at $\lambda = 1.55 \text{ }\mu\text{m}$. Considering buried and regular slot channel structures, the inner channel state is less exposed to the medium resulting in smaller change of wavelength while index changes. The suspended structure

is designed such that both inner and outer states are exposed to the medium change, therefore the sensitivity is higher than two aforementioned structures. These devices sensitivity ($\Delta\lambda/\Delta n \frac{nm}{RIU}$) of these devices are comparable. The most sensitive sensor is the buried slot channel structure in the flat range ($\epsilon_r = 1.85$ to 2.4) with $342 \frac{nm}{RIU}$ and the second sensitive sensor is the suspended slot channel in the range of ($\epsilon_r = 1$ to 2.4) with $277 \frac{nm}{RIU}$.

6.2 Future work

Some aspects of this work will require further research and examination. One aspect is the computational solution of the master equation within the cylindrical coordinates in terms of the Fourier-Fourier–Bessel expansions. In this work the medium is assumed to be non-magnetic, further development of the method could result in including the magnetic properties and complex relative permittivity of different structures. This will provide the feature of considering the solutions to a broader range of designs and permit the examination of plasmonic structures. Another aspect is the slot channel sensor configurations. The challenge is implementing these structures along with the coupling light into the channels. The fabrication difficulties may be overcome using a rolled up silicon membrane [32,33] or through a number of film deposition, lithography and etch steps. Also by further examination of these three novel structures for higher index values more interesting features might be observed.

Appendices

Appendix A Derivatives of the magnetic field components

The magnetic field is composed of three components. Examination of the derivatives required in Appendix B indicates that the following derivatives for the H_r component are required:

$$\frac{\partial H_r}{\partial \varphi} = \sum_r \kappa^r J_o \left(\rho_{p_r} \frac{r}{R} \right) \frac{\partial e^{jq_r \varphi}}{\partial \varphi} e^{jG_{n_r} z} = \sum_r \kappa^r (jq_r) J_o \left(\rho_{p_r} \frac{r}{R} \right) e^{jq_r \varphi} e^{jG_{n_r} z} \quad (\text{A.1})$$

$$\frac{\partial^2 H_r}{\partial \varphi^2} = \sum_r \kappa^r J_o \left(\rho_{p_r} \frac{r}{R} \right) \frac{\partial^2 e^{jq_r \varphi}}{\partial \varphi^2} e^{jG_{n_r} z} = \sum_r \kappa^r (-q_r^2) J_o \left(\rho_{p_r} \frac{r}{R} \right) e^{jq_r \varphi} e^{jG_{n_r} z} \quad (\text{A.2})$$

$$\frac{\partial H_r}{\partial z} = \sum_r \kappa^r J_o \left(\rho_{p_r} \frac{r}{R} \right) e^{jq_r \varphi} \frac{\partial e^{jG_{n_r} z}}{\partial z} = \sum_r \kappa^r (jG_{n_r}) J_o \left(\rho_{p_r} \frac{r}{R} \right) e^{jq_r \varphi} e^{jG_{n_r} z} \quad (\text{A.3})$$

$$\frac{\partial^2 H_r}{\partial z^2} = \sum_r \kappa^r J_o \left(\rho_{p_r} \frac{r}{R} \right) e^{jq_r \varphi} \frac{\partial^2 e^{jG_{n_r} z}}{\partial z^2} = \sum_r \kappa^r (-G_{n_r}^2) J_o \left(\rho_{p_r} \frac{r}{R} \right) e^{jq_r \varphi} e^{jG_{n_r} z} \quad (\text{A.4})$$

$$\frac{\partial^2 H_r}{\partial r \partial z} = \sum_r \kappa^r \frac{\partial J_o \left(\rho_{p_r} \frac{r}{R} \right)}{\partial r} e^{jq_r \varphi} \frac{\partial e^{jG_{n_r} z}}{\partial z} = \sum_r \kappa^r \left(-\frac{\rho_{p_r}}{R} \right) (jG_{n_r}) J_1 \left(\rho_{p_r} \frac{r}{R} \right) e^{jq_r \varphi} e^{jG_{n_r} z} \quad (\text{A.5})$$

$$\frac{\partial^2 H_r}{\partial r \partial \varphi} = \sum_r \kappa^r \frac{\partial J_o \left(\rho_{p_r} \frac{r}{R} \right)}{\partial r} \frac{\partial e^{jq_r \varphi}}{\partial \varphi} e^{jG_{n_r} z} = \sum_r \kappa^r \left(-\frac{\rho_{p_r}}{R} \right) (jq_r) J_1 \left(\rho_{p_r} \frac{r}{R} \right) e^{jq_r \varphi} e^{jG_{n_r} z} \quad (\text{A.6})$$

Appendix C indicates that the following derivatives for the H_φ component are required:

$$\frac{\partial H_\varphi}{\partial r} = \sum_\varphi \kappa^\varphi \left(-\frac{\rho_{p_\varphi}}{R} \right) J_1 \left(\rho_{p_\varphi} \frac{r}{R} \right) e^{jq_\varphi \varphi} e^{jG_{n_\varphi} z} \quad (\text{A.7})$$

$$\frac{\partial H_\varphi}{\partial \varphi} = \sum_\varphi \kappa^\varphi (jq_\varphi) J_o \left(\rho_{p_\varphi} \frac{r}{R} \right) e^{jq_\varphi \varphi} e^{jG_{n_\varphi} z} \quad (\text{A.8})$$

$$\frac{\partial H_\varphi}{\partial z} = \sum_\varphi \kappa^\varphi (jG_{n_\varphi}) J_o \left(\rho_{p_\varphi} \frac{r}{R} \right) e^{jq_\varphi \varphi} e^{jG_{n_\varphi} z} \quad (\text{A.9})$$

$$\frac{\partial^2 H_\varphi}{\partial z^2} = \sum_\varphi \kappa^\varphi J_o \left(\rho_{p_\varphi} \frac{r}{R} \right) e^{jq_\varphi \varphi} \frac{\partial^2 e^{jG_{n_\varphi} z}}{\partial z^2} \quad (\text{A.10})$$

$$\frac{\partial^2 H_\varphi}{\partial z \partial \varphi} = \sum_\varphi \kappa^\varphi J_o \left(\rho_{p_\varphi} \frac{r}{R} \right) \frac{\partial e^{jq_\varphi \varphi}}{\partial \varphi} \frac{\partial e^{jG_{n_\varphi} z}}{\partial z} \quad (\text{A.11})$$

$$\frac{\partial^2 H_\varphi}{\partial r^2} = \sum_\varphi \kappa^\varphi \left(-\frac{\rho_{p_\varphi}}{R} \right) \frac{\partial J_1 \left(\rho_{p_\varphi} \frac{r}{R} \right)}{\partial r} e^{jq_\varphi \varphi} e^{jG_{n_\varphi} z} \quad (\text{A.12})$$

In order to evaluate this derivative, the derivative of the first order Bessel function is required. The derivative of a Bessel function in generic form is:

$$\frac{\partial J_1(r)}{\partial r} = J_0(r) - \frac{J_1(r)}{r} \quad (\text{A.13})$$

To make use of (A.13) a change of variable is required, $r = \frac{R}{\rho_{p\varphi}}\zeta$, and as a result

$\partial r = \frac{R}{\rho_{p\varphi}} \partial \zeta$, then the derivative suitable for (A.13) is:

$$\frac{\partial J_1\left(\rho_{p\varphi} \frac{r}{R}\right)}{\partial r} = \frac{\partial J_1(\zeta)}{\frac{R}{\rho_{p\varphi}} \partial \zeta} = \frac{\rho_{p\varphi}}{R} \left(J_0(\zeta) - \frac{J_1(\zeta)}{\zeta} \right) = \frac{\rho_{p\varphi}}{R} \left(J_0\left(\rho_{p\varphi} \frac{r}{R}\right) - \frac{R}{\rho_{p\varphi}} \frac{J_1\left(\rho_{p\varphi} \frac{r}{R}\right)}{r} \right) \quad (\text{A.14})$$

$$\frac{\partial J_1\left(\rho_{p\varphi} \frac{r}{R}\right)}{\partial r} = \left(\frac{\rho_{p\varphi}}{R} \right) J_0\left(\rho_{p\varphi} \frac{r}{R}\right) - \frac{J_1\left(\rho_{p\varphi} \frac{r}{R}\right)}{r} \quad (\text{A.15})$$

Making use of equation (E-12) results in two terms for the second derivative:

$$\frac{\partial^2 H_\varphi}{\partial r^2} = -\sum_\varphi \kappa^\varphi \left(\frac{\rho_{p\varphi}}{R} \right)^2 J_0\left(\rho_{p\varphi} \frac{r}{R}\right) e^{jq_\varphi \varphi} e^{jG_{n\varphi} z} + \sum_\varphi \kappa^\varphi \left(\frac{\rho_{p\varphi}}{rR} \right) J_1\left(\rho_{p\varphi} \frac{r}{R}\right) e^{jq_\varphi \varphi} e^{jG_{n\varphi} z} \quad (\text{A.16})$$

$$\frac{\partial^2 H_\varphi}{\partial z^2} = \sum_\varphi \kappa^\varphi \left(-G_{n\varphi}^2 \right) J_0\left(\rho_{p\varphi} \frac{r}{R}\right) e^{jq_\varphi \varphi} e^{jG_{n\varphi} z} \quad (\text{A.17})$$

$$\frac{\partial^2 H_\varphi}{\partial \varphi \partial r} = \sum_\varphi \kappa^\varphi \left(-\frac{\rho_{p\varphi}}{R} \right) (jq_\varphi) J_1\left(\rho_{p\varphi} \frac{r}{R}\right) e^{jq_\varphi \varphi} e^{jG_{n\varphi} z} \quad (\text{A.18})$$

$$\frac{\partial^2 H_\varphi}{\partial z \partial \varphi} = \sum_\varphi \kappa^\varphi (-q_\varphi) \left(G_{n\varphi} \right) J_0\left(\rho_{p\varphi} \frac{r}{R}\right) e^{jq_\varphi \varphi} e^{jG_{n\varphi} z} \quad (\text{A.19})$$

which can be simplified to:

$$\frac{\partial^2 H_\varphi}{\partial z \partial \varphi} = \sum_\varphi \kappa^\varphi \left(-q_\varphi G_{n\varphi} \right) J_0\left(\rho_{p\varphi} \frac{r}{R}\right) e^{jq_\varphi \varphi} e^{jG_{n\varphi} z} \quad (\text{A.20})$$

Also following derivatives for the H_z component are required:

$$\frac{\partial H_z}{\partial r} = \sum_z \kappa^z \left(-\frac{\rho_{pz}}{R} \right) J_1\left(\rho_{pz} \frac{r}{R}\right) e^{jq_z \varphi} e^{jG_{nz} z} \quad (\text{A.21})$$

$$\frac{\partial H_z}{\partial \varphi} = \sum_z \kappa^z (jq_z) J_0\left(\rho_{pz} \frac{r}{R}\right) e^{jq_z \varphi} e^{jG_{nz} z} \quad (\text{A.22})$$

$$\frac{\partial^2 H_z}{\partial r \partial \varphi} = \sum_z \kappa^z \left(-\frac{\rho_{p_z}}{R} \right) (j q_z) J_1 \left(\rho_{p_z} \frac{r}{R} \right) e^{j q_z \varphi} e^{j G_{n_z} z} \quad (\text{A.23})$$

$$\frac{\partial^2 H_z}{\partial z \partial \varphi} = \sum_z \kappa^z (j q_z) (j G_{n_z}) J_0 \left(\rho_{p_z} \frac{r}{R} \right) e^{j q_z \varphi} e^{j G_{n_z} z} = \sum_z \kappa^z (-q_z G_{n_z}) J_0 \left(\rho_{p_z} \frac{r}{R} \right) e^{j q_z \varphi} e^{j G_{n_z} z} \quad (\text{A.24})$$

$$\frac{\partial^2 H_z}{\partial \varphi^2} = \sum_z \kappa^z (-q_z^2) J_0 \left(\rho_{p_z} \frac{r}{R} \right) e^{j q_z \varphi} e^{j G_{n_z} z} \quad (\text{A.25})$$

$$\frac{\partial^2 H_z}{\partial r^2} = -\sum_z \kappa^z \left(\frac{\rho_{p_z}}{R} \right)^2 J_0 \left(\rho_{p_z} \frac{r}{R} \right) e^{j q_z \varphi} e^{j G_{n_z} z} + \sum_z \kappa^z \left(\frac{\rho_{p_z}}{r R} \right) J_1 \left(\rho_{p_z} \frac{r}{R} \right) e^{j q_z \varphi} e^{j G_{n_z} z} \quad (\text{A.26})$$

$$\frac{\partial^2 H_z}{\partial r \partial z} = \sum_z \kappa^z \left(-\frac{\rho_{p_z}}{R} \right) (j G_{n_z}) J_1 \left(\rho_{p_z} \frac{r}{R} \right) e^{j q_z \varphi} e^{j G_{n_z} z} \quad (\text{A.27})$$

Appendix B Curl of curl in cylindrical coordinate

This appendix details the mathematical steps required to obtain the curl of the curl in cylindrical coordinates. In generic form the curl of a vector field $\vec{A}$ is:

$$\nabla \times \vec{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) \hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\varphi} + \frac{1}{r} \left(\frac{\partial(rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi} \right) \hat{z} \quad (\text{B.1})$$

The result is redefined in terms of a vector $\vec{C}$ as:

$$\nabla \times \vec{A} = C_r \hat{r} + C_\varphi \hat{\varphi} + C_z \hat{z} \quad (\text{B.2})$$

$$C_r = \left(\frac{1}{r} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) \quad (\text{B.3})$$

$$C_\varphi = \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \quad (\text{B.4})$$

$$C_z = \frac{1}{r} \left(\frac{\partial(rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi} \right) \quad (\text{B.5})$$

Note that the components for $\vec{E}$ field (with no derivatives of the inverse of relative permittivity, $\Omega = \frac{1}{\varepsilon_r(\vec{r})}$) can be extracted from the following $\vec{H}$ field components. The $\vec{E}$ field equation is simpler as the inverse of relative permittivity tensor is external to the double curl operator.

For the $\vec{H}$ field analysis the first curl operation is multiplied by the inverse of the dielectric. Since the dielectric is a scalar quantity it can be multiplied into the definition of the vector $\vec{C}$. It must be retained that the inverse of relative permittivity is a function of the coordinates and derivatives with respect to the inverse of relative permittivity must be retained. The next step is to perform a second curl, now on the vector $\vec{C}$:

$$\nabla \times \vec{C} = \left(\frac{1}{r} \frac{\partial C_z}{\partial \varphi} - \frac{\partial C_\varphi}{\partial z} \right) \hat{r} + \left(\frac{\partial C_r}{\partial z} - \frac{\partial C_z}{\partial r} \right) \hat{\varphi} + \frac{1}{r} \left(\frac{\partial(rC_\varphi)}{\partial r} - \frac{\partial C_r}{\partial \varphi} \right) \hat{z} \quad (\text{B.6})$$

The next set below should be used for the magnetic field wave equation:

$$\nabla \times \vec{C} =$$

$$\begin{aligned} & \left(\frac{1}{r} \frac{\partial}{\partial \varphi} \left(\frac{1}{\epsilon_r(\vec{r})} r \left(\frac{\partial(rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi} \right) \right) - \frac{\partial}{\partial z} \left(\frac{1}{\epsilon_r(\vec{r})} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \right) \right) \hat{r} + \left(\frac{\partial}{\partial z} \left(\frac{1}{\epsilon_r(\vec{r})} \left(\frac{1}{r} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) \right) - \frac{\partial}{\partial r} \left(\frac{1}{\epsilon_r(\vec{r})} r \left(\frac{\partial(rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi} \right) \right) \right) \hat{\varphi} + \\ & \frac{1}{r} \left(\frac{\partial}{\partial r} \left(\frac{1}{\epsilon_r(\vec{r})} \left(r \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \right) \right) - \frac{\partial}{\partial \varphi} \left(\frac{1}{\epsilon_r(\vec{r})} \left(\frac{1}{r} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) \right) \right) \hat{z} \end{aligned} \quad (\text{B.7})$$

Once again the result of the second curl can be expressed by a vector $\vec{W}$ in the form:

$$\nabla \times \vec{C} = \vec{W} \quad (\text{B.8})$$

With each component expressed as:

$$W_r = \left(\frac{1}{r} \frac{\partial}{\partial \varphi} \left(\frac{1}{\epsilon_r(\vec{r})} r \left(\frac{\partial(rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi} \right) \right) - \frac{\partial}{\partial z} \left(\frac{1}{\epsilon_r(\vec{r})} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \right) \right) \quad (\text{B.9})$$

$$W_\varphi = \left(\frac{\partial}{\partial z} \left(\frac{1}{\epsilon_r(\vec{r})} \left(\frac{1}{r} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) \right) - \frac{\partial}{\partial r} \left(\frac{1}{\epsilon_r(\vec{r})} r \left(\frac{\partial(rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi} \right) \right) \right) \quad (\text{B.10})$$

$$W_z = \frac{1}{r} \left(\frac{\partial}{\partial r} \left(\frac{1}{\epsilon_r(\vec{r})} \left(r \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \right) \right) - \frac{\partial}{\partial \varphi} \left(\frac{1}{\epsilon_r(\vec{r})} \left(\frac{1}{r} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) \right) \right) \quad (\text{B.11})$$

$$\nabla \times \vec{C} = \vec{W} \quad (\text{B.8}) \quad \Leftrightarrow \quad \vec{\nabla} \times \frac{1}{\epsilon_r(\vec{r})} \vec{\nabla} \times \vec{H} = \left(\frac{\omega}{c} \right)^2 \vec{H} \quad (2.13)$$

Radial component:

$$\begin{aligned} W_r &= \left(\frac{1}{r} \frac{\partial}{\partial \varphi} \left(\frac{1}{\epsilon_r(\vec{r})} r \left(\frac{\partial(rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi} \right) \right) - \frac{\partial}{\partial z} \left(\frac{1}{\epsilon_r(\vec{r})} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \right) \right) \Leftrightarrow W_r = \left(\frac{1}{r} \frac{\partial}{\partial \varphi} \left(\frac{1}{\epsilon_r(\vec{r})} r \left(\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial \varphi} \right) \right) - \right. \\ & \left. \frac{\partial}{\partial z} \left(\frac{1}{\epsilon_r(\vec{r})} \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) \right) \right) \end{aligned} \quad (\text{B.12})$$

Expanding out the derivatives by one order gives:

$$W_r = \frac{1}{r} \frac{\partial}{\partial \varphi} \left[\frac{1}{\varepsilon_r(\vec{r})} \left[\frac{1}{r} \left(\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial \varphi} \right) \right] + \frac{1}{r} \frac{1}{\varepsilon_r(\vec{r})} \frac{1}{r} \frac{\partial \left(\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial \varphi} \right)}{\partial \varphi} - \frac{\partial}{\partial z} \left[\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) \right] - \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right)}{\partial z} \right] \quad (\text{B.13})$$

Expanding out the derivatives by another order gives:

$$W_r = \frac{1}{r} \frac{\partial}{\partial \varphi} \left[\frac{1}{\varepsilon_r(\vec{r})} \left[H_\varphi + r \frac{\partial H_\varphi}{\partial r} - \frac{\partial H_r}{\partial \varphi} \right] \right] + \frac{1}{r^2 \varepsilon_r(\vec{r})} \left[\frac{\partial H_\varphi}{\partial \varphi} + r \frac{\partial^2 H_\varphi}{\partial \varphi \partial r} - \frac{\partial^2 H_r}{\partial \varphi^2} \right] - \frac{\partial}{\partial z} \left[\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) \right] - \frac{1}{\varepsilon_r(\vec{r})} \left[\frac{\partial^2 H_r}{\partial z^2} - \frac{\partial^2 H_z}{\partial z \partial r} \right] \quad (\text{B.14})$$

Expanding all terms:

$$W_r = \frac{H_\varphi}{r^2} \frac{\partial}{\partial \varphi} \frac{1}{\varepsilon_r(\vec{r})} + \frac{1}{r} \frac{\partial H_\varphi}{\partial r} \frac{\partial}{\partial \varphi} \frac{1}{\varepsilon_r(\vec{r})} - \frac{1}{r^2} \frac{\partial H_r}{\partial \varphi} \frac{\partial}{\partial \varphi} \frac{1}{\varepsilon_r(\vec{r})} + \frac{1}{r^2 \varepsilon_r(\vec{r})} \frac{\partial H_\varphi}{\partial \varphi} + \frac{1}{r \varepsilon_r(\vec{r})} \frac{\partial^2 H_\varphi}{\partial \varphi \partial r} - \frac{1}{r^2 \varepsilon_r(\vec{r})} \frac{\partial^2 H_r}{\partial \varphi^2} - \frac{\partial H_r}{\partial z} \frac{\partial}{\partial z} \frac{1}{\varepsilon_r(\vec{r})} + \frac{\partial H_z}{\partial r} \frac{\partial}{\partial z} \frac{1}{\varepsilon_r(\vec{r})} - \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial^2 H_r}{\partial z^2} + \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial^2 H_z}{\partial z \partial r} \quad (\text{B.15})$$

A further simplification in the notation is possible by defining the inverse of the relative permittivity as:

$$\Omega = \frac{1}{\varepsilon_r(\vec{r})} \quad (\text{3.1})$$

Then (B.15) becomes:

$$W_r = \frac{H_\varphi}{r^2} \frac{\partial \Omega}{\partial \varphi} + \frac{1}{r} \frac{\partial H_\varphi}{\partial r} \frac{\partial \Omega}{\partial \varphi} - \frac{1}{r^2} \frac{\partial H_r}{\partial \varphi} \frac{\partial \Omega}{\partial \varphi} + \frac{\Omega}{r^2} \frac{\partial H_\varphi}{\partial \varphi} + \frac{\Omega}{r} \frac{\partial^2 H_\varphi}{\partial \varphi \partial r} - \frac{\Omega}{r^2} \frac{\partial^2 H_r}{\partial \varphi^2} - \frac{\partial H_r}{\partial z} \frac{\partial \Omega}{\partial z} + \frac{\partial H_z}{\partial r} \frac{\partial \Omega}{\partial z} - \Omega \frac{\partial^2 H_r}{\partial z^2} + \Omega \frac{\partial^2 H_z}{\partial z \partial r} \quad (\text{B.16})$$

For $\vec{E}$ field:

$$W_r = \frac{E_\varphi}{r^2} + \frac{1}{r} \frac{\partial E_\varphi}{\partial r} - \frac{1}{r^2} \frac{\partial E_r}{\partial \varphi} + \frac{\Omega}{r^2} \frac{\partial E_\varphi}{\partial \varphi} + \frac{\Omega}{r} \frac{\partial^2 E_\varphi}{\partial \varphi \partial r} - \frac{\Omega}{r^2} \frac{\partial^2 E_r}{\partial \varphi^2} - \frac{\partial E_r}{\partial z} + \frac{\partial E_z}{\partial r} - \Omega \frac{\partial^2 E_r}{\partial z^2} + \Omega \frac{\partial^2 E_z}{\partial z \partial r} \quad (\text{B.17})$$

Angular component:

$$W_\varphi = \left(\frac{\partial \frac{1}{\varepsilon_r(\vec{r})} \left(\frac{1}{r} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right)}{\partial z} - \frac{\partial \frac{1}{\varepsilon_r(\vec{r})} \frac{1}{r} \left(\frac{\partial(rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi} \right)}{\partial r} \right) \Leftrightarrow \left(\frac{\partial \frac{1}{\varepsilon_r(\vec{r})} \left(\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} \right)}{\partial z} - \frac{\partial \frac{1}{\varepsilon_r(\vec{r})} \frac{1}{r} \left(\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial \varphi} \right)}{\partial r} \right) \quad (\text{B.18})$$

Expanding out the derivatives by one order gives:

$$W_\varphi = \left(\frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial z} \left[\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} \right] + \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial \left(\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} \right)}{\partial z} - \frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial r} \left[\frac{1}{r} \left(\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial \varphi} \right) \right] - \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial \left(\frac{1}{r} \left(\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial \varphi} \right) \right)}{\partial r} \right) \quad (\text{B.19})$$

Expanding out the derivatives by another order gives:

$$W_\varphi = \left(\frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial z} \left[\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} \right] + \frac{1}{\varepsilon_r(\vec{r})} \frac{1}{r} \frac{\partial^2 H_z}{\partial z \partial \varphi} - \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial^2 H_\varphi}{\partial z^2} - \frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial r} \left[\frac{1}{r} \left(\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial \varphi} \right) \right] - \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial \left(\frac{1}{r} H_\varphi \right)}{\partial r} - \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial^2 H_\varphi}{\partial r^2} + \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial \left(\frac{1}{r} \frac{\partial H_r}{\partial \varphi} \right)}{\partial r} \right) \quad (\text{B.20})$$

Expanding out the derivatives by another order gives:

$$W_\varphi = \left(\frac{1}{r} \frac{\partial H_z}{\partial \varphi} \frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial z} - \frac{\partial H_\varphi}{\partial z} \frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial z} + \frac{1}{\varepsilon_r(\vec{r})} \frac{1}{r} \frac{\partial^2 H_z}{\partial z \partial \varphi} - \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial^2 H_\varphi}{\partial z^2} - \frac{1}{r} \frac{\partial(rH_\varphi)}{\partial r} \frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial r} + \frac{1}{r} \frac{\partial H_r}{\partial \varphi} \frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial r} - \frac{1}{\varepsilon_r(\vec{r})} H_\varphi \frac{\partial \left(\frac{1}{r} \right)}{\partial r} - \frac{1}{\varepsilon_r(\vec{r})} \frac{1}{r} \frac{\partial H_\varphi}{\partial r} - \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial^2 H_\varphi}{\partial r^2} + \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial H_r}{\partial \varphi} \frac{\partial \left(\frac{1}{r} \right)}{\partial r} + \frac{1}{\varepsilon_r(\vec{r})} \frac{1}{r} \frac{\partial^2 H_r}{\partial r \partial \varphi} \right) \quad (\text{B.21})$$

A further simplification in the notation is possible by defining the inverse of the relative permittivity as:

$$\Omega = \frac{1}{\varepsilon_r(\vec{r})} \quad (3.1)$$

Then (B.21) Becomes:

$$W_\varphi = \left(\frac{1}{r} \frac{\partial H_z}{\partial \varphi} \frac{\partial \Omega}{\partial z} - \frac{\partial H_\varphi}{\partial z} \frac{\partial \Omega}{\partial z} + \frac{\Omega}{r} \frac{\partial^2 H_z}{\partial z \partial \varphi} - \Omega \frac{\partial^2 H_\varphi}{\partial z^2} - \frac{H_\varphi}{r} \frac{\partial \Omega}{\partial r} - \frac{\partial H_\varphi}{\partial r} \frac{\partial \Omega}{\partial r} + \frac{1}{r} \frac{\partial H_r}{\partial \varphi} \frac{\partial \Omega}{\partial r} + \frac{\Omega}{r^2} H_\varphi - \right. \\ \left. \frac{\Omega}{r} \frac{\partial H_\varphi}{\partial r} - \Omega \frac{\partial^2 H_\varphi}{\partial r^2} - \frac{\Omega}{r^2} \frac{\partial H_r}{\partial \varphi} + \frac{\Omega}{r} \frac{\partial^2 H_r}{\partial r \partial \varphi} \right) \quad (\text{B.22})$$

For $\vec{E}$ field:

$$W_\varphi = \left(\frac{1}{r} \frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z} + \frac{\Omega}{r} \frac{\partial^2 E_z}{\partial z \partial \varphi} - \Omega \frac{\partial^2 E_\varphi}{\partial z^2} - \frac{E_\varphi}{r} - \frac{\partial E_\varphi}{\partial r} + \frac{1}{r} \frac{\partial E_r}{\partial \varphi} + \frac{\Omega}{r^2} E_\varphi - \frac{\Omega}{r} \frac{\partial E_\varphi}{\partial r} - \Omega \frac{\partial^2 E_\varphi}{\partial r^2} - \right. \\ \left. \frac{\Omega}{r^2} \frac{\partial E_r}{\partial \varphi} + \frac{\Omega}{r} \frac{\partial^2 E_r}{\partial r \partial \varphi} \right) \quad (\text{B.23})$$

Z directed component:

$$W_z = \frac{1}{r} \left(\frac{\partial \frac{1}{\varepsilon_r(\vec{r})} \left(r \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \right)}{\partial r} - \frac{\partial \frac{1}{\varepsilon_r(\vec{r})} \left(\frac{1}{r} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right)}{\partial \varphi} \right) \Leftrightarrow \frac{1}{r} \left(\frac{\partial \frac{1}{\varepsilon_r(\vec{r})} \left(r \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) \right)}{\partial r} - \frac{\partial \frac{1}{\varepsilon_r(\vec{r})} \left(\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} \right)}{\partial \varphi} \right) \quad (\text{B.24})$$

Expanding out the derivatives by one order gives:

$$W_z =$$

$$\frac{1}{r} \left(\frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial r} \left[r \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) \right] + \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial \left(r \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) \right)}{\partial r} - \frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial \varphi} \left[\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} \right] - \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial \left(\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} \right)}{\partial \varphi} \right) \quad (\text{B.25})$$

Expanding out the derivatives by another order gives:

$$W_z = \frac{1}{r} \left(\left[r \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) \right] \frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial r} + \frac{1}{\varepsilon_r(\vec{r})} \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) + \frac{r}{\varepsilon_r(\vec{r})} \frac{\partial \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right)}{\partial r} - \left[\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} \right] \frac{\partial \frac{1}{\varepsilon_r(\vec{r})}}{\partial \varphi} - \frac{1}{\varepsilon_r(\vec{r})} \frac{1}{r} \frac{\partial^2 H_z}{\partial \varphi^2} + \frac{1}{\varepsilon_r(\vec{r})} \frac{\partial^2 H_\varphi}{\partial \varphi \partial z} \right) \quad (\text{B.26})$$

Expanding out the derivatives by another order gives:

$$\begin{aligned}
W_z = \frac{1}{r} & \left(\left[r \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) \right] \frac{\partial \frac{1}{\epsilon_r(\vec{r})}}{\partial r} + \frac{1}{\epsilon_r(\vec{r})} \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) + \frac{r}{\epsilon_r(\vec{r})} \frac{\partial^2 H_r}{\partial r \partial z} - \frac{r}{\epsilon_r(\vec{r})} \frac{\partial^2 H_z}{\partial r^2} - \right. \\
& \left. \left[\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} \right] \frac{\partial \frac{1}{\epsilon_r(\vec{r})}}{\partial \varphi} - \frac{1}{\epsilon_r(\vec{r})} \frac{1}{r} \frac{\partial^2 H_z}{\partial \varphi^2} + \frac{1}{\epsilon_r(\vec{r})} \frac{\partial^2 H_\varphi}{\partial \varphi \partial z} \right) \quad (B.27)
\end{aligned}$$

A further simplification in the notation is possible by defining the inverse of the dielectric as:

$$\Omega = \frac{1}{\epsilon_r(\vec{r})} \quad (3.1)$$

Then (B.27) Becomes:

$$\begin{aligned}
W_z = & \left(\frac{\partial H_r}{\partial z} \frac{\partial \Omega}{\partial r} - \frac{\partial H_z}{\partial r} \frac{\partial \Omega}{\partial r} + \frac{\Omega}{r} \frac{\partial H_r}{\partial z} - \frac{\Omega}{r} \frac{\partial H_z}{\partial r} + \Omega \frac{\partial^2 H_r}{\partial r \partial z} - \Omega \frac{\partial^2 H_z}{\partial r^2} - \frac{1}{r^2} \frac{\partial H_z}{\partial \varphi} \frac{\partial \Omega}{\partial \varphi} + \frac{1}{r} \frac{\partial H_\varphi}{\partial z} \frac{\partial \Omega}{\partial \varphi} - \right. \\
& \left. \frac{\Omega}{r^2} \frac{\partial^2 H_z}{\partial \varphi^2} + \frac{\Omega}{r} \frac{\partial^2 H_\varphi}{\partial \varphi \partial z} \right) \quad (B.28)
\end{aligned}$$

For $\vec{E}$ field:

$$\begin{aligned}
W_z = & \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} + \frac{\Omega}{r} \frac{\partial E_r}{\partial z} - \frac{\Omega}{r} \frac{\partial E_z}{\partial r} + \Omega \frac{\partial^2 E_r}{\partial r \partial z} - \Omega \frac{\partial^2 E_z}{\partial r^2} - \frac{1}{r^2} \frac{\partial E_z}{\partial \varphi} + \frac{1}{r} \frac{\partial E_\varphi}{\partial z} - \frac{\Omega}{r^2} \frac{\partial^2 E_z}{\partial \varphi^2} + \frac{\Omega}{r} \frac{\partial^2 E_\varphi}{\partial \varphi \partial z} \right) \quad (B.29)
\end{aligned}$$

Appendix C Derivatives of the inverse of relative permittivity

The inverse of relative permittivity has expansion taken as:

$$\Omega = \sum_{\Omega} \kappa^{\Omega} J_0 \left(\rho_{p\Omega} \frac{r}{R} \right) e^{jq_{\Omega}\varphi} e^{jG_{n\Omega}z} \quad (3.3)$$

The derivative with respect to the radial coordinate is evaluated first:

$$\frac{\partial \Omega}{\partial r} = \sum_{\Omega} \kappa^{\Omega} \frac{\partial J_0 \left(\rho_{p\Omega} \frac{r}{R} \right)}{\partial r} e^{jq_{\Omega}\varphi} e^{jG_{n\Omega}z} \quad (C.1)$$

The derivative of the lowest order Bessel function in generic form is:

$$\frac{\partial J_0(r)}{\partial r} = -J_1(r) \quad (C.2)$$

To make use of (C.2) a change of variable is required, $r = \frac{R}{\rho_{p\Omega}} \zeta$, and as a result $\partial r =$

$\frac{R}{\rho_{p\Omega}} \partial \zeta$, then the derivative suitable for (C.1) is:

$$\frac{\partial J_0 \left(\rho_{p\Omega} \frac{r}{R} \right)}{\partial r} = -\frac{\rho_{p\Omega}}{R} J_1 \left(\rho_{p\Omega} \frac{r}{R} \right) \quad (C.3)$$

The radial dependent derivative of the inverse of relative permittivity expression is:

$$\frac{\partial \Omega}{\partial r} = \sum_{\Omega} \kappa^{\Omega} \left(-\frac{\rho_{p\Omega}}{R} \right) J_1 \left(\rho_{p\Omega} \frac{r}{R} \right) e^{jq_{\Omega}\varphi} e^{jG_{n\Omega}z} \quad (3.4)$$

The derivative with respect to the angular coordinate is evaluated second:

$$\frac{\partial \Omega}{\partial \varphi} = \sum_{\Omega} \kappa^{\Omega} J_0 \left(\rho_{p\Omega} \frac{r}{R} \right) \frac{\partial e^{jq_{\Omega}\varphi}}{\partial \varphi} e^{jG_{n\Omega}z} \quad (C.4)$$

This derivative is easily evaluated and is:

$$\frac{\partial \Omega}{\partial \varphi} = \sum_{\Omega} \kappa^{\Omega} (jq_{\Omega}) J_0 \left(\rho_{p\Omega} \frac{r}{R} \right) e^{jq_{\Omega}\varphi} e^{jG_{n\Omega}z} \quad (3.5)$$

The derivative with respect to the z coordinate is evaluated third:

$$\frac{\partial \Omega}{\partial z} = \sum_{\Omega} \kappa^{\Omega} J_0 \left(\rho_{p\Omega} \frac{r}{R} \right) e^{jq_{\Omega}\varphi} \frac{\partial e^{jG_{n\Omega}z}}{\partial z} \quad (C.5)$$

This derivative is easily evaluated and is:

$$\frac{\partial \Omega}{\partial z} = \sum_{\Omega} \kappa^{\Omega} (jG_{n_{\Omega}}) J_o \left(\rho_{p_{\Omega}} \frac{r}{R} \right) e^{jq_{\Omega} \varphi} e^{jG_{n_{\Omega}} z} \quad (3.6)$$

Appendix D Term by Term Development of the expression relating the components (r, φ, z) of the master equation (2.13)

Note: FFB_o and FFB_1 are defined in (3.4) and (3.8) respectively.

D.1 Term by Term Development of the expression relating the radial components of the master equation (2.13)

D.1.1 Term 1 of (3.43)(Z)

$$\frac{H_{\varphi}}{r^2} \frac{\partial \Omega}{\partial \varphi} = \frac{1}{r^2} [\sum_{\varphi} \kappa^{\varphi} FFB_o(\varphi)] [\sum_{\Omega} \kappa^{\Omega} (jq_{\Omega}) FFB_o(\Omega)] \quad (D.1)$$

$$\frac{H_{\varphi}}{r^2} \frac{\partial \Omega}{\partial \varphi} = \left[\sum_{\varphi \Omega} \left(\frac{jq_{\Omega} \kappa^{\varphi} \kappa^{\Omega}}{r^2} \right) FFB_o(\varphi) FFB_o(\Omega) \right] \quad (D.2)$$

D.1.2 Term 2 of (3.43)(Z)

$$\frac{1}{r} \frac{\partial H_{\varphi}}{\partial r} \frac{\partial \Omega}{\partial \varphi} = \frac{1}{r} \left[\sum_{\varphi} \kappa^{\varphi} \left(-\frac{\rho_{p\varphi}}{R} \right) FFB_1(\varphi) \right] [\sum_{\Omega} \kappa^{\Omega} (jq_{\Omega}) FFB_o(\Omega)] \quad (D.3)$$

$$\frac{1}{r} \frac{\partial H_{\varphi}}{\partial r} \frac{\partial \Omega}{\partial \varphi} = \left[\sum_{\varphi \Omega} \left(-\frac{j\rho_{p\varphi} q_{\Omega} \kappa^{\varphi} \kappa^{\Omega}}{rR} \right) FFB_1(\varphi) FFB_o(\Omega) \right] \quad (D.4)$$

D.1.3 Term 3 of (3.43)(Z)

$$-\frac{1}{r^2} \frac{\partial H_r}{\partial \varphi} \frac{\partial \Omega}{\partial \varphi} = -\frac{1}{r^2} [\sum_r \kappa^r (jq_r) FFB_o(r)] [\sum_{\Omega} \kappa^{\Omega} (jq_{\Omega}) FFB_o(\Omega)] \quad (D.5)$$

$$-\frac{1}{r^2} \frac{\partial H_r}{\partial \varphi} \frac{\partial \Omega}{\partial \varphi} = \left[\sum_{r \Omega} \left(\frac{q_r q_{\Omega} \kappa^r \kappa^{\Omega}}{r^2} \right) FFB_o(r) FFB_o(\Omega) \right] \quad (D.6)$$

D.1.4 Term 4 of (3.43)(Z)

$$\frac{\Omega}{r^2} \frac{\partial H_{\varphi}}{\partial \varphi} = \frac{1}{r^2} [\sum_{\varphi} \kappa^{\varphi} (jq_{\varphi}) FFB_o(\varphi)] [\sum_{\Omega} \kappa^{\Omega} FFB_o(\Omega)] \quad (D.7)$$

$$\frac{\Omega}{r^2} \frac{\partial H_\varphi}{\partial \varphi} = \left[\sum_{\varphi\Omega} \left(\frac{j q_\varphi \kappa^\varphi \kappa^\Omega}{r^2} \right) FFB_o(\varphi) FFB_o(\Omega) \right] \quad (D.8)$$

D.1.5 Term 5 of (3.43)(Z)

$$\frac{\Omega}{r} \frac{\partial^2 H_\varphi}{\partial \varphi \partial r} = \frac{1}{r} \left[\sum_{\varphi} \kappa^\varphi \left(-\frac{\rho p_\varphi}{R} \right) (j q_\varphi) FFB_1(\varphi) \right] [\sum_{\Omega} \kappa^\Omega FFB_o(\Omega)] \quad (D.9)$$

$$\frac{\Omega}{r} \frac{\partial^2 H_\varphi}{\partial \varphi \partial r} = \left[\sum_{\varphi\Omega} \left(-\frac{j q_\varphi \rho p_\varphi \kappa^\varphi \kappa^\Omega}{rR} \right) FFB_1(\varphi) FFB_o(\Omega) \right] \quad (D.10)$$

D.1.6 Term 6 of (3.43)(Z)

$$-\frac{\Omega}{r^2} \frac{\partial^2 H_r}{\partial \varphi^2} = -\frac{1}{r^2} [\sum_r \kappa^r (-q_r^2) FFB_o(r)] [\sum_{\Omega} \kappa^\Omega FFB_o(\Omega)] \quad (D.11)$$

$$-\frac{\Omega}{r^2} \frac{\partial^2 H_r}{\partial \varphi^2} = \left[\sum_{r\Omega} \left(\frac{q_r^2 \kappa^r \kappa^\Omega}{r^2} \right) FFB_o(r) FFB_o(\Omega) \right] \quad (D.12)$$

D.1.7 Term 7 of (3.43)(\varphi)

$$-\frac{\partial H_r}{\partial z} \frac{\partial \Omega}{\partial z} = -[\sum_r \kappa^r (j G_{n_r}) FFB_o(r)] [\sum_{\Omega} \kappa^\Omega (j G_{n_\Omega}) FFB_o(\Omega)] \quad (D.13)$$

$$-\frac{\partial H_r}{\partial z} \frac{\partial \Omega}{\partial z} = [\sum_{r\Omega} (G_{n_r} G_{n_\Omega} \kappa^r \kappa^\Omega) FFB_o(r) FFB_o(\Omega)] \quad (D.14)$$

D.1.8 Term 8 of (3.43) (\varphi)

$$\frac{\partial H_z}{\partial r} \frac{\partial \Omega}{\partial z} = \left[\sum_z \kappa^z \left(-\frac{\rho p_z}{R} \right) FFB_1(z) \right] [\sum_{\Omega} \kappa^\Omega (j G_{n_\Omega}) FFB_o(\Omega)] \quad (D.15)$$

$$\frac{\partial H_z}{\partial r} \frac{\partial \Omega}{\partial z} = \left[\sum_{z\Omega} \left(-\frac{j G_{n_\Omega} \rho p_z \kappa^z \kappa^\Omega}{R} \right) FFB_1(z) FFB_o(\Omega) \right] \quad (D.16)$$

D.1.9 Term 9 of (3.43) (\varphi)

$$-\Omega \frac{\partial^2 H_r}{\partial z^2} = -[\sum_r \kappa^r (-G_{n_r}^2) FFB_o(r)] [\sum_{\Omega} \kappa^\Omega FFB_o(\Omega)] \quad (D.17)$$

$$-\Omega \frac{\partial^2 H_r}{\partial z^2} = [\sum_{r\Omega} (G_{n_r}^2 \kappa^r \kappa^\Omega) FFB_o(r) FFB_o(\Omega)] \quad (D.18)$$

D.1.10 Term 10 of (3.43) (\varphi)

$$\Omega \frac{\partial^2 H_z}{\partial z \partial r} = \left[\sum_z \kappa_{p_z q_z n_z}^z \left(-\frac{\rho p_z}{R} \right) (j G_{n_z}) FFB_1(z) \right] [\sum_{\Omega} \kappa^\Omega FFB_o(\Omega)] \quad (D.19)$$

$$\Omega \frac{\partial^2 H_z}{\partial z \partial r} = \left[\sum_{z\Omega} \left(-\frac{j G_{n_z} \rho p_z \kappa^z \kappa^\Omega}{R} \right) FFB_1(z) FFB_o(\Omega) \right] \quad (D.20)$$

D.2 Term by Term Development of the expression relating the angular components of the master equation (2.13)

D.2.1 Term 1 of (3.45) (r)

$$\frac{1}{r} \frac{\partial H_z}{\partial \varphi} \frac{\partial \Omega}{\partial z} = \frac{1}{r} [\sum_z \kappa^z (j q_z) FFB_o(z)] [\sum_\Omega \kappa^\Omega (j G_{n_\Omega}) FFB_o(\Omega)] \quad (D.21)$$

$$\frac{1}{r} \frac{\partial H_z}{\partial \varphi} \frac{\partial \Omega}{\partial z} = \left[\sum_{z\Omega} \left(-\frac{q_z G_{n_\Omega} \kappa^z \kappa^\Omega}{r} \right) FFB_o(z) FFB_o(\Omega) \right] \quad (D.22)$$

D.2.2 Term 2 of (3.45) (r)

$$-\frac{\partial H_\varphi}{\partial z} \frac{\partial \Omega}{\partial z} = - \left[\sum_\varphi \kappa^\varphi (j G_{n_\varphi}) FFB_o(\varphi) \right] [\sum_\Omega \kappa^\Omega (j G_{n_\Omega}) FFB_o(\Omega)] \quad (D.23)$$

$$-\frac{\partial H_\varphi}{\partial z} \frac{\partial \Omega}{\partial z} = \left[\sum_{\varphi\Omega} \left(G_{n_\varphi} G_{n_\Omega} \kappa^\varphi \kappa^\Omega \right) FFB_o(\varphi) FFB_o(\Omega) \right] \quad (D.24)$$

D.2.3 Term 3 of (3.45) (r)

$$\frac{\Omega}{r} \frac{\partial^2 H_z}{\partial z \partial \varphi} = \frac{1}{r} [\sum_z \kappa^z (-q_z G_{n_z}) FFB_o(z)] [\sum_\Omega \kappa^\Omega FFB_o(\Omega)] \quad (D.25)$$

$$\frac{\Omega}{r} \frac{\partial^2 H_z}{\partial z \partial \varphi} = \left[\sum_{z\Omega} \left(-\frac{q_z G_{n_z} \kappa^z \kappa^\Omega}{r} \right) FFB_o(z) FFB_o(\Omega) \right] \quad (D.26)$$

D.2.4 Term 4 of (3.45) (r)

$$-\Omega \frac{\partial^2 H_\varphi}{\partial z^2} = - \left[\sum_\varphi \kappa^\varphi (-G_{n_\varphi} G_{n_\varphi}) FFB_o(\varphi) \right] [\sum_\Omega \kappa^\Omega FFB_o(\Omega)] \quad (D.27)$$

$$-\Omega \frac{\partial^2 H_\varphi}{\partial z^2} = \left[\sum_{\varphi\Omega} \left(G_{n_\varphi}^2 \kappa^\varphi \kappa^\Omega \right) FFB_o(\varphi) FFB_o(\Omega) \right] \quad (D.28)$$

D.2.5 Term 5 of (3.45) (Z)

$$-\frac{H_\varphi}{r} \frac{\partial \Omega}{\partial r} = -\frac{1}{r} [\sum_\varphi \kappa^\varphi FFB_o(\varphi)] \left[\sum_\Omega \kappa^\Omega \left(-\frac{\rho p_\Omega}{R} \right) FFB_1(\Omega) \right] \quad (D.29)$$

$$-\frac{H_\varphi}{r} \frac{\partial \Omega}{\partial r} = \left[\sum_{\varphi\Omega} \left(\frac{\rho p_\Omega \kappa^\varphi \kappa^\Omega}{r R} \right) FFB_o(\varphi) FFB_1(\Omega) \right] \quad (D.30)$$

D.2.6 Term 6 of (3.45) (Z)

$$-\frac{\partial H_\varphi}{\partial r} \frac{\partial \Omega}{\partial r} = -\left[\sum_\varphi \kappa^\varphi \left(-\frac{\rho_{p\varphi}}{R}\right) FFB_1(\varphi)\right] \left[\sum_\Omega \kappa^\Omega \left(-\frac{\rho_{p\Omega}}{R}\right) FFB_1(\Omega)\right] \quad (D.31)$$

$$-\frac{\partial H_\varphi}{\partial r} \frac{\partial \Omega}{\partial r} = \left[\sum_{\varphi\Omega} \left(-\frac{\rho_{p\varphi} \rho_{p\Omega} \kappa^\varphi \kappa^\Omega}{R^2}\right) FFB_1(\varphi) FFB_1(\Omega)\right] \quad (D.32)$$

D.2.7 Term 7 of (3.45) (Z)

$$\frac{1}{r} \frac{\partial H_r}{\partial \varphi} \frac{\partial \Omega}{\partial r} = \frac{1}{r} [\sum_r \kappa^r (jq_r) FFB_o(r)] \left[\kappa^\Omega \left(-\frac{\rho_{p\Omega}}{R}\right) FFB_1(\Omega)\right] \quad (D.33)$$

$$\frac{1}{r} \frac{\partial H_r}{\partial \varphi} \frac{\partial \Omega}{\partial r} = \left[\sum_{r\Omega} \left(-\frac{jq_r \rho_{p\Omega} \kappa^r \kappa^\Omega}{rR}\right) FFB_o(r) FFB_1(\Omega)\right] \quad (D.34)$$

D.2.8 Term 8 of (3.45) (Z)

$$\frac{\Omega}{r^2} H_\varphi = \frac{1}{r^2} [\sum_\varphi \kappa^\varphi FFB_o(\varphi)] [\sum_\Omega \kappa^\Omega FFB_o(\Omega)] \quad (D.35)$$

$$\frac{\Omega}{r^2} H_\varphi = \left[\sum_{\varphi\Omega} \left(\frac{\kappa^\varphi \kappa^\Omega}{r^2}\right) FFB_o(\varphi) FFB_o(\Omega)\right] \quad (D.36)$$

D.2.9 Term 9 of (3.45) (Z)

$$-\frac{\Omega}{r} \frac{\partial H_\varphi}{\partial r} = -\frac{1}{r} \left[\sum_\varphi \kappa^\varphi \left(-\frac{\rho_{p\varphi}}{R}\right) FFB_1(\varphi)\right] [\sum_\Omega \kappa^\Omega FFB_o(\Omega)] \quad (D.37)$$

$$-\frac{\Omega}{r} \frac{\partial H_\varphi}{\partial r} = \left[\sum_{\varphi\Omega} \left(\frac{\rho_{p\varphi} \kappa^\varphi \kappa^\Omega}{rR}\right) FFB_1(\varphi) FFB_o(\Omega)\right] \quad (D.38)$$

D.2.10 Term 10 of (3.45) (Z)

$$-\Omega \frac{\partial^2 H_\varphi}{\partial r^2} = -\left[-\sum_\varphi \kappa^\varphi \left(\frac{\rho_{p\varphi}}{R}\right)^2 FFB_o(\varphi) + \sum_\varphi \kappa^\varphi \left(\frac{\rho_{p\varphi}}{rR}\right) FFB_1(\varphi)\right] [\sum_\Omega \kappa^\Omega FFB_o(\Omega)] \quad (D.39)$$

$$-\Omega \frac{\partial^2 H_\varphi}{\partial r^2} = \left[\sum_{\varphi\Omega} \left(\frac{\rho_{p\varphi}^2 \kappa^\varphi \kappa^\Omega}{R^2}\right) FFB_o(\varphi) FFB_o(\Omega) + \sum_{\varphi\Omega} \left(-\frac{\rho_{p\varphi} \kappa^\varphi \kappa^\Omega}{rR}\right) FFB_1(\varphi) FFB_o(\Omega)\right] \quad (D.40)$$

D.2.11 Term 11 of (3.45) (Z)

$$-\frac{\Omega}{r^2} \frac{\partial H_r}{\partial \varphi} = -\frac{1}{r^2} [\sum_r \kappa^r (jq_r) FFB_o(r)] [\sum_\Omega \kappa^\Omega FFB_o(\Omega)] \quad (D.41)$$

$$-\frac{\Omega}{r^2} \frac{\partial H_r}{\partial \varphi} = \left[\sum_{r\Omega} \left(-\frac{jq_r \kappa^r \kappa^\Omega}{r^2}\right) FFB_o(r) FFB_o(\Omega)\right] \quad (D.42)$$

D.2.12 Term 12 of (3.45) (Z)

$$\frac{\Omega}{r} \frac{\partial^2 H_r}{\partial r \partial \varphi} = \frac{1}{r} \left[\sum_r \kappa^r \left(-\frac{\rho p_r}{R} \right) (j q_r) FFB_1(r) \right] [\sum_{\Omega} \kappa^{\Omega} FFB_o(\Omega)] \quad (D.43)$$

$$\frac{\Omega}{r} \frac{\partial^2 H_r}{\partial r \partial \varphi} = \left[\sum_{r\Omega} \left(-\frac{j q_r \rho p_r \kappa^r \kappa^{\Omega}}{r R} \right) FFB_1(r) FFB_o(\Omega) \right] \quad (D.44)$$

D.3 Term by Term Development of the expression relating the azimuthal components of the master equation (2.13)

D.3.1 Term 1 of (3.47) (φ)

$$\frac{\partial H_r}{\partial z} \frac{\partial \Omega}{\partial r} = [\sum_r \kappa^r (j G_{n_r}) FFB_o(r)] \left[\sum_{\Omega} \kappa^{\Omega} \left(-\frac{\rho p_{\Omega}}{R} \right) FFB_1(\Omega) \right] \quad (D.45)$$

$$\frac{\partial H_r}{\partial z} \frac{\partial \Omega}{\partial r} = \left[\sum_{r\Omega} \left(-\frac{j G_{n_r} \rho p_{\Omega} \kappa^r \kappa^{\Omega}}{R} \right) FFB_o(r) FFB_1(\Omega) \right] \quad (D.46)$$

D.3.2 Term 2 of (3.47) (φ)

$$-\frac{\partial H_z}{\partial r} \frac{\partial \Omega}{\partial r} = - \left[\sum_z \kappa^z \left(-\frac{\rho p_z}{R} \right) FFB_1(z) \right] \left[\sum_{\Omega} \kappa^{\Omega} \left(-\frac{\rho p_{\Omega}}{R} \right) FFB_1(\Omega) \right] \quad (D.47)$$

$$-\frac{\partial H_z}{\partial r} \frac{\partial \Omega}{\partial r} = \left[\sum_{z\Omega} \left(-\frac{\rho p_z \rho p_{\Omega}}{R^2} \kappa^z \kappa^{\Omega} \right) FFB_1(z) FFB_1(\Omega) \right] \quad (D.48)$$

D.3.3 Term 3 of (3.47) (φ)

$$\frac{\Omega}{r} \frac{\partial H_r}{\partial z} = \frac{1}{r} [\sum_r \kappa^r (j G_{n_r}) FFB_o(r)] [\sum_{\Omega} \kappa^{\Omega} FFB_o(\Omega)] \quad (D.49)$$

$$\frac{\Omega}{r} \frac{\partial H_r}{\partial z} = \left[\sum_{r\Omega} \left(\frac{j G_{n_r} \kappa^r \kappa^{\Omega}}{r} \right) FFB_o(r) FFB_o(\Omega) \right] \quad (D.50)$$

D.3.4 Term 4 of (3.47) (φ)

$$-\frac{\Omega}{r} \frac{\partial H_z}{\partial r} = -\frac{1}{r} \left[\sum_z \kappa^z \left(-\frac{\rho p_z}{R} \right) FFB_1(z) \right] [\sum_{\Omega} \kappa^{\Omega} FFB_o(\Omega)] \quad (D.51)$$

$$-\frac{\Omega}{r} \frac{\partial H_z}{\partial r} = \left[\sum_{z\Omega} \left(\frac{\rho p_z \kappa^z \kappa^{\Omega}}{r R} \right) FFB_1(z) FFB_o(\Omega) \right] \quad (D.52)$$

D.3.5 Term 5 of (3.47) (φ)

$$\Omega \frac{\partial^2 H_r}{\partial r \partial z} = \left[\sum_r \kappa^r \left(-j G_{n_r} \frac{\rho p_r}{R} \right) FFB_1(r) \right] [\sum_{\Omega} \kappa^{\Omega} FFB_o(\Omega)] \quad (D.53)$$

$$\Omega \frac{\partial^2 H_r}{\partial r \partial z} = \left[\sum_{r\Omega} \left(-\frac{j G_{nr} \rho p_r \kappa^r \kappa^\Omega}{R} \right) FFB_1(r) FFB_o(\Omega) \right] \quad (D.54)$$

D.3.6 Term 6 of (3.47) (φ)

$$-\Omega \frac{\partial^2 H_z}{\partial r^2} = - \left[-\sum_z \kappa^z \left(\frac{\rho p_z}{R} \right)^2 FFB_o(z) + \sum_z \kappa^z \left(\frac{\rho p_z}{rR} \right) FFB_1(z) \right] [\sum_\Omega \kappa^\Omega FFB_o(\Omega)] \quad (D.55)$$

$$-\Omega \frac{\partial^2 H_z}{\partial r^2} = \left[\sum_{z\Omega} \kappa^z \kappa^\Omega \left(\frac{\rho p_z}{R} \right)^2 FFB_o(z) FFB_o(\Omega) + \sum_{z\Omega} \kappa^z \kappa^\Omega \left(-\frac{\rho p_z}{rR} \right) FFB_1(z) FFB_o(\Omega) \right] \quad (D.56)$$

D.3.7 Term 7 of (3.47) (r)

$$-\frac{1}{r^2} \frac{\partial H_z}{\partial \varphi} \frac{\partial \Omega}{\partial \varphi} = -\frac{1}{r^2} [\sum_z \kappa^z (jq_z) FFB_o(z)] [\sum_\Omega \kappa^\Omega (jq_\Omega) FFB_o(\Omega)] \quad (D.57)$$

$$-\frac{1}{r^2} \frac{\partial H_z}{\partial \varphi} \frac{\partial \Omega}{\partial \varphi} = \left[\sum_{z\Omega} \left(\frac{q_z q_\Omega \kappa^z \kappa^\Omega}{r^2} \right) FFB_o(z) FFB_o(\Omega) \right] \quad (D.58)$$

D.3.8 Term 8 of (3.47) (r)

$$\frac{1}{r} \frac{\partial H_\varphi}{\partial z} \frac{\partial \Omega}{\partial \varphi} = \frac{1}{r} \left[\sum_\varphi \kappa^\varphi \left(j G_{n\varphi} \right) FFB_o(\varphi) \right] [\sum_\Omega \kappa^\Omega (jq_\Omega) FFB_o(\Omega)] \quad (D.59)$$

$$\frac{1}{r} \frac{\partial H_\varphi}{\partial z} \frac{\partial \Omega}{\partial \varphi} = \left[\sum_{\varphi\Omega} \left(-\frac{G_{n\varphi} q_\Omega \kappa^\varphi \kappa^\Omega}{r} \right) FFB_o(\varphi) FFB_o(\Omega) \right] \quad (D.60)$$

D.3.9 Term 9 of (3.47) (r)

$$-\frac{\Omega}{r^2} \frac{\partial^2 H_z}{\partial \varphi^2} = -\frac{1}{r^2} [\sum_z \kappa^z (-q_z)^2 FFB_o(z)] [\sum_\Omega \kappa^\Omega FFB_o(\Omega)] \quad (D.61)$$

$$-\frac{\Omega}{r^2} \frac{\partial^2 H_z}{\partial \varphi^2} = \left[\sum_{z\Omega} \left(\frac{q_z^2 \kappa^z \kappa^\Omega}{r^2} \right) FFB_o(z) FFB_o(\Omega) \right] \quad (D.62)$$

D.3.10 Term 10 of (3.47) (r)

$$\frac{\Omega}{r} \frac{\partial^2 H_\varphi}{\partial \varphi \partial z} = \frac{1}{r} \left[\sum_\varphi \kappa^\varphi \left(jq_\varphi j G_{n\varphi} \right) FFB_o(\varphi) \right] [\sum_\Omega \kappa^\Omega FFB_o(\Omega)] \quad (D.63)$$

$$\frac{\Omega}{r} \frac{\partial^2 H_\varphi}{\partial \varphi \partial z} = \left[\sum_{\varphi\Omega} \left(-\frac{q_\varphi G_{n\varphi} \kappa^\varphi \kappa^\Omega}{r} \right) FFB_o(\varphi) FFB_o(\Omega) \right] \quad (D.64)$$

Appendix E (S, T, U, V) integrals

The eigen-matrix in (3.49) has three row blocks and three column blocks. The order of a block is equal to the number of basis functions used to decompose the inverse of relative permittivity and express the field components. An individual element of a block can be generated using the expressions given below. It is important to link the matrix element location in a block to the correct expansion coefficient in the column matrix.

Within the summations are contained integrals of three Bessel function written as $(S_1, S_o, S_{-1}, T_1, T_o, U_o, U_1, V_1)$ which are expressed over the normalized radial range,

$$\left(\zeta = \frac{r}{R} \right):$$

$$S_1 = \int_0^1 J_o(\rho_i \zeta) J_o(\rho_\Omega \zeta) J_o(\rho_i^* \zeta) \zeta d\zeta \quad (\text{E.10})$$

$$S_o = \int_0^1 J_o(\rho_i \zeta) J_o(\rho_\Omega \zeta) J_o(\rho_i^* \zeta) d\zeta \quad (\text{E.11})$$

$$S_{-1} = \int_0^1 J_o(\rho_i \zeta) J_o(\rho_\Omega \zeta) J_o(\rho_i^* \zeta) \frac{1}{\zeta} d\zeta \quad (\text{E.12})$$

$$T_1 = \int_0^1 J_1(\rho_i \zeta) J_o(\rho_\Omega \zeta) J_o(\rho_i^* \zeta) \zeta d\zeta \quad (\text{E.13})$$

$$T_o = \int_0^1 J_1(\rho_i \zeta) J_o(\rho_\Omega \zeta) J_o(\rho_i^* \zeta) d\zeta \quad (\text{E.14})$$

$$U_o = \int_0^1 J_0(\rho_i \zeta) J_1(\rho_\Omega \zeta) J_o(\rho_i^* \zeta) d\zeta \quad (\text{E.15})$$

$$U_1 = \int_0^1 J_0(\rho_i \zeta) J_1(\rho_\Omega \zeta) J_o(\rho_i^* \zeta) \zeta d\zeta \quad (\text{E.16})$$

$$V_1 = \int_0^1 J_1(\rho_i \zeta) J_1(\rho_\Omega \zeta) J_o(\rho_i^* \zeta) d\zeta \quad (\text{E.17})$$

These are calculated once, tabulated and stored as look up tables for generating the matrix elements of all cylindrically symmetric dielectric profiles examined using the Fourier-Bessel technique presented here.

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